REMARKS ON MULTIPLICITIES

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A) Preliminaries and notations

Let A be a local ring*, m its maximal ideal, and $k = A/m$. Let M be an A-module*, and set $d = \dim(M)$. Let q be an ideal of A such that $\ell_A(M/qM)<\infty$ and $x = (x_1,\dots,x_r) \in A^r$ such that $\ell_A M/xM < \infty$ (hence $r \geq d$). Then $P_{M,q}(n) = \ell_A(M/q^nM)$ is a polynomial in n for n >> 0 (Hilbert-Samuel polynomial) and its degree is d. As usual we set $e_q(M)$ to denote the coefficient of $n^d/d!$ in $P_{M,q}(n)$, so that $e_q(M) = \Delta^d P_{M,q}(n)$, where for any sequence $a = \{a_n\}$ we write $\Delta a(n) = a_{n+1}-a_n$.

We also define $\mu_x(M) = \Delta^r P_{M,(x)}(n)$, so that

$$\mu_x(M) = \begin{cases} e_{(x)}(M) > 0 & \text{if } r = d \\ 0 & \text{if } r > d \end{cases}$$

Now Serre's main theorem on multiplicities states the following:

Theorem 0

$$\mu_x(M) = \chi(x;M) =: \sum_{i\geq 0} (-1)^i \ell_A H_i(K.(x;M)),$$

where K.(x;M) is the Koszul complex of x with coefficients in M. ##

Remark. Serre's original proof [S] uses the spectral sequence associated to the Koszul complex; for a more direct proof, see [A-B], or [X].

B) Agreement between Northcott and Serre's multiplicities

In [N] Northcott gives a recursive definition of multiplicity as follows:

$$e_A(x_1;M) = \ell_A(M/x_1M) - \ell_A(\ker(x_1;M)), \text{ for } r = 1, \text{ and for } r \geq 2,$$

$$e_A(x_1,\dots,x_r;M) = e_A(x_2,\dots,x_r;M/x_1M) - e_A(x_2,\dots,x_r;\ker(x_1;M))^{**}.$$

* Except when otherwise stated, rings are assumed to be commutative, noetherian, and local, and A-modules are assumed to be finitely generated

** $\ker(a;M) =: (0:a)_M$

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Afterwards Northcott proves that $e_A(x_1,\dots,x_r;M) = \mu_x(M)$. Since his argument is rather long, it may have some interest to give a proof of this result by proving that $e_A(x_1,\dots,x_r;M) = \chi(x_1,\dots,x_r;M)$ and then using theorem 0. The goal of this section is providing a direct proof of this equality.

Lemma 1 (See [N], pp. 367-368)

If $x_1 \notin Z(M)$ then there exists an exact sequence

$$0 \to K.(x;M) \longrightarrow K.(x';M) \longrightarrow K.(\bar{x};\bar{M})(-1) \to 0 ,$$

where if $x = (x_1,\dots,x_r)$ we set $x' = (1,x_2,\dots,x_r)$, $\bar{x} = (x_2,\dots,x_r)$ and $\bar{M} = M/x_1M$. ##

Remark. Since the sequence x' contains a unit, $H_i(K.(x';M)) = 0$ for all i and consequently $\chi(x;M) = \chi(\bar{x};\bar{M})$. Notice also that lemma 1 is true if we replace x_1 for any x_i that is not a zero-divisor in M.

Lemma 2. For $n \gg 0$,

$$\chi(x;M) = \chi(\bar{x};M/x_1M+\ker(x_1^n;M)) = \chi(\bar{x};\bar{M}) - \chi(\bar{x};\ker(x_1;M)).$$

Proof

Put $N = M/\ker(x_1^n;M)$ for $n \gg 0$, so that $\ker(x_1;N) = 0$. Then from the exact sequence $0 \to \ker(x_1^n;N) \longrightarrow M \to N \to 0$ we get $\chi(x;M) = \chi(x;N)$, because x_1 is nilpotent on $\ker(x_1^n;M)$. Hence, by lemma 1, $\chi(x;N) = \chi(\bar{x};N/x_1N)$, and so $\chi(x;M) = \chi(\bar{x};M/x_1M+\ker(x_1^n;M))$. Now one sees immediately that this last expression is equal to $\chi(x;\bar{M})-\chi(x;\ker(x_1^n;M)/x_1M\cap\ker(x_1^n;M))$. From this, the equality $x_1M\cap\ker(x_1^n;M) = x_1\cdot\ker(x_1^n;M)$, and the exact sequence

$$0 \to \ker(x_1;M) \to \ker(x_1^n;M) \xrightarrow{x_1} \ker(x_1^n;M) \to \ker(x_1^n;M)/x_1\ker(x_1^n;M) \to 0,$$

