WIT: A symbolic system for computations in IT and EG (programmed in pure Python)

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Index II: Basic intersection theory

- Preliminairies
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- Examples of intersection rings
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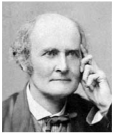
Preliminaries

Heroes, II. Conventions. Parameter spaces (moduli). Conditions.


Chasles


Grassmann


Cayley


Halphen


Pieri


C. Segre


Severi


Gambelli


E. Noether


Zariski


B. Segre


Todd


Samuel


Hirzebruch

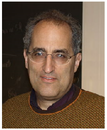

Murre


Mumford


Manin


Kleiman


Piene


Witten


Grayson


Colley


Aluffi


Miret


Götsche


Rosselló


Faber


Vakil

Michel **Chasles** (1793-1880), Hermann **Grassmann** (1809-1877), Arthur **Cayley** (1821-1895), George-Henri **Halphen** (1844-1889), Mario **Pieri** (1860-1913), Corrado **Segre** (1863-1924), Francesco **Severi** (1879-1961);

Giovanni **Giambelli** (1879-1935), Emmy **Noether** (1882-1935), Oscar **Zariski** (1899-1986), Beniamino **Segre** (1903-1977), John A. **Todd** (1908-1994), Pierre **Samuel** (1921-2009), Friedrich **Hirzebruch** (1927-2012);

Jakob **Murre** (1928-), David **Mumford** (1937-), Yuri **Manin** (1937-), Steven **Kleiman** (1942-), Ragni **Piene** (1947-), Edward **Witten** (1951-), Daniel R. **Grayson** (1952-);

Susan **Colley** (1959-), Paolo **Aluffi** (1960-) Josep M. **Miret** (1960-), Lothar **Götsche** (1961-), Francesc A. **Rosselló** (1961-), Carel **Faber** (1962-), Rahul **Pandharipande** (1969-), Ravi **Vakil** (1970-)

- *Prerequisites*. Basic concepts of algebraic geometry, say first two chapters of Hartshorne's book [8], henceforth **H77** (except sections 8 and 9 of chapter II). Shafarevich book [16] is another standard reference (chapters I and II, and § 1 of chapter 3). A good survey of the required notions can be found in appendix B of Fulton's book [5], henceforth **F98**.

- *References*. [Ch. 1][12] (Murre-Nagel-Peters-2013, *Lectures on the theory of pure moitives*). Note also the general references cited at the beginning, particularly **F98**. We will also use parts of [17] (*Using intersection theory*).

- *Ground field*: $\boldsymbol{k}$, an algebraically closed field. *Note*: Most concepts and results can be adapted when this hypothesis is not true.

- *Varieties*. $\boldsymbol{k}$-schemes of finite type and separated (most of the time, quasiprojective varieties, usually smooth and irreducible). The structural sheaf of a variety X will be denoted $\mathcal{O}_X$. Morphisms of varieties will (also) be called *maps*.

- A variety X whose points are in one-to-one correspondence with the set $\{F\}$ of figures F of some kind is a *parameter space* for those figures.

- *Projective space* $\mathbf{P} = \mathbf{P}(V)$. It is a parameter space for the set of linear subspaces of dimension 1 of V (a $\boldsymbol{k}$-vector space). Usually denoted simply by $\mathbf{P}^n$ when $\dim(V) = n+1$.

- *Grassman variety* $\mathrm{Gr}(k, \mathbf{P}) = \mathrm{Gr}(k+1, V)$. It is a parameter space for the set of linear varieties of dimension k in $\mathbf{P}$ (or of linear subspaces of dimension $k+1$ in V).

Instead of $\mathrm{Gr}(k, \mathbf{P}^n)$ we just write $\mathrm{Gr}(k, n)$. $\mathrm{Gr}(k, \mathbf{P})$ is the closed subvariety of $\mathbf{P}(\Lambda^{k+1} V)$ whose rational points have the form $[v_0 \wedge \cdots \wedge v_k]$, where $v_0, \ldots, v_k \in V$ are linearly independent over $\boldsymbol{k}$ (*Plücker embedding*).

- *Flag varieties*. The variety of *complete flags* of $\mathbf{P} = \mathbf{P}(V)$, denoted $\mathbb{F} = \mathbb{F}(\mathbf{P})$, is the subvariety of

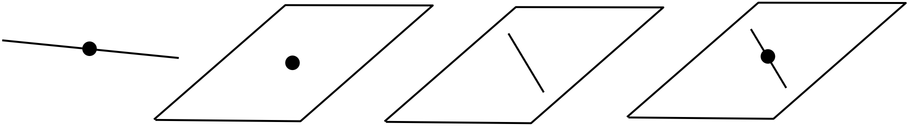
$$\mathrm{Gr}(0, \mathbf{P}) \times \mathrm{Gr}(1, \mathbf{P}) \times \cdots \times \mathrm{Gr}(n-1, \mathbf{P})$$

whose points are the n-tuples $L_0, \ldots, L_{n-1}$ such that $L_{i-1} \subset L_i$ for $i = 1, \ldots, n-1$.

If $0 \leqslant k_1 < \cdots < k_m \leqslant n-1$, the *variety of partial flags of type* $(k_1, ..., k_m)$, $\mathbb{F}_{k_1,\ldots,k_m}(\mathbf{P})$ or $\mathbb{F}_{k_1,\ldots,k_m}$, is the subvariety of

$$\mathrm{Gr}(k_1, \mathbf{P}) \times \mathrm{Gr}(k_2, \mathbf{P}) \times \cdots \times \mathrm{Gr}(k_m, \mathbf{P})$$

whose points are the m-tuples $M_1, \ldots, M_m$ such that $M_{i-1} \subset M_i$ for $i = 2, \ldots, m$.



$F_{0,1}$ $F_{0,2}$ $F_{1,2}$ $F_{0,1,2}$

Hypersurfaces. Hypersurfaces of degree d in $\mathbf{P}$ are parameterized, as point sets, by $\mathbf{P}(S^d V^*)$.

- $S^1 V^* = V^*$ and $\mathbf{P}^* = \mathbf{P}(V^*)$, the *dual projective* space of $\mathbf{P}$, parameterizes the hyperplanes of $\mathbf{P}$.

- Since by definition the *quadric* hypersurfaces are the degree 2 hypersurfaces, we see that quadric hypersurfaces are parameterized by $\mathbf{P}(S^2 V^*)$.

- If $\dim(V) = n+1$ then $S^d V^*$ has dimension $\binom{n+d}{d}$ and so the hypersurfaces of degree d in $\mathbf{P}^n$ form a projective space $\mathbf{P}^N$, where $N = \binom{n+d}{d} - 1$. In particular, quadric hypersurfaces in $\mathbf{P}^n$ form a projective space $\mathbf{P}^N$, $N = n(n+3)/2$.

Veronese embedding. The mapping $V^* \to S^d V^*$ such that $w \mapsto w^d$ induces $\mathbf{P}(V^*) \hookrightarrow \mathbf{P}(S^d(V^*))$ that transforms a given hyperplane H of $\mathbf{P}(V)$ (that is, a point of $\mathbf{P}(V^*)$) into the hypersurface of degree d in $\mathbf{P}(V)$ that consists of H repeated d times.

The correspondence through a parameterization between points x of a variety X and figures F in the set $\{F\}$ should be natural, in the folowing sense: the points x corresponding to figures F that satisfy some specific projective relation (*condition*) with a given figure G (*datum*) is a subvariety X_G of X.

- *Example*. The set of hypersurfaces of degree d that go through a point of $\mathbf{P}$ is a hyperplane of the projective space $\mathbf{P}^N$, $N = \binom{n+d}{d} - 1$, that parameterizes hypersurfaces of degree d of $\mathbf{P}$. Here the condition is *passing through a point* and the datum is the specified point.

In $\mathrm{Gr}(k, \mathbf{P})$, the points corresponding to linear varieties L of dimension k in $\mathbf{P}$ that meet a given linear variety G form a closed subvariety of $\mathrm{Gr}(k, \mathbf{P})$. Here the condition is the *incidence* of L and G and the datum is the given linear variety G.

Cycles

Vocabulary and examples. Pushforward. Intersection product. Transversality. Functorialities. Weil's diagonal formula

- An *algebraic cycle* on a variety X is a formal finite integral linear combination $Z = \sum n_i Z_i$ of irreducible subvarieties Z_i of X. If all the Z_i have the same codimension r we say that Z is a *codimension r cycle*. These cycles form an abelian group that is denoted $\mathcal{Z}^r X$, or $\mathcal{Z}_{d-r} X$ if we want to emphasize the dimension $d - r$ (for this to make sense, X has to be pure dimensional).

- Codimension 1 cycles are called (Weil) *divisors*. The group $\mathcal{Z}^1 X$ is also denoted by $\mathrm{Div}(X)$.

- Each hypersurface $F = 0$ of $\mathbf{P}^n$ defines a *positive* divisor. Its *irreducible components* and their *multiplicities* correspond to the irreducible factors and their multiplicities in the complete factorization of F.

- If X has pure dimension d, $\mathcal{Z}^d X = \mathcal{Z}_0 X$ is the group of 0-cycles. Its elements are finite integral linear combinations of points. There is a homomorphism $\deg : \mathcal{Z}_0 X \to \mathbf{Z}$ such that $\deg(P) = 1$ for each point P ($= [k(P) : k]$ when $\boldsymbol{k}$ is not algebraically closed).

- If Y is a subscheme of X with irreducible components Y_i of dimension d_i, it defines the cycle $[Y] = \sum_i n_i Y_i$, where $n_i = \mathrm{len}(\mathcal{O}_{Y,Y_i})$ (see **F98**, §1.5). If Y is irreducible and reduced, we also say that Y is a *prime* cycle.

