COMPUTING CHOW GROUPS

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To Dian Fossey, in memoriam

1. Introduction

In this paper we prove two results concerning Chow groups. The first gives information about the Chow groups of schemes (see the conventions below) that have a "sufficiently nice" filtration (see **Theorem 1**). This theorem implies, in particular, that the Chow groups of a scheme that possesses a cellular decomposition are free with basis the closures of the cells (see **Corollary** to **Theorem 1**). This result seems to be well known in characteristic 0; we include a proof in general because we have not found one in the literature (see, for instance, Fulton [1984], 1.9.1, where it is proved that the cells *generate* the Chow groups)

The second result (**Theorem 2**) gives information about the Chow groups of "nice fibrations" in terms of the Chow groups of the base and the Chow groups of the fiber. Part (i) of this theorem generalizes substantially the statement 1 10.2 in Fulton [1984], while part (ii) gives a tool for computing Chow groups that seems to be more effective than alternative methods that are available, such as those derived from the results of Bialynicki-Birula [1973,1976] about actions of the multiplicative group on complete smooth schemes with finitely many fixed points. These methods have been used by Ellingsrud and Stromme [1984] to compute the Chow groups of $Hilb^kP^2$, the Hilbert scheme of k-tuples of P^2, for all integers k (for the case of $Hilb^3P^2$ see Elencwajg and Le Barz [1985a]).

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One of the motivations of the present work was the study of $\mathrm{Cop}^k\mathbf{P}^n$ (and in particular $\mathrm{Hilb}^3\mathbf{P}^n$), the scheme which parametrizes, in the sense of Hilbert scheme, k-tuples of coplanary points in $\mathbf{P}^n$ The aim of such an study is to establish enumerative formulae for multisecant planes similar to the multisecant formulae for lines, especially those obtained by Le Barz. But it turns out that $\mathrm{Cop}^k\mathbf{P}^n$, for k>3, among other pathologies, is singular, and that $\widetilde{\mathrm{Cop}}^k\mathbf{P}^n$, the scheme that parametrizes pairs formed by a k-tuple of points in $\mathbf{P}^n$ and a plane that contains it, is a natural desingularization of $\mathrm{Cop}^k\mathbf{P}^n$, and so many of the formulae we are seeking can already be obtained from the knowledge of the Chow groups of $\widetilde{\mathrm{Cop}}^k\mathbf{P}^n$.

The computation of these groups using the method of Bialynicki-Birula [1973,1976] appears to be much more intricate than for the case treated by Ellingsrud and Strømme, which makes it desirable to have a more convenient method at hand. This computation has been done, using the methods introduced in this paper, by Rosselló [1986], regarding $\widetilde{\mathrm{Cop}}^k\mathbf{P}^n$ as a fibration over the Grassmannian of planes. In this paper we give an independent computation of the Chow groups of $\mathrm{Hilb}^3\mathbf{P}^n$ (see **Theorem 3**) which has interest in itself. As far as the determination of the multiplicative structure of $A.(\mathrm{Hilb}^3\mathbf{P}^3)$ goes, as well as enumerative applications of it, they will appear elsewhere.

For the relevance in enumerative geometry of knowing that certain Chow groups are finitely generated and free, we refer to the articles of Kleiman [1976,1979].

2. Notations and conventions

By scheme we shall understand an algebraic **k**-scheme of finite type which can be embedded as a closed subscheme of a smooth **k**-scheme of finite type, where **k** is an algebraically closed field. The hypothesis of finite type for the schemes is in order to apply the intersection theory as developed in Fulton [1984]. Our restriction to a field **k** comes from the fact that in our arguments we use an homology theory satisfying properties (**a**) to (**d**) below. In the characteristic 0 case, it is the homology with locally finite supports, or the Borel-Moore homology (see Fulton [1984], Ch. 19; Fulton-MacPherson [1981], Ch. III; Iversen [1986], Ch. 10), and if **k** has positive characteristic p then the homology theory is defined as some suitable relative l-adic cohomology, l a prime number different from p (see Iversen [1986], Ch. 9 and Laumon [1976]). If the quoted properties of intersection and homology theories could be guaranteed under more general hypothesis, then our proofs would be also valid in such a generality. Notice also that (**i**) of **Theorem 2** does not involve any homology arguments

By a closed filtration of a scheme we shall understand a finite filtration by closed subschemes.

