Systems, patterns and data engineering with geometric calculi (GC&DL)

(Born in Campinas, Brazil, on the occasion of AGACSE 2018, which was an early satellite of AGACSE 2018)

First developed as a language for physics, recently there has been an explosion of applications of **Geometric Calculus** in a great variety of areas, like general relativity, cosmology, robotics, computer graphics, computer vision, molecular geometry, quantum computing, etc.

The goal of the mini-symposium is to overview the basic ideas of GC, to report on some relevant applications, and to explore the bearing of the formalism in novel approaches to deep learning.

Geometric Calculus Techniques in Science and Engineering (Sebastià Xambó-Descamps)

Bringing New Perspectives to Robotics and Computer Science (Isiah Zaplana)

Geometric Algebra and Distance Geometry (Carlile Lavor)

Embedded Coprocessors for Native Execution of Geometric Algebra Operations (Salvatore Vitabile)

Hypercomplex Algebras for Art Investigation (Srđan Lazendić)

Conformal Geometric Algebra for Medical Imaging (Salvatore Vitabile)

Bio-inspired geometric deep learning (Eduardo U. Moya Sánchez)

Geometric calculus meets deep learning (SXD)

https://mat-web.upc.edu/people/sebastia.xambo/ICIAM2019/GC&DL.html (abstracts, references, and slides)

Geometric Calculus Techniques in Science and Engineering

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- **Grassmann algebra**
- **Synopsis of Geometric algebra (GA)**
- **Versors, pinors, spinors and rotors**
- **The Dirac operator**
- **References**

Grassmann algebra

E real vector space of finite dimension n.

$(\wedge E, \wedge)$ Grassmann's *exterior algebra* of E.

It is *unital*, *associative* and *skew-commutative*:
$x \wedge x' = -x' \wedge x$ for all $x, x' \in E$.

In particular $x \wedge x = x^{\wedge 2} = 0$ for all $x \in E$.

$\wedge^k E \subset \wedge E$ (k-th exterior power of E):
subspace of $\wedge E$ generated by all *k-blades*, which are the non-zero exterior products $x_1 \wedge \cdots \wedge x_k$ $(x_1, \ldots, x_k \in E)$.

By convention, $\wedge^0 E = \mathbf{R}$ and clearly $\wedge^1 E = E$.

The elements of $\wedge^k E$ are called *k-vectors*.

Special names: $k = 0$, *scalars*; $k = 1$, *vectors*; $k = 2$, *bivectors*;
$k = n - 1$, *pseudovectors* (dim n); $k = n$, *pseudoscalars* (dim 1).

◇ $\wedge E = \oplus_{k=0}^{k=n} \wedge^k E$ (*grading* of $\wedge E$)

The elements $a \in \wedge E$ are called *multivectors* and we have a unique decomposition $a = a_0 + a_1 + \cdots + a_n$, with $a_k \in \wedge^k E$.

$N = \{1, \ldots, n\}$ set of *indices*.

$\mathcal{J}$ set of subsets of N: Its elements are called *multiindices*.

$\mathcal{J}_k \subset \mathcal{J}$ subset of multiindices of cardinal k.

Let $\boldsymbol{e} = e_1, \ldots, e_n$ be a basis of E.

If $K = k_1, \ldots, k_m$ is a sequence of indices, we set $e_{\hat{K}} = e_{k_1} \wedge \cdots \wedge e_{k_m}$. Note that $e_{\hat{K}} = 0$ if K has repeated indices (it occurs when $m > n$).

◇ $\{e_{\hat{I}}\}_{I \in \mathcal{J}_k}$ *is a basis of* $\wedge^k E$.

Thus $\dim \wedge^k E = \binom{n}{k}$ and $\dim \wedge E = 2^n$.

Parity involution. The linear automorphism $E \to E$, $e \mapsto -e$, induces a linear automorphism of $\wedge E$ that is denoted $a \mapsto \hat{a}$.