- Even if the variety X_G defined on page 10 is a prime cycle for generic G, it may appear as a *composite* cycle for special G. Poncelet's argument for the number of lines meeting four lines in $\mathbf{P}^3$ (WIT-1, page 27), or the relation $l^2 = p + \pi$ we met there on page 29, provide examples.

- If $f : X \to Y$ is a morphism of $\boldsymbol{k}$-varieties, there is a natural homomorphism $f_* : \mathbb{Z}_k X \to \mathbb{Z}_k Y$. For an irreducible subvariety Z of X, $f_* Z = 0$ if $\dim f(Z) < \dim Z$ and otherwise $s[f(Z)]$, where $s = [k(Z) : k(f(Z))]$.

Assume X is smooth and let V, V' be subvarieties of codimensions r, r', respectively. Let Z be an irreducible component of $V \cap V'$. Then it codimension s satisfies $s \geqslant r + r'$ and Z is said to be a *proper component* of $V \cap V'$ if equality holds, $s = r + r'$. Let, in this case, $A = \mathcal{O}_{X,Z}$ be the local ring of X at Z, and $\mathcal{I}, \mathcal{I}'$ the ideals of V and V' in A. Then the *intersection multiplicity* $i_Z(V, V')$ is defined by the formula

$$i_Z(V, V') = \sum_j (-1)^j \mathrm{len}_A(\mathrm{Tor}_j^A(A/\mathcal{I}, A/\mathcal{I}')).$$

Finally, the *intersection product* $V \cdot V'$ is defined as $\sum_Z i_Z(V, V')$, where the sum runs over all proper components of $V \cap V'$.

This definition is due to Serre [15, p. 144] and coincides with earlier definitions due to Chevalley, van de Waerden, Weil and Samuel. In any case, this formula can be seen as the zeroth term, namely $\mathrm{len}_A((A/\mathcal{I}) \otimes (A/\mathcal{I}')) = \mathrm{len}_A(A/(\mathcal{I} + \mathcal{I}'))$, which sufficies in some cases, with 'corrections' expressed in terms of the higher Tor functors.

It is important to have criteria that guarantee that an intersection is *transversal*. One of the more useful is the following version of Kleiman's *transversality theorem* (see [9, Kleiman 1974]). Let $\mathcal{T}$ be an algebraic group and assume that it acts algebraically and **transitively** on a variety X. Assume that Y and Z are locally closed irreducible subvarieties of X. Then there exists a non-empty open set U of $\mathcal{T}$ such that the intersection $X \cap \tau(Y)$ is proper for $\tau \in U$, and *transversal* if $p = 0$, in which case all components in the intersection have multiplicity one (this is proven, for example, in Hartshorne [8, Hartshorne 1977, Theorem 10.5]). If $p > 0$, then U can be chosen in such a way that the multiplicities of all components of $X \cap \tau(Y)$ are equal, and this common value is a power of p.

Remark. If we have a condition, expressed as the family of subvarieties X_G, one for each datum G, then $\tau(X_G) = X_{\tau(G)}$ provided that $\mathcal{T}$ also acts on $\{G\}$ and that the condition is equivariant. In this case, $Y \cap \tau(X_G) = Y \cap X_{\tau(G)}$, and hence the *generic* condition $X_{\tau(G)}$ meets Y transversally.

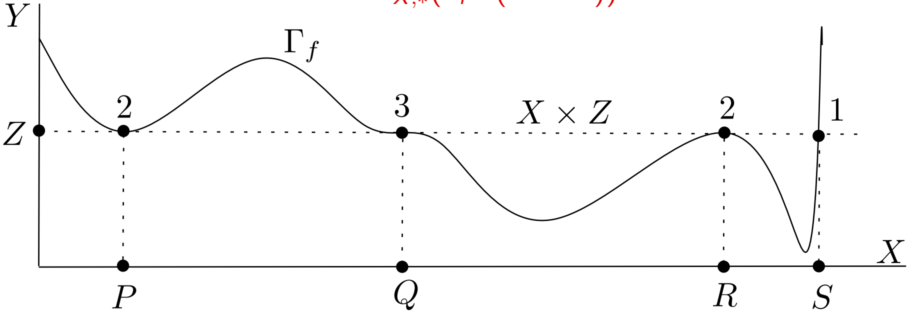
But let's not forget the transitivity condition. Let $X = \mathbf{P}^5$ is the space of conics in $\mathbf{P}^2$. Then the condition of being tangent to a conic G, expressed as X_G, is irreducible if G is non-degenerate, has three components if G is a pair of distinct lines, and is all X if G is a double line. Since the projective group acts transitively on the open set X^0 of non-degenerate conics, the intersection of X_G and the generic $X_{\tau(G)}$ is transversal on X^0, but not on X, because any X_G contains the Veronese surface V_2 of double lines. These ideas were at the core of Halphen's approach to enumerative geometry, with his distinction of *proper* solutions (those that move when we move the datum) from the improper ones, that are insensitive to that motion, like the double lines in the case conics. See, for example, [4, 3], for a thourough analysis in terms of the singularities of the X_G along V_2.

- There is a natural homomorphism $\mathcal{Z}^r X \times \mathcal{Z}^{r'} X' \to \mathcal{Z}^{r+r'}(X \times X')$.

- Given a morphism of smooth varieties $f : X \to Y$, there is a natural homomorphism $f^* : \mathcal{Z}^r Y \to \mathcal{Z}^r X$, which is called the *pullback of f*.

To define it, it suffices to define f^*Z, where Z is a codimension r subvariety of Y. In this case, the graph $\Gamma_f \subseteq X \times Y$ of f and $X \times Z$ intersect properly on $X \times Y$, and f^*Z is defined by the formula

$$f^*Z = \pi_{X,*}(\Gamma_f \cdot (X \times Z)).$$



$$f^*Z = 2P + 3Q + 2R + S$$

Let $\Delta : X \to X \times X$ be the *diagonal embedding* of X. Let V, W be subvarieties of X meeting properly. Then

$$V \cdot W = \Delta^*(V \times W).$$

Equivalence relations

Adequate relations. Rational equivalence. Computing Chow groups. Bézout's theorem. Byalinicki-Birula theorem. Axioms for intersection theory. Uniqueness for smooth quasi-porjective varieties. Serre's positivity conjecture. A conjecture of Diaz-Harris.

An equivalence relation $\sim$ on $\mathcal{Z}(X) = \oplus_r \mathcal{Z}^r X$ is called *adequate* if it is compatible with grading and addition and satisfies the following properties, which express compatibility with products, intersections and projections:

1. If $Z \sim 0$ on X, $X \times Y \sim 0$ in $\mathcal{Z}(X \times Y)$.

2. If $Z \sim 0$ and the intersection $Z \cap Z'$ is proper, then $Z \cdot Z' \sim 0$.

3. If $Z \sim 0$ on $X \times Y$, $\pi_{X,*} Z \sim 0$ on X.

In addition it has to satisfy the following *moving lemma*:

4. Given $Z, W_1, \ldots, W_m \in Z(X)$, there exists $Z' \sim Z$ such that the intersections $Z' \cap W_j$ are proper for all j.

For such a relation, we have the groups $\mathcal{Z}^j_\sim X = \{Z \in \mathcal{Z}^j X : Z \sim 0\}$, $\mathcal{C}^j_\sim X = \mathcal{Z}^j X / \mathcal{Z}^j_\sim X$, and $\mathcal{C}_\sim X = \oplus_j \mathcal{C}^j_\sim X$.

Theorem

1. $\mathcal{C}_\sim X$ is a commutative ring with the product induced from the intersection of cycles. The total class $1_X = [X]$ is its unit.

2. For any morphism $f : X \to Y$, f_* and f^* induce (well defined) group homomorphisms $f_* : \mathcal{C}_\sim X \to \mathcal{C}_\sim Y$ and $f^* : \mathcal{C}_\sim Y \to \mathcal{C}_\sim X$ and in fact *f^* is a ring homomorphism*.

Remark:

$$f^*(V \cdot W) = f^*\Delta_Y^*(V \times W) = \Delta_X^*(f^*V \times f^*W) = f^*V \cdot f^*W.$$

Now we will look at the most relevant adequate relations.

- Given an irreducible divisor Z of X, consider the ring $\mathcal{O} = \mathcal{O}_{X,Z}$. Since it has dimension 1, there is a function $\mathrm{ord}_Z : \mathcal{O} \to \mathbf{Z}$ given by $\mathrm{ord}_Z(f) = \mathrm{len}_{\mathcal{O}}(\mathcal{O}/(f))$. This function extends to the field of fractions $k(\mathcal{O}) = k(X)$ and hence we have $\mathrm{ord}_Z : k(X) \to \mathbf{Z}$.

- The divisor of a rational function $f \in k(X)$, $\mathrm{div}(f)$, is defined by $\mathrm{div}(f) = \sum_Z \mathrm{ord}_Z(f) Z$, where the sum is extended to all irreducible divisors Z of X.

- The group $\mathcal{Z}^j_{\mathrm{rat}}X$ is the group generated by the cicles $\mathrm{div}(f)$, where f is a non-zero rational function on an irreducible subvariety Y of codimension $j-1$.

- The *Chow groups* of X are $A^jX = \mathcal{Z}^j/\mathcal{Z}^j_{\mathrm{rat}}X = \mathcal{C}^j_{\mathrm{rat}}X$ (they are also denoted CH^jX). The graded ring $A(X)$ is the *Chow ring*, or *intersection ring*, of X.

