EULER AND THE DYNAMICS OF RIGID BODIES

Sebastià Xambó Descamps
Sebastia.xambo@upc.edu

0.- Notations and conventions.

The material in sections 0-6 of this paper is an adaptation of parts of the Mechanics chapter of XAMBÓ (2007). The mathematical language used is rather standard. The reader will need a basic knowledge of linear algebra, of Euclidean geometry (see, for example, XAMBÓ (2001)), and of basic calculus.

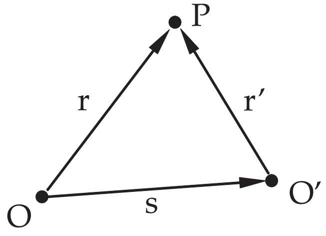
0.1.- If we select an origin O in Euclidean 3-space E_3, each point P can be specified by a vector $\boldsymbol{r} = P - O = \overline{OP} \in V_3$, where V_3 is the Euclidean vector space associated with E_3. This sets up a one-to-one correspondence between points P and vectors $\boldsymbol{r}$. The inverse map is usually denoted $P = O + \boldsymbol{r}$.

Usually we will speak of "the point $\boldsymbol{r}$", instead of "the point P", implying that some point O has been chosen as an origin. Only when conditions on this origin become relevant we will be more specific. From now on, as it is fitting to a mechanics context, any origin considered will be called an *observer*.

When points are assumed to be moving with time, their movement will be assumed to be smooth. This includes observers O for which we do not put any restriction on its movement (other than it be smooth). This generality, which we find necessary for our analysis, is not considered in the classical mechanics texts, where O is allowed to have a uniform movement or to be some special point of a moving body. Another feature of our presentation is that it is coordinate-free. Coordinate axes are used only as an auxiliary means in cases where it makes possible a more accessibly proof of a coordinate free statement (for an example, see §4.2).

0.2.- The derivative $\dot{\boldsymbol{r}} = d\boldsymbol{r}/dt$ is the *velocity*, or *speed*, *of* P *relative to* O. Similarly, $\ddot{\boldsymbol{r}} = d\dot{\boldsymbol{r}}/dt = d^2\boldsymbol{r}/dt^2$ is the *acceleration of* P *relative to* O.

Let us see what happens to speeds and accelerations when they are referred to another observer, say $O' = O + \boldsymbol{s}$, where $\boldsymbol{s}$ is any (smooth) function of t. If $\boldsymbol{r}' = P - O'$ (the position vector of P with respect to O'), then $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{s}$ and hence $\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}' + \dot{\boldsymbol{s}}$. In other words, the velocity with respect to O is the (vector) sum of the velocity with respect to O' and the velocity of O' with respect to O. Taking derivative once more, we see that $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}' + \ddot{\boldsymbol{s}}$, which means that the acceleration with respect to O is the (vector) sum of the acceleration with respect to O' and the acceleration of O' with respect to O.



As a corollary we see that the velocity (acceleration) of P with respect to O is the same as the velocity (acceleration) of P with respect O' if and only if $\dot{\boldsymbol{s}} = 0$ $(\ddot{\boldsymbol{s}} = 0)$. Note that the condition for O' to be at rest with respect to O for some temporal interval is that $\dot{\boldsymbol{s}} \equiv 0$ in that interval. Similarly, $\ddot{\boldsymbol{s}} \equiv 0$ for some temporal interval if and only if $\boldsymbol{u} = \dot{\boldsymbol{s}}$ is constant on that interval, or $\boldsymbol{s} = \boldsymbol{u}t + \boldsymbol{p}$, where $\boldsymbol{p}$ is also constant. In other words, $\ddot{\boldsymbol{s}} \equiv 0$ for a temporal interval means that the movement of O' relative to O is *uniform* for that interval.

1.- The momentum principle.

Since we refer points to an observer O, velocities, accelerations and other vector quantities defined using them (like momentum, force and energy) *will also be relative to* O. Our approach is non-relativistic, as masses are assumed to be invariable and speeds are not bounded.

1.1.- Consider a system Σ of point masses $m_1,\ldots,m_N$ located at the points $\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N$. The *total mass* of Σ is $m = m_1 + \ldots + m_N$. The *velocity* of m_k is $\dot{\boldsymbol{r}}_k = d\boldsymbol{r}_k/dt$ and its (linear) *momentum* is $\boldsymbol{p}_k = m_k\dot{\boldsymbol{r}}_k$. The *acceleration* of m_k is $\ddot{\boldsymbol{r}}_k = d^2\boldsymbol{r}_k/dt^2$. The *force* acting on $\boldsymbol{m}_k$ is $\boldsymbol{F}_k = m_k\ddot{\boldsymbol{r}}_k$ (all observers accept *Newton's second law*). In particular we have that $\boldsymbol{F}_k \equiv \boldsymbol{0}$ for some temporal interval if and only if the movement of m_k relative to O is *uniform* on that interval (cf. §0.2). This is *Galileo's inertia principle*, or *Newton's first law*, relative to O.