- For $a \in \wedge E$, we have $\hat{a} = \sum_k (-1)^k a_k$.
- $\widehat{a \wedge b} = \hat{a} \wedge \hat{b}$ for all $a, b \in \wedge E$ (*algebra automorphism*).

Reverse involution. Exchanging the order of exterior products yields a linear *antiautomorphism* of $\wedge E$, $a \mapsto \tilde{a}$.

Since reversing a k-blade amounts to $\binom{k}{2}$ sign changes, and since this number has the same parity as $k /\!\!/ 2$ (the integer quotient of k by 2), we have

- $\tilde{a} = \sum_k (-1)^{k /\!\!/ 2} a_k$.
- $\widetilde{a \wedge b} = \tilde{b} \wedge \tilde{a}$ for all $a, b \in \wedge E$ (*algebra antiautomorphism*).

Let q be a *metric* on E: a non-degenerate quadratic form of E.

The metric is also regarded as a bilinear non-degenerate form:

$$2q(x,x') = q(x+x') - q(x) - q(x'), \quad q(x) = q(x,x).$$

A vector x is said to be *positive*, *negative* or *null* (or *isotropic*) according to whether $q(x) > 0$, $q(x) < 0$ or $q(x) = 0$.

The basis $\boldsymbol{e}$ is said to be *orthogonal* if $q(e_j, e_k) = 0$ for $j \neq k$.

The basis is *orthonormal* if in addition $q(e_i) = \pm 1$.

The *signature* (r, s) of q is obtained by counting the numbers r and s of positive and negative vectors in any orthogonal basis.

$(E, q) = E_{r,s}$: *orthogonal geometry* of signature (r, s).

Fundamental goal: To understand the group $\mathrm{O}_{r,s}$ of *isometries* of $E_{r,s}$, the subgroup $\mathrm{SO}_{r,s}$ of *proper isometries* (or *rotations*), and the subgroup $\mathrm{SO}^0_{r,s}$ of *rotations connected to the identity*.

Examples

Euclidean space: $E_n = E_{n,0}$ (signature $(n,0)$).
E_2 (Euclidean plane), E_3 (ordinary Euclidean space).

Antieuclidean space $\bar{E}_n = E_{\bar{n}} = E_{0,n}$ (signature $(0,n)$).

Minkowski space: $(E,\eta) = E_{1,3}$. In this case a convenient notation for an orthonormal basis is e_0, e_1, e_2, e_3, where e_0 is positive and e_1, e_2, e_3 negative. $O_{1,3}$ is the group of *Lorentz transformations*.

$E_3^c = E_{3+1,1}$: Conformal space. In this case, a convenient basis is formed by adding null vectors e_0 and e_∞ that are orthogonal to E_3, and such that $e_0 \cdot e_\infty = -1$, to an orthonormal basis e_1, e_2, e_3 of E_3.

Note that *the signature of the plane* $\langle e_0, e_\infty \rangle$ *is* $(1,1)$: the vectors $e^+ = (e_0 - e_\infty)/\sqrt{2}$ and $e^- = (e_0 + e_\infty)/\sqrt{2}$ are orthogonal and $q(e^+) = 1$, $q(e^-) = -1$.

$\diamond$ There is a unique metric on $\wedge E$, still denoted q, such that the spaces $\wedge^k E$ are pairwise q-orthogonal and with $q(x_1 \wedge \cdots \wedge x_k, x'_1 \wedge \cdots \wedge x'_k) = \det((q(x_i, x'_j)))$ $(i, j = 1, \ldots, k)$.

If follows that the basis $\{e_{\hat{I}}\}_{I \in \mathcal{J}}$ is orthogonal (orthonormal) if $\boldsymbol{e}$ is orthogonal (orthonormal).

Exercise. The signature of this metric is $(2^n, 0)$ if $s = 0$ (so $r = n$), and $(2^{n-1}, 2^{n-1})$ otherwise. In particular $(\wedge E, q)$ is:

- non-degenerate when (E, q) is non-degenerate;
- Euclidean when (E, q) is Euclidean;
- has signature $(8, 8)$ for the Minkowski space;
- has signature $(16, 16)$ for the conformal space.