- *Alternative definition of rational equivalence*. A cicle $Z \in \mathcal{Z}^j X$ is rationally equivalent to 0 if and only if there exists $W \in \mathcal{Z}^j(X \times \mathbf{P}^1)$ and $a, b \in \mathbf{P}^1$ such that $W(a) = 0$ and $W(b) = Z$, where $W(t) = \pi_{X,*}(W \cdot (X \times t))$.

- The rational equivalence of divisors is *linear equivalence*, and hence $A^1X = \mathrm{Pic}(X)$, the *Picard group* of X (the group of isomorphism classes of invertible sheaves on X with the operation induced by tensor product).

So the methods for calculating A^1 may sometimes be of use for the calculation of Pic groups.

Remark. On a smooth projective surface X, the intersection pairing $A^1(X) \times A^1(X) \to A^2(X)$, composed with the degree map $\deg : A^2(X) \to \mathbf{Z}$, yields the customary intersection pairing of $\mathrm{Pic}(X)$. For this, see [11] (Mumford 1966) and **H77**.

- $A(\mathbf{A}^n) = \mathbf{Z}[\mathbf{A}^n] = \mathbf{Z}\,1_{\mathbf{A}^n} \simeq \mathbf{Z}$. So $A^0(\mathbf{A}^n) = A_n(\mathbf{A}^n) = \mathbf{Z}$, generated by the class $[\mathbf{A}^n]$, is the only non-zero Chow group of $\mathbf{A}^n$.

More generally, if X is any smooth quasi-projective variety, then the map $\pi_X^* : A^*(X) \to A^*(X \times \mathbf{A}^n)$ is an isomorphism (follows easily from the proof of proposition 1.9 in **F98**).

- *A basic exact sequence*. If Y is a closed subvariety of a variety X and $U = X - Y$, then there is a short exact sequence

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \to 0$$

(see **F98**, Proposition 1.8). Here i is the inclusion of Y in X, so i is a proper map, and j is the inclusion of U in X.

- *The intersection ring of* $\mathbf{P}^n$. If $h \in A^1\mathbf{P}^n$ denotes the class of a hyperplane, then h^r is the class of any linear variety of dimension r, $A^r(\mathbf{P}^n) = \langle h^r \rangle_{\mathbf{Z}} \simeq \mathbf{Z}$ and $A(\mathbf{P}^n) = \mathbf{Z}[h]/(h^{n+1})$.

Note that the basic exact sequence, together with the intersection ring of $\mathbf{A}^n$, tell us that $A_n(\mathbf{P}^n) = A_n(\mathbf{A}^n) = \mathbf{Z}$ and $A_k(\mathbf{P}^{n-1}) = A_k(\mathbf{P}^n)$ for all $r < n$, and the claims follow easily by induction.

- *Degree*. If Z is dimension k cicle of $\mathbf{P}^n$, then its degree is the integer d such that $[Z] = dh^{n-k}$. Clearly, d is the degree of the intersection of Z with a general linear variety of codimension k.

Example. Let $\mathbf{P}^N$ be the projective space that parameterizes quadric hypersurfaces of $\mathbf{P}^n$ (so $N = n(n+3)/2$) and let W_n be the open set in $\mathbf{P}^N$ corresponding to the non-singular ones. Then
$\mathrm{Pic}(W_n) \simeq \mathbf{Z}/(n+1)$.

Indeed, $\mathbf{P}^N - W_n$ is the set of degenerate quadric hypersurfaces and so it is equal to the hypersurface Δ_n which is the zero locus of the determinant of the symmetric matrix representing a quadric hypersurface. Since $A^1(\mathbf{P}^N) = \mathrm{Pic}(\mathbf{P}^N) = \mathbf{Z}$, generated by the class h of a hyperplane, we see that $A^1(W_n)$ is the quotient of $\mathbf{Z}$ by the image of $A^0(\Delta_n)$ in $A^1(\mathbf{P}^N) = \mathbf{Z}$, which is $(n+1)$ because Δ_n has degree $n+1$ in $\mathbf{P}^N$.

Theorem. Let $V_1, \ldots, V_m$ be pure dimensional varieties of $\mathbf{P}^n$ of degrees $d_1, \ldots, d_m$. Assume that the intersection $V_1 \cap \cdots \cap V_m$ is proper. Then the degree of the cycle $V_1 \cdots V_m$ is the product $d = d_1 \cdots d_m$.

Indeed, if V_i has dimension k_i, then $[V_i] = d_i[L_{k_i}]$, where L_{k_i} is any linear variety of dimension k_i. Now the product of the right hand side is $d\,[L]$, where L is any linear space of dimension $k_1 + \cdots + k_m - n(m-1)$, while the product on the left hand side is $[V_1 \cdots V_m]$.

Note that if $V_1, \ldots, V_m$ are not assumed to intersect properly then it is still true that the product of classes $[V_1] \cdots [V_m]$ is $d[L]$ (same notations as above). At times this has originated some confusion. Let us illustrate this with a couple of examples.

Plane conics are parameterized by a $\mathbf{P}^5$ and it is easy to see that the conics that are tangent to a line L form a (rank 3) quadric hypersurface $Q_L \subset \mathbf{P}^5$. Thus we see that if $L_1, \ldots, L_5$ are lines then the degree of $\prod_{i=1}^5 [Q_{L_i}]$ is 32, while from elementary projective geometry we know that there is a unique smooth conic which is tangent to 5 lines if no three of them are concurrent. Here the point is that Q_L contains the Veronese surface of double lines and so the intersection $\cap_{i=1}^5 Q_{L_i}$ is far from proper.

The second example is about the number of conics that are tangent to 5 conics in general position in $\mathbf{P}^2$. The correct number is 3264 (for $p \neq 2$, determined by Chasles), but the first 'determination' was done by De Jonquières and Steiner through an incorrect application of Bézout's theorem that led to the number $6^5 = 7776$ (much as if they had concluded that the number of conics that are tangent to five lines is 32).

Actually it is not hard to see that the conics that are tangent to a given non-singular conic C form a sextix hypersurface H_C in the $\mathbf{P}^5$ that parameterizes conics. Therefore if $C_1, \ldots, C_5$ are 5 non-singular conics in general position in $\mathbf{P}^2$ then the product of the classes $[H_{C_i}]$ is $7776[c]$, where c is a point in $\mathbf{P}^5$. But the hypersurfaces H_{C_i} do not intersect properly on $\mathbf{P}^5$, for all of them contain the Veronese surface of double lines, and so the product of the classes $[H_{C_i}]$ does not give any information about the intersection of the H_{C_i}.

For the case of n divisors $D_1, \ldots, D_n$ in $\mathbf{P}^n$ whose intersection is finite, Bézout's theorem just says that the number of intersection points, each counted according to the corresponding intersection multiplicity, is the product of the degrees of the divisors.

Assume that X is a complete variety/$\mathbf{C}$ and that the multiplicative group $\mathcal{T} = \mathbf{C}^*$ acts on X in such a way that only finitely many points $x_1, \ldots, x_s$ are fixed. Let

$$X_i = \{x \in X \mid \lim_{t \to 0} tx = x_i\}\ . \tag{1}$$

Note that X_i contains x_i. Then the following holds (see [1, 2], Bialynicki-Birula 1973, 1976]):

a) X_i is $\mathcal{T}$ invariant, locally closed and isomorphic to an affine space $\mathbf{A}^{n_i}$.

b) X is the disjoint union of the X_i.

c) The classes $\xi_i \in A_{n_i}(X)$ of the closure of X_i in X form a free basis of $A_*(X)$.

Example. The relation

$$t[x_0, x_1, x_2, x_3] = [x_0, tx_1, t^2x_2, t^4x_3]$$

defines an action of G on $\mathbf{P}^3$. The fixed points are the coordinate points $P_0 = [1, 0, 0, 0]$, $P_1 = [0, 1, 0, 0]$, $P_2 = [0, 0, 1, 0]$ and $P_3 = [0, 0, 0, 1]$. The corresponding locally closed sets are $L_3 - L_2$, $L_2 - L_1$, $L_1 - L_0$ and L_0, where $L_0 = \{P_0\}$, $L_1 = P_0 \vee P_1$, $L_2 = P_0 \vee P_1 \vee P_2$ and $L_3 = \mathbf{P}^3$.

The action in question induces an action on $\mathrm{Gr}(1, 3)$ which is given by the following relation:

$$t[p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}] = [p_{01}, tp_{02}, t^3p_{03}, t^2p_{12}, t^4p_{13}, t^5p_{23}]\ ,$$

where the p_{ij} are the Plücker coordinates of lines. So there are 6 fixed points, the coordinate points of $\mathbf{P}^5$. The corresponding locally closed sets are the *Schubert cells* of $\mathrm{Gr}(1, 3)$.

Further reading: Rosselló [1986, 88, 90], Rosselló–Xambó [1987, 91].

A few properties of the intersection product are enough to make it unique. To see how this happens, and also as a summary of the properties we have seen so far, let us introduce the notion of intersection theory (see Grothendieck [1958 b] or the appendix A in **H77**).

Let $\mathcal{V}$ be a given class of varieties which is closed under products. Assume that we have a pairing

$$A^r(X) \times A^s(X) \to A^{r+s}(X)\ , \quad (\alpha, \beta) \mapsto \alpha \cdot \beta \qquad (*)$$

for each $X \in \mathcal{V}$ and for all integers r and s. We say that $(*)$ is an *intersection theory for* $\mathcal{V}$ if properties (1)-(6) below hold.

(1) For any $X \in \mathcal{V}$ the product $(*)$ makes $A^*(X)$ into a commutative associative graded ring with a multiplicative unit.