The force with respect to the observer $O' = P + \boldsymbol{s}$ is

$$\boldsymbol{F}'_k = \boldsymbol{F}_k - m_k\ddot{\boldsymbol{s}}$$

as $\boldsymbol{F}'_k = m_k\ddot{\boldsymbol{r}}'_k = m_k(\ddot{\boldsymbol{r}}_k - \ddot{\boldsymbol{s}}) = m_k\ddot{\boldsymbol{r}}_k - m_k\ddot{\boldsymbol{s}} = \boldsymbol{F}_k - m_k\ddot{\boldsymbol{s}}$.

1.2.- The *centre of mass of* Σ is the point $G = O + r_G$ with

$$\boldsymbol{r}_G = \frac{1}{m}(m_1\boldsymbol{r}_1 + \ldots + m_N\boldsymbol{r}_N) \;.$$

The point G is also called *inertia centre* or *barycenter*. It *does not depend on the observer* O, and hence it is a point intrinsically associated with Σ. Indeed, if $O' = O + \boldsymbol{s}$ is another observer, then $\boldsymbol{r}'_k = \boldsymbol{r}_k - \boldsymbol{s}$ is the position vector of m_k with respect to O', and

$$\frac{1}{m}\sum_k m_k\boldsymbol{r}_k' = \frac{1}{m}\sum_k m_k\boldsymbol{r}_k - \frac{1}{m}\sum_k m_k\boldsymbol{s} = \boldsymbol{r}_G - \boldsymbol{s} \;,$$

which just says that

$$O' + \frac{1}{m}\sum_k m_k r_k' = O + \boldsymbol{s} + (\boldsymbol{r}_G - \boldsymbol{s}) = O + \boldsymbol{r}_G = G$$

Note that the argument also shows that the position vector of G with respect to O' is

$$\boldsymbol{r}_G' = \boldsymbol{r}_G - \boldsymbol{s} \,.$$

The velocity V of G is

$$\boldsymbol{V} = \dot{\boldsymbol{r}}_G = \frac{1}{m}(m_1\dot{\boldsymbol{r}}_1 + \ldots + m_N\dot{\boldsymbol{r}}_N) \cdot$$

The relation $\boldsymbol{r}'_G = \boldsymbol{r}_G - \boldsymbol{s}$ implies that if $\boldsymbol{V}'$ denotes the velocity of G with respect to another origin O', then $\boldsymbol{V}' = \boldsymbol{V} - \boldsymbol{v}$, with $\boldsymbol{v} = \dot{\boldsymbol{s}}$ the velocity of O' with respect to O.

1.3.- The (linear) *momentum* of Σ is $\boldsymbol{P} = \sum_k \boldsymbol{p}_k = \sum_k m_k\dot{\boldsymbol{r}}_k$. By §1.2 we can write

$$\boldsymbol{P} = m\dot{\boldsymbol{r}}_G = m\boldsymbol{V} \,.$$

Note that if $\boldsymbol{P}'$ denotes the momentum referred to another observer $O' = O + \boldsymbol{s}$, and v is the velocity of O' with respect to O, then

$$\boldsymbol{P}' = \boldsymbol{P} - m\boldsymbol{v} \;.$$

Indeed, $\boldsymbol{P}' = m\boldsymbol{V}' = m(\boldsymbol{V} - \boldsymbol{v}) = m\boldsymbol{V} - m\boldsymbol{v} = \boldsymbol{P} - m\boldsymbol{v}$.

1.4.- If F_k denotes the force acting on m_k, the *total* or *resultant force* acting on Σ is defined as

$$\boldsymbol{F} = \sum_k \boldsymbol{F_k} .$$

From the definitions it follows immediately that

$$\dot{\boldsymbol{P}} = \boldsymbol{F} .$$

In fact, $\boldsymbol{F} = \sum_K \boldsymbol{F_k} = \sum_k m_k \ddot{\boldsymbol{r}}_k = \frac{d}{dt}\sum_k m_k \dot{\boldsymbol{r}}_k = \dot{\boldsymbol{P}}$.

