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FRANCESCO SEVERI
and
THE PRINCIPLE OF CONSERVATION OF NUMBER

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Preface. Ever since the beginning of geometry, many interesting problems seek to construct the figures that satisfy some given requirements. For examples: the common tangents to two circles; the conics going through 3 points and touching two lines; the lines meeting four lines in space; the twisted cubics contained in a non-singular quadric and going through five general points on it; and so on without end.

In all such kinds of problems, the question of how the solutions depend on the data involved in the requirements has fascinated geometers of all times. According to M. Kline [1972, p. 843], Leibniz had stated, as early as 1687, that when we move a little the data of some construction, the solutions of the construction will also move just a little, and so, in particular, its number will remain unchanged. Poncelet [1822]

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named a similar assertion the *principle of continuity* and later Schubert introduced other flavors by the names of *principle of special position* (Schubert [1874 a]) and *principle of conservation of number* ("Prinzip der Erhaltung der Anzahl", Schubert [1876, 79]). As we shall see, the basic use of such principles was as a device to get the *number of solutions* of geometrical problems without having to perform the required construction, a rather magic sort of operation that was to be called *geometric elimination* by Halphen [1885].

But the point is that without further qualification the principle of continuity was plainly false: on moving just a little apart two tangent circles a transition from three to four common tangents occurs, while on moving them so that the centers become just a little closer, then the transition is to two common tangents. Thus there was in the early nineteenth century a real conflict between the observed behavior of configurations in Euclidean space, the mathematical structure that seems to embody our primordial geometrical experience, and what the principle was stating. Since the conflict continued for a long time, the question is: Why did Poncelet and his successors insist on believing the principle?

With the benefit of hindsight we can say that faith in the principle played the role, in the mathematical physicists terminology of recent decades, of an *Ansatz*: instead of an axiom or a conjecture, as it has been considered sometimes, it is rather a bet about how things, or our understanding of things, should be; in sum, a desirable feature that geometrical theories should have built in. Thus the principle was a guiding thread in the search for a better world in which the number of solutions of a given construction *does indeed remain the same* when we jiggle just a little the data out of which the construction is supposed to be done. Still more, it has been, to this day, one of the shaping forces of projective algebraic geometry as we understand it today: if the principle is to hold, where are the missing solutions in examples like the common tangents to two circles, and what are they? Thus the quest has been precisely to find the right conceptual frameworks in which the principle holds.

These roles for the principle of continuity have some resemblance with those played by the principle of conservation of energy in physics, stated by Mayer in 1842 and with no known exception so far. In fact the principle of conservation of energy can be seen as a perpetual prompt to look for not yet observed forms of energy to explain the energy losses or gains that take place in a process after having into account all the already known ones. From this point of view, we see that the principle of conservation of energy has driven physics along paths that guarantee the forging of conceptual frameworks in which the principle holds. (1)

In this essay we will delve into the historical development of the principle of conservation of number from the time it was first formulated by Poncelet until today, paying especial attention to Severi's fundamental contribution of 1912 and its influence on subsequent research.

(1) The reader is referred to chapter 4 of the first volume of *The Feymann's Lectures on Physics* (Addison-Wesley, 1963), for a few interesting and lively pages about the meaning of the principle of conservation of energy from the point of view of a pure physicist.

Acknowledgements. My thanks to the organizers of the conference for the wonderful job they did in bringing together a good number of people interested in the magnificent history of algebra and geometry in Italy. The inspiring atmosphere in the "palazzone", that idyllic Tuscan balcony over the lands north of lake Trasimeno; the closeness to Arezzo, the town where Severi was born on 13 April of the year in which Schubert's book was published; the quality of the lectures and conversations concerning algebra and geometry, and not only their history; even the reminiscence in so many unexpected ways of Hannibal's feats; ... everything was so vivid and splendid that over one year after it looks as if it were yesterday.

I am also indebted to Prof. J. P. Murre for fruitful discussions concerning the history of rational and algebraic equivalence; to Prof. S. Kleiman for his generous comments: they allowed me to improve several points, the most important being the present version of footnote 8 so as to better assess the enumerative accomplishments of Schubert after 1880; and to Prof. E. Casas for his comments, and especially for calling my attention on the paper Segre [1892] (see footnote 7).

Finally, even if far from the narrow subject of this paper, I would like to thank Profs. A. Brigaglia, C. Ciliberto and P. Nastasi for sharing their knowledge, ideas, views and opinions about the impact and significance of Severi in the Italian geometry landscape, especially in the period between the two world wars, all of which has driven me to the forging of a more realistic picture of Severi's institutional role. I only can hope that such matters will get the deserved attention in the forthcoming book by Ciliberto and Brigaglia on the history of Italian Algebraic Geometry.

1 Early history (before circa 1870). In his *Traité*, Poncelet [1822] worded the principle of continuity in a rather obscure and vague form (cf. Zeuthen–Pieri [1915], § 5; Struik [1967], p. 162). For our purposes, however, it will be enough to recall the phrasing reported in Kline [1972], p. 843:

> If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure.