We shall let H_i denote an homology theory, that is, a functor from schemes to abelian groups that is covariant for proper maps and contravariant for open embeddings, and which, moreover, satisfies the following statements (see [F], Ch. 19):

(a) Let X be a scheme, Y a closed subscheme and $U = X - Y$ Then there exists a long exact sequence

$$\to H_{i+1}(U) \to H_i(Y) \to H_i(X) \to H_i(U) \to$$

(b) For any finite disjoint union of schemes $\cup X_i$, and for all k,

$$H_k(\cup X_i) = \oplus H_k(X_i).$$

(c) For all schemes X and all integers k there exists a map

$$cl_X : A_k(X) \to H_{2k}(X)$$

that commutes with push-forward by proper morphisms and with restrictions to open sets.

In characteristic 0 we shall say cl_X that is an isomorphism if cl_X is an isomorphism and $H_{2k+1}(X)=0$ for all k. In characteristic $p > 0$ we shall say that cl_X is an isomorphism if

$$cl_X : A_k(X) \otimes \mathbf{Z}_l \to H_{2k}(X)$$

is an isomorphism for all k, and $H_{2k+1}(X) = 0$ for all k.

(We do not know whether "cl_X is an isomorphism for all k" implies "$H_{2k+1}(X) = 0$ for all k")

(d) If X is a scheme such that cl_X is an isomorphism, then given any projective bundle

$$\mathbf{P} \to X$$

the map $cl_{\mathbf{P}}$ is an isomorphism.

For convenience of the exposition we shall first develop in detail the characteristic 0 case and in Section 6 we will explain the slight modifications of the proofs that are required in characteristic $p>0$.

Now combining properties (a)-(d) we prove a simple lemma which plays a key role in the proof of our theorems.

Lemma

Let X be a scheme such that cl_X is an isomorphism. Then for any fiber bundle

$$E \to X$$

we have that cl_E is an isomorphism.

Proof

We shall use induction on n. The case n = 0 is a direct consequence of the hypothesis and the fact that $X_0 = Z_0$.

Assume now that the theorem is true for n-1. Then we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H_{2k}(X_{n-1}) & \to & H_{2k}(X_n) & \to & H_{2k}(Z_n) & \to 0 \\ & \uparrow cl^k_{X_{n-1}} & & \uparrow cl^k_{X_n} & & \uparrow cl^k_{Z_n} & \\ & A_k(X_{n-1}) & \to & A_k(X_n) & \to & A_k(Z_n) & \to 0 \end{array}$$

In this diagram, $cl_{X_{n-1}}$ is an isomorphism by induction and cl_{Z_n} is an isomorphism by hypothesis, so we see, by (a) and the definition, that the top row is exact. The bottom row is also exact. Therefore, cl_{X_n} is an isomorphism and hence we have en exact sequence

$$(*) \qquad 0 \to A_k(X_{n-1}) \to A_k(X_n) \to A_k(Z_n) \to 0.$$

In this exact sequence, by induction, $A_k(X_{n-1})$ is a finitely generated free group such that the classes of the images in X_{n-1} of cycle representatives of given bases of the $A_k(Z_i)$, for i < n, form a basis Since $A_k(Z_n)$ is free by hypothesis, the exact sequence (*) is split and from this the theorem follows. ◆

We say (see Fulton [1984], Ex. 1.9.1) that a scheme X has a cellular decomposition if there exists a closed filtration

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \varnothing$$

such that

$$Z_i = X_i - X_{i-1}$$

is a disjoint union of locally closed subschemes Z_{ij} isomorphic to affine spaces $A^{m_{ij}}$. The Z_{ij} will be referred to as cells of the cellular decomposition. These notations will be used henceforth.