Unfortunately the equation $\dot{\boldsymbol{P}} = \boldsymbol{F}$ is not very useful, as we do not yet have information on $\boldsymbol{F}$ other than the formal definition using Newton's second law.

1.5.- In the analysis of the force, it is convenient to set $\boldsymbol{F_k} = \boldsymbol{F_k^e} + \boldsymbol{F_k^i}$, where $\boldsymbol{F_k^e}$ and $\boldsymbol{F_k^i}$ denote the *external* and *internal forces* acting on m_k, so that $\boldsymbol{F} = \boldsymbol{F}^e + \boldsymbol{F}^i$, with $\boldsymbol{F}^e = \sum_k \boldsymbol{F_k^e}$ and $\boldsymbol{F}^i = \sum_k \boldsymbol{F_k^i}$.

The internal force is construed as due to some form of "interaction" between the masses, and is often represented as $\boldsymbol{F_k^i} = \sum_j \boldsymbol{F_{kj}}$, with $\boldsymbol{F_{kj}}$ the force "produced" by m_j on m_k (with the convention $\boldsymbol{F_{kk}} = 0$). Here we further assume that $\boldsymbol{F_{kj}}$ only depends on $\boldsymbol{r_j} - \boldsymbol{r_k}$, which implies that $\boldsymbol{F}^i$ does not depend on the observer. One example of interaction is the given by *Newton's law for the gravitation force*, for which

$$\boldsymbol{F_{kj}} = G\frac{m_k m_j}{\left|\boldsymbol{r}_j - \boldsymbol{r}_k\right|^3}(\boldsymbol{r_j} - \boldsymbol{r_k}) .$$

The external force usually models interaction forces of m_k with systems that are "external" to Σ, as, for example, the Earth gravitation force on the particle m_k.

Since the internal force is independent of the observer, the transformation law for the external force is the same as for the force and total force:

$$\boldsymbol{F}\,^e_k = \boldsymbol{F_k^e} - m_k\ddot{\boldsymbol{s}} \ , \ \boldsymbol{F}\,^e = \boldsymbol{F}^e - m\ddot{\boldsymbol{s}} .$$

We may say that observer O sees that the movement of O' is driven by

the force $m\ddot{\boldsymbol{s}}$ and therefore that O' will experience this force as an external force $-m\ddot{\boldsymbol{s}}$. In particular we see that $\boldsymbol{F}\,^e = \boldsymbol{F}^e$ at a given instant (or temporal interval) if and only if $\ddot{\boldsymbol{s}} = 0$ ($\ddot{\boldsymbol{s}} \equiv 0$ in that interval, which means that O' is moving uniformly with respect to O).

1.6.- We will say that the system is *Eulerian* if $\boldsymbol{F}^i = 0$. It is thus clear that for Eulerian systems $\boldsymbol{F} = \boldsymbol{F}^e$, and so they satisfy the law

$$\dot{\boldsymbol{P}} = \boldsymbol{F}^e .$$

This law is called the *momentum principle*. In particular we see that for Eulerian systems *the momentum* $\boldsymbol{P}$ *is constant if there are no external forces*, and this is the *principle of conservation of momentum*. Notice, however, that external forces may vanish for an observer but not for another (cf. §1.5).

1.7.- We will say that a discrete system is *Newtonian* if $\boldsymbol{F_k^i} = \sum_j \boldsymbol{F_{kj}}$ (cf. §1.5) and $\boldsymbol{F_{kj}} = c_{kj}(\boldsymbol{r_j} - \boldsymbol{r_k})$, where c_{kj} are real quantities such that $c_{kj} = c_{jk}$ for any k, j. Note that this implies that $\boldsymbol{F_{kj}} = -\boldsymbol{F_{jk}}$ for all pairs $j,k \in \{1,...,N\}$, which is what we expect if $\boldsymbol{F_{kj}}$ is thought as the force "produced" by m_j on m_k and *Newton's third law* is correct.