$\chi(x;\ker(x_1;M)/x_1M\cap\ker(x_1^n;M)) = \chi(x;\ker(x_1;M))$. This proves the lemma. ##

Remark. Equality between the first and third expressions in the lemma was proved by Auslander-Buchsbaum in [A-B], Th. 3.3; the proof above is simpler and shorter.

Theorem 1

$$\chi(x;M) = e_A(x_1,\dots,x_r;M)$$

Proof

By lemma 2 both expressions satisfy the same recursive formula, so it is enough to see the equality for $r = 1$, in which case it is immediate from the definitions. ##

C) Generalization of a formula of Boda-Vogel

We fix the following notations. If $N \hookrightarrow M$ is a submodule of M, we set $U(N) = \bigcap Q$, where Q runs through the primary submodules of M belonging to N

such that $\dim(M/N) = \dim(M/Q)$. (Henceforth Q will always denote a primary module.) We define p_Q by $\mathrm{Ass}(M/Q) = \{p_Q\}$.

Given a system of parameters (sop) $x_1,\ldots,x_d$ for M ([S], III-10), we set $q = (x_1,\ldots,x_d)$ and $q_i = (x_1,\ldots,x_i)$. Finally we set $M_o = 0$ and for $i = 1,\ldots,d$, $M_i = x_iM + U(M_{i-1})$.

With these notations, the Boda-Vogel formula ([B-V], Prop. 1), which expresses the multiplicity of an ideal in a local ring as a length, can be generalized to modules as follows:

Proposition 1

$$e_q(M) = \ell_A(M/M_d)$$

Proof

There is no loss of generality if we replace A by $A/\mathrm{Ann}\,M$ and so we can assume that $\dim(M) = d$, that M is a faithful A-module, and that $(x_1,\ldots,x_d)$ is also a system of parameters for A.

In order to prove the poposition we will use induction on d. First we establish the case $d = 1$ ($d=0$ is immediate). Since we have seen that $e(x_1;M) = \ell_A(M/x_1M+\ker(z;M))$, $z = x_1^n$, $n \gg 0$, it suffices to prove that $U(0) = \ker(z;M)$.

To begin with, we know that $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(M/\ker(z;M)) \cup \mathrm{Ass}(\ker(z;M))$ and that equality holds for minimal primes (these are the generic points of the components of $\mathrm{Supp}(M)$). Next, if $p \in \mathrm{Spec}(A)$, then $h(p) = 0$ iff $x_1 \notin p$, and so $\{p \in \mathrm{Ass}(M) \mid x_1 \notin p\} = \{p \in \mathrm{Ass}(M/\ker(z;M)) \mid x_1 \notin p\}$ (by [A-B], lemma 4.5) $= \mathrm{Ass}(M/\ker(z;M))$. So $\ker(z;M) = \bigcap Q$, where Q runs through the primary modules such that $x_1 \notin p_Q$ and $\ker(z;M) \subseteq Q$.

Now from the fact that if $x_1 \notin p_Q$ then already $\ker(z;M) \subseteq Q$, and that $h(p_Q) = 0$ iff $\dim(M/Q) = 1$ * we finally get that $\ker(z;M) = \bigcap Q$ (for $\dim M/Q = 1$), $= U(0)$.

So suppose that $d > 1$. From the previous section we know that $e_A(x_1,\ldots,x_d;M) = e_A(x_2,\ldots,x_d;M')$, $M' = M/(x_1M+\ker(z;M))$, $z = x_1^n$, $n \gg 0$.

Claim: $\dim(M') = d-1$, and $(x_2,\ldots,x_d)$ is a sop for M'.

Indeed, since $M'/q'M'$, $q' = (x_2,\ldots,x_d)$, is a quotient of M/qM, $\dim(M'/q'M') = 0$ and so we only have to prove that $\dim(M') = d-1$. But this follows from the fact that $\mathrm{Ann}(M/\ker(z;M)) = (0:z)$, so that $\dim(M/\ker(z;M)) = d$, and the fact that $(x_1,\ldots,x_d)$ is still a sop for $M/\ker(z;M)$. This proves the claim.