It is derived from the (left) contraction operator i_x $(x \in E)$:

$i_x(x_1 \wedge \cdots \wedge x_k) = \sum_j (-1)^{j-1} q(x, x_j) x_1 \wedge \cdots \wedge x_{j-1} \wedge x_{j+1} \wedge \cdots \wedge x_k.$

The result is a bilinear product $a \cdot b$ $(a, b \in \wedge E)$ uniquely determined by the following properties:

- $a \cdot b = 0$ if a or b is a scalar;
- $x \cdot b = i_x(b)$ if $x \in E$; so $x \cdot y = q(x, y)$ for $x, y \in E$.
- $a \cdot b = (-1)^{jk+m} b \cdot a$ $(m = \min(j, k))$ if $a \in \wedge^j E, b \in \wedge^k E$. In particular, $a \cdot x = (-1)^{j+1} i_x(a)$ if x is a vector and $j \geq 1$.
- $(x_1 \wedge \cdots \wedge x_{j-1} \wedge x_j) \cdot b = (x_1 \wedge \cdots \wedge x_{j-1}) \cdot (x_j \cdot b)$ if $b \in \wedge^k E$ and $2 \leq j \leq k$.
- If $a = x_1 \wedge \cdots \wedge x_k$ and $b = x'_1 \wedge \cdots \wedge x'_k$, $a \cdot b = (-1)^{k /\!/ 2} q(a, b)$. In general, $a \cdot b = q(\tilde{a}, b)$ if $a, b \in \wedge^k E$, $k \geq 1$.

There is also a *Laplace formula* for the inner product $a \cdot b$ when $a = x_1 \wedge \cdots \wedge x_j$ and $b = x'_1 \wedge \cdots \wedge x'_k$. Its general expression can be easily guessed from the following example: $(x_1 \wedge x_2) \cdot (x'_1 \wedge x'_2 \wedge x'_3) = ((x_1 \wedge x_2) \cdot (x'_1 \wedge x'_2))x'_3 - ((x_1 \wedge x_2) \cdot (x'_1 \wedge x'_3))x'_2 + ((x_1 \wedge x_2) \cdot (x'_2 \wedge x'_3))x'_1$.

◇ For all $a, b \in \bigwedge E$,

$$\widehat{a \cdot b} = \hat{a} \cdot \hat{b} \quad \text{and} \quad \widetilde{a \cdot b} = \tilde{b} \cdot \tilde{a}$$

Synopsis of GA

The GA of $(E, q) = E_{r,s}$, denoted $\mathcal{G} = \mathcal{G}_q = \mathcal{G}_{r,s}$, can be constructed by enriching $\wedge E$ with the *geometric product* ab (Clifford). It is unital, bilinear and associative. Moreover,

- For any vectors $x, x' \in E$, $xx' = x \cdot x' + x \wedge x'$ (*Clifford relations*).
- Thus $xx' = -x'x$ iff $x \cdot x' = 0$ (anticommuting property) and $x^2 = q(x)$ (*Clifford reduction*).
- If $q(x) \neq 0$ (*non-isotropic*, or *non-null* vector), $x^{-1} = x/q(x)$.
- For $x \in E$ and $a \in \wedge E$,

$$xa = x \cdot a + x \wedge a = (i_x + \mu_x)(a).$$

$$ax = a \cdot x + a \wedge x$$

- If $a \in \mathcal{G}^j$ and $b \in \mathcal{G}^k$, then $(ab)_i$ is 0 unless i is in the range $|j-k|, |j-k|+2, \ldots, j+k-2, j+k$, and

$$(ab)_{|j-k|} = a \cdot b \text{ for } j, k > 0, \quad \text{and} \quad (ab)_{j+k} = a \wedge b.$$

- For any $a, b \in \mathcal{G}$, $\widehat{ab} = \hat{a}\hat{b}$ and $\widetilde{ab} = \widetilde{b}\,\widetilde{a}$.
- Riesz formulas $2x \wedge a = xa + \hat{a}x$, $2x \cdot a = xa - \hat{a}x$
- The metric in terms of the geometric product: For all $a, b \in \mathcal{G}$,
$$q(a, b) = (\tilde{a}b)_0 = (a\tilde{b})_0.$$
- In particular we have
$$q(a) = (\tilde{a}a)_0 = (a\tilde{a})_0$$

for all $a \in \mathcal{G}$.