The main point here is that *Newtonian systems are Eulerian*, for

$$\boldsymbol{F}^i = \sum_k \boldsymbol{F_k^i} = \sum_k \sum_j \boldsymbol{F_{kj}} = \sum_{k<j} (\boldsymbol{F_{kj}} + \boldsymbol{F_{jk}}) = 0 .$$

1.8.- A (discrete) *rigid body* is a Newtonian system Σ in which the distances $d_{kj} = \left|\boldsymbol{r_j} - \boldsymbol{r_k}\right|$ are constant. The intuition for this model is provided by situations in which we imagine that the force of m_j on m_k is produced by some sort of inextensible massless rod connecting the two masses. The inextensible rod ensures that the distance between m_k and m_j is constant. The force $\boldsymbol{F_{kj}}$ has the form $c_{kj}(\boldsymbol{r_j} - \boldsymbol{r_k})$ because that force is parallel to the rod, and $c_{kj} = c_{jk}$ by Newton's third law and the identity $\boldsymbol{r_j} - \boldsymbol{r_k} = -(\boldsymbol{r_k} - \boldsymbol{r_j})$. For real rigid bodies, atoms play the role of particles and inter-atomic electric forces the role of rods.

Since a rigid body is Newtonian, it is also Eulerian. Therefore *a rigid body satisfies the momentum principle* (cf. §1.6).

2.- The angular momentum principle.

2.1.- With the same notations as in the Section 1, we define the *angular momentum* $\boldsymbol{L}_k$ *of* m_k *relative to* O by

$$\boldsymbol{L}_k = \boldsymbol{r}_k \times \boldsymbol{p}_k = m_k \boldsymbol{r}_k \times \dot{\boldsymbol{r}}_k,$$

and the *angular momentum* $\boldsymbol{L}$ *of* Σ *with respect to* O by

$$\boldsymbol{L} = \sum_k \boldsymbol{L}_k = \sum_k \boldsymbol{r}_k \times \boldsymbol{p}_k = \sum_k m_k \boldsymbol{r}_k \times \dot{\boldsymbol{r}}_k .$$

If we choose another observer $O' = O + s$, the angular momentum $\boldsymbol{L}'_k$ of m_k with respect to O' is related to $\boldsymbol{L}_k$ as follows:

$$\boldsymbol{L}'_k = m_k \boldsymbol{r}'_k \times \dot{\boldsymbol{r}}'_k = m_k(\boldsymbol{r}_k - \boldsymbol{s}) \times (\dot{\boldsymbol{r}}_k - \dot{\boldsymbol{s}})$$

$$= \boldsymbol{L}_k - \boldsymbol{s} \times (m_k \dot{\boldsymbol{r}}_k) + \dot{\boldsymbol{s}} \times (m_k \boldsymbol{r}_k) + m_k \boldsymbol{s} + \dot{\boldsymbol{s}} .$$

Summing with respect to k we obtain that

$$\boldsymbol{L}' = \boldsymbol{L} - m\boldsymbol{s} \times \dot{\boldsymbol{r}}_G + m\dot{\boldsymbol{s}} \times \boldsymbol{r}_G + m\boldsymbol{s}\,\dot{}\,.$$

Note that

[*] $$\dot{\boldsymbol{L}}' = \dot{\boldsymbol{L}} - m\boldsymbol{s} \times \ddot{\boldsymbol{r}}_G + m\ddot{\boldsymbol{s}} \times \boldsymbol{r}_G + m\boldsymbol{s} \times \ddot{\boldsymbol{s}} .$$

Note also that $\dot{\boldsymbol{L}}' = \dot{\boldsymbol{L}} - m\boldsymbol{s} \times \ddot{\boldsymbol{r}}_G$ if either O' has a uniform motion with respect to O or else $O' = G$ (in this case $\boldsymbol{s} = \boldsymbol{r}_G$).

2.2.- The *moment* or *torque* $\boldsymbol{N}_k$ of the force $\boldsymbol{F}_k$ with respect to O is defined by

$$\boldsymbol{N}_k = \boldsymbol{r}_k \times \boldsymbol{F}_k,$$

and the *total* (or *resultant*) *moment* or *torque* $\boldsymbol{N}$ of the forces by

$$\boldsymbol{N} = \sum_k \boldsymbol{N}_k = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k .$$

If we choose another observer $O' = O + \boldsymbol{s}$, then

$$\begin{aligned}\boldsymbol{N}'_k &= \boldsymbol{r}'_k \times \boldsymbol{F}'_k = (\boldsymbol{r}_k - \boldsymbol{s}) \times m_k(\ddot{\boldsymbol{r}}_k - \ddot{\boldsymbol{s}}) \\ &= \boldsymbol{N}_k - \boldsymbol{s} \times \boldsymbol{F}_k - \ddot{\boldsymbol{s}} \times (m_k \boldsymbol{r}_k) + m_k \boldsymbol{s} \times \ddot{\boldsymbol{s}} .\end{aligned}$$

Summing with respect to k, we get

[*] $$\boldsymbol{N}' = \boldsymbol{N} - \boldsymbol{s} \times \boldsymbol{F} + m\ddot{\boldsymbol{s}} \times \boldsymbol{r}_G + m\boldsymbol{s} \times \ddot{\boldsymbol{s}} .$$

Note that $\boldsymbol{N}' = \boldsymbol{N} - \boldsymbol{s} \times \boldsymbol{F}$ if either O' has a uniform movement with respect to O or if $O' = G$ (in this case $\boldsymbol{s} = \boldsymbol{r}_G$).

