A Clifford perspective
on
Klein's Geometry

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ABSTRACT

Klein's Erlangen Program provided an organizing principle for geometry based on the notion of group of transformations and the study of its invariants. It allowed, in particular, to think of projective geometry as a unifying framework for affine, metric and hyperbolic geometries (or, in Cayley's motto, "Projective geometry is all geometry").

The impact of these ideas on the teacher's training was developed in detail one century ago in the volume "Geometrie", the second of "Elementarmathematik vom höheren Standpunk aus".

In the talk we will present, after a brief outline of the main features of the book and of how it can account for "newer" geometries, such as the special theory of relativity, a proposal of guidelines for its updating using "geometric algebra" (first introduced by Clifford, as a generalization of Grassmann's algebra and Hamilton's quaternions, it has developed into a unified basis for geometry and its applications, not only across pure mathematics, but also in physics, engineering and computation).
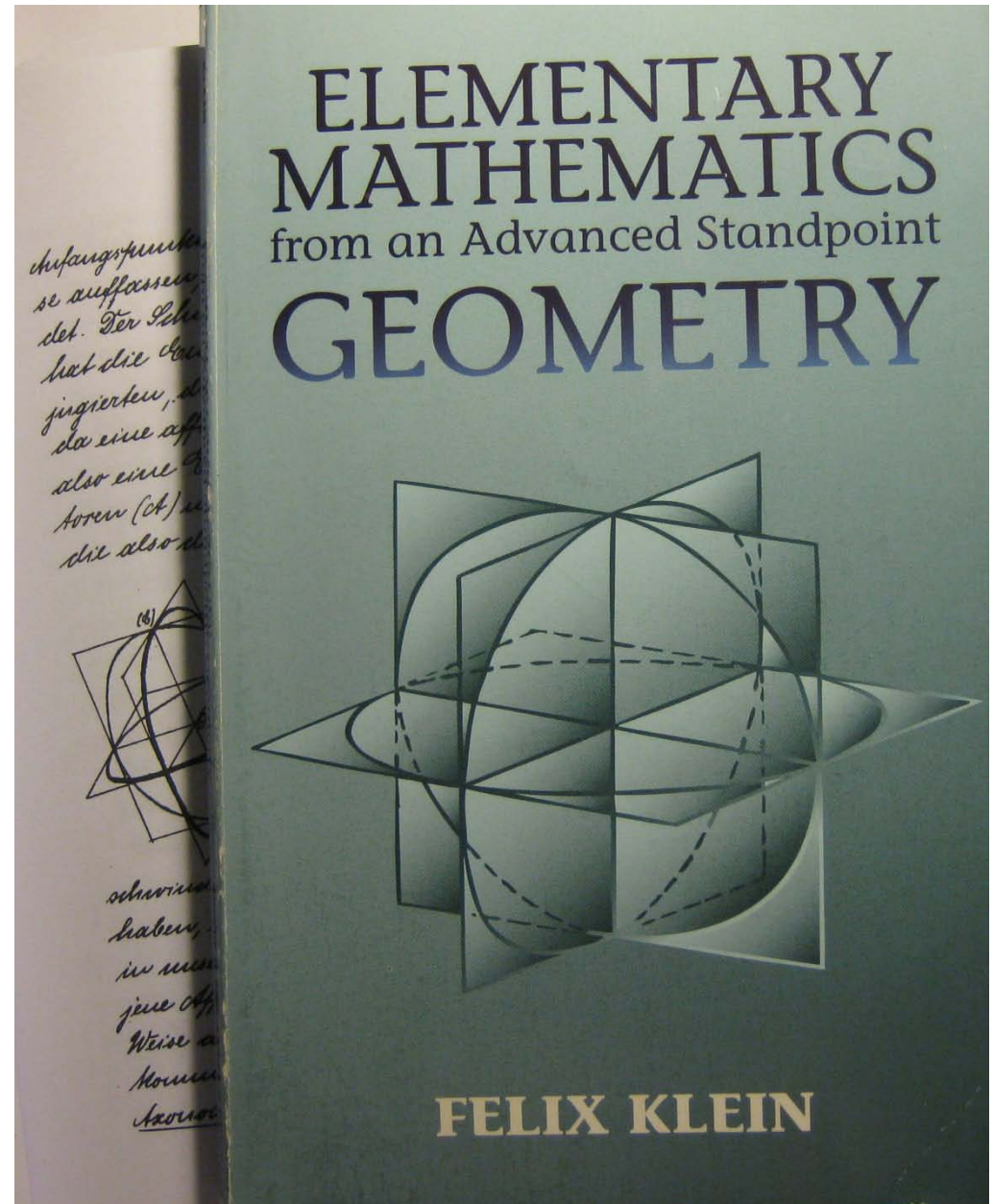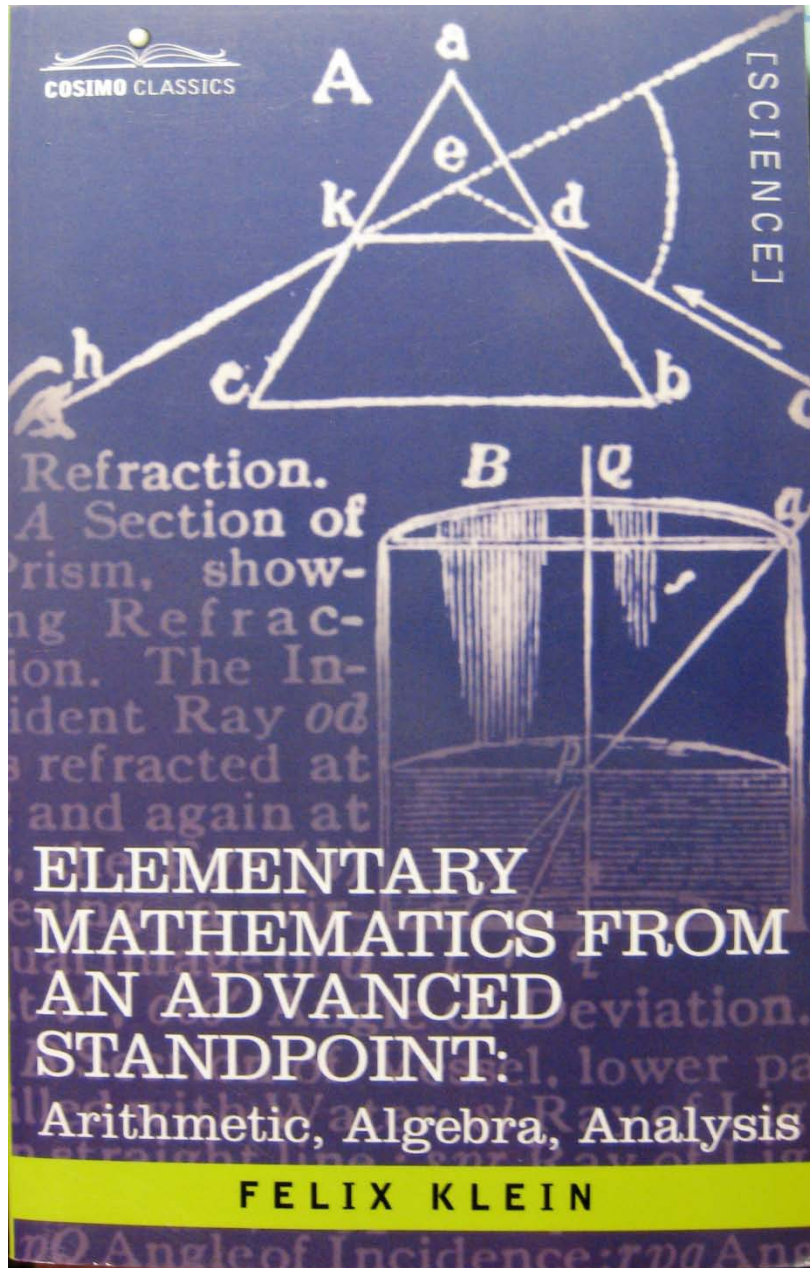
Main Points