The principle was criticized by Cauchy for not having a sound logical foundation. Cauchy's judgment, according to Kleiman [1976], led to some prejudice, but the fact is that it did not prevent the use of the principle to solve many problems (see Zeuthen–Pieri [1915], §§ 5-7). Of course, Poncelet did not explain, as some unwritten constitutions do with government principles, how "as general as the former" is to be interpreted, nor what kinds of figures and processes are allowed for the principle to hold.

Whatever it be, the continuity Ansatz led, guided by the special case of the solutions of an algebraic univariate equation, (2) and by a great deal of examples that can be reduced to that case through analytical geometry and algebraic elimination, to the realization that:

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(2) The fundamental theorem of algebra was proved by Gauss in 1799 and it asserts that the number of solutions of a degree n univariate complex polynomial equation is exactly n, provided each solution is counted with its multiplicity.

1. The solutions have to be counted with suitable multiplicities when the data take special positions;
2. Some elements, possibly all, of a solution have to be allowed to move to infinity, and
3. Some elements, possibly all, of a solution have to be allowed to become imaginary. (3)

The first point turned out to be a delicate matter, if not in most examples encountered in the day-to-day algebro-geometric practice, certainly from a theoretical point of view. The fundamental theorem of algebra and its consequences in analytical geometry led Poncelet to say that the principle could be demonstrated easily by means of algebra, although he never proposed such a demonstration. On the contrary, he apparently was satisfied by expressing the view that one should refrain from such a proof because the principle *ought to be seen in a purely geometric light*. We will come to that point again later.

The second and third points were easier to understand, and so they were progressively in the period 1822-1870, in the sense that the leading geometers agreed on *complex projective space* as the *unifying* ambient space. This space was, and is, the result of complexifying the projective closure of real affine space, a process that through the practice of years (4) was clarified when it was realized, especially after the work of von Staudt, (5) that the synthetic and analytic methods in projective geometry were two sides of the same coin. Indeed, in the final analysis, the addition of points at infinity to the real affine space and of the imaginary points afterwards depends on a clear understanding of how to use coordinates and how to recognize what does not depend on coordinates (6)

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To illustrate the three points above, let us recall some very simple examples from elementary projective geometry. All of them deal with problems that can be reduced, by means of analytic geometry, to a suitable equation whose solutions and their multiplicities are then interpreted as the solutions and corresponding multiplicities to the problem. In practice this was not the cause of worry because a multiple solution of multiplicity m often can be visualized to correspond to the coalescing onto a degenerate figure of precisely m solutions to the general problem.

Plane conics are parameterized by the points of a $\mathbf{P}^5$ and the conics through a point form a hyperplane of that $\mathbf{P}^5$. The hyperplanes corresponding to five distinct points turn out to be independent provided no four of them are collinear, in which

(3) Zeuthen–Pieri [1915], § 5: "Les exemples donnés par J. V. Poncelet montrent bien qu'il ne songait à appliquer son principe que dans le cas où un simple renvoi à la représentation analytique, sans aucune exécution de calculs, suffit pour obtenir une démonstration parfaite." Here it is also to be noted that many others used the principle in Poncelet's fashion: Chasles, Steiner, Salmon, De Jonquières, Cremona, Cayley, to name just a few.

(4) See, for example, Salmon's *Conic Sections* (1848) and *Higher plane curves* (1852).

(5) *Geometrie der Lage* (1847) and *Beiträge zur Geometrie der Lage* (1856 and 1860).

(6) In present day terms, of how to use linear algebra to model projective geometry and how to reconstruct back linear algebra from projective geometry.

case we see that there is a unique conic going through the five points (it is still easier to analyze what happens when four of the points are collinear).

The construction of the conics through 4 points and tangent to a line can be reduced to find the solutions of a quadratic equation and so there always are two such conics, provided we allow that they may coincide or become imaginary according to whether the equation has a double root or two conjugate imaginary roots. Here is an example that they may become imaginary: the conics through the points of a projective reference are of the form $axy + byz + czx = 0$, $a + b + c = 0$, and, as we learn after some calculations, one such conic is tangent to the unit line $x + y + z = 0$ if and only if $a^2 + b^2 + c^2 = 0$, which leads to two imaginary, complex conjugate, solutions. On the other hand the same conic is tangent to the line $x = 0$ if and only if $c^2 = 0$, in which case the only solution is the conic $xy - yz = 0$, which is a pair of lines to be counted, because of the exponent 2, as a double solution. By duality, there are two conics that are tangent to four lines and go through a given point.

A similar analysis leads to the conclusion that there are four conics going through three points and tangent to two lines (or tangent to three lines and going through two points). But the solutions may coincide, or there can be one that is double, or two may become imaginary, and so on. For example, there is only one conic through the vertices of the reference triangle that is tangent to the lines $x = 0$ and $x + y + z = 0$, namely $x(y + z) = 0$, and therefore this solution is to be assigned multiplicity 4.