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* In fact $\dim(M/Q) = \dim(M) - h(p_Q)$. This equality will be used several times in the sequel

So if we set $M_1' = 0$ and $M_i' = x_iM'+U(M_{i-1}')$, for $2\le i\le d$, then by induction we have $e_{q'}(M') = \ell_A(M'/M_d')$. Hence $e_q(M) = \ell_A(M'/M_d')$, by lemma 2. Now define $M_1^* = x_1M+\ker(z;M)$ and $M_i^* = x_iM+U(M_{i-1}^*)$, for $2\le i\le d$. We shall prove that

$$\ell_A(M'/M_d') \overset{(1)}{=} \ell_A(M/M_d^*) \overset{(2)}{=} \ell_A(M/M_d)$$

Here, for further reference we state the following easy lemma.

Lemma 3

If $S \hookrightarrow N \hookrightarrow M$, then $U(N)/S = U(N/S)$. ##

Proof of (1): Using lemma 3 one sees by induction that $M_i' = M_i^*/M_1^*$. From this equality (1) follows immediately.

Proof of (2): We only need to see that $U(M_1) = U(M_1^*)$. To begin with, from $\ker(z;M) \subseteq U(0)$ we get that $M_1 \subseteq M_1^*$. Next, we observe that $\dim(M/M_1) = d-1$, because $\dim(M/U(0)) = d$ and $x_1,\dots,x_d$ is still a sop for $M/U(0)$. Finally equality (2) follows from the following claim: if $M_1^* \subseteq Q$ and $h(p_Q) = 1$, then $M_1 \subseteq Q$. To see this claim we only need to prove that $(M_1^*)_p \supseteq (M_1)_p$ for p a prime ideal with $h(p) = 1$, inasmuch as the p_Q satisfying the hypothesis of the claim are the minimal prime ideals of $\mathrm{Ass}(M/M_1^*)$. Since $x_1 \in p$, we want to prove that $U(0)_p = \ker(z;M)_p$. To see this notice that $x_1 \in pA_p$ is a system of parameters for M_p and therefore $\ker(z;M)_p = \ker(z;M_p) = U(0\cdot A_p) = U(0)_p$. ##

Next theorem, also a generalization of a result of Boda-Vogel ([B-V], Th. 1) makes it easier to apply last proposition. Lets first recall a definition.

Definition ([A-B]). A sequence $x_1,\dots,x_d$ of elements of A is said to be an M-reducing sop for A iff (a) $x_1,\dots,x_d$ is a sop for M, and (b) for all $p \in \mathrm{Ass}(M/q_{i-1}M)$ such that $\dim(A/p) = d-i$ we have that $x_i \notin p$.

Theorem 2

The following statements are equivalent:

(i) $e_q(M) = \ell_A(M/x_dM+U(q_{d-1}M))$

(ii) $x_1,\dots,x_d$ is an M-reducing sop for M

(iii) $U(q_iM) = U(M_i)$, for $1 \le i \le d-1$

(iv) $U(q_{d-1}M) = U(M_{d-1})$

Proof

It is clear that (iii) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (i) is true by proposition 1. (i) $\Rightarrow$ (ii). Put $q' = (x_1,\dots,x_{d-1},x_d^n)$, where $n >> 0$. From $\dim(M/q_{d-1}M) = 1$ we get that $\ker(x_d^n; M/q_{d-1}M) = U(\bar{0})$ (case d=1 of proposition 1), and so

$U(\bar{0}) \cap x_d^n \cdot (M/q_{d-1}M) = 0$. From lemma 3 we infer that $U(q_{d-1}M) \cap q'M = q_{d-1}M$. Moreover, since $x_d \not\in Z(M/U(q_{d-1}M))$,

$$\begin{aligned}
e_{q'}(M) = n \cdot e_q(M) &= n \cdot e_A(x_d; M/U(q_{d-1}M)) \\
&= \ell_A(M/x_d^n M + U(q_{d-1}M)) \\
&= \ell_A(M/q'M) - \ell_A\, (x_d^n M + U(q_{d-1}M))/q'M \\
&= \ell_A(M/q'M) - \ell_A U(q_{d-1}M)/U(q_{d-1}M) \cap q'M \\
&= \ell_A(M/q'M) - \ell_A U(q_{d-1}M)/q_{d-1}M \\
&= \ell_A(M/q'M) - \ell_A(U(\bar{0})) \\
&= \ell_A(M/q'M) - \ell_A \ker(x_d^n; M/q'M) \; .
\end{aligned}$$

By [A-B], Cor. 4.8, we see that $(x_1, \ldots, x_{d-1}, x_d^n)$ is an M-reducing sop for M and thus so is $(x_1, \ldots, x_d)$.