- Context and motivations
- Klein's *Geometry*
- Geometry after Klein
- Desiderata for enlightened paradigm
- The potential of Geometric algebra
- Ending remarks

CONTEXT AND MOTIVATIONS

➢ Teaching

- *Geometry* (3rd Semester)

- *Mathematical models of physics* (6th Semester)

- Preparing a program for

  Multilinear Algebra and Projective Geometry

  (3rd Semester —after *Linear Algebra*, in 1st Semester, and

  Affine and Euclidean Geometry in 2nd Semester)

➢ Life and work of Sir Michael Francis Atiyah

(exponent of geometric innovation: deep results and

connections between geometry, analysis, topology and

mathematical physics; also concerned with education)

➢ The "Funchal challenge"

(how best to meet the needs of prospective teachers of

mathematics —ambitious, worthwhile)

KLEIN´S *GEOMETRY*

ELEMENTARMATHEMATIK VOM HÖHEREN STANDPUNKTE AUS

AAA

- Concern with the different ways in which the problem of instruction can be presented to the mathematician.
- Subject matter of instruction: endeavor to put before the teacher, as well as the maturing student, from the viewpoint of modern science, but in a manner as simple, stimulating, and convincing as possible, both the content and the foundations of the topics of instruction, with due regard for the current methods of teaching.
- Mental spur, not a detailed handbook.
- Endeavor, as always, to ***combine geometric intuition with the precision of arithmetic formulas***, and … following historical developments of the various theories.

- Appreciation of the growing significances of applied mathematics for modern school instruction [physics and engineering]
- Emphasize *mutual connection between problems in the various fields*, a thing which is not brought out sufficiently in the usual lecture course, and the relation of these problems to those of school mathematics.
- Function concept at the very center of instruction … with constant use of graphical method.
- Strong development of space perception, […] will always be a prime consideration.

GEOMETRY

Comprehensive view of the field of geometry, of such a range as I should wish every teacher in a high school to have; the discussions about geometric *instruction* were pushed into the background and were placed in connected form at the end.

Unified presentation.

[Yaglom, in *F. Klein and S. Lie*] The basic principles expounded by Klein were: more of the graphical element in teaching, greater attention to the functional viewpoint in algebra and analysis, and application of geometric transformations in the teaching of geometry.

He called for the elimination of the "China wall" separating different mathematical subjects, for taking into account the needs of related fields in mathematics courses, and for decreasing the gap between mathematical education and contemporary science.

CONTENTS

Part One: The simplest geometric manifolds

Part Two: Geometric transformations

Part Three: Systematic discussion of geometry and its foundations

Part One: The simplest geometric manifolds

1. Line-segment, area, volume, as relative magnitudes
2. The Grassmann determinant principle for the plane
3. The Grassmann principle for space

   [*system of magnitudes* defines new geometric entity if *its expression is the same in all coordinate systems*, i.e. $x_1 y_1 + x_2 y_2 + x_3 y_3$ or
   $(x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$]

4. Classification of elementary configurations of space according to their behavior under transformations of rectangular coordinates.

   [*geometric property* ↔ *invariance under change of coordinates*]

5. Derivative manifolds

Interesting points: discussion of *Amsler's Polar Planimeter*; [one-sided] surfaces with no area/volume; a [coordinate dependent] version of Grassmann's approach to extended geometrical quantities; applications to statics (geometry of null systems, for example).