Two conics meet at four points, again with all sorts of possibilities for coalescing or becoming imaginary. This in particular applies to the intersection of two circles, and since this intersection has at most two real points, it must have at least to imaginary conjugate points. Actually any circle goes through two fixed imaginary points at infinity (the *cyclic points* $[1, \pm i, 0]$ of Poncelet), which are therefore to be counted in the intersection of any two circles. The other two intersection points may be real, possibly coincident if the circles are tangent, or imaginary. When the circles become concentric those two points coincide with the cyclic points, so that the intersection consists only of the cyclic points, each counted twice. Incidentally, the construction of the common tangents to two circles is equivalent, by taking the dual conics, to the construction of the intersection of two conics, and this transformation explains that there are always four common tangents, provided that we admit that a pair of them, or all four, may become imaginary. When the circles are tangent, there is a repeated real tangent, and when the circles are concentric there are two repeated complex tangents (the tangents at the cyclic points, each counted twice).

Given four lines in $\mathbf{P}^3$, skew in pairs, there are two lines that meet the four, possibly confounded or imaginary. Indeed, as learned in elementary projective geometry the lines meeting three given lines, any two of them skew, is a ruled non-singular quadric which is met by the fourth line in two points, possibly confounded or imaginary.

In contrast with Poncelet's view according to which the continuity principle ought to be seen in a purely geometric light, it is interesting to notice that one of the applications of Gauss fundamental theorem is what ever since 1864 has been called *Chasles' coincidence formula:* (7) the number of fixed points of an algebraic corre-

(7) C. Segre set up, in Segre [1892], a detailed and well documented case about the 'inesattezza sto-

spondence of type (m,n) between two copies of $\mathbf{P}^1$ is $m+n$, if each is counted with a suitable multiplicity. Indeed, such a correspondence is given by a non-zero polynomial $f(x_0,x_1,y_0,y_1)$ which is homogeneous of degree m in $[x_0,x_1]$ and homogeneous of degree n in $[y_0,y_1]$ and its fixed points correspond to the homogeneous solutions $[x_0,x_1]$ of the homogeneous polynomial of degree $m+n$ given by setting $y_0=x_0$ and $y_1=x_1$. Recall that to a point $[a_0,a_1]$ on the first copy of $\mathbf{P}^1$ there correspond on the second copy the n points $[y_0,y_1]$ that are solutions of the degree n homogeneous equation $f(a_0,a_1,y_0,y_1)=0$, and similarly that to a point $[b_0,b_1]$ on the second copy of $\mathbf{P}^1$ there correspond on the first the m points $[x_0,x_1]$ that are solutions of the degree m homogeneous equation $f(x_0,x_1,b_0,b_1)=0$.

To illustrate how Chasles' coincidence formula was applied, let us consider the Braikenridge problem, which was stated in 1733 and asks to find the locus described by the vertex C of a triangle ABC when A and B are constrained to move on fixed lines a and b, respectively, and the sides AB, BC and CA are constrained to go through fixed points R, P and Q, respectively. We shall see that the locus is a conic. For that purpose, take a general line ℓ and set up a self-correspondence on it as follows: if X lies on ℓ, let X_a be the intersection of XQ with a, let X_b be the intersection of X_aR with b, and let $Y\in\ell$, the point corresponding to X, be the intersection of X_bP with ℓ. It is clear that the correspondence is algebraic and of type $(1,1)$, hence it has 2 fixed points. But these two fixed points are precisely the intersection of the locus we are seeking with ℓ.

2 The Golden Age of Enumerative Geometry. One important and quite general advance in enumerative geometry in the last century, in fact the one which is more relevant for our story, was due to Schubert. His fundamental contributions were published in the decade 1870-1880, culminating in the book Schubert [1879], which doubtless is a major milestone in the history of algebraic geometry. (8)

Schubert's starting point is the work of Chasles and De Jonquières in the previous decade, the 1860's. Here it has to be said that Halphen also made deep contributions to enumerative geometry around the same decade 1870-1880 and that the early ones, like his formula $aa'+bb'$ for the number of lines in $\mathbf{P}^3$ common to two congruences (see Halphen [1869, 1871, 1872]), or his observation about the "factorization" of the formula of Chasles expressing the number of conics that satisfy 5 conditions (see Halphen [1873 a, b]), were among the results that inspired the symbolic calculus of

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rica' of this attribution, the main fact being that 'tre anni prima dello Chasles il sig. De Jonquières, e seguendo il suo esempio il sig. Cremona, avevano pubblicate parecchie applicazioni [to those that Chasles presented to the Academy on June 27, 1864] di quel prinzipio.

(8) His best enumerative work in the 1880's and 90's deals mainly with non-trivial applications of his symbolic calculus. Aside from his work on triangles, one basic theme is the extension to any projective space $\mathbf{P}^n$ of concepts and results established previously for $\mathbf{P}^2$ and $\mathbf{P}^3$, like the determination of the intersection ring of a Grassmannian, a feat by which he is best known (Schubert calculus), the research on the characteristic numbers of complete quadrics, or his theory of general coincidence formulas.