Corollary

Let X be a scheme and assume that X admits a cellular decomposition. Then $A_k(X)$ is, for all k, a finitely generated free group for which the classes of the closures of the k-dimensional cells form a basis.

Proof

Let $\overline{P} = P(E\oplus 1)$ be the projective completion of E, so that we have an open embedding

$$j : E \to \overline{P}$$

such that $\overline{P}-E = P$, where P is the projective bundle associated to E. Therefore we have, for all k, a commutative diagram

$$\begin{array}{ccccccccc} 0 \to H_{2k+1}(E) \to & H_{2k}(P) & \to & H_{2k}(\overline{P}) & \to & H_{2k}(E) & \to 0 \\ & \uparrow cl_P^k & & \uparrow cl_{\overline{P}}^k & & \uparrow cl_E^k & \\ 0 \to & A_k(P) & \to & A_k(\overline{P}) & \to & A_k(E) & \to 0 \end{array}$$

in which the rows are exact and cl_P^k, $cl_{\overline{P}}^k$ are isomorphisms. For the latter we use **(d)**, which then implies that the odd homology groups of **P** (and of $\overline{P}$) are zero, so that by **(a)** we get the exactness of the top row. For the exactness of the bottom row, see Fulton [1984], 1.8. Moreover, the map $A_k(P) \to A_k(\overline{P})$ is injective (See Fulton [1984], Theorem 3.3 and its proof.) Now by a little diagram chasing we easily get that cl_E^k is an isomorphism and that $H_{2k+1}(E) = 0$. ◆

3. Good closed filtrations

Theorem 1

Let X *be a scheme and let*

$$X = X_n \supset X_{n-1} \supset \ldots \supset X_0 \supset X_{-1} = \varnothing$$

be a closed filtration of X. *Set*

$$Z_i = X_i - X_{i-1}$$

and assume that for all i

(i) $A_k(Z_i)$ *is a finitely generated free group, and*

(ii) cl_{Z_i} *is an isomorphism.*

Then cl_X *is an isomorphism and, for all* k, $A_k(X)$ *is finitely generated free group. Moreover, the union of the classes of the images in* X *of representative cycles of free bases of* $A_k(Z_i)$ *is a free basis of* $A_k(X)$.

Proof

With the notations explained before, we shall show that the conditions (i) and (ii) of **Theorem 1** are fullfilled.

That (i) is satisfied is a direct consequence of the definitions and Fulton [1984], 1.3.1 and 1.9. Moreover, the classes of the k-dimensional cells of Z_i form a basis of $A_k(Z_i)$. Similarly, the lemma above and the properties of *cl* imply that cl_{Z_i} is an isomorphism. ◆

Remark 1

This proof also gives that cl_X is an isomorphism under the conditions of the corollary, which is the statement (b) in Ex. 19.1.11 of Fulton [1984].

Remark 2

The above **Corollary** implies Theorem 4.5 in Bialynicki-Birula [1973], which says that the number of cells of given dimension in any two cellular decompositions of X is the same. Let us also remark that the proof in Bialynicki-Birula [1973] is purely combinatorial.

4. The Chow group of some fibre spaces

Theorem 2

Let X be a scheme which admits a cellular decomposition and let

$$f : X' \rightarrow X$$

be a morphism such that for all cells Z_{ij} *of the decomposition*

$$f^{-1}(Z_{ij}) \cong Z_{ij} \times F$$

where F is a fixed scheme. Then

(i) *For all* k *there exists an epimorphism*

$$(*) \qquad \bigoplus_{r+s=k} A_r(X) \otimes A_s(F) \rightarrow A_k(X')$$

(ii) *If* cl_F *is an isomorphism and* $A_k(F)$ *is free for all* k, *then* (*) *is an isomorphism for all* k *and* $cl_{X'}$ *is an isomorphism.*

Proof

Let $X'_i = f^{-1}(X_i)$, so that

$$X' = X'_n \supset X'_{n-1} \supset \ldots \supset X'_0 \supset X'_{-1} = \emptyset$$

is a closed filtration of X'. We shall write $Z'_i = X'_i - X'_{i-1} = f^{-1}(Z_i)$.