(2) Given varieties $X, X' \in \mathcal{V}$ and a map $f : X \to X'$, the map $f^* : A^*(X) \to A^*(X')$ defined by the formula

$$f^*(\alpha') = p_* \left([\Gamma_f] \cdot ([X] \times \alpha')\right) \ ,$$

where $\Gamma_f \subset X \times X'$ is the graph of f and $p : X \times X' \to X$ is the projection, is a homomorphism of rings. Moreover, if $X'' \in \mathcal{V}$ and $g : X' \to X''$ is a map, then $f^* g^* = (gf)^*$.

(3) *Projection formula:* If $X, X' \in \mathcal{V}$ and $f : X \to X'$ is a proper map, then

$$f_*(\alpha \cdot f^* \alpha') = f_*(\alpha) \cdot \alpha' \ .$$

(4) *Reduction to the diagonal:* If $X \in \mathcal{V}$ and $\alpha, \beta \in A^*(X)$, then

$$\alpha \cdot \beta = \delta^*(\alpha \times \beta) \ ,$$

where $\delta : X \to X \times X$ is the diagonal map.

(5) *Local nature:* If $X \in \mathcal{V}$ and W, W' are subvarieties of X that intersect properly (that is, every irreducible component of $W \cap W'$ has codimension equal to $\operatorname{codim}(W) + \operatorname{codim}(W')$), then

$$[W] \cdot [W'] = \textstyle\sum_C j_C(W, W')[C] \,,$$

where the sum runs over all irreducible components C of $W \cap W'$ and where $j_C(W, W')$ is an integer which only depends on the ideals of W and W' in $\mathcal{O}_{X,C}$.

(6) *Normalization:* If W is a subvariety of $X \in \mathcal{Z}$ and D is a Cartier divisor on X which intersects W properly, then

$$[D] \cdot [W] = [D \cap W] \,.$$

The intersection product for the category of smooth quasi-projective varieties is an intersection theory. Let us see now that it is unique. Indeed, let $\alpha, \beta \in A^*(X)$, where X is smooth and quasi-projective. We want to see that if $(*)$ is an intersection theory for quasi-projective varieties then $\alpha \cdot \beta$ is necessarily the intersection product. To that end, we may assume, by the moving lemma, that α and β are represented by cycles z and w that intersect properly on X. We may even assume, without loss of generality, that z and w are irreducible cycles. Now reduction to the diagonal and the definition of δ^* reduce the question to the product $[\Delta] \cdot [z \times w]$. Now Δ is a local complete intersection and so the local nature of an intersection theory allows us to assume that Δ is a complete intersection of divisors. Finally the claim results from repeated application of the normalization axiom.

Serre's Tor formula makes sense in more generality than the geometric context we studied above and here we will describe the simplest of such generalizations. Let A be a regular local ring and M and M' finitely generated A-modules. Let n, d and d' be the dimensions of A, $\mathrm{supp}(M)$ and $\mathrm{supp}(M')$, respectively. Assume that $\mathrm{supp}(M) \cap \mathrm{supp}(M') = \{\boldsymbol{m}\}$, where $\boldsymbol{m}$ is the closed point of $\mathrm{Spec}(A)$, in which case $d + d' \leqslant n$ (Serre [1965]). Then the A-modules $\mathrm{Tor}_i^A(M, M')$ have finite length, because its support is contained in $\{\boldsymbol{m}\}$, and are zero if $i > n$, because the projective dimension of any A-module is bounded above by n. Serre's positivity conjecture states that the Tor-characteristic of M and M', defined as

$$\chi(M, M') = \textstyle\sum_{i \geqslant 0} (-1)^i \ell \left(\mathrm{Tor}_i^A(M, M') \right) ,$$

is positive if and only if $d + d' = n$ (ℓ denotes the length function). This conjecture is a major one in the foundations of intersection theory and arithmetic algebraic geometry. It is known to be true when A is unramified (Serre [1965]) and a graded version of the conjecture is also known (Peskine–Szpiro [1974]). Moreover, it turns out that $\chi(M, M') = 0$ if $d + d' < n$ (Roberts [1985], Gillet–Soulé [1985, 87]).

Let $V_{d,\delta}$ be the Severi variety of irreducible plane curves of degree d with exactly δ nodes as singularities. Diaz and Harris [1986, 88] conjectured that $\mathrm{Pic}(V_{d,\delta})$ is a torsion group. The case $\delta = 0$ is easy: if $\mathbf{P}^N$ is the projective space parameterizing plane curves of degree d ($N = d(d+3)/2$), then $\mathbf{P}^N - V_{d,0}$ is the hypersurface of singular curves (the discriminant locus), which has degree $3(d-1)^2$, and from this it follows that $\mathrm{Pic}(V_{d,0})$ is a cyclic group of order $3(d-1)^2$. The case $\delta = 1$ has been established in Miret–Xambó [1992]: the order of $\mathrm{Pic}(V_{d,1})$ is $6(d-2)(d^2-3d+1)$, the group being cyclic if d is odd and the product of $\mathbf{Z}_2$ and a cyclic group of order $3(d-2)(d^2-3d+1)$ if d is even. The methods of proof here are an elaboration of some of the ideas used in Miret–Xambó [1990, 89, 91], but they do not dependent on them.

Chern classes

Axiomatics. Top Chern class. Calculating Chern classes. The splitting principle and applications.

Sources. The essential ideas are taken from [6, 7] (Grothendieck 1958), but we have also used [10] (Kleiman 1977, section II.B) and **H77** (Appendix A).

Variety means non-singular irreducible quasi-projective variety.

Locally free sheaves and vector bundles. On a variety X, the category of locally free sheaves of rank r is equivalent to the category of vector bundles of rank r. If E is a vector bundle, we will write $\mathcal{O}_X(E)$ to denote its associated locally free sheaf, which by definition is the sheaf of sections of E.

If D is a divisor on X, we will write $\mathcal{O}_X(D)$ to denote the invertible sheaf associated to D, and L_D to denote the corresponding line bundle, so that $\mathcal{O}_X(L_D) = \mathcal{O}_X(D)$.

The trivial vector bundle on X whose fiber is the vector space V will be denoted $V|X$.

We will assume that for any variety X, any vector bundle E on X and any integer k we have classes $c_k(E) \in A^k(X)$. Thus $c_k(E) = 0$ if $k \notin [0, n]$, $n = \dim(X)$, for $A^k(X) = 0$ for such indices k. For a locally free sheaf $\mathcal{E}$, we will write $c_k(\mathcal{E}) = c_k(E)$ if $\mathcal{O}_X(E) = \mathcal{E}$.

The classes $c_0(E), \ldots, c_n(E)$ will be called *Chern classes* of E, and

$$c(E) = c_0(E) + c_1(E) + \ldots + c_n(E) \in A^*(X)$$

the *total Chern class* of E, if the properties below (*normalization*, *functoriality*, *additivity*) are satisfied (and we have a similar definition for a locally free sheaf).

Normalization

If L is a line bundle, and D is a divisor such that $L \simeq L_D$, where L_D denotes the line bundle associated to D, then

$$c(L) = 1 + [D] \ .$$

In particular we have $c_0(L) = 1$, $c_1(L) = [D] \in A^1(X)$ and $c_k(L) = 0$ for $k > 1$. Note that $[D]$ only depends on L, for if D' is another divisor such that $L \simeq L_{D'}$ then D and D' are linearly equivalent and hence $[D] = [D']$.

The vanishing of $c_1(L)$ is equivalent to say that L is trivial, for if $L \simeq L_D$ and $[D] = [0]$, then $L \simeq L_0$.

Note that since L_{-D} is the dual of L_D we have

$$c_1(L^{\vee}) = -c_1(L) \tag{2}$$

for any line bundle L on X. Similarly, since $L_D \otimes L_{D'} \simeq L_{D+D'}$, D' another divisor, we have that if L' is another line bundle then

$$c_1(L \otimes L') = c_1(L) + c_1(L') \ .$$

In terms of invertible sheaves we have that $c_1(\mathcal{O}_X(D)) = 1 + [D]$.

For example, in $\mathbf{P}^n$ the sheaf $\mathcal{O}_{\mathbf{P}^n}(m)$, m an integer, is the invertible sheaf associated to the divisor mH, where H is a hyperplane. Hence

$$c_1(\mathcal{O}_{\mathbf{P}^n}(m)) = m[H] \ .$$

A last point is that the map c_1, with values in $A^1(X)$, is the same as the map cl, with values in $\mathrm{Pic}(X)$, when we identify $A^1(X)$ and $\mathrm{Pic}(X)$ via the canonical map $A^1(X) \to \mathrm{Pic}(X)$. Indeed, given an invertible sheaf $\mathcal{L}$, $cl(\mathcal{L}) \in \mathrm{Pic}(X)$ is, by definition, the isomorphism class of $\mathcal{L}$, while $c_1(\mathcal{L}) = [D]$ for any divisor D such that $\mathcal{L} \simeq \mathcal{O}_X(D)$, and so the claim follows because $[D] \in A^1(X)$ is mapped to the isomorphism class of $\mathcal{O}_X(D)$.

Functoriality

If X and X' are varieties, $f : X \to X'$ is a map, and E' is a vector bundle on X', then

$$c_k(f^*E') = f^*(c_k E') \tag{3}$$

for all k. Equivalently,

$$c(f^*E') = f^*(c(E'))\ .$$

Note that this property is consistent with normalization, for if L' is a line bundle on X' and $L' \simeq L_{D'}$, D' a divisor on X', which we may assume to satisfy $f(X) \not\subseteq D'$, then

$$c_1(f^*L') = c_1(f^*L_{D'}) = c_1(L_{f^*D'}) = [f^*D'] = f^*[D'] = f^*(c_1 L')\ .$$

Note that f^*D' is defined because $f(X)$ is not contained in (the support of) D'.