Schubert.(9) Halphen's understanding of enumerative geometry and a description of his main results can be found in Halphen [1885] (cf. also Xambó [1993]).

Let us go back to Schubert. In the first chapter of the *Kalkül*, Schubert explains his most basic ideas, and in particular the principle of conservation of number. He starts with some kind of figures Γ (Gebilde) and considers conditions K (Bedingungen) imposed on those figures, paying attention to the number of parameters, or degrees of freedom, of which the figures depend (Constantenzahl) and to the number of degrees of freedom that are lost after imposing a condition (Dimension einer Bedingung). Most conditions K arise from imposing some relation with another kind of figure Γ',(10) which here we will call the *data* of K. He also considers systems of figures and the number of parameters on which the figures in the system depend (Stufe eines Systems). With this terminology, the basic question of enumerative geometry is to find the number N of figures in a system Σ that satisfy a given condition K, provided this number is finite.

The principle of conservation of number (Erhaltung der Anzahl) is stated in § 4:

> *The number N is unaltered or becomes infinite when the datum Γ' of the condition K is moved or specialized.*(11)

That the number is unchanged when Γ' is moved is a phrasing of the principle of continuity, but to say that it is also unchanged when Γ' is specialized is a stronger statement that Schubert called, in his early writings, the *principle of special position* (see Schubert [1874 a, b]). For a proof of the principle Schubert goes on to say that interpreting the principle algebraically it just says that any change of the coefficients of a polynomial equation does not affect the number of roots if the changes do not make the polynomial identically zero. Let us mention in passing that Chasles and de Jonquières suggested in 1866 that the principle of continuity is valid because of the principle of *analytic continuation*.

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Before we continue, here are a few examples to illustrate Schubert's use of the principle. Since a plane curve of degree n can be transformed continuously into a figure consisting of n lines, the number of points of intersection of two plane curves of degrees m and n, respectively, is mn (Bezout's theorem).

To get the number of lines that meet four lines in space, Schubert's principle allows to move one of the four lines until it meets another of the four, so that we may assume that two of the four lines meet at a point P, and then it is clear that the solutions are the line through P that meets the other two lines, and the line in the plane of the two lines meeting at P which joins the points of intersection of the other two lines with that plane.

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(9) See the Literatur-Bemerkungen in Schubert [1879], especially the introduction to the Erster Abschnitt and the note **Lit. 3** on page 333.

(10) Such conditions are called *räumliche Bedingungen* by Schubert and *condizione variabile* by Severi (see Severi [1912]).

(11) Schubert in fact states the principle in four forms, but for our purposes the one adopted here is enough inasmuch it captures the essence of them all.

Another example is how Schubert deals with the number N of curves in a one-dimensional system Σ that are tangent to a curve C of degree n and class m: he specializes C to a figure that point-wise is composed of a line L counted n times and that tangent-wise is composed of m pencils of lines with foci on L, thus getting $N = n\nu + m\mu$, where μ and ν are the characteristics of Σ (by definition μ and ν are the number of curves in the system that go through a point and are tangent to a line, respectively). The fact that the preceding specialization exists can be seen by quite elementary reasoning. Take, for example, the hyperbola $y^2 - \epsilon x^2 = \epsilon$ (say for real $\epsilon > 0$). As $\epsilon \to 0$, the point-wise limit is the conic $y^2 = 0$, that is, the x-axis counted twice. To see what becomes in the limit of the tangent lines to the hyperbola, introduce homogeneous coordinates X, Y, Z (so $x = X/Z$ and $y = Y/Z$) and dual projective coordinates U, V, W. Then the conic dual to the projective conic corresponding to our hyperbola is $U^2 - \epsilon V^2 - W^2 = 0$, and its limit as $\epsilon \to 0$ is $U^2 - W^2 = 0$. So the dual conic degenerates into the pair of dual lines $U \pm W = 0$, which correspond by duality to the two pencils of lines with foci $[\pm 1, 0, 1]$, or $(\pm 1, 0)$ in non-homogeneous coordinates.

Let us also show how to 'solve' the last problem in the first paragraph of the Preface. If we specialize the points so that four of them become complanary, then a cubic going through the five points breaks up into the conic section of the quadric by the plane containing the four points and one of the two rulings of the quadric through the fifth point. Hence there are 'two' solutions.

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In the last two sections of the first chapter of the *Kalkül* Schubert introduces the symbolic calculus of conditions. Here is the description of it made by Kleiman in the introduction to the 1979 Springer reprint:

> A geometric condition is represented by an algebraic symbol. If two independent conditions with symbols x, y are given, then the new condition of imposing one or the other is represented by the sum $x + y$ and the new condition of imposing both simultaneously is represented by the product xy. Now, by a minor abuse of notation, a symbol x representing a condition is also used to denote the number of figures satisfying the condition, if finite. Then two symbols x, y are considered to be equal if they represent conditions that are the same for enumerative purposes, that is, if the numbers xw and yw are equal for every auxiliary condition w such that both numbers are finite. In this way, the symbols form a torsion-free, commutative, associative ring with 1.