For all i, j we fix an isomorphism

$$h_{ij} : Z_{ij} \times F \cong f^{-1}(Z_{ij})$$

To prove (i) we shall proceed by induction on n. If n = 0 then

$$\begin{aligned} A_k(X'_0) &= A_k(f^{-1}(Z_0)) \\ &= \oplus_j A_k(f^{-1}(Z_{0j})) \xleftarrow[\sim]{\oplus h_{0j*}} \oplus A_k(Z_{0j} \times F) \\ &= \oplus_j A_{m_{0j}}(Z_{0j}) \otimes A_{k-m_{0j}}(F) \\ &= \oplus_j \left(\oplus_{r+s=k} A_r(Z_{0j}) \otimes A_s(F) \right) \\ &= \bigoplus_{j,\, r+s=k} A_r(Z_{0j}) \otimes A_s(F) \\ &= \bigoplus_{r+s=k} A_r(X_0) \otimes A_s(F). \end{aligned}$$

Notice that the resulting isomorphism

$$h_0 : \bigoplus_{r+s=k} A_r(X_0) \otimes A_s(F) \to A_k(X'_0)$$

is such that

$$h_0([Z_{0j}] \otimes [V]) = [h_{0j}(Z_{0j} \times V)],$$

for all j and any [V] in $A_{k-m_{0j}}(F)$.

Assume now by induction that (i) is true for n-1, i. e., that we have an epimorphism

$$h_{n-1}: \bigoplus_{r+s=k} A_r(X_{n-1}) \otimes A_s(F) \to A_k(X'_{n-1})$$

such that

$$h_{n-1}([Z_{i'j'}] \otimes [V]) = [\overline{h_{i'j'}(Z_{i'j'} \times V)}]$$

for i' < n, any j' and any [V] in $A_{k-m_{i'j'}}(F)$.

Now reasoning as in the case n=0 we see that there exists an isomorphism

$$g_n : \bigoplus_{r+s=k} A_r(Z_n) \otimes A_s(F) \to A_k(Z'_n)$$

such that

$$g_n([Z_{nj}] \otimes [V]) = [h_{nj}(Z_{nj} \times V)]$$

for any j and any [V] in $A_{k-m_{nj}}(F)$. From these facts and the exact sequence

$$A_k(X'_{n-1}) \to A_k(X'_n) \to A_k(Z'_n) \to 0$$

one can easily construct an epimorphism

$$h_n : \bigoplus_{r+s=k} A_r(X_n) \otimes A_s(F) \to A_k(X'_n)$$

such that

$$h_n([Z_{i'j'}] \otimes [V]) = [\overline{h_{i'j'}(Z_{i'j'} \times V)}]$$

for i' < n, any j' and any [V] in $A_{k-m_{i'j'}}(F)$. This ends the proof of (i).

In order to see (ii) we shall prove that the filtration

$$X' = X'_n \supset X'_{n-1} \supset \ldots \supset X'_0 \supset X'_{-1} = \varnothing$$

satisfies the statements (i) and (ii) of **Theorem 1**.

The statement (i) has already been proved above (isomorphisms g). Notice, moreover, that if for all k we have bases

$$\{\Sigma\, m_{j,\alpha}[V_{j,\alpha}]\}_{\alpha}$$

of $A_k(F)$ then we have bases

$$\{\Sigma\, m_{j,\alpha}[h_{ij'}(Z_{ij'}\times V_{j,\alpha})]\}_{\alpha,j'}$$

of $A_k(Z_i)$.