Additivity

If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of vector bundles on a variety X, then

$$c(E) = c(E')c(E'') \; .$$

This relation is also called *Whitney formula* and is equivalent to the relations

$$c_k(E) = \sum_{j=0}^{k} c_j(E')c_{k-j}(E'') \quad (k \in \mathbf{Z}) \; .$$

In particular we see that $c(E) = c(E')c(E'')$ if $E = E' \oplus E''$. From this it follows that $c(E) = 1$ if E is a trivial bundle.

It turns out that $c_k(E) = 0$ if $k > r$, $r = \mathrm{rank}(E)$. Because of this, $c_r(E)$ is called the *top Chern class* of E. For a line bundle L, the top Chern class is $c_1(L)$, which by normalization is equal to $[D]$, D any divisor such that $L \simeq L_D$. But D can be recovered from L_D as the zero scheme $V(\sigma)$ of some section σ of L_D. This description also holds for the top Chern class, in the following sense. Let σ be a non-zero section of E, a vector bundle of rank r. Then the zero scheme of σ, $V(\sigma)$, has codimension at most r, because locally $V(\sigma)$ is given as the vanishing of r functions. The *top Chern class formula* holds for any section σ of E such that $V(\sigma)$ has codimension r and it says that

$$c_r(E) = [V(\sigma)]\,.$$

If X is a variety, the classes $c_k(T_X)$, T_X the tangent bundle of X, will be called *Chern classes* of X, and to simplify notation we will write $c_k(X)$ instead of $c_k(T_X)$. The *total Chern class* of X, $c(X)$, is the class $c(T_X)$. If X is projective of dimension n, then the *Euler characteristic* of X, $\chi(X)$, is given by the Gauss–Bonet formula

$$\chi(X) = \int_X c_n(X) .$$

Chern classes of $\mathbf{P}^n$

Since there is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbf{P}^n} \to 0 \ ,$$

the so called Euler exact sequence, the additivity formula implies that

$$c(\mathbf{P}^n) = (1+h)^{n+1} \ ,$$

where $h = [H]$ is the class of a hyperplane H. Hence

$$c_k(\mathbf{P}^n) = \binom{n+1}{k} h^k \ .$$

In particular, $c_n(\mathbf{P}^n) = (n+1)h^n$ and so $\chi(\mathbf{P}^n) = n+1$, because $\deg(h^n) = 1$.

Euler characteristic of a curve

If X is a projective curve of genus g, then $\chi(X) = 2 - 2g$, for $c_1(X) = c_1(T_X) = -c_1(\omega_X) = -[K]$, $[K]$ the canonical class on X, and $\deg(K) = 2g - 2$.

Adjunction formula

Let $i : X \to Y$ be a closed embedding of codimension r, where X and Y are varieties, and let $N = N_{X/Y}$ be the corresponding normal bundle. Then, by the definition of N, we have an exact sequence

$$0 \to T_X \to i^* T_Y \to N \to 0\ .$$

Additivity and functoriality yield that

$$c(X) = i^*(c(Y))/c(N)\ .$$

Assume now that X is the complete intersection of r divisors, $X = D_1 \cap \ldots \cap D_r$. Then, setting $L_i = L_{D_i}$,

$$N = i^*(L_1 \oplus \ldots \oplus L_r)$$

and so

$$c(N) = i^*((1+[D_1])\ldots(1+[D_r]))\ .$$

Using the preceding formulas, pushing forward with i and using projection formula, we get the following *adjunction formula*:

$$i_*(c(X)) = c(Y)\prod_{i=1}^{r}\frac{[D_i]}{1+[D_i]}\ .$$

(Note that $[D_i]/(1+[D_i]) = [D_i] - [D_i]^2 + \ldots)$.

For example, if $Y = \mathbf{P}^n$ and $\deg(D_i) = d_i$, then

$$c_1(X) = (n+1-\sum d_i)h\ ,$$

where h is the class of a hyperplane section of X.

In case $r = 1$, so that X is a (smooth) divisor in Y, the preceding formula is equivalent to the relations

$$i_*(c_k(X)) = \sum_j (-1)^j c_{k-j}(Y) \cdot [X]^{j+1} \quad (k \in \mathbf{Z}) \; . \tag{4}$$

Taking $Y = \mathbf{P}^{n+1}$ and X a smooth hypersurface of degree d, it is straightforward to show that

$$\chi(X) = \sum_{j=0}^{n} (-1)^j \binom{n+2}{n-j} d^{j+1} \; . \tag{5}$$

More generally, taking $Y = \mathbf{P}^{n+r}$ and X a smooth complete intersection of hypersurfaces $D_1, \ldots, D_r$ of degrees $d_1, \ldots, d_r$, let σ_j $(0 \leqslant j \leqslant n)$ be the j-th symmetric polynomial in $d_1, \ldots, d_r$ (notice that $\sigma_r = d_1 \cdots d_r$ is the degree of X and that, by convention, $\sigma_j = 0$ for $r < j \leqslant n$), and let $s_0, s_1, \ldots, s_n$ be the sequence determined recursively by $s_0 = 1$ and

$$s_j + s_{j-1}\sigma_1 + \ldots + s_1\sigma_{j-1} + \sigma_j = 0\ .$$

Then

$$\chi(X) = d\sum_{j=0}^{n} \binom{n+r+1}{n-j} s_j\ .$$

Self-intersection formula

Again let $i : X \to Y$ be a codimension r embedding of varieties and set $N = N_{X/Y}$ to denote the corresponding normal bundle. Then (see Lascu–Mumford–Scott [1975])

$$i^* i_*[X] = c_r(N) \ .$$

Pushing forward with i and using the projection formula we get the 'self-intersection formula'

$$[X]^2 = i_* c_r(N) \ .$$

Applying the self-intersection formula to the diagonal inclusion $\delta : X \to X \times X$ and taking into account that T_X is isomorphic to $N_{X/X\times X}$, we get:

$$\int [\Delta]^2 = \int c_n(N_{X/X\times X}) = \int c_n(T_X) = \int c_n(X) = \chi(X) \ ,$$

which is *Lefschetz formula* for the Euler characteristic of X.

Chern classes of a filtered bundle

Assume

$$0 = E_0 \subseteq E_1 \subseteq \ldots \subseteq E_r = E$$

is a filtration of E by subbundles E_i and set $Q_i = E_i/E_{i-1}$ $(1 \leqslant i \leqslant r)$. Then

$$c(E) = c(Q_1) \cdot \ldots \cdot c(Q_r) \ .$$

Indeed, by definition of Q_i and additivity, $c(E_i) = c(E_{i-1})c(Q_i)$, and by induction $c(E_i) = c(Q_1) \cdot \ldots \cdot c(Q_i)$ $(1 \leqslant i \leqslant r)$.

Assume now that the Q_i are line bundles, in which case we say that E *splits into line bundles*. Then if we set $\alpha_i = c_1(Q_i)$ we have

$$c(E) = (1+\alpha_1)\cdots(1+\alpha_r)\,.$$

This relation is equivalent to the relations

$$c_i(E) = \sigma_i(\alpha_1,\dots,\alpha_r) \quad (i \in \mathbf{Z})$$

where $\sigma_i(\alpha_1,\dots,\alpha_r)$ is the i-th symmetric polynomial in $\alpha_1,\dots,\alpha_r$.

Virtual bundles

If E and F are vector bundles on a variety X, the *total Chern class* $c(E-F)$ of the 'virtual' bundle $E-F$ (this is a just formal difference) is defined as follows:

$$c(E-F) = c(E)/c(F)\ .$$

Hence $c_1(E-F) = c_1(E) - c_1(F)$,
$c_2(E-F) = c_2(E) - c_1(E)c_1(F) + c_1(F)^2 - c_2(F)$, and so on.

This is better understood using the Grothendieck group $K^0(X)$ of the category $Vect_X$ of vector bundles on X, that is, the quotient of the free abelian group generated by the isomorphism classes e of vector bundles E by the subgroup generated by the elements $e-e'-e''$, one for each exact sequence $0 \to E' \to E \to E'' \to 0$ of vector bundles.

Indeed, since the total Chern class c is an additive function from $Vect_X$ with values in the abelian group $1 + A^+(X)$, where $A^+(X) = A^1X + \ldots + A^nX$ ($n = \dim(X)$), it extends to a unique group homomorphism $c : K^0X \to 1 + A^+X$, and it is clear that $c(e - f) = c(E - F)$.

Note that K^0 is a contravariant functor with values in the category of associative commutative rings with unit: the product in $K^0(X)$ is induced by the tensor product of vector bundles and the contravariant map $f^* : K^0(X') \to K^0(X)$ corresponding to a map $f : X \to X'$ is induced by the pullback of vector bundles E' on X' to vector bundles $f^*(E')$ on X.

Chern classes of coherent sheaves

Let $K_0(X)$ denote the Grothendieck group of the category Coh_X of coherent sheaves on the variety X. By definition it is the quotient of the free abelian group generated by the isomorphism classes f of coherent sheaves $\mathcal{F}$ by the subgroup generated by the elements $f - f' - f''$, one for each exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$.

Note that K_0 is a covariant functor of the category of (smooth and quasi-projective) varieties with proper maps with values in the category of abelian groups: the covariant map $f_! : K_0(X) \to K_0(X')$ corresponding to a proper map $f : X \to X'$ is induced by mapping the isomorphism class f of a coherent sheaf $\mathcal{F}$ to the alternating sum of the isomorphism classes $r^i f_*(\mathcal{F})$ of the higher direct images $R^i f_*(\mathcal{F})$ of $\mathcal{F}$. Here the key points are that the higher direct images $R^i f_*(\mathcal{F})$ are coherent sheaves on X' and the cohomology exact sequence of the higher direct images associated to a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of coherent sheaves.