The whole *Kalkül* is the result, beyond the accepted practice at the time in the arts of analytical projective geometry, of a skillful combined use of the symbolic calculus and the principle of conservation of number. But it is to be noted that the absence of suitable foundations for both aspects of the theory is not the only drawback, for another unsatisfactory issue is that computations become soon rather cumbersome and with apparently no incidence on the production of new ideas. Better times in which Schubert's quest could cast roots in a much richer soil and blossom in full were still far ahead in the future.

3 Crisis interlude. The symbolic calculus of conditions, together with the principle of conservation of number, worked somehow amazingly well, but the mathematical status of the calculus was unclear and no real proof had ever been proposed or found for the principle. So while it continued to be used enthusiastically by many, it was also criticized, rightly, by others. Hilbert, for example, who could not admit to rely on geometric intuition nor on general principles whose validity conditions were scarcely made precise, included this problem, as number 15, in his 1900 list:

> To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of so-called ***principle of special position***, or ***conservation of number***, by means of the ***enumerative calculus*** developed by him. [Hilbert 1900; italics are ours]

After 1900, some authors published a number of 'counterexamples' designed to show that the principle of conservation of number was wrong (Kohn [1903], Study [1905]), others proposed restrictions for the validity of the principle (Giambelli [1904],(12) Brill [1905]), and yet others defended it by saying that the proposed counterexamples were not really such, because the 'dimension accounting' of Schubert ruled them out (Sturm [1907], mostly exposing Kohn's faulty understanding, under a strangely gruff language, of Schubert's theory).

Study objected about the way Schubert formulated the principle, about the lack of precision in most of the applications, and was skeptical about some of the results.(13) Moreover, his main counterexample to the principle is easy to explain. Let the objects on which we will impose conditions be homographies α of $\mathbf{P}^1$. Let the data for the conditions be (unordered) tetrads of distinct points, τ, of $\mathbf{P}^1$. Given τ, let K_τ be the condition that α permutes τ. Then it is well known (see, for example, Semple–Roth [1949], § 6.5) that the number of homographies α satisfying K_τ is 4, 8 or 12 according to whether τ is general, harmonic of equianharmonic. So the number is not preserved when we vary continuously the data τ of the condition.

We will not deal here with Kohn's 'counterexamples' of [1903], for they are quite artificial as the already quoted paper of Sturm [1907] aptly pointed out (Kohn introduces, for instance, conditions which are not algebraic, or that do not have the right codimension, and so on; see also Severi [1912], § I.3). Instead we will recall a couple of counterexamples that exhibit interesting phenomena which we will comment later on.

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The first is transcribed from Fulton [1984], example 10.2.4, and is close to a slightly more complicated counterexample of Study that we will explain afterwards. Take the cone Y in $\mathbf{P}^3$, with vertex at $[1,0,0,0]$, defined by the equation $x_3^2 = x_1^2 + x_2^2$. Let L be the line $x_1 = 0$, $x_2 = x_3$. Let $T = \mathbb{A}^1$, with coordinate t, and let $X \subseteq Y \times T$ be the family of Cartier divisors $x_2 = tx_0$. Then X_t is, for $t \neq 0$, an irreducible conic

(12) For extensive information about Giambelli's work and life, see D. Laksov's paper in this volume.

(13) Here it is to be noted that no real errors have ever been found in Schubert's book, although a sizable part of it has not been vindicated yet. Already Severi [1912] said: "...nesun errore si è riscontrato finora nelle applicazioni che non furon fatte con intento critico".

meeting L at just one point, but X_0 is the union of two lines, each meeting L at P. So by continuous variation we make a transition from one solution to two solutions.

Another example which we would like to explain is due to Zobel [1961] (cf. Fulton [1984], example 10.2.5). Let $Y \subseteq \mathbf{P}^4$ be the cone $x_1 x_4 = x_2 x_3$ and let $L \subseteq Y \times \mathbb{A}^1$ be the family of lines $x_1 = 0, x_3 = 0, x_2 = t x_0$. Let M be the plane $x_1 = x_2, x_3 = x_4$. Then, for general t, L_t and M are disjoint, while L_0 and M meet at the vertex of the cone. So here a continuous transition is made from no solutions to one solution.

The counterexample of Study we referred to above deals with the number of intersection points of two twisted cubics on a quadric cone away from the vertex. This number turns out to be 4. Moreover, one can show that it is possible to degenerate algebraically one of the cubics into three rulings of the cone, whereupon the other cubic meets the degenerated one at three points away from the vertex. According to Study, this is another counterexample to the principle of conservation of number.

4 Severi's era. At some point it was necessary to come to terms with questions like the following: How does one deal with the figures one is interested in, and with its degenerate forms? How does one deal with conditions imposed to figures? What processes are to be allowed, and under what assumptions, for the principle of conservation of number to hold?