Now to see that statement (ii) is satisfied, notice that by the lemma $cl_{Z_{ij}\times F}$ is an isomorphism because cl_F is an isomorphism. From this it follows that $cl_{f^{-1}(Zij)}$ is an isomorphism and so

$$cl_{Z_i^!} = \oplus cl_{f^{-1}(Zij)}$$

is an isomorphism. Thus we can apply **Theorem 1**, which gives that $cl_{X'}$ is an isomorphism and that $A_k(X')$ is finitely generated free group for which the elements

$$\{\Sigma\, m_{j,\alpha}[\overline{h_{ij'}(Z_{ij'}\times V_{j,\alpha})}]\}_{\alpha,i,j'}$$

form a basis. This implies that the epimorphism h_n is an isomorphism. ♦

Corollary (of the proof)

In **Theorem 2 (ii)** if we assume that cl_F is an isomorphism, then (*) is an isomorphism for all k, up to torsion, and $cl_{X'}$ is an isomorphism. ♦

5. A.(Hilb³Pⁿ)

Let Gr(2,n) be the Grassmannian of planes in projective n-space $\mathbf{P}^n$, n≥3. Let $Al^3\mathbf{P}^n$ be the subscheme of $Hilb^3\mathbf{P}^n$ that parametrizes triples of colinear points and let i be the closed embedding of Al^3P^n in $Hilb^3P^n$. Let $U' = Hilb^3P^n - Al^3P^n$. We have that the map which sends a given triple of non-colinear points to the unique plane that contains it, $U' \to Gr(2,n)$, is locally trivial with fiber

$$U = Hilb^3P^2 - Al^3P^2$$

Proposition

The Chow groups of U *are given by the following table:*

i	0	1	2	3	4	5	6
A_i	0	$Z/3Z$	$Z^2 \oplus Z/3Z$	Z^3	Z^3	Z	Z

and cl_U *is an isomorphism. Moreover, these groups are determined by the following table of generators and (abelian) relations* (we use notations and conventions of Elencwajg and Le Barz [1985b], which for convenience of the reader we list at the end of this paper):

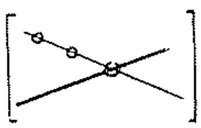
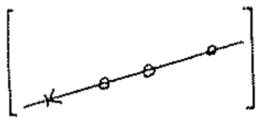
i	*Generators*	*Relations*
6	U	
5	H	
4	H^2, h, p	
3	H^3, Hh, β	
2	H^2h, h^2, hp	$3H^2h - 6h^2 - 6hp = 0$
1	Hhp	$3Hhp = 0$

Proof

The sequence

$$(*) \qquad A_k(Al^3P^2) \xrightarrow{i_*} A_k(Hilb^3P^2) \to A_k(U) \to 0$$

is exact by Fulton [1984], 1.8. Now we first compute $A_\cdot(Al^3P^2)$ using the fact that **Proposition 2** in Le Barz [1987] actually gives **Z**-bases of $A_k(Al^3P^2)$. Consider the divisors

V =  V' = 

on Al^3P^2. Then we have that

$A_5 = Z$, with basis $[Al^3P^2]$,

$A_4 = Z^2$, with a basis given by V and V',

$A_3 = Z^3$, with a basis given by V^2, VV' and V'^2,

$A_2 = \mathbf{Z}^3$, with a basis V^3, V^2V', VV'^2

$A_1 = \mathbf{Z}^2$, with a basis $V^3V', V^2V'^2$, and

$A_0 = \mathbf{Z}$.