Now we have a canonical map of abelian groups $K^0(X) \to K_0(X)$, induced by mapping the isomorphism class of a vector bundle E to the isomorphism class of the locally free sheaf $\mathcal{O}_X(E)$. The wonderful fact about this map is that it is an isomorphism, the reason being that on a smooth quasi-projective variety any coherent sheaf admits a finite homological resolution by locally free sheaves. This means that we have a total Chern class homomorphism $c : K_0X \to 1 + A^+X$, just by composing the isomorphism $K^0(X) \simeq K_0(X)$ with the total Chern class homomorphism $c : K^0X \to 1 + A^+X$. This yields, in particular, Chern classes for coherent sheaves.

According to the definitions, if $\mathcal{F}$ is a coherent sheaf, then

$$c(\mathcal{F}) = \prod_{i=0}^{m} c(\mathcal{E}_i)^{(-1)^i}, \qquad (6)$$

the alternating product of the total Chern classes of the locally free sheaves $\mathcal{E}_i$ of a projective resolution

$$0 \to \mathcal{E}_m \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

of $\mathcal{F}$.

Of course, the formula also works if $\mathcal{F}$ is a locally free sheaf, thus providing a means of calculating its Chern classes if it happens that we know the Chern classes of the locally free sheaves $\mathcal{E}_i$ of the resolution.

The Chern classes satisfy the following 'splitting principle':

> *Given a vector bundles $E_1, \ldots, E_m$ on a variety X, there exists a variety P and map $f : P \to X$ such that $f^* : A^*X' \to A^*X$ is injective and so that f^*E_i splits into line bundles for all i.*

In this section we will use this principle to calculate the Chern classes of various bundles that appear in rather concrete geometrical questions. The idea is this: if $f : P \to X$ is a map such that f^*E splits into line bundles, say $L_1, \ldots, L_r$, then

$$f^*c(E) = c(f^*E) = (1 + \alpha_1) \ldots (1 + \alpha_r) \,,$$

where $\alpha_i = c_1(L_i)$. In other words,

$$f^*c_k(E) = \sigma_k(\alpha_1, \ldots, \alpha_r) \,.$$

If in addition $f^*: A^*X \to A^*P$ is injective, the last relation determines $c_k(E)$. Actually it says that $\sigma_k(\alpha_1, \ldots, \alpha_r)$ lies in $f^*(A^*X)$ and hence if we identify A^*X with $f^*(A^*X)$ via f^*, the relation in question can be written $c_k(E) = \sigma_k(\alpha_1, \ldots, \alpha_r)$. This just says that *to handle the Chern classes of any given finite number of vector bundles we can just pretend that they split into line bundles*.

If E splits into line bundles $L_1, \ldots, L_r$, then the L_i will be called *root bundles* of E and the $\alpha_i = c_1(L_i)$ the *Chern roots* of E.

Chern classes of $E^\vee$

If E has Chern roots $\alpha_1, \ldots, \alpha_r$, then it is clear that $E^\vee$ has Chern roots $-\alpha_1, \ldots, -\alpha_r$ and so

$$c_k(E^\vee) = \sigma_k(-\alpha_1, \ldots, -\alpha_r) = (-1)^k c_k(E)\ .$$

Chern classes of $E \otimes F$

Let $\alpha_1, \ldots, \alpha_r$ and $\beta_1, \ldots, \beta_s$ be the Chern roots of vector bundles E and F, respectively. Then by the bilinearity of the tensor product it follows that the Chern roots of $E \otimes F$ are $\alpha_i + \beta_j$ $(1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s)$. Hence

$$c(E \otimes F) = \prod_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}} (1 + \alpha_i + \beta_j)$$

and so we can find the partial Chern classes by expressing the right hand side, which is symmetric in the α's and in the β's, as a polynomial in the elementary symmetric functions $\sigma_i(\alpha_1, \ldots, \alpha_r)$ and $\sigma_j(\beta_1, \ldots, \beta_s)$, that is, as a polynomial in the $c_i(E)$ and $c_j(F)$.

An explicit expression for the resulting polynomial was found by Lascoux (see **F98**, example 14.5.2).

Here we only look at the case when F is a line bundle L. If $c_1(L) = \beta$, then

$$c(E \otimes L) = (1 + \alpha_1 + \beta) \ldots (1 + \alpha_r + \beta)$$
$$= \sum_{i=0}^{r} (1 + \beta)^{r-i} c_i(E) \; .$$

For the partial Chern classes we find:

$$c_k(E \otimes L) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) \beta^{k-i} \; .$$

In particular the first and top Chern classes of $E \otimes L$ is

$$c_1(E \otimes L) = r\beta + c_1(E) \quad \text{and} \quad c_r(E \otimes L) = \sum c_{r-i}(E) \beta^i \; .$$

Chern classes of $\Lambda^p E$ and $S^p E$

If $0 \to L \to E \to E' \to 0$ is an exact sequence of vector bundles, L a line bundle, then there is a derived short exact sequence

$$0 \to (\Lambda^{p-1} E') \otimes L \to \Lambda^p E \to \Lambda^p E' \to 0\ .$$

Using this fact inductively, we see that if $L_1, \ldots, L_r$ are the root line bundles of E, then the root line bundles of $\Lambda^p E$ are of the form $L_{i_1} \otimes \cdots \otimes L_{i_p}$, where $1 \leqslant i_1 < \ldots < i_r \leqslant r$. Hence the Chern roots of $\Lambda^p E$ are $\alpha_{i_1} + \ldots + \alpha_{i_r}$, with the same conditions on the indices and $\alpha_1, \ldots, \alpha_r$ being the Chern roots of E. So

$$c(\Lambda^p E) = \prod_{1 \leqslant i_1 < \ldots < i_r \leqslant r} (1 + \alpha_{i_1} + \ldots + \alpha_{i_r})\ ,$$

which allows us to find the Chern classes of $\Lambda^p E$ by writing the polynomial on the right hand side, which is symmetrical in the $\alpha_1, \ldots, \alpha_r$, as a polynomial in the $\sigma_i(\alpha_1, \ldots, \alpha_r) = c_i E$. Note, for example, that

$$c_1(\Lambda^r E) = c_1 E. \qquad (7)$$

In a similar way we find that

$$c(S^pE) = \prod_{\substack{m_1+\ldots+m_r=p \\ m_1,\ldots,m_r\geqslant 0}} (1 + m_1\alpha_1 + \cdots + m_r\alpha_r) \,.$$

(see **F98**, example 3.2.6).

Riemann Roch

Segre classes. Chern character and the Todd class. Hirzebruch Riemann-Roch formula. Grothendick Riemann-Roch. Porteous formula.

The *total Segre class* $s(E)$ of a vector bundle E on a variety X is defined as the inverse of $c(E)$:

$$s(E) = c(E)^{-1} \quad \text{or} \quad s(E)c(E) = 1 \ .$$

If X has dimension n, then $s(E) = s_0(E) + s_1(E) + \cdots + s_n(E)$, where $s_k(E) \in A^k(X)$. From the definition we see that $s_0(E) = 1$ and that for $k > 0$

$$s_k(E) + s_{k-1}c_1(E) + \ldots + s_1(E)c_{k-1}(E) + c_k(E) \ ,$$

which allows us to calculate $s_k(E)$ recursively. Thus $s_1(E) = -c_1(E)$, $s_2(E) = -s_1(E)c_1(E) - c_2(E) = c_1(E)^2 - c_2(E)$, and so on.

Any symmetrical polynomial with integer coefficients $f(\alpha_1, \ldots, \alpha_r)$ in the Chern roots $\alpha_1, \ldots, \alpha_r$ of a vector bundle E can be written as a polynomial in the Chern classes of E. If we write $f(E)$ to denote this polynomial, $f(E) \in A^*(X)$, then we can write

$$f(E) = f_0(E) + f_1(E) + \ldots + f_n(E)$$

with $f_k(E) \in A^k(X)$ and $n = \dim(X)$. We will also set $f(X)$ and $f_k(X)$ to denote the classes $f(T_X)$ and $f_k(T_X)$, respectively.

If instead f has rational coefficients, $f_k(E)$, $f_k(X) \in A^k(X) \otimes \mathbf{Q}$.

Two important examples of this procedure for constructing classes associated to a vector bundle are the *Chern character*, $ch(E)$, and the *Todd class*, $td(E)$:

$$ch(\alpha_1, \ldots, \alpha_r) = e^{\alpha_1} + \ldots + e^{\alpha_r},$$

$$td(\alpha_1, \ldots, \alpha_r) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}} .$$

Note that if α is a Chern root, then both e^{α} and $\alpha/(1 - e^{-\alpha})$ are polynomials in α with rational coefficients, the reason being that $\alpha^N = 0$ for sufficiently large N.

From the definitions it is clear that

$$ch(E \oplus E') = ch(E) + ch(E') \quad \text{and} \quad ch(E \otimes E') = ch(E) \cdot ch(E')$$

and that

$$td(E \oplus E') = td(E) \cdot td(E') .$$

These relations, together with the fact that

$$td_n(\mathbf{P}^n) = h^n$$

(h the hyperplane class) are sufficient to determine the polynomials ch and td (see Hirzebruch [1966]; below we check the cases $n = 2$ and $n = 3$).