In present day terms, which to a great extend are due to the Italian school of algebraic geometry, and especially to Severi (cf. Severi [1912, 33, 40]), the figures Γ are *parameterized* by the points of some irreducible algebraic variety X, whose dimension d is the number of degrees of freedom of the figures, and both the *conditions* and the *systems* are subvarieties K and Σ of X. The codimension of K corresponds to the 'Dimension' of Schubert, the dimension of Σ corresponds to the 'Stuffe', and the number of figures of the system that satisfy the condition corresponds to the number, N, of points in the intersection of K and Σ, assuming it is finite and, as always, counting each point P with its intersection multiplicity $i_P(\Sigma, K)$. Moreover, a condition that depends on some data Γ' gives rise to an *algebraic family* of subvarieties $K_{\Gamma'}$ of X, so that we can identify K with a subvariety of $X \times T$ (an *algebraic correspondence*, to use Severi's terminology), where T is the variety parameterizing the figures Γ'. A condition K will be said to be irreducible if the corresponding subvariety of $X \times T$ is irreducible.

The variety X is often non-singular and complete, and the 'non-degenerate' figures correspond to the points of some open set X_0 in X. The most favorable cases for enumerative geometry occur when $X = X_0$. This happens, for instance, if the figures are linear spaces of a given dimension, or flags of linear spaces of some given dimensions. When $X - X_0$ in non-empty, it often is the union of non-singular divisors that meet transversely, or can be arranged to be that way without modifying X_0. One lovely example of this is the variety of complete conics, or, more generally, of complete quadrics in $\mathbf{P}^n$.

In general, the intersection of K and Σ will consist of a discrete set of points on X_0 (14) and of a variety on the boundary $X - X_0$ which often has components of

(14) This is usually proved by a suitable transversality criterion, such as the transversality theorem

positive dimension, and the number one really is looking for is the number of discrete solutions on X_0. The determination of such numbers usually involves an analysis of the behavior of K and Σ along the boundary.

Now we can address the contributions of Severi toward the clarification of the principle of conservation of number (cf. Severi [1912, 33, 40]) and those of later workers to establish them on a firm basis (this is a long story: see Semple–Roth [1949], §§ 6.6-6.7; B. Segre [1962], § I; Roth [1963], § II; van der Waerden [1970]; Fulton [1984], Notes and References to chapters 4, 11 and 12). We rely basically on Severi's formulation in his memoir of 1940, for in it assumptions that in the 1912 paper were tacit, like the non-singularity and completeness of the varieties involved, are made explicit.

To explain the key facts discovered by Severi in relation to the principle of conservation of number let us use the contemporary language of algebraic geometry. Assume X is a complete irreducible non-singular variety of dimension n, that T is an irreducible non-singular variety, and that $K \subseteq X \times T$ is an irreducible subvariety of codimension n such that K dominates T. For each closed point $t \in T$, let K_t be the cycle on X defined as the image in X by the isomorphism $X \times \{t\} \simeq X$ of the intersection cycle $(X \times \{t\}) \cdot K$ on $X \times T$, which consists of the set of isolated points of $(X \times \{t\}) \cap K$, each such point P being counted with the intersection multiplicity $i_P(X \times \{t\}, K)$. With this terminology the key theorem of Severi (see Severi [1940], § 16, and the historical discussion in van der Waerden [1970]) can be stated as follows:

> *The $t \in T$ for which $(X \times \{t\}) \cap K$ has dimension 0, that is, it contains only isolated points, is a non-empty open set T_0 of T and the degree of K_t is constant for $t \in T_0$.*

The generalization of this basic theorem to the case in which K is possibly reducible, but with all its components of codimension n and dominating T, is straightforward (such correspondences are called *pure* by Severi). Let us mention here that van der Waerden [1927] gave an algebraic proof of Severi's theorem under the assumption that X is rational; that about five years later Severi was aware of the need to assume X non-singular (see Severi [1932]); that van der Waerden [1934] gave an algebraic proof of Severi's theorem which surely influenced Severi [1940] and whose spirit was embodied and refined by A. Weil [1946]; and that today's intersection multiplicities are understood through a lot of further work by, among many others, Samuel, Chevalley, Northcott, Chow, Serre, Grothendieck and Fulton.

Severi claims that his theorem can be applied to justify all uses of the principle of conservation of number done by Schubert, but our view is that this is not so because it yields no theoretical results to deal with two kinds of common difficulties. (15) The first difficulty is the evaluation of the multiplicities of the points in the support of K_{t_0}, for a particular value of t_0. For generic t one can often use general principles, like

of generic translates (see Kleiman [1973]).

(15) The reader is invited to note that to this day the same difficulties remain in spite of repeated claims that some theoretical construct has settled Hilbert's 15th problem for good, be it some intersection ring of cycle classes, as in Severi and others, or the homology ring, as in van der Waerden [1930].

some form of transversality, to infer, say, that all the points in K_t have multiplicity one, but for special values t_0 of t all one knows is that the multiplicities of the points in K_{t_0} add up to the number of points in K_t, for generic t. The way Schubert evaluated non-trivial multiplicities is not clear, because often results are quoted with no indication of how they were obtained, but it seems likely that he used, at least for some cases, a dynamic interpretation of the intersection multiplicities. In other cases he seems to have determined them by writing enough independent relations involving the sought multiplicity. For modern tools on this issue we refer to Fulton [1984].