Table on page 46

	Translation	Rotation	Inversion	Rescaling
Free Vector	X	$a_1 X + b_1 Y + c_1 Z$	$-X$	λX
Couple	L	$a_1 L + b_1 M + c_1 N$	L	$\lambda^2 L$
Free Plane-Magnitude	$\mathfrak{L}$	$a_1 \mathfrak{L} + b_1 \mathfrak{M} + c_1 \mathfrak{N}$	$\mathfrak{L}$	$\lambda^2 \mathfrak{L}$

(X, Y, Z) polar vector, (L, M, N) couple, $(\mathfrak{L}, \mathfrak{M}, \mathfrak{N})$ free plane magnitude (axial vector).

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \text{ orthogonal matrix.}$$

The terminology which is usual today in vector calculation comes historically [...] from **Hamilton**'s quaternion calculus and from **Grassmann**'s theory of extension. The developments of Grassmann were hard to read and remained unknown to German physicists; for a long time they formed a sort of esoteric doctrine for small mathematical groups. The ideas of Hamilton [...] made their way into English physics, mainly through **Maxwell**. [...] in his opinion it was desirable [...] to avoid the introduction of coordinates and to draw attention instantly to a point in space instead of to its three coordinates, and to the direction and magnitude of force rather than to its three components. That which today is called vector calculus of the physicist is derived from the work of [...] **Heaviside** and [...] **Gibbs**. Although [...] Hamiltonian at the start, they both took over Grassmann's ideas into their calculus. [...] The first book that introduced the vector calculus into the circle of German physicists, and that after the manner of Heaviside, was A. Föppl's *Einführung in die Maxwell'sche Theorie*, which appered in 1894. [pp. 52 & 53]

- Unnatural way [through coordinates] of understanding Grassmann's principle. Certain ambivalence between "change of coordinates" and "geometric transformations".

- Lack of linear algebra does not favor proving formulas for area such as

$$A(e_1, e_2)^2 = \begin{vmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{vmatrix} = \det \mathbf{e}^T \cdot \mathbf{e} = |e_1 \wedge e_2|^2,$$

  and similarly for volume, $V(e_1, e_2, e_3)$ (generalizing $|e|^2 = e \cdot e$).

- No reference to the seminal paper of W. K. Clifford (Exeter, 1845 – Funchal, 1879), *Applications of Grassmann's Extensive Algebra*: Amer. J. Math. 1, No. 4 (1978), pp. 350-358. Under the name of *geometric algebra*, Clifford proposes a synthesis of Grassmann and Hamilton systems.

- Gibbs and Heaviside focused on the vector algebra of Grassmann, but devoted little attention to the "extended magnitudes".

Part Two: Geometric transformations

1. Affine transformations

     [relation of affine transformations to parallel projection; theory of
       conjugate diameters for ellipses; applications to drawing (cf. p. 83)]

2. Projective transformations

     [relation to affine transformations; cross-ratio; only one conic]

3. Higher point transformations

     [inversions and Peaucellier inversor; map projections; topology]

4. Transformations with change of space element

     [duality; contact transformations; applications to gear design]

5. Theory of the imaginary

     [a circle is a conic through the cyclic points; geometric involutions]

Part Three: Systematic discussion of geometry and its foundations

1. The systematic discussion
   - Affine group = projective transformations that leave invariant the improper line (case of plane) or plane (case of space);
   - Euclidean group = affine transformations that leave invariant the cyclic points (or the imaginary circle at infinity);
   - The role of invariants and methods for finding them (all);
2. Foundations of geometry
   - Many axiomatic systems are possible;
   - Own system of axioms emphasizing transformations;
   - Cayley-Klein model for hyperbolic geometry (pp. 180-184):

$$r = k \operatorname{acos} \frac{\xi_1 \xi_2 + \eta_1 \eta_2 - \tau_1 \tau_2}{\sqrt{\xi_1^2 + \eta_1^2 - \tau_1^2}\sqrt{\xi_2^2 + \eta_2^2 - \tau_2^2}}$$

(k pure imaginary, $[\xi_1, \eta_1, \tau_1]$ and $[\xi_2, \eta_2, \tau_2]$ two points in homogeneous coordinates).