- If a is a k-blade, then $\tilde{a}a$ is already a scalar and
$$q(a) = \tilde{a}a = a\tilde{a} = (-1)^{k/\!\!/2}a^2$$

In particular we see that a is invertible if and only if $a^2 \neq 0$, or if and only if $q(a) \neq 0$, and if this is the case, then we have
$$a^{-1} = a/a^2 = \tilde{a}/q(a).$$

Let $\boldsymbol{e} = e_1, \ldots, e_n$ be a basis of E and $N = \{1, \ldots, n\}$ the *set of indices*.

If $K = k_1, \ldots, k_m$ is a sequence of indices, set

$$e_K = e_{k_1} \cdots e_{k_m}.$$

- $\{e_I\}_{I \in \mathcal{I}}$ *is a basis of* $\mathcal{G} = \wedge E$.

Remark that if $I \in \mathcal{J}_k$, then in general $e_I = e_{\hat{I}} +$ lower grade terms, like $e_{12} = e_1 e_2 = e_1 \wedge e_2 + e_1 \cdot e_2 = e_{\widehat{12}} + e_1 \cdot e_2$.

- If $\boldsymbol{e}$ is orthogonal, then $e_I = e_{\hat{I}}$, as

$$x_1 \cdots x_k = x_1 \wedge \cdots \wedge x_k$$

when $x_1, \ldots, x_k$ are pair-wise orthogonal vectors.

- Artin's formula: If I, J are multiindices, then

$$e_I e_J = (-1)^{t(I,J)} q_{I\cap J} e_{I\Delta J}$$

where $t(I,J)$ is the number of inversions in the squence I, J, $I\Delta J$ is the symmetric difference of I and J, and $q_K = q\left(e_{k_1}\right)\cdots q\left(e_{k_m}\right)$.

- In particular, $e_J^2 = (-1)^{|J|/\!\!/2} q_J$

Examples

- In E_2, $e_{12}^2 = -1$ (as $2/\!\!/2 = 1$ and $q_{12} = 1$).
- In E_3, $e_{123}^2 = -1$ (as $3/\!\!/2 = 1$ and $q_{123} = 1$).
- In $E_{1,3}$, $e_{0123}^2 = -1$ (as $4/\!\!/2 = 2$ and $q_{0123} = (-1)^3 = -1$).
- In $\bar{E}_4$, $e_{1234}^2 = 1$ (as $4/\!\!/2 = 2$ and $q_{1234} = (-1)^4 = 1$).

$\mathcal{G}_2 = \langle 1, e_1, e_2, e_{12} = \boldsymbol{i} \rangle$, $\boldsymbol{i}^2 = -1$ (Gauss algebra).

$\mathcal{G}_2^+ = \langle 1, \boldsymbol{i} \rangle \simeq \mathbf{C}$

$\mathcal{P} = \mathcal{G}_3 = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} = \boldsymbol{i} \rangle$, $\boldsymbol{i}^2 = -1$ (Pauli).

$e_{23} = \boldsymbol{i} e_1 = e_1 \boldsymbol{i}$, $e_{31} = \boldsymbol{i} e_2 = e_2 \boldsymbol{i}$, $e_{12} = \boldsymbol{i} e_3 = e_3 \boldsymbol{i}$.

General element: $(\alpha + \beta \boldsymbol{i}) + (x + y\boldsymbol{i})$ ($\alpha, \beta \in \mathbf{R}$, $x, y \in E_3$).

$G_2^+ = \{\alpha + x\boldsymbol{i}\} = \mathbf{H}$ (quaternion field).