2.3.- We have the equation

$$\dot{\boldsymbol{L}} = \boldsymbol{N} .$$

Indeed, since $\dot{\boldsymbol{r}}_k \times \dot{\boldsymbol{r}}_k = 0$,

$$\dot{\boldsymbol{L}} = \frac{d}{dt} \sum_k m_k \boldsymbol{r}_k \times \dot{\boldsymbol{r}}_k = \sum_k m_k \boldsymbol{r}_k \times \ddot{\boldsymbol{r}}_k = \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k = \boldsymbol{N} .$$

2.4.- Now $\boldsymbol{N} = \boldsymbol{N}^e + \boldsymbol{N}^i$, where $\boldsymbol{N}^e$ and $\boldsymbol{N}^i$ are the momenta of the external and internal forces, respectively. We will say that Σ is *strongly Eulerian* if it is Eulerian and $\boldsymbol{N}^i = 0$. For strongly Eulerian systems we have (§2.3) the equation

$$\dot{\boldsymbol{L}} = \boldsymbol{N}^e,$$

which is called the *angular momentum principle*.

Newtonian systems, hence in particular rigid bodies, are strongly Eulerian. Indeed, since they are Eulerian, it is enough to see that $\boldsymbol{N}^i = 0$, and this can be shown as follows:

$$\begin{aligned}\boldsymbol{N}^i &= \sum_k \boldsymbol{r}_k \times \boldsymbol{F}_k^i = \sum_k \boldsymbol{r}_k \times \sum_j \boldsymbol{F}_{kj} = \sum_{k<j} (\boldsymbol{r}_k \times \boldsymbol{F}_{kj} + \boldsymbol{r}_j \times \boldsymbol{F}_{jk}) \\ &= \sum_{k<j} (\boldsymbol{r}_k \times \boldsymbol{F}_{kj} - \boldsymbol{r}_j \times \boldsymbol{F}_{kj}) = \sum_{k<j} (\boldsymbol{r}_k - \boldsymbol{r}_j) \times \boldsymbol{F}_{kj} = 0 .\end{aligned}$$

We have used that $\boldsymbol{F}_{kj} = c_{kj}(\boldsymbol{r}_k - \boldsymbol{r}_j)$, with $c_{kj} = c_{jk}$, for a Newtonian system.

Remark. If we consider the observer $O' = O + \boldsymbol{s}$, then of course we have $\dot{\boldsymbol{L}}' = \boldsymbol{N}'^e$, and here we point out that this equation is, for an strongly Eulerian system, consistent with the relations $[*]$ in §2.1 and §2.2. Since these relations are

$$\dot{\boldsymbol{L}}' = \dot{\boldsymbol{L}} - m\boldsymbol{s}\times\ddot{\boldsymbol{r}}_G + m\ddot{\boldsymbol{s}}\times\boldsymbol{r}_G + m\boldsymbol{s}\times\ddot{\boldsymbol{s}},$$

$$\boldsymbol{N}'^e = \boldsymbol{N}^e - \boldsymbol{s}\times\boldsymbol{F}^e + m\ddot{\boldsymbol{s}}\times\boldsymbol{r}_G + m\boldsymbol{s}\times\ddot{\boldsymbol{s}},$$

the consistency amounts to the relation $m\boldsymbol{s}\times\ddot{\boldsymbol{r}}_G = \boldsymbol{s}\times\boldsymbol{F}^e$, which is true because an Eulerian system satisfies $m\ddot{\boldsymbol{r}}_G = \boldsymbol{P} = \boldsymbol{F}^e$ (momentum principle).

3.- Energy.

3.1.- Kinetic energy.