A straightforward computation yields the following expressions, where to simplify notation we set $c_k = c_k(E)$:

$$ch_0(E) = \mathrm{rank}(E) \qquad td_0(E) = 1$$

$$ch_1(E) = c_1 \qquad td_1(E) = \frac{1}{2}c_1$$

$$ch_2(E) = \frac{1}{2}(c_1^2 - 2c_2) \qquad td_2(E) = \frac{1}{12}(c_1^2 + c_2)$$

$$ch_3(E) = \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) \qquad td_3(E) = \frac{1}{24}c_1c_2\ .$$

Let us check that $td_2(\mathbf{P}^2) = h^2$ and that $td_3(\mathbf{P}^3) = h^3$. If c_1 and c_2 are the Chern classes of $\mathbf{P}^2$, then $td_2(\mathbf{P}^2) = (c_1^2 + c_2)/12$. But $c(\mathbf{P}^2) = (1+h)^3 = 1 + 3h + 3h^2$ and so

$$td_2(\mathbf{P}^2) = (9h^2 + 3h^3)/12 = h^2.$$

As for $\mathbf{P}^3$, $c(\mathbf{P}^3) = (1+h)^4 = 1 + 4h + 6h^2 + \cdots$ and so $td_3(\mathbf{P}^3) = (c_1 c_2)/24 = (4h \cdot 6h^2)/24 = h^3$.

Note that the properties of ch say that it induces a ring homomorphism

$$ch : K^0 X \to A^* X \ .$$

Given a vector bundle E on a smooth projective variety X of dimension n, the *Euler characteristic* of E, $\chi(E)$, is defined as follows:

$$\chi(E) = \sum_{i=0}^{n} (-1)^i h^i(X, E) ,$$

where $h^i(X, E)$ is the dimension over $\boldsymbol{k}$ of the cohomology space $H^i(X, E)$. Then

$$\chi(E) = \int_X ch(E) \cdot td(X) .$$

Here if $\alpha \in A^*(X)$ we set $\int_X \alpha = \int_X \alpha_n$.

For example, if D is a divisor on X and we set $\chi(D) = \chi(L_D)$, then

$$\chi(D) = \int_X \left(1 + [D] + [D]^2/2 + \ldots\right) \cdot \left(1 + c_1/2 + (c_1^2 + c_2)/12 + \ldots\right) ,$$

where $c_i = c_i(X)$. Since $c_1 = -[K]$, on a curve $(n = 1)$ of genus g we get

$$\chi(D) = \int_X ([D] - [K]/2) = \deg(D) + 1 - g.$$

On a surface of arithmetic genus p_a

$$\chi(D) = \int_X \big([D]^2/2 - ([D]\cdot[K])/2 + ([K]^2 + c_2)/12 \big)$$
$$= \frac{1}{2}\int_X [D]\cdot([D]-[K]) + \frac{1}{12}\int_X ([K]^2 + c_2)/12 \,.$$

In particular $\int_X ([K]^2 + c_2)/12 = \chi(0) = \chi(\mathcal{O}_X) = 1 + p_a$.

Another example is the case of an abelian variety X of dimension n. In this case T_X is trivial and so $td(X) = 1$. Thus we have, for any divisor D on X,

$$\chi(D) = \frac{1}{n!}\int_X [D]^n \,.$$

Reference: Hirzebruch [1966].

Let $f : X \to X'$ be a smooth projective morphism of non-singular projective varieties and T_f the relative tangent bundle. Then for any $\alpha \in K_0(X) = K^0(X)$ the following relation holds in $A^*(X') \otimes \mathbf{Q}$:

$$ch(f_!\alpha) = f_*\left(ch(\alpha) \cdot td(T_f)\right) \ .$$

This relation yields, when applied to the structural constant map $\pi : X \to \mathrm{Spec}(\boldsymbol{k})$ and with α (the class of) a vector bundle E, the Hirzebruch–Riemann–Roch formula, for on one hand $\pi_!\alpha = \chi(E)$ and ch is the identity of $\mathbf{Z}$, and on the other $T_\pi = T_X$.

References: Borel–Serre [1958], Grothendieck [1971 a], Fulton [1984].

Examples of intersection rings

Projective bundle $\mathrm{P}(E)$. Grassmanians. Flags. Blowups.

Tautological line bundle on $\mathbf{P}(E)$

Let E be a vector bundle of rank r on a variety X. Let $P = \mathbf{P}(E)$ and $p : P \to X$ the projection.

Given a non-zero $v \in E_x$, the fiber of p^*E over the point $[v] \in P$ is E_x,

$$(p^*E)_{[v]} = E_x \ .$$

But E_x contains the line $\langle v \rangle$ spanned by v and so we can consider the line subbundle of $p^*E = E|P$ whose fiber over $[v]$ is $\langle v \rangle$.

The *dual* of this line bundle is called the *tautological line bundle* of P and is denoted $L = L_E$ and so p^*E contains the line subbundle $L^\vee = L_E^\vee$. The quotient bundle $Q = Q_E = p^*E/L^\vee$ will be called the *tautological quotient* of P. Thus we have, by definition, a *tautological exact* sequence

$$0 \to L^\vee \to E|P \to Q \to 0 \ .$$

The invertible sheaf $\mathcal{O}_P(L)$ is denoted $\mathcal{O}_P(1)$, or just $\mathcal{O}(1)$, and is also called the *tautological invertible sheaf* on P.

- *The hyperplane class*. The first Chern class $\xi = \xi_E$ of L will be called the *hyperplane class*, or the *characteristic class*, of P. Note that if E is trivial, say $E = V|X$, then $P = X \times \mathbf{P}(V)$. Moreover, the tautological line bundle is $L_{X \times H}$, where H is a hyperplane of $\mathbf{P}(V)$. Hence $\xi = [X \times H]$. When X is a point, we see that the $\xi = [H]$.

Since p_* drops codimension by $r - 1$, we see that

$$p_*(\xi^i) = 0 \quad \text{for} \quad 0 \leqslant i \leqslant r - 2 \,.$$

On the other hand it is clear that

$$p_*(\xi^{r-1}) = [X]$$

for over a non-empty open set U of X on which E is trivial, say $U \times V$ (V a $\boldsymbol{k}$-vector space of dimension r), we can represent ξ by the cycle $[U] \times [H]$, H a hyperplane of $\mathbf{P}(V)$, and so ξ^{r-1} is represented by $[U] \times [pt]$, where pt is a point of $\mathbf{P}(V)$. Since $U \times \{pt\}$ has degree 1 over U, the claim follows.

Functoriality

The tautological data defined so far are functorial in the following sense. Let $f : X \to X'$ be a map of varieties, let E' be a vector bundle on X' and set $P' = \mathbf{P}(E')$, $\xi' = \xi_{E'}$, $Q' = Q_{E'}$, $E = f^*E'$, $P = \mathbf{P}(E)$, $\xi = \xi_E$ and $Q = Q_E$. Then P is the inverse image of P' under f, so that there is a unique map $g : P \to P'$ making the diagram

$$\begin{pmatrix} P & \xrightarrow{g} & P' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow{f} & X' \end{pmatrix}$$

a fiber square, and the inverse image of the tautological exact sequence for P' under the map g is the tautological exact sequence for P. In particular we have that $g^*(\xi') = \xi$. When X is a point of X' we get that the restriction of the hyperplane class to a fiber of P is the class of the hyperplane of that fiber.

Intersection ring of $P = \mathbf{P}(E)$

The classes $1, \xi, \ldots, \xi^{r-1}$ are linearly independent over $A^*(X)$, where $A^*(P)$ is considered as an $A^*(X)$-module via the ring homomorphism $p^* : A^*X \to A^*P$.

Indeed, assume

$$\sum_{i=0}^{r-1} p^*(\alpha_i)\xi^i = 0 \;, \quad \alpha_i \in A^iX \;.$$

Applying p_* to this relation, and using the projection formula, together with (3.2.a) and (3.2.b), we get $\alpha_{r-1} = 0$. So the claim follows upon multiplying the displayed relation repeatedly by ξ. Note that this implies that p^* is injective.

Now it turns out that the classes $1, \xi, \ldots, \xi^{r-1}$ also span $A^*(P)$ as an $A^*(X)$-module (see Grothendieck [1958 b], § 6, corollaire 2; the idea is that the classes $1, \xi, \ldots, \xi^{r-1}$ restricted to any fiber of $\mathbf{P}(E)$ generate, by Bézout's theorem, the intersection ring of that fiber considered as an abelian group; see also Fulton [1984], theorem 3.3). So we see that there are unique classes $c_i = c_i(E) \in A^i X$ such that

$$\xi^r + p^*(c_1)\xi^{r-1} + \ldots + p^*(c_r) = 0 \qquad (8)$$

and consequently (Grothendieck's theorem)

$$A^*(\mathbf{P}(E)) = A^*(X)[T]/\left(f_E(T)\right),$$

where T is an indeterminate and

$$f_E(T) = T^r + p^*(c_1)T^{r-1} + \ldots + p^*c_r = 0\,.$$

Interpretation of the Segre classes

From Whitney formula we have, with the same notations as before, that

$$c(E|P) = (1-\xi)c(Q)\ .$$

Hence $c(Q) = (\sum_{i\geqslant 0}\xi^i)p^*(c(E))$. Since $c_r(Q) = 0$ because Q has rank $r-1$, this relation implies that

$$\xi^{r+\ell-1} + \xi^{r+\ell-2}p^*(c_1) + \ldots + \xi^{\ell-1}p^*(c_r) = 0$$

for any $\ell \geqslant 1$. Applying p_* and using the projection formula we get, defining

$$s_i = p_*(\xi^{i+r-1})\ ,$$

that

$$s_\ell + s_{\ell-1}c_1 + \ldots + s_{\ell-r}c_r = 0$$

for any $\ell \geqslant 1$. Since $s_0 = 1$, these relations just say that the s_i are the Segre classes of E.