$\diamond$ $q(\alpha + x\boldsymbol{i}) = (\alpha + x\boldsymbol{i})(\alpha + x\boldsymbol{i})\tilde{} = \alpha^2 + x^2$

Hamilton units: $\boldsymbol{I} = e_{12} = e_3 \boldsymbol{i}$, $\boldsymbol{J} = e_{31} = e_2 \boldsymbol{i}$, $\boldsymbol{K} = e_{23} = e_1 \boldsymbol{i}$.

(Yes, in that order if we want that the Hamilton's original relations $\boldsymbol{I}^2 = \boldsymbol{J}^2 = \boldsymbol{K}^2 = \boldsymbol{IJK} = -1$ are satisfied).

$E_{1,3} = \langle e_0, e_1, e_2, e_3 \rangle$. In $\mathcal{D} = \mathcal{G}_{1,3}$ (Dirac algebra), set: $\boldsymbol{i} = e_{0123}$, $\boldsymbol{\sigma}_k = e_k e_0$. Then $\boldsymbol{i}^2 = -1$, $\boldsymbol{i}$ anticommutes with vectors, and

$\mathcal{D} = \langle 1, e_0, e_1, e_2, e_3, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{i\sigma}_1, \boldsymbol{i\sigma}_2, \boldsymbol{i\sigma}_3, e_0\boldsymbol{i}, e_1\boldsymbol{i}, e_2\boldsymbol{i}, e_3\boldsymbol{i}, \boldsymbol{i} \rangle$.

A general element has the form $(\alpha + \beta\boldsymbol{i}) + (x + y\boldsymbol{i}) + (\boldsymbol{E} + \boldsymbol{iB})$, $(\alpha, \beta \in \mathbf{R}$, $x, y \in E_{1,3}$, $\boldsymbol{E}, \boldsymbol{B} \in \mathcal{E} = \langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3 \rangle)$.

$\mathcal{D}^+ = \langle 1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{i\sigma}_1, \boldsymbol{i\sigma}_2, \boldsymbol{i\sigma}_3, \boldsymbol{i} \rangle \simeq \mathcal{P}(\mathcal{E})$.

Its elements have the form $(\alpha + \beta\boldsymbol{i}) + (\boldsymbol{E} + \boldsymbol{iB})$.

$\diamond$ $\boldsymbol{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$.

Versors

Let $E^\times$ be the set on non-isotropic vectors of E.

If $x \in E$, define the linear automorphism $\underline{x} : \mathcal{G} \to \mathcal{G}$ by

$$\underline{x}(a) = -xax^{-1} = \hat{x}ax^{-1}.$$

◇ For a vector y, $\underline{x}(y)$ *is the reflection of y along x* (or *across* $x^\perp$).
Proof: $\underline{x}(x) = -x$ and $\underline{x}(y) = y$ if $y \in x^\perp$.

The map $\underline{x}$ is not an algebra automorphism, but satisfies:

$$\underline{x}(ab) = -x(ab)x^{-1} = -xax^{-1}xbx^{-1} = -\underline{x}(a)\underline{x}(b).$$

It follows that $\underline{x}$ is grade-preserving. Moreover, it is an *isometry*:

$$\begin{aligned} q(\underline{x}(a)) &= ((-xax^{-1})\tilde{}(-xax^{-1}))_0 \\ &= (x^{-1}\tilde{a}x^2ax^{-1})_0 = (x\tilde{a}ax^{-1})_0 = q(a). \end{aligned}$$

◇ Let $x_1, \ldots, x_k \in E^\times$ and $v = x_1 \cdots x_k$. Then

$$(\underline{x}_1 \cdots \underline{x}_k)(a) = \hat{x}_1 \cdots \hat{x}_k a x_k^{-1} \cdots x_1^{-1} = \hat{v} a v^{-1}.$$

The expressions v form a group under the geometric product. We denote it by $\mathcal{V} = \mathcal{V}_{r,s}$ and its elements are called *versors*.

◇ *Any isometry $f : E \to E$ has the form $\underline{v}$ for some versor v. Moreover, if $\underline{v} = \underline{v}'$, then $v' = \lambda v$ for some scalar λ.*

A *unit versor* (also called a *pinor*) is a versor v such that $v\tilde{v} = \pm 1$.