Now we know a basis of $A.(\mathrm{Hilb}^3\mathbf{P}^2)$ (see Elencwajg and Le Barz [1985b]),

i	A_i	*Bases*
5	$\mathbf{Z}^2$	H, A
4	$\mathbf{Z}^5$	H^2,HA,h,a,p
3	$\mathbf{Z}^6$	H^3, H^2A, Hh, Ha, α, β
2	$\mathbf{Z}^5$	H^2h, H^2a, h^2,ha, hp
1	$\mathbf{Z}^2$	Hha, Hhp
0	$\mathbf{Z}$	h^3

An straightforward computation shows us that

$$i_*([Al^3\mathbf{P}^2]) = A.$$

$$i_*(V) = HA, \quad i_*(V') = a.$$

$$i_*(V^2) = H^2A, \quad i_*(VV') = Ha, \quad i_*(V'^2) = \alpha.$$

$$i_*(V^3) = 3H^2h + 6H^2a - 6h^2 - 18ha - 6hp.$$

$$i_*(V^2V') = H^2a, \quad i_*(VV'^2) = ha.$$

$$i_*(V^3V') = 3Hhp + 3Hha, \quad i_*(V^2V'^2) = Hha.$$

$$i_*(V^3V'^3) = h^3.$$

From these relations we infer on the one hand that $A_k(U)$ are the groups given in the statement, and on the other that i_* is a monomorphism. Finally by an argument similar to that used in the proof of the **Lemma in Section 2** we conclude that cl_U is an isomorphism. ♦

Remark 3

The scheme U provides an example of a scheme in which cl_U is an isomorphism and $A.(U)$ has torsion.

Unfortunately, in order to compute the Chow groups of U', it is not possible to apply **Theorem 2** as it stands. Notice, however, that the **Corollary** to **Theorem 2** applies and so

$$A_k(U')_{\mathbf{Q}} \cong \oplus_{i=0:k} A_i(U)_{\mathbf{Q}} \otimes A_{k-i}(Gr(2,n))_{\mathbf{Q}}$$

and $cl_{U'}$ is an isomorphism, whence

$$0 \to A_k(Al^3\mathbf{P}^n)_Q \to A_k(Hilb^3\mathbf{P}^n)_Q \to A_k(U')_Q \to 0$$

is a split exact sequence, hence

$$b_k(Hilb^3\mathbf{P}^n) = b_k(Al^3\mathbf{P}^n)+\Sigma_{i=0:k}b_i(U).b_{k-i}(Gr(2,n))$$

$$=\Sigma_{i=0:k}b_i(\mathbf{P}^3).b_{k-i}(Gr(1,n))+\Sigma_{i=0:k}(b_i(Hilb^3\mathbf{P}^2)-b_i(Al^3\mathbf{P}^2)).b_{k-i}(Gr(2,n))$$

Finally A $(Hilb^3\mathbf{P}^n)$ is free (apply Bialynicki-Birula [1973]) and so we have obtained a formula for $rg_Z A_k(Hilb^3\mathbf{P}^n)$. For a combinatorial expression of this formula, see Rosselló [1986].

Remark 4

The expression of $b_k(Hilb^3\mathbf{P}^n)$ given in Rosselló [1986] is different from the one given above, but it is not difficult to see, again using **Theorem 2**, that they agree.

Remark 5

Theorem 2 can also be applied to determine the Betti numbers of varieties of ordered triangles in projective space. Define $W_n{}^*$, for $n\geq 3$, as the closure in

$$(\mathbf{P}^n)^3\times Gr(1,n)^3\times Gr(2,n)\times Gr(2,\mathbf{P}(Sym^2E_n{}^*))$$

of

$$\{(x_1,x_2,x_3,l_1,l_2,l_3,\pi,\Sigma) \mid x_1,x_2,x_3 \text{ distinct points, } x_i\in l_j \text{ for all } j\neq i,$$
$$\pi \text{ the plane spanned by } x_1,x_2,x_3, \text{ and } \Sigma \text{ the 2-dimensional system}$$
$$\text{of conics in } \pi \text{ that contain the points } x_1,x_2,x_3\}.$$