This is also a quantity that depends on the observer O and which can be defined for general systems Σ. The *kinetic energy* of Σ, as measured by O, is

$$T = \frac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k^2 .$$

If $O' = O + s$ is another observer, and T' is the kinetic energy of Σ as measured by O', then we have:

$$T' = T + \frac{1}{2}m\dot{\boldsymbol{s}}^2 - \boldsymbol{P}\cdot\dot{\boldsymbol{s}},$$

where $\boldsymbol{P} = m\dot{\boldsymbol{r}}_G$ (m times the speed of G with respect to O). Indeed, with the usual notations,

$$\dot{\boldsymbol{r}}'_k = \dot{\boldsymbol{r}}_k - \dot{\boldsymbol{s}},\ \dot{\boldsymbol{r}}'^2_k = \dot{\boldsymbol{r}}_k^2 - 2\dot{\boldsymbol{r}}_k\dot{\boldsymbol{s}} + \dot{\boldsymbol{s}}^2, \text{ and}$$

$$T' = \frac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}'^2_k = \frac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k^2 + \frac{1}{2}\sum_k m_k \dot{\boldsymbol{s}}^2 - \sum_k m_k \dot{\boldsymbol{r}}_k\cdot\dot{\boldsymbol{s}},$$

which yields the claim because the first summand is T, the second is $1/2\, m\dot{\boldsymbol{s}}^2$ and in the third $\sum_k m_k\dot{\boldsymbol{r}}_k = m\dot{\boldsymbol{r}}_G = \boldsymbol{P}$.

The *power* Π of the forces $\boldsymbol{F}_k$, as measured by O, is defined as $\sum_k \dot{\boldsymbol{r}}_k\cdot\boldsymbol{F}_k$.

If we define in an analogous way the power Π^e of the external forces and the power Π^i of the internal forces, then the instantaneous variation of T is given by

$$\dot{T} = \Pi = \Pi^e + \Pi^i.$$

The proof is a short computation:

$$\dot{T} = \frac{d}{dt}\left(\frac{1}{2}\sum_k m_k\dot{\boldsymbol{r}}_k^2\right) = \sum_k m_k\dot{\boldsymbol{r}}_k\cdot\ddot{\boldsymbol{r}}_k = \sum_k \dot{\boldsymbol{r}}_k\cdot\boldsymbol{F}_k = \Pi = \Pi^e + \Pi^i.$$

3.2.- Conservative systems.

The system Σ is said to be *conservative* if there is a smooth function $V = V(\boldsymbol{r}_1,\dots,\boldsymbol{r}_N)$ that depends only of the differences $\boldsymbol{r}_j - \boldsymbol{r}_k$ and such that $\boldsymbol{F}_k^i = -\boldsymbol{\partial}_{rk}V$, where $\boldsymbol{\partial}_{rk}V$ denotes the gradient of V as a function of $\boldsymbol{r}_k$. The function V is independent of the observer and it is called the *potential* of Σ.

Example. The Newtonian gravitational forces

$$\boldsymbol{F}_{kj} = G\frac{m_k m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^3}(\boldsymbol{r}_j - \boldsymbol{r}_k)$$

are conservative, with potential

$$V = \frac{1}{2}G\sum_{k<j}\frac{m_k m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_k|} = G\sum_{k\neq j}\frac{m_k m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_k|},$$

as $-\partial_{rk}\dfrac{m_k m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_k|} = \dfrac{m_k m_j}{|\boldsymbol{r}_j - \boldsymbol{r}_k|^3}(\boldsymbol{r}_j - \boldsymbol{r}_k)$.

Conservative systems satisfy the relation

$$\Pi^i = -\dot{V}.$$

Indeed, $\Pi^i = \sum_k \dot{\boldsymbol{r}}_k\cdot\boldsymbol{F}_k^i = -\sum_k \dot{\boldsymbol{r}}_k\cdot\boldsymbol{\partial}_{rk}V = -\dot{V}$ (the latter equality is by the chain rule).

For a conservative system Σ, the sum $E = T + V$ is called the *energy*. As a corollary of §3.1 and §3.2, we have that

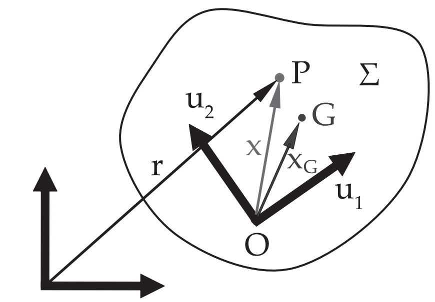
$$\dot{E} = \Pi^{e}.$$

In particular E is a conserved quantity if there are no external forces. This is the case, for example, for a system of particles with only gravitational interaction.

4.- Kinematics of rigid bodies.

To study the kinematics of a rigid body Σ, it is convenient to modify a little the notations of the previous sections.

4.1.- Angular velocity.