(ii) $\Rightarrow$ (iii). We will proceed by induction on i. The case $i = 0$ is trivial, so assume $1 \le i \le d-1$, and that (iii) is true for $1, \ldots, i$. From $q_{i+1}M \subseteq M_{i+1}$ and $\dim(M/q_{i+1}M) = \dim(M/M_{i+1}) = d-i-1$ we get that $U(q_{i+1}M) \subseteq U(M_{i+1})$. Then if $(x_1, \ldots, x_d)$ is an M-reducing system of parameters, $U(q_iM) \subseteq U(q_{i+1}M)$, as we show below, so that $M_{i+1} = x_{i+1}M + U(M_i) = x_{i+1}M + U(q_iM) \subseteq U(q_{i+1}M)$, and so $U(M_{i+1}) \subseteq U(q_{i+1}M)$, which ends the proof.

To see that $U(q_iM) \subseteq U(q_{i+1}M)$, take $m \not\in U(q_{i+1}M)$. Then there exists a primary submodule Q of M with $\dim(M/Q) = d-i-1$, $q_{i+1}M \subseteq Q$, and $(Q:m) \subseteq p_Q$. So $(q_iM:M) \subseteq (q_{i+1}M:m) \subseteq (Q:m) \subseteq p_Q$, which implies that $x_{i+1} \not\in p_Q$. But being $x_1, \ldots, x_d$ an M-reducing sop we have therefore that $p_Q \not\in \operatorname{Ass}(M/q_iM)$, which implies that there exists $p_i \in \operatorname{Ass}(M/q_iM)$ such that $(q_iM:M) \subseteq p_i \subsetneq p_Q$ and $h(p_i) = i$. Thus if Q' is the p_i-primary submodule belonging to q_iM we get finally that $m \not\in Q'$. ##

<u>Remark</u>. The range of application of theorem 2 is actually broader than it would seem at first glance, because by [A-B], proposition 4.9, any ideal of A which is generated by a sop for A can actually be generated by an M-reduxing system of parameters. Recall that if we replace A by A/Ann(M) there is one-to-one correspondence between sop's for A and sop's for M.

C) <u>On two theorems of Serre</u>

In this section we give short proofs of two theorems of Serre related to multiplicities.

<u>Theorem 3</u> ([S], Th. V-4)

Let A be a regular local ring with $\dim(A) = n$. Let M, N be two A-modules such that $\ell_A(M \otimes N) < \infty$. Then $d(M)+d(N)-n = j(M,N)$, where

$j(M,N) =: \sup\{i \mid Tor_i^A(M,N) \neq 0\}$ and $d(M) = $ depth (M).

Proof

By induction on $d(M)$. If $d(M) = 0$, then $m \in Ass(M)$ and so there exists an exact sequence $0 \to A/m \to M \to \bar{M} \to 0$. From the corresponding Tor exact sequence we get the exact sequence $Tor_{j+1}^A(\bar{M},N) \to Tor_j^A(A/m,N) \to Tor_j^A(M,N)$, from which we easily see that $j(M,N) = pd_A(N)$. So in this case the relation we want to establish is equivalent to the Auslander-Buchsbaum formula $pd_A(N)+d(N) = n$.