Let $\Gamma = \mathrm{Gr}(k, n)$ be the Grassmannian of k-planes in $\mathbf{P}^n = \mathbf{P}(V)$ and $d = \dim(\Gamma) = (k+1)(n-k)$.

Let $T \subset \Gamma \times V$ be the *tautological* subbundle, so that the fiber $T_\gamma \subset V$ is the subspace of vectors representing points of $\gamma \in \Gamma$. Thus T has rank $k+1$.

Similarly, let $Q = (V|\Gamma)/T$, the *tautological quotient bundle*. Its rank is $d - k - 1$.

Let $c_1, c_2, \ldots, c_{k+1}$ be the Chern classes of T and $C = [c_1, c_2, \ldots, c_{k+1}]$. The $A^*(\Gamma)$ is isomorphic to the ring $\mathbf{Z}[c_1, \ldots, c_{k+1}]/R$, where R is the ideal generated by the polynomials in the list

$$R = \mathrm{invert\_vector}(C, n+1)[-k-1:].$$

Remark: c_{k+1}^{n-k} is a point.

A sample of solutions

Lines meeting 4 lines in $\mathbf{P}^3$, 6 planes in $\mathbf{P}^4$,... Lines in a cubic in $\mathbf{P}^3$. Lines in a quintic in $\mathbf{P}^4$. Conics meeting 8 lines in $\mathbf{P}^3$. Conics tangent to 5 conics in $\mathbf{P}^2$. Conics in a quintic in $\mathbf{P}^4$.

- *Moduli*: $X = \mathrm{Gr}(1,3)$, $d = \dim(X) = (1+1)(3-1) = 4$.
- *Setting*: $N = \deg(X_{L_1} \cdot X_{L_2} \cdot X_{L_3} \cdot X_{L_4})$, L_j generic lines.
- *Intersection ring*: $A^*(X)$.
- *Relevant condition*: X_L, lines meeting a line L, and $\ell = [X_L] \in A^1(X)$ (the same for all L!).
- *Theoretical solution*: $N = \deg(\ell^4) = \int_X \ell^4$ (because the projective group acts transitively on X).
- *Algebra*: $A^*(X) = \mathbf{Z}[c_1, c_2]/R$, R the ideal generated by $c_1^3 - 2c_1c_2$, $c_1^4 - 3c_1^2c_2 + c_2^2$.
- *Geometry*: $\ell = c_1$ and $c_2 = \pi = [X_{plane}]$.
- *Effective solution*: $\ell^4 = c_1^4 = 3c_1^2c_2 - c_2^2 = 3\ell^2\pi - \pi^2$, so

$$N = \int_X \ell^4 = 3\int_X \ell^2\pi - \int_X \pi^2 = 3 - 1 = 2.$$

- *Moduli*: $X = \mathrm{Gr}(1,4)$, $d = \dim(X) = (1+1)(4-1) = 6$.
- *Setting*: $N = \deg(X_{L_1} \cdot X_{L_2} \cdot X_{L_3} \cdot X_{L_4} \cdot X_{L_5} \cdot X_{L_6})$, L_j generic planes.
- *Intersection ring*: $A^*(X)$.
- *Relevant condition*: X_L, lines meeting a plane L, and

  $\ell = [X_L] \in A^1(X)$ (the same for all L!).
- *Theoretical solution*: $N = \deg(\ell^6) = \int_X \ell^6$ (because the projective group acts transitively on X).
- *Algebra*: $A^*(X) = \mathbf{Z}[c_1, c_2]/R$, R the ideal generated by

  $c_1^4 - 3c_1^2c_2 + c_2^2$ and $-c_1^5 + 4c_1^3c_2 - 3c_1c_2^2$
- *Geometry*: $\ell = c_1$ and $c_2 = \pi = [X_{3-plane}]$.
- *Effective solution*: $\ell^6 = 4(3c_1^2c_2^2 - c_2^3) - 3c_1^2c_2^2 = 9c_1^2c_2^2 - 4c_2^3$, so

$$N = \int_X \ell^6 = 9\int_X \ell^2\pi^2 - 4\int_X \pi^3 = 9 - 4 = 5.$$

- *Moduli*: $X = \mathrm{Gr}(1,4)$, $d = \dim(X) = (1+1)(4-1) = 6$.
- *Setting*: A generic cubic f lives in S^3V^*. If we let T be the tautological subbundle of $X \times V$, the locus we are looking at is $Z(\sigma)$, where $\sigma = \bar{f}$ is the image in S^3T^* of the constant section f of $X \times S^3V^*$ ($\sigma(x)$ is the restriction of f to the vector subspace of V representing the points of the line x). Therefore, $N = \deg(Z(\sigma))$.
- *Intersection ring*: $A^*(X)$.
- *Theoretical solution*: $[Z(\sigma)] = c_{\max}(S^3T^*) = c_4(S^3T^*) \in A^4(X)$.
- *Computation of c_4*: Using the expression on page 67,

$$c_4 = (3\alpha_1)(2\alpha_1 + \alpha_2)(\alpha_1 + 2\alpha_2)(3\alpha_2) = 18c_1^2c_2 + 9c_2^2.$$

- *Geometry*: $c_1 = -c_1(T) = -\ell$ and $c_2 = c_2(T) = \pi$.
- *Effective solution*: $N = 27$.

- *Moduli*: $X = \mathrm{Gr}(1,4)$, $d = \dim(X) = (1+1)(4-1) = 6$.

- *Setting*: A generic quintic f lives in S^5V^*. If we let T be the tautological subbundle of $X \times V$, the locus we are looking at is $Z(\sigma)$, where $\sigma = \bar{f}$ is the image in S^5T^* of the constant section f of $X \times S^3V^*$ ($\sigma(x)$ is the restriction of f to the vector subspace of V representing the points of the line x). Therefore, $N = \deg(Z(\sigma))$.

- *Intersection ring*: $A^*(X)$.

- *Theoretical solution*: $[Z(\sigma)] = c_{\max}(S^3T^*) = c_6(S^5T^*) \in A^6(X)$.

- *Computation of c_6*: Using the expression on page 67,

$$c_6 = 25c_2(4\alpha_1 + \alpha_2)(3\alpha_1 + 2\alpha_2)(2\alpha_1 + 3\alpha_2) = 25c_2(\ldots) = \ldots$$

- *Geometry*: $c_1 = -c_1(T) = -\ell$ and $c_2 = c_2(T) = \pi$.

- *Effective solution*: $N = 2875$.

- *Moduli*: $\mathcal{C} = \mathbf{P}(S^2 T^*)$, where T is the (rank 3) tautological subbundle of $\mathbf{P}^* \times S^2 V^*$.
- *Setting*: $N = \deg(\cap_{1\leqslant j\leqslant 8} \mathcal{C}_{L_j})$, the L_j lines in general position.
- *Intersection ring*: $A^*(\mathcal{C})$.
- *Relevant condition*: $\mathcal{C}_L$, lines meeting a line L, and

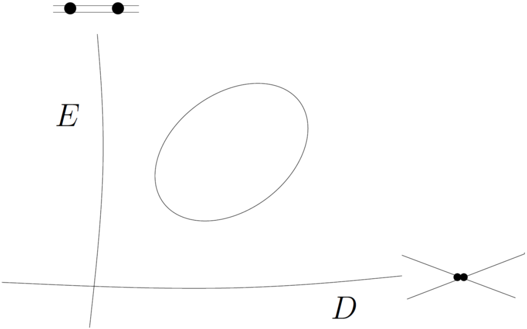
  $\lambda = [\mathcal{C}_L] \in A^1(\mathcal{C})$ (the same for all L!).
- *Theoretical solution*: $N = \int_{\mathcal{C}} \lambda^8 = \int_{\mathbf{P}^*} \pi_*(\lambda^8)$.
- *Algebra*: $A^*(\mathcal{C}) = A^*(\mathbf{P}^*)[\sum_0^6 c_j^* \xi^{6-j}]$, $\xi = c_1(L_{\mathcal{C}})$, $L_{\mathcal{C}}$ the tautological subbundle of $\mathcal{C}$, $c_j^* = \pi^* c_j(T^*) \in A^j(\mathcal{C})$. $A^1(\mathcal{C}) = \langle c_1^*, \xi \rangle$.
- *Geometry*: $A^*(\mathbf{P}^*) = \mathbf{Z}[p]/(p^4)$, $c_1 = p$, $c_2 = p^2$, $c_3 = p^3$ (p the condition that the plane of the conic is incident with a given point). It turns out that $\lambda = 2c_1^* + \xi$.

The computation of $\pi_*(2c_1^* + \xi)^8$ is reduced to the computation of $\pi_*((c_1^*)^j \xi^{8-j}) = c_1^j \pi_*(\xi^{8-j})$. This vanishes if $j > 3$ and otherwise $c_1^j \pi_*(\xi^{8-j}) = p^j s_{3-j}(S^2 T^*)$ (s_k Segre classes).

- *Result*: $N = 92$.

This is solved by blowing up the Veronese surface of double lines in the space $X = \mathbf{P}^5$ of plane conics.

By Bézout, the hypersurface X_C of conics tangent to a given conic C has class $6H$ in X, where H is the class of a hyperplane in $\mathbf{P}^5$. The class of the strict transform of X_C on $\tilde{X}$ turns out to be $6H - 2D$, where D is the class of the exceptional divisor. Thus $N = \int_{\tilde{X}}(6H - 2D) = 3264$.



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