We will let O denote an observer fixed in relation to the body (not necessarily a point of the body) and let $\boldsymbol{x} = \boldsymbol{x}_p$ be the position vector with respect to O of any moving point P, so that $P = O + \boldsymbol{x}$. If we fix a positively oriented orthonormal basis (also called a Cartesian reference) $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ to the body at O, then

$$\boldsymbol{x} = x_1\boldsymbol{u_1} + x_2\boldsymbol{u_2} + x_3\boldsymbol{u_3}$$

$$= (x_1, x_2, x_3)\begin{pmatrix}\boldsymbol{u_1}\\ \boldsymbol{u_2}\\ \boldsymbol{u_3}\end{pmatrix} = x\boldsymbol{u} \text{ (matrix notation),}$$

with $x_i = x_i(t)$, $i = 1,2,3$. We define the *velocity of* $\boldsymbol{x}$ (or of P) *with respect to* Σ as the vector

$$\boldsymbol{x}' = \dot{x}_1\boldsymbol{u_1} + \dot{x}_2\boldsymbol{u_2} + \dot{x}_3\boldsymbol{u_3} = \dot{x}\boldsymbol{u}.$$

It is easy to see that $\boldsymbol{x}'$ does not depend on the Cartesian basis $\boldsymbol{u}$ used to define it, or on the observer O fixed with respect to the body. Indeed, let $\boldsymbol{v}$ be another Cartesian reference fixed to the body, and A the matrix of $\boldsymbol{v}$ with respect to $\boldsymbol{u}$ (defined so that $\boldsymbol{v} = A\boldsymbol{u}$). Let $\xi = (\xi_1, \xi_2, \xi_3)$ be the

components of $\boldsymbol{x}$ with respect to $\boldsymbol{v}$. Then $x = \xi A$, for $\boldsymbol{x} = x\boldsymbol{u} = \xi\boldsymbol{v} = \xi A\boldsymbol{u}$, and $\dot{\xi}\boldsymbol{v} = \dot{\xi}A\boldsymbol{u} = (\dot{\xi A})\boldsymbol{u} = \dot{x}\boldsymbol{u} = \boldsymbol{x}'$. This shows that $\boldsymbol{x}'$ does not depend on the Cartesian reference used. That it does not depend on the observer at rest with respect to the body is because two such observers differ by a vector that has constant components with respect to a Cartesian basis fixed to the body and so it disappears when we take derivatives.

Now a key fact is that there exists $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ such that

$$[*] \qquad \dot{\boldsymbol{x}} = \boldsymbol{x}' + \boldsymbol{\omega} \times \boldsymbol{x}.$$

To establish this, note first that

$$\dot{\boldsymbol{x}} - \boldsymbol{x}' = x_1\dot{\boldsymbol{u}}_1 + x_2\dot{\boldsymbol{u}}_2 + x_3\dot{\boldsymbol{u}}_3 = x\dot{\boldsymbol{u}}.$$

Since $\boldsymbol{u}$ is Cartesian, we have that $\boldsymbol{u} \cdot \boldsymbol{u}^T = I_3$ (the identity matrix of order 3), and on taking derivatives of both sides we get

$$\dot{\boldsymbol{u}} \cdot \boldsymbol{u}^T + \boldsymbol{u} \cdot \dot{\boldsymbol{u}}^T = 0.$$

Thus $\Omega = \dot{\boldsymbol{u}} \cdot \boldsymbol{u}^T$ is a skew-symmetric matrix, because $\boldsymbol{u} \cdot \dot{\boldsymbol{u}}^T = (\dot{\boldsymbol{u}} \cdot \boldsymbol{u}^T)^T = \Omega^T$. Therefore

$$\Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

(the signs are chosen for later convenience), where $\omega_j = \omega_j(t)$. Since the rows of Ω are the components of $\dot{\boldsymbol{u}}_1, \dot{\boldsymbol{u}}_2, \dot{\boldsymbol{u}}_3$ with respect to $\boldsymbol{u}$, we can write $\dot{\boldsymbol{u}} = \Omega\boldsymbol{u}$ and consequently

$$\dot{\boldsymbol{x}} - \boldsymbol{x}' = x\dot{\boldsymbol{u}} = x\Omega\,\boldsymbol{u} = (x_2\omega_3 - x_3\omega_2, x_3\omega_1 - x_1\omega_3, x_1\omega_2 - x_2\omega_1)\begin{pmatrix}\boldsymbol{u_1}\\ \boldsymbol{u_2}\\ \boldsymbol{u_3}\end{pmatrix} = \boldsymbol{\omega} \times \boldsymbol{x},$$

with $\boldsymbol{\omega} = \omega_1\boldsymbol{u_1} + \omega_2\boldsymbol{u_2} + \omega_3\boldsymbol{u_3}$. Here we have used that the components of $\boldsymbol{\omega} \times \boldsymbol{x}$ with respect to a Cartesian basis are $(x_2\omega_3 - x_3\omega_2, x_3\omega_1 - x_1\omega_3, x_1\omega_2 - x_2\omega_1)$.