GEOMETRY AFTER KLEIN

(MOSTLY IN RELATION TO TOPOLOGY, ANALYSIS, MATHEMATICAL PHYSICS, …)

H. Weyl: Gravitation und Elektrizität (*Sitzber. Preuss. Akad. Wiss.*, 1918)

O. Klein: Quantentheorie und fünfdimensionale Relativitätstheorie
(*Z. Phys*, 1926)

P. M. Dirac: The quantum theory of the electron (*Proc. Royal Soc.*, 1928)

R. Brauer, H. Weyl: Spinors in n-dimensions (*AJM*, 1935)

E. Cartan: *The theory of spinors* (Hermann, 1937; Dover, 1981)

N. Steenrod: *The topology of fiber bundles* (Princeton Univ. Press, 1951)

R. Thom: Quelques propriétés globales des variétés différentiables
(L'Enseignement Math., 1954)

E. Artin: *Geometric Algebra* (Interscience, 1957)

M. F. Atiyah, F. Hirzebruch: Riemann-Roch theorems for differentiable manifolds (*BAMS*, 1959)

J. Milnor: *Morse theory* (Princeton Univ. Press, 1963)

M. F. Atiyah, I. Singer: The index of elliptic operators (*BAMS*, 1963)

M. F. Atiyah, R. Bott, A. Shapiro: Clifford modules (*Topology*, 1964)

M. F. Atiyah: *Geometry of the Yang-Mills fields* (1979)

M. Freedman: The topology of 4-manifolds (J. Diff. Geo., 1982)

R. Penrose, W. Rindler: *Spinors and space-time, I & II* (CUP, 1984 & 1986)

B. Lawson, M-L. Michelson: *Spin Geometry* (Princeton Univ. Press, 1989)

M. F. Atiyah: *The geometry and physics of knots* (1990)

M. F. Atiyah: The Dirac equation and geometry (In *Paul Dirac: the man and his* work, CUP, 1996)

DESIDERATA FOR AN ENLIGHTENED PARADIGM

- To dissolve dichotomy between synthetic and analytic methods.
- Algebraic language should allow, favor and facilitate coordinate-free representations and computations of objects and transformations.
- The deep algebro-geometric nature of complex numbers and quaternions should emerge early in a natural way.
- Linear algebra, vector calculus, differential forms and spinors should be naturally integrated in the paradigm. In particular, the geometric basis of concepts such as polar and axial vectors should be easily clarified.
- The system should enhance and promote the connection of geometry with the rest of mathematics, and with physics (mechanics, electromagnetism, relativity, quantum theory) and engineering.
- The paradigm should facilitate discussion of the strengths and weaknesses of past and present trends. In particular, should throw light into how to proceed with respect to elementary instruction.

THE POTENTIAL OF GEOMETRIC ALGEBRA (GA)

The honing of GA (in the full sense of Grassmann and Clifford) as a viable basis for a paradigm shift has been championed in the last 40 years by David **Hestenes**, a physicist/mathematician at Arizona State University, with important help and insights from others.

On the occasion of his being awarded the Oersted Medal (2002), he wrote three very interesting papers that give an honest perspective of the main developments and major applications, the philosophy that inspired them, and some views for the future:

Reforming the Mathematical Language of Physics (Oersted Medal Lecture 2002, 43 pages)

Spacetime Physics with Geometric Algebra (*Amer. J. Physics*, 2003)