Assume $d(M) > 0$ and take $a \in m$, $a \notin Z(M)$. From the exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$ we get the exact sequence $Tor_{j+1}^A(M/aM) \to Tor_j^A(M,N) \xrightarrow{a} Tor_j^A(M,N)$ and so if $j = j(M,N)$ we see $\ker(a;Tor_j^A(M,N)) \neq 0$ (apply A.2.1 in [F]), from which we deduce that $j(M/aM,N) = j+1$. Since $d(M/aM) = d(M)-1$ the proof is complete. ##

Remark. From theorem 3 it is easy to see that $Tor_i^A(M,N) = 0$ for $i > 0$ iff M and N are Cohen-Macaulay modules satisfying $\dim(M) + \dim(N) = n$, and in this case $\chi(M,N) = \ell_A(M \otimes N)$, so there is agreement between Serre's definition of multiplicity and that of Gröbner. Serre Conjectured ([S], V.20) that this is the only case when agreement can occur. Recently Hochster [H] has solved this affirmatively, with hypotheses more general than in the geometric case.

We shall end this note by giving a rather direct and elementary proof of the basic properties of Serre's intersection multiplicities on algebraic varieties. This proof is closely related to the proof of the agreement between Serre's multiplicities and those of Samuel. We will also obtain a proof of the dimension theorem for intersections on algebraic varieties.

Let $(V,\mathcal{O}_V)$ be an algebraic variety over a field k. Let X, Y be subvarieties ov V, and let Z be a simple component of $X \cap Y$, that is to say, Z is an irreducible component of $X \cap Y$ and $Z \not\subset Sing(V)$. As usual, we will say that X and Y intersect properly along Z if $cod_V(Z) = cod_V(X) + cod_V(Y)$. Let p_X and p_Y be the prime ideals of $\mathcal{O} =: \mathcal{O}_{Z,V}$ defined by X and Y, respectively. Then $m_{Z,V}$ is minimal over p_X+p_Y and so $\ell_{\mathcal{O}}(\mathcal{O}/p_X+p_Y) < \infty$. The intersection multiplicity of X and Y at Z, relative to V (in the sense of Serre) is defined to be the integer $I_Z^V(X,Y) =: \chi^{\mathcal{O}}(\mathcal{O}/p_X,\mathcal{O}/p_Y) =: \Sigma(-1)^i \ell_{\mathcal{O}} Tor_i^{\mathcal{O}}(\mathcal{O}/p_X,\mathcal{O}/p_Y)$. When V can be understood we simply write $I_Z(X,Y)$.

Theorem 4 ([S], Theorème1, p. V-13)

(i) $I_Z(X,Y) \geq 0$,

(ii) $I_Z(X,Y) = 0$ iff X and Y do not intersect properly at Z,

(iii) $cod_V(Z) \leq cod_V(X) + cod_V(Y)$.

Proof

First assume that X is a complete intersection at Z, i.e. $p_X=(x_1,\ldots,x_{n-r})$, where $n=\dim(V)$, $r=\dim(X)$. Then $K.(x_1,\ldots,x_{n-r};\mathcal{O}) \to \mathcal{O}/p_X \to 0$ becomes a free resolution of $\mathcal{O}/p_X$ and so $\chi^{\mathcal{O}}(\mathcal{O}/p_X,\mathcal{O}/p_Y)=\chi(x;\mathcal{O}/p_Y) = \mu_x(\mathcal{O}/p_Y)$, where we set $x=(x_1,\ldots,x_{n-r})$. Thus in this case $I_Z(X,Y) \geq 0$ and $I_Z(X,Y)=0$ iff $\dim(V)-\dim(X) > \dim(Y)-\dim(Z)$, that is, iff $\mathrm{cod}_V(Z) < \mathrm{cod}_V(X)+\mathrm{cod}_V(Y)$. In any evert we have $\dim(V)-\dim(X) \geq \dim(Y)-\dim(Z)$ (cf. section A). This proves the theorem in the case when X is a local complete intersection at Z.

Now let's consider the general case. We can suppose that V is non-singular affine variety. Let A denote the algebra of regular functions of V and set $B = A\otimes_k A$, the algebra of regular functions of $V\times V$. If $B \to A$ is the multiplication epimorphism and I is its kernel, then I is the ideal of the diagonal Δ of $V\times V$, $A=B/I$, and $\mathrm{Tor}_i^A(M,N) = \mathrm{Tor}_i^B(M\otimes_k N,A)$. From this one deduces immediately the formula of "reduction to the diagonal",

$$I_Z^V(X,Y) = I_{Z^\Delta}^{V\times V}(X\times Y,\Delta),$$

where Z^Δ is the image of Z in $V\times V$ by the diagonal morphism. But on a non-singular variety Δ is a local complete intersection, so the theorem follows by the local complete intersection case. ##

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