The second difficulty, usually also the more serious, is that we often are not interested in all the points of K_t, but only on those that lie in some open set X_0 of X, that is, in $K_t \cap X_0$ (as said before, X_0 usually is the subset of X parameterizing 'non-degenerate' figures), but it is not the case that $K_t \cap X_0$ and $K_{t_0} \cap X_0$ have the same degree, and so the final accounting of what is going on with the points in K_t when we make the transition $t \to t_0$ may be quite complicated. (16) In fact the analysis of such questions turns out to require a mixture of intersection theory and singularity theory, and it is in this direction that we foresee interesting advances in enumerative geometry. For a sample of examples in which difficulties of the kind alluded to above are encountered the reader may consult Halphen [1873 a, b], Schubert [1879] (especially the fourth chapter, devoted to the degeneration method), Casas–Xambó [1986], Kleiman-Strømme-Xambó [1987], Miret–Xambó [1988, 1991], Procesi–Xambó [1991] or Miret–Xambó [1992]. For an historical perspective see Xambó [1993].

▼

Let us illustrate the preceding ideas with some examples. As in Severi's work, let us first consider Study's homography example. The homography of $\mathbf{P}^1$ defined by the matrix $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (hence $ad - bc \neq 0$) can be mapped to the point $[a, b, c, d] \in \mathbf{P}^3$, and so we can identify the homographies with the open set X_0 of $X = \mathbf{P}^3$ that is the complement of the non-singular quadric $ad - bc = 0$. Since tetrads of distinct points of $\mathbf{P}^1$ can be identified with monic polynomials of degree four with distinct roots, we can take as variety T the open set of the 4-dimensional affine space whose points $\tau = (t_1, \ldots, t_4)$ are those such that the polynomial $Z^4 + t_1 Z^3 + t_2 Z^2 + t_3 Z + t_4$ has non-zero discriminant. Then the condition considered by Study is a correspondence $K \subseteq X_0 \times T$. Since X_0 is not complete, Severi's theorem cannot be applied, but this is not all there is to say. To begin with, the projection of K to X_0, let us call it S, is the union of three surfaces, $S = S_2 \cup S_3 \cup S_4$, where the points in S_2 are the involutions, the points in S_3 are the cyclic homographies of order 3, and the points in S_4 are the cyclic homographies of order 4. So K is reducible, for it is the union of the three correspondences K_i determined by the S_i, $K = K_2 \cup K_3 \cup K_4$. Still more, K_2 is itself reducible: given an involution α, there are two kinds of $\tau \in T$ left invariant by α, namely the tetrads formed by the two fixed points of α together with one pair of mates of α, and the tetrads formed by two pairs of mates of α. From this one can infer that K_2 has two irreducible components K' and K'' which correspond to the

(16) So to speak, some may remain in X_0, but others may run to the boundary $X - X_0$ of 'degenerate' figures.

two kinds of tetrads just described. Notice that the projection of K' to T is the 3-dimensional subvariety H of T formed by the harmonic tetrads, while K'' dominates T; hence K' has dimension 3 and K'' has dimension 4 (in Severi's terminology, K_2 is impure).

As far as the other examples we have considered, of course they do not contradict Severi's theorem: a quadric cone, albeit complete, is singular, and if we take away the singular point (the vertex) then the resulting surface is not complete anymore. On the other hand we can make sense of what is going on, using Severi's framework, by blowing up the vertex of the cone. For instance, in Study's example of the intersection of two twisted cubics lying on a quadric cone, one finds that when we go onto the degenerate cubic that consists of three lines then one of the four intersection points moves onto the exceptional divisor.

▲

Severi returned to enumerative geometry throughout his life and to a great extend his equivalence theories (rational, algebraic and numerical) were introduced in the quest for a suitable theoretical framework that would be useful to analyze such kinds of problems (cf. the 'Notes and references' section of the first chapter of Fulton [1984]). How highly he placed enumerative geometry can be realized on inspecting the chapter on 'Nozioni introduttorie' in Severi [1926]: he was planning to devote the eighth and last volume of the Trattato to that discipline. Incidentally, there we learn that one aspect he included in enumerative geometry was to "valutare la molteplicità delle singole soluzioni".

As Semple and Roth [1949] put it, all this shows that

> From the indications we have given it will be clear that any analysis of the methods of enumerative geometry must involve deep questions of algebraic geometry.