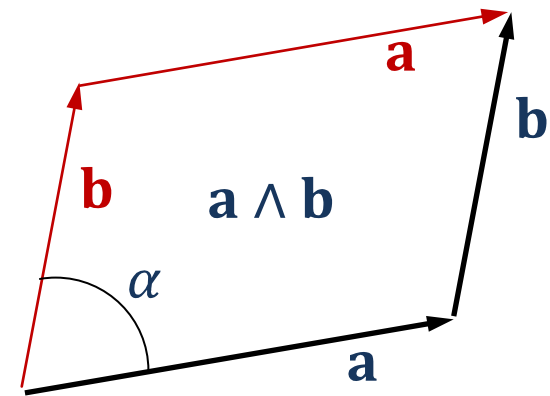Gauge theory gravity with Geometric Calculus (*Found. Phys*, 2005)

"Spacetime algebra (STA) simplifies, extends and integrates the mathematical methods of classical, relativistic and quantum physics while elucidating geometric structure of the theory".
"The entire physics curriculum can be unified and simplified by adopting STA as the standard mathematical language".

Glimpses into the essence of GA

If **a** and **b** are vectors (say in the plane or space), its *inner product* is $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \alpha$, where $\alpha \in [0, \pi]$ is the angle $\widehat{\mathbf{a}, \mathbf{b}}$.

The *outer* or *exterior product* of Grassmann, the *bivector* $\mathbf{a} \wedge \mathbf{b}$**,** represents the oriented parallelogram defined by **a** and **b**. It is *bilinear, associative,* and *vanishes if **a** and **b** are parallel.* In particular $\mathbf{b} \wedge \mathbf{a} = -\mathbf{a} \wedge \mathbf{b}.$

Clifford defined the *geometric product*, **ab**, by

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \text{ (Yes, a scalar plus a bivector!)}$$

It is *bilinear, associative,* and

$$\mathbf{ab} = \begin{cases} \mathbf{a} \cdot \mathbf{b} = \mathbf{ba} & \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are parallel} \\ \mathbf{a} \wedge \mathbf{b} = -\mathbf{ba} & \text{if } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \end{cases}$$

In particular, $\mathbf{a}^2 = \mathbf{aa} = |\mathbf{a}|^2$, and so $\mathbf{a}^{-1} = \mathbf{a}/|\mathbf{a}|^2$ (Yes, non-zero vectors are invertible!). Moreover, $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$ and $\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba})$
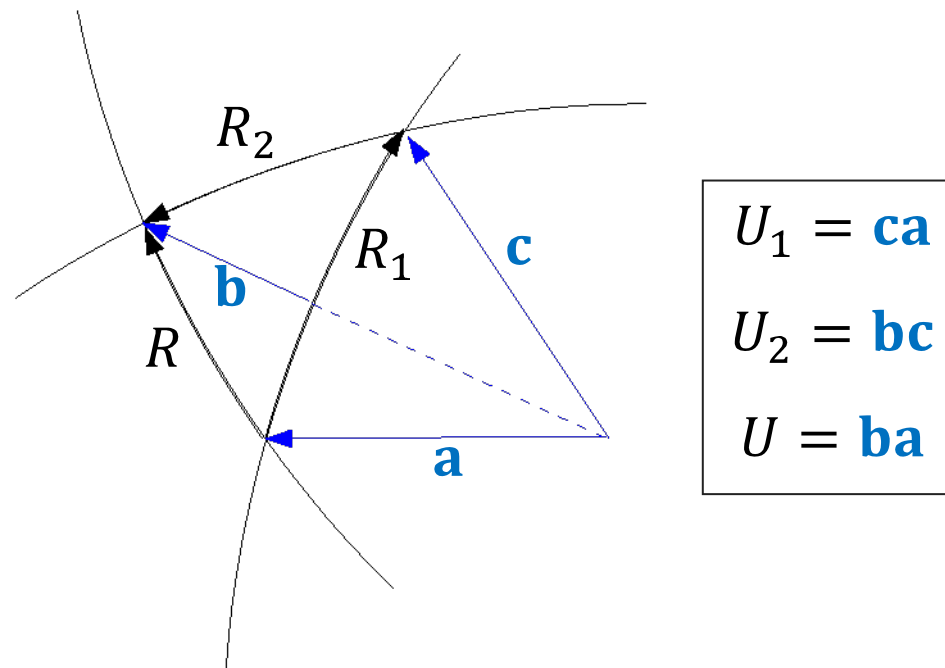
Example. In the plane, if $\mathbf{u}_1, \mathbf{u}_2$ is an orthonormal basis, the elements of the geometric algebra have the form $x = \alpha + \lambda\mathbf{u}_1 + \mu\mathbf{u}_2 + \beta i$, where $i = \mathbf{u}_2\mathbf{u}_1 = \mathbf{u}_2 \wedge \mathbf{u}_1$. This bivector, which is often called the *pseudoscalar unit,* satisfies $i^2 = -1$, and rotates vectors by $\pi/2$: $\mathbf{u}_2 = i\mathbf{u}_1$ and $-\mathbf{u}_1 = i\mathbf{u}_2$. So we may write $x = (\alpha + \beta i) + (\lambda + \mu i)\mathbf{u}_1$. Thus the geometric algebra of the Euclidean plane contains the complex numbers (as $\{\alpha + \beta i | \alpha, \beta \in \mathbb{R}\}$). Note: $\mathbf{ab} = |\mathbf{a}||\mathbf{b}|(\cos\alpha + i\sin\alpha) = |\mathbf{a}||\mathbf{b}|e^{i\alpha}$.