◇ Any unit versor is the product of unit vectors (and conversely).

◇ Any isometry $f : E \to E$ has the form $\underline{v}$ for some unit versor v. Moreover, if $\underline{v} = \underline{v}'$ (v' also a unit versor), then $v' = \pm v$.

The even unit versors are called *spinors*. They form a subgroup $\mathcal{S}_{r,s}$ of $\mathcal{V}_{r,s}$. The *rotors* are the spinors v such that $v\tilde{v} = 1$. They form a subgroup $\mathcal{R}_{r,s}$ of $\mathcal{S}_{r,s}$. In the Euclidean space, any spinor is a rotor, but this is not true in general.

$\diamond$ *Any proper isometry (also called rotation) has the form* $\underline{v}$ *for some spinor* v*. If the rotation is connected to the identity, then it has the form* $\underline{v}$ *for some rotor* v.

Example. Let u and u' to unit linearly independent vectors of E_n and $\theta = \angle(u, u')$. *Then the rotation $\underline{v}$ produced by the rotor $v = u'u$ is the rotation in the plane $P = \langle u, u' \rangle$ of amplitude 2θ*.

Indeed, since $\underline{v}$ is the identity on $P^\perp$, it amounts to a rotation in P. Let $\boldsymbol{i} = \boldsymbol{i}_P$ be the unit area of P. Then u and $u^\perp = u\boldsymbol{i}$ form an orthonormal basis of P and $u' = u \cos\theta + u^\perp \sin\theta$. Hence $v = u'u = \cos\theta - \boldsymbol{i}\sin\theta = e^{-\boldsymbol{i}\theta}$. Finally,
$\underline{v}(u) = vu\tilde{v} = e^{-\boldsymbol{i}\theta} u e^{\boldsymbol{i}\theta} = ue^{2\boldsymbol{i}\theta} = u\cos 2\theta + u^\perp \sin 2\theta$.

$\diamond$ *If $b \in \mathcal{G}^2$, $R = e^b = \sum_{k\geq 0} \frac{1}{k!} b^k$ satisfies $R\tilde{R} = e^b e^{-b} = 1$. If $n \leq 5$, then R is a rotor*.

The Hestenes embedding $E_3 \to E^0_{3,1}$, $x \mapsto X$:

$$X = e_0 + x + \kappa(x)e_\infty, \quad \kappa(x) = \tfrac{1}{2}x^2.$$

The isometry group $\mathrm{O}_{4,1}$ acts on $E^0_{3,1}$ and hence on E_3. These actions are conformal and any conformal map of E_3 can be obtained in this way.

The similarities are induced by the isometries leaving e_∞ fixed.

Using the general construction of rotors, we can produce similarities (sufficient for robotics) tailored to our needs.

Transformation	**Conformal rotor**
Rotation	$e^{-\boldsymbol{i}\theta}$
Translation	$e^{-ve_\infty/2}$
Dilation	$e^{\alpha e_0 e_\infty/2}$

The Dirac operator

Let e_j be a basis of E and e^j its *reciprocal*, defined by the relations $e^j \cdot e_k = \partial^j_k$.

Examples. For an orthonormal basis of E_n, $e^j = e_j$ for all j. In the Minkowski space, $e^0 = e_0$ and $e^j = -e_j$ for $j = 1, 2, 3$.

The *Dirac operator* can be defined by the expression $\partial = e^j \partial_j$ (sum wrt j implied by Einstein's convention), where $\partial_j = \partial/\partial x^j$, x^j the coordinate functions wrt to e_j (so $x = x^j e_j$ for $x \in E$).

There are thee actions of ∂ on a *multivector field* $a = a^I(x)e_I$:

- $\partial a = \partial_j a^I e^j e_I$
- $\partial \cdot a = \partial_j a^I e^j \cdot e_I$
- $\partial \wedge a = \partial_j a^I e^j \wedge e_I$.