5 The principle in modern intersection theory. The widespread accusations of lack of rigor in the classical Italian school of algebraic geometry, and Severi's forceful defense of it, are well known (see Severi [1949], and also the discussions by Severi, *Problèmes résolus et problèmes nouveaux dans la théorie des systèmes d'équivalence*, and van der Waerden, *On the definition of rational equivalence cycles on a variety*, in the proceedings of the 1954 Amsterdam International Congress). Some acknowledged their indebtedness to Severi, or to the Italian school in general. For instance, in van der Waerden [1971] we find

> The Italian school [...] erected an admirable structure, but its logical foundation was shaky. The notions were not well-defined, the proofs were insufficient. And yet, as Bernard Shaw puts it: "There is an Olympian ring in it. It must be true, for it is fine art."

And W.-L. Chow [1956], § 1, speaking about his 'systematic treatment' of equivalence classes of cycles, says:

> Just as the treatment in the book of Hodge and Pedoe, (17) our theory is also based on the ideas of Severi and van der Waerden, and in this sense we do not claim any

(17) Methods of Algebraic Geometry, vol. II, Cambridge, 1952.

> great originality; nevertheless, [...] a discerning reader will easily perceive that such a systematic treatment does involve substantially more than can be readily obtained from the papers of Severi and van der Waerden.

But some went to the extreme of pretending to ignore the predecessors, as reminded by the following authoritative paragraph from the "Notes and References" to the first chapter in Fulton [1984]:

> In 1958 Chevalley's seminar focused on rational equivalence. In the notes [...] the theory is developed from first principles, with no reference to, or even mention of, previous work on the subject; there the ring of rational equivalence classes was named "Chow ring". [...] It would be unfortunate, however, if Severi's pioneering work in this area were forgotten; and if incompleteness or the presence of errors are grounds for ignoring Severi's work, few of the subsequent papers on rational equivalence would survive.

Fortunately many others rooted their work in the Italian school of thought, adapting it to the new standards of rigor and deepening its results (see Fulton [1984], *loc. cit*). The ideas of Severi about intersection multiplicities, about rational, algebraic and numerical equivalence, and so on, were taken by workers such as van der Waerden, B. Segre, Todd, Hodge–Pedoe (see footnote 17), Weil (see Weil [1954]), Samuel (see Samuel [1956]), Chow (loc. cit.), Chevalley, Grothendieck, and many others, and were made available to an ever increasing number of workers in algebraic geometry. The result is an intellectual landscape shaped to a great extent as Severi had conceivably dreamed.

One point to make, which I hope will help some to appreciate this prized legacy, is that some important matters that once were obscure, and the source of conflict and bitter disputes, have become apparently trivial because they are now settled at the level of existence theorems, like the existence and functorialities of the rational or algebraic intersection rings of any non-singular quasi-projective variety.

Take, for example, the simplest form of the principle of conservation of number in present day terms. The basic fact is that the push-forward by a proper map preserves rational equivalence and from this it follows immediately that two zero-cycles that are rationally equivalent on a complete (not necessarily smooth) variety have the same degree. A stronger result is that two zero cycles on a complete variety have the same degree if they are algebraically equivalent, and there are still more refined forms of the principle which are valid on varieties that need not be complete and which follow from the powerful intersection theory available today (see Fulton [1984]).

Another example that illustrates our point is Bezout's theorem. Here one computes by standard methods that the codimension k group of rational equivalence classes of $\mathbf{P}^n$ is isomorphic to $\mathbf{Z}$ and that the integer corresponding to a variety of codimension k is its degree. Now if V and V' intersect properly on $\mathbf{P}^n$, then $[V \cdot V'] = [V] \cdot [V']$, which shows that the degree of a proper intersection is the product of the degrees of its factors.

To finish let us say that the continuity Ansatz also played a role in the Fulton–Macpherson intersection theory, in the following sense: the existence of a flat deformation of a regular embedding to the zero section embedding of the corresponding normal cone suggested that it should be possible to define algebraic intersections by a construction on the normal cone. And this turned out to work perfectly well, just as it did, in the long run, Poncelet's faith:

> The existence of such a deformation, together with the "principle of continuity", helps explain the key role of normal cones in the construction of intersection products. (Fulton [1984], p. 3)

Let us mention that one of the bonus features of this approach is that it requires no previous foundation of the 'intersection theory + moving lemma' sort. But this approach, if it only were because its spirit is close to B. Segre's [1953], certainly belongs to the main stream of Severi's heritage.

A chronological list of geometers. J. Poncelet (1788-1867), M. Chasles (1793-1880), J. Steiner (1796-1863), J. Plücker (1801-1868), G. Salmon (1819-1904), De Jonquières (1820-1901), A. Cayley (1821-1895), L. Cremona (1830-1903), H. Zeuthen (1839-1920), M. Nöther (1841-1935), G. Halphen (1844-1889), H. Schubert (1848-1911), C. Segre (1863-1924), G. Castelnuovo (1864-1952), F. Enriques (1871-1946), **F. Severi (1879-1961)**, O. Zariski (1899-1985), B. Segre (1903-1977), B. L. van der Waerden (1903-), A. Weil (1906-), J. A. Todd (1908-), C. Chevalley (1909-1984), W. Chow (1911-), P. Samuel (1921-), M. Eger (1922-1952), J. P. Serre (1926-), A. Grothendieck (1928-).

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