Proposition. If $\mathbf{a}$ is a unit vector, the map $\mathbf{x} \mapsto -\mathbf{a}\mathbf{x}\mathbf{a}$ is the symmetry with respect to $\mathbf{a}^\perp$ (a line in the plane or a plane in space).

Proof. It is linear, $\mathbf{a} \mapsto -\mathbf{a}\mathbf{a}\mathbf{a} = -\mathbf{a}$, and if $\mathbf{x} \in \mathbf{a}^\perp$, $\mathbf{x} \mapsto \mathbf{x}\mathbf{a}\mathbf{a} = \mathbf{x}$.

Corollary. If $\mathbf{a}$ and $\mathbf{b}$ are two unit vectors with $\alpha = \widehat{\mathbf{a}, \mathbf{b}}$, and we let $R = \mathbf{b}\mathbf{a}$, then the map $\mathbf{x} \mapsto R\mathbf{x}R^t$, where $R^t = \mathbf{a}\mathbf{b}$, is a rotation of amplitude 2α (we say that R is the *rotor* defined by $\mathbf{a}$ and $\mathbf{b}$).

Graphical representation of rotations in Euclidean 3-space and interpretation of the composition of rotations $R = R_2 R_1$:

Example. In the Euclidean 3-space, if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is an orthonormal basis, a basis of the geometric algebra is

1	$\{\mathbf{u}_i\}$	$\{\mathbf{u}_i\mathbf{u}_j = \mathbf{u}_i \wedge \mathbf{u}_j\}$	$I = \mathbf{u}_1\mathbf{u}_2\mathbf{u}_3$
1 scalar	3 vectors	3 bivectors	1 trivector

Note that $I^2 = -1$ and $\mathbf{u}_i\mathbf{u}_j = I\mathbf{u}_k$ if $\{i, j, k\}$ is a cyclic permutation of $\{1,2,3\}$. So I gives an isomorphism between the space of vectors (polar vectors) and the space of bivectors (axial vectors). Setting

$$i = I\mathbf{u}_1, j = I\mathbf{u}_2, k = I\mathbf{u}_3 ,$$

we see that i, j, k are rotors that rotate space with respect to the axes $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ by a straight angle (axial symmetries), and that

$$i^2 = j^2 = k^2 = ijk = -1.$$

Thus the even subalgebra $\{\alpha + \beta i + +\gamma j + \delta k \mid \alpha, \beta, \gamma, \delta \in \mathbb{R}\}$ of the geometric algebra of 3-space is the quaternion algebra of Hamilton.