◇ $\partial a = \partial \cdot a + \partial \wedge a$ (as $e^j e_I = e^j \cdot e_I + e^j \wedge e_I$).

Let $a = a^i e_i$ be a *vector field*. Then

- $\partial \cdot a = \partial_j a^j$ (*divergence*).
- $\partial \wedge a = \sum_{i<j} (q_i \partial_i a^j - q_j \partial_j a^i) e_{ij}$.

In E_3, $\partial = \nabla = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3$:

- $\nabla \cdot a = \partial_1 a^1 + \partial_2 a^2 + \partial_3 a^3 = \mathrm{div}(a)$.
- $$\begin{aligned} \nabla \wedge a &= (\partial_1 a^2 - \partial_2 a^1) e_{12} + (\partial_1 a^3 - \partial_3 a^1) e_{13} + (\partial_2 a^3 - \partial_3 a^2) e_{23} \\ &= (\partial_2 a^3 - \partial_3 a^2) e_1 \boldsymbol{i} + (\partial_3 a^1 - \partial_1 a^3) e_2 \boldsymbol{i} + (\partial_1 a^2 - \partial_2 a^1) e_3 \boldsymbol{i} \\ &= (\nabla \times a) \boldsymbol{i} = \mathrm{curl}(a) \boldsymbol{i}. \end{aligned}$$

A bivector of $\mathcal{D}$ has the form $F = \boldsymbol{E} + \boldsymbol{i}\boldsymbol{B}$ ($\boldsymbol{E}, \boldsymbol{B} \in \mathcal{E}$) and can be used to encode the electromagnetic field (*Faraday bivector*).

Let $\rho = \rho(x, t)$ be the scalar function representing the *charge density* and $\boldsymbol{j} \in \mathcal{E}$ the vector representing the *current density*. The $J = \rho e_0 + \boldsymbol{j}$ is the *current vector*.

◇ The equation $\partial F = J$ is equivalent to the Maxwell equations for the electromagnetic field generated by ρ and $\boldsymbol{j}$.

◇ If we multiply $\partial F = J$ by ∂ on the left, we obtain $\Box F = \partial \cdot J + \partial \wedge J$, where $\Box = \partial^2 = \partial_0^2 - (\partial_1^2 + \partial_2^2 + \partial_3^2)$ (*dalambertian*).

Since the left side is a bivector ($\Box$ preserves grades), the scalar part of the right-hand side expression must vanish: $\partial \cdot J = 0$. This is the charge conservation equation, as it is equivalent to the *continuity equation* $\partial_t \rho + \nabla \cdot j = 0$.

The original Dirac equations were written in terms of 4×4 complex matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ that provided a matrix representation of $\mathcal{D}$ determined by $e_i \mapsto \gamma_i$. The space on which these matrices act, $\mathbf{C}^4$, was the space of *Dirac spinors*; the *wave function* was map $\psi : E_{1,3} \to \mathbf{C}^4$; and the Dirac equation was derived as a "relativistic Schrödinger equation for the electron wave function" (Klein-Gordon equation).

It turns out, however, that GC shows that *the complex matrices are superfluous, as the only crucial fact required is that they satisfy Clifford's relations*. And after that, the analysis reveals that the role of $\mathbf{C}^4$ must be played by the space $\mathcal{D}^+$ ($\simeq \mathcal{P}$), which has complex dimension 4, and hence that the wave function is to be thought as a *spinor field*, the name for a function $\psi : E_{1,3} \to \mathcal{D}^+$.

As is customary, instead of e_0, e_1, e_2, e_3 used so far, we will use $\gamma_0, \gamma_1, \gamma_2, \gamma_3$.

The final conclusion is that the *Dirac equation* is morphed into the following equation for the spinor field ψ:

$$\partial\psi i\hbar - \frac{e}{c}A\psi = m_e c\psi\gamma_0,$$

where c is the speed of light, e is the electron charge and m_e its mass. In this equation i is not $\sqrt{-1}$, but the bivector $i = \gamma_{21}$, and A is the *electromagnetic potential*, a vector field such that $\partial \wedge A = F$ and $\partial \cdot A = 0$.