Then one can see that the projection from $W_n{}^*$ to $Gr(2,n)$ is a fibration that satisfies the hypothesis of **Theorem 2**, with fiber the triangle variety W^* of Schubert, Semple, and Roberts and Speiser, and thus one can obtain that $A_k(W_n{}^*)$ is a finitely generated free abelian group of rank

$$b_k(W_n{}^*) = \Sigma_{i=0:k}\, b_i(Gr(2,n)b_{k-i}(W^*)$$

and the class map is an isomorphism for all k. In particular one easily sees that

$$b_k(W_n{}^*) = b_{3n-k}(W_n{}^*).$$

Here is a table comparing Betti numbers for W_3^* and for $\mathrm{Hilb}^3\mathbf{P}^3$:

k	0	1	2	3	4	5	6	7	8	9
$b_k(W_3^*)$	1	8	25	47	63	63	47	25	8	1
$b_k(\mathrm{Hilb}^3\mathbf{P}^3)$	1	2	6	10	13	13	10	6	2	1

6. The case of positive characteristic

We shall indicate briefly how to modify the proofs of the characteristic 0 case when the charateristic of the ground field is p>0. We only need to take care of the proof of **Theorem 1** (**Section 3**) and the **Lemma** (**Section 2**) because these are the elements used in the proof of **Theorem 2**.

As far as the lemma goes, it is enough to consider, instead of the diagram in the proof of the **Lemma**, the diagram

$$\begin{array}{ccccccc} 0 \to H_{2k+1}(E) \to H_{2k}(P) & \to & H_{2k}(\overline{P}) & \to & H_{2k}(E) & \to & 0 \\ \uparrow cl_P^k & & \uparrow cl_{\overline{P}}^k & & \uparrow cl_E^k & & \\ 0 \to A_k(P)\otimes Z_l & \to & A_k(\overline{P})\otimes Z_l & \to & A_k(E)\otimes Z_l & \to & 0 \end{array},$$

and reason in the same way as there, but using the definition of "*cl* isomorphism" given for the positive characteristic case.

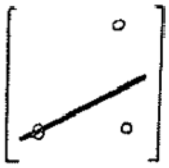
For the proof of **Theorem 1**, notice that step 0 of the induction is still valid. If n>0, let K denote the kernel of the map $A_k(X_{n-1}) \to A_k(X_n)$, which is free because by the inductive hypothesis the group $A_k(X_{n-1})$ is free. Consider the diagram

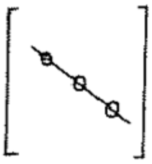
$$\begin{array}{ccccccccc} & & 0 \to H_{2k}(X_{n-1}) & \to & H_{2k}(X_n) & \to & H_{2k}(Z_n) & \to & 0 \\ & & \uparrow cl_{X_{n-1}}^k & & \uparrow cl_{X_n}^k & & \uparrow cl_{Z_n}^k & & \\ 0 \to K\otimes Z_l & \to & A_k(X_{n-1})\otimes Z_l & \to & A_k(X_n)\otimes Z_l & \to & A_k(Z_n)\otimes Z_l & \to & 0. \end{array}$$

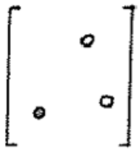
Now the same argument as in the proof of **Theorem 1** shows that the middle vertical arrow is an isomorphism and that $K\otimes Z_l$ is 0. Hence $K = 0$ and the proof can be continued as in the characteristic 0 case.

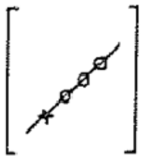
List of notations and conventions (after Elencwajg and Le Barz)

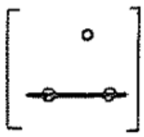
A bold line (resp. point) stands for a fixed line (resp. for a point of the triple). An ordinary line stands for a variable line, and a small circle for a variable point of the triple. A cross denotes a fixed point of the plane.

H = {triples of P^2 with one of its points on a given line} = 

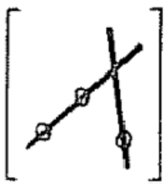
$A = [Al^3P^2] =$ 

h = {triples with a fixed point} = 

a = {triples that are colinear with a given point} = 

p = {triples with two points on a given line} = 

α = {triples on a fixed line} = 

β = {one point on a fixed line and the other two on another} = 

$b_i(X)$ = Betti number of X = rank $A_i(X)$

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