[**Pauli** notations: σ_i instead of $\mathbf{u}_i$ (1926)]

Example. Consider an orthonormal basis $\{\gamma_\mu\}$ of a space-time (**Dirac** notations), that is, a real 4-space of signature $(1,3)$: $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. In this case the geometric algebra satisfies that $\gamma_0^2 = 1$ and $\gamma_k^2 = -1$ for $k = 1,2,3$. More generally,

$$2\,\gamma_\mu \cdot \gamma_\nu = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\,.$$

These are the defining relations of the Dirac matrix algebra, which, therefore, provide a *representation* of the geometric algebra of the spacetime (STA algebra). Letting $I = \gamma_0\gamma_1\gamma_2\gamma_3$ (unit pseudo-scalar), and $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$, it is an exercise to check that $I^2 = -1$, that

$$2\,\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k + \boldsymbol{\sigma}_k\boldsymbol{\sigma}_j = 2\delta_{jk}$$

(the geometric algebra of Euclidean 3-space, in Pauli's notation), and that $I = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$. Moreover, $I\gamma_\mu = -\gamma_\mu I$ (I anticommutes with all vectors).

We cannot go further here, but today Klein would most certainly explain the GPS systems as an application of this example.

For an interesting and novel application of this algebra, see the following paper by A. N. Lasenby and C. J. L. Doran, of the Cavendish Laboratory:

Geometric Algebra, Dirac Wavefunctions and Black Holes

In P.G. Bergmann and V. de Sabbata (eds.): *Advances in the Interplay Between Quantum and Gravity Physics*, 251-283, Kluwer (2002).

For a broad range of other developments and applications, see the additional references at the end.

ENDING REMARKS

- GA permits reasoning about scalars, segments, areas, volumes,… in a consistent, unified framework. Interesting geometric transformations are produced by the geometric product.

- GA allows to see multidimensional analogies and phenomena hidden to the direct intuition.

- Just as F. Klein does not mention Clifford, the descendants of Clifford seldom mention Klein (exception: Hestenes and Sobczyk, on page xi of *Clifford Algebra to Geometric Calculus*).

- The phrasing of Klein's *Geometry* in present day terms is quite standard, but the thorough presentation with GA does not seem to be available in full.

- GA yields, by its very nature, invariants of the relevant groups. The question of how to get all the invariants has to be translated from Klein's coordinate approach to a coordinate-free formulation.

- Advantages of GA (after Hestenes):

1. Coordinate-free formulation and computation.

2. Simple algebraic composition.

3. Geometric depiction of rotors as directed arcs.

4. Rotor products depicted as addition of directed arcs.

5. Integration of rotations and reflections in a single method.

6. Efficient parameterizations.

7. Smooth articulation with matrix methods.

8. Rotational kinematics without matrices.

9. The approach generalizes directly to Lorentz transformations.

"The picture of mathematics that emerges from Klein's book does not so much resemble the unified perspective that he had achieved in his own mind as a precarious journey from bridge to bridge over the cracks and crevasses of mathematical thought. For he was limited in his ability to express his coceptions by the heterogeneous mathematical formalisms of his age. To adequately represent Klein's vision of mathematics as a seamless whole and make his hard-earned insights common property of the mathematical community, the diverse symbolic systems of mathematics must be modified, coordinated and ultimately united in a single mathematical language."

Hestenes & Sobczyk,

Clifford Algebra and Geometric Calculus (p. xi)

To finish, I would like to point out explicitly that I am not saying whether elementary geometric algebra has to be taught at the secondary level, and if it had to, I am not saying how (all this ought to be discussed in detail by taking into account the more relevant points of view). But I am certainly asserting that it would be positive, in the spirit of "elementary mathematics from an advanced viewpoint", that mathematics teachers had a reasonable knowledge of its basic ideas and applications.

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