As found and expressed by D. Hestenes, this equation "reveals geometric structure in the Dirac theory that is so deeply hidden [even inaccessible] in the matrix version that it remains unrecognized by QED experts to this day".

See [**?**, §3.3], [116], [117, §6.2 and §6.3], which include a comprehensive survey of applications.

- $i = \gamma_2\gamma_1 = \boldsymbol{i}\gamma_3\gamma_0 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$.

This 2-area element is a geometric imaginary unit that replaces the (ungeometric) imaginary unit $\sqrt{-1}$ in the original Dirac equation.

The first important advantage of the GC formulation of the Dirac equation is that $\psi(x)$ admits a decomposition of the form

$$\psi = \rho^{1/2} e^{\boldsymbol{i}\beta/2} R,$$

where $\rho = \rho(x)$ is a positive real number, $\beta = \beta(x) \in [0, 2\pi)$ and $R = R(x)$ is a *rotor* (that is, $R\widetilde{R} = 1$). Note that this expression has eight degrees of freedom: $1 + 1 + 6$.

Define $e_\mu = e_\mu(x) = R\gamma_\mu \widetilde{R}$ (*comoving frame*). Since R is a rotor, this is an orthonormal frame field in $E_{1,3}$ with the same orientation and temporal orientation as the reference frame γ_μ.

Note that $\psi\gamma_\mu\widetilde{\psi} = \rho e_\mu$, because $\boldsymbol{i}$ anticommutes with vectors and $\widetilde{\boldsymbol{i}} = \boldsymbol{i}$:

$$\psi\gamma_\mu\widetilde{\psi} = \rho e^{\boldsymbol{i}\beta/2} R\gamma_\mu \widetilde{R} e^{\boldsymbol{i}\beta/2} = \rho e^{\boldsymbol{i}\beta/2} e^{-\boldsymbol{i}\beta/2} R\gamma_\mu \widetilde{R} = \rho e_\mu.$$

In particular, $\psi\gamma_0\widetilde{\psi} = \rho v$, where $v = e_0$, is the *Dirac current*.

The vector

$$s = \tfrac{\hbar}{2} R\gamma_3 \widetilde{R} = \tfrac{\hbar}{2} e_3 \qquad (1)$$

is the *spin vector*.

The rotor R transforms the unit i to $\iota = Ri\widetilde{R}$, which is the (comoving) space-like plane quantity e_2e_1 and $S = \frac{\hbar}{2}\iota$ can be called the *spin bivector*. The relation to the spin vector is as follows:

$$S = \boldsymbol{i}sv.$$

Proof $\boldsymbol{i}sv = \frac{\hbar}{2}\boldsymbol{i}R\gamma_3\widetilde{R}R\gamma_0\widetilde{R} = \frac{\hbar}{2}R\boldsymbol{i}\gamma_3\gamma_0\widetilde{R} = \frac{\hbar}{2}Ri\widetilde{R} = \frac{\hbar}{2}\iota = S.$ □

With $R = e^{i(k\cdot x)}$, we have a 'monochromatic spinor' (yes, $\boldsymbol{i}$ and i)

$$\psi = \rho^{1/2} e^{\boldsymbol{i}\beta/2} e^{i(k\cdot x)}.$$

A straightforward computation shows that the condition for this wave to satisfy the real Dirac equation is that

$$\hbar k = m_e c v e^{-\boldsymbol{i}\beta}$$

This implies that $\cos(\beta) = \pm 1$. As for monochromatic electromagnetic waves, the condition for constant phase in the moving frame is $v \cdot x = c\tau$, and so

$$\hbar k \cdot x = \pm m_e c (v \cdot x) = \pm m_e c^2 \tau$$

which yields the de Broglie frequency $m_e c^2/\hbar$ of the electron.

A closer analysis shows that the vector e_1 turns in the plane ι with frequency $2m_e c^2/\hbar$, which is the *zitterbewegung* frequency of Schrödinger, with period 4.0466×10^{-21}s.

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