The formula $[*]$ says that the instantaneous variation of $\boldsymbol{x}$ is the sum of the instantaneous variation of $\boldsymbol{x}$ with respect to Σ and, assuming $\boldsymbol{\omega} \neq 0$, the velocity of $\boldsymbol{x}$ under the rotation of angular velocity ω (the modulus of $\boldsymbol{\omega}$) about the axis $0+\langle\boldsymbol{\omega}\rangle$ (this will be explained later in a different way). The vector ω is called the *rotation velocity* of Σ and $0+\langle\boldsymbol{\omega}\rangle$, if $\boldsymbol{\omega}\neq 0$, the *rotation axis* relative to O. The points P that are at rest with respect to Σ (i.e. with $\boldsymbol{x}'=0$) lie on the rotation axis (i.e. $\boldsymbol{x}\in\langle\boldsymbol{\omega}\rangle$) if and only if they are at instantaneous rest with respect to O, for $\boldsymbol{x}\in\langle\boldsymbol{\omega}\rangle \Leftrightarrow \boldsymbol{\omega}\times\boldsymbol{x}=0$.

If we let $\boldsymbol{r}$ be the position vector of P with respect to an unspecified observer (you may think about it as a worker in the lab), so that

$$\boldsymbol{r}=\boldsymbol{r_0}+\boldsymbol{x},$$

and set $\boldsymbol{v}=\dot{\boldsymbol{r}}$, $\boldsymbol{v_0}=\dot{\boldsymbol{r}}_{\boldsymbol{0}}$, we have

$$\boldsymbol{v}=\boldsymbol{v_0}+\boldsymbol{x}'+\boldsymbol{\omega}\times\boldsymbol{x}.$$

4.2.- The inertia tensor.

The *inertia tensor* of Σ with respect to O is the linear map $I:V_3\to V_3$ defined with respect to any Cartesian basis $\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}$ by the matrix

$$\sum m_k\begin{pmatrix} x_{k2}^2+x_{k3}^2 & -x_{k1}x_{k2} & -x_{k1}x_{k3}\\ -x_{k2}x_{k1} & x_{k1}^2+x_{k3}^2 & -x_{k2}x_{k3}\\ -x_{k3}x_{k1} & -x_{k3}x_{k2} & x_{k1}^2+x_{k2}^2\end{pmatrix},$$

where (x_{k1},x_{k2},x_{k3}) are the components of the position vector $\boldsymbol{x_k}$ of m_k with respect to $\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}$. This does not depend on the Cartesian basis $\boldsymbol{u_1},\boldsymbol{u_2},\boldsymbol{u_3}$, because it is easy to check that the matrix in the expression is the matrix of the linear map $V_3\to V_3$ such that $\boldsymbol{y}\mapsto\boldsymbol{x_k^2 y}-(\boldsymbol{x_k}\cdot\boldsymbol{y})\boldsymbol{x_k}$. Notice, for example, that for $\boldsymbol{y}=\boldsymbol{u_1}$ we get

$$\boldsymbol{x_k^2 u_1}-(\boldsymbol{x_k}\cdot\boldsymbol{u_1})\boldsymbol{x_k}\equiv(x_{k1}^2+x_{k2}^2+x_{k3}^2,0,0)-x_{k1}(x_{k1},x_{k2},x_{k3}),$$

which is the first row of the matrix. In particular we have a coordinate-free description of I, namely

$$I(\boldsymbol{y})=\sum_k m_k(\boldsymbol{x_k^2 y}-(\boldsymbol{x_k}\cdot\boldsymbol{y})\boldsymbol{x_k})\cdot$$

The main reason for introducing the inertia tensor is that it relates the angular momentum relative to O, $\boldsymbol{L}$, and $\boldsymbol{\omega}$:

$$\boldsymbol{L}=I(\boldsymbol{\omega})\cdot$$

Indeed, since $\boldsymbol{x_k}$ is fixed with respect to Σ (i.e., $\boldsymbol{x'}_{\boldsymbol{k}}=0$),

$$\boldsymbol{L}=\sum_k m_k\boldsymbol{x_k}\times\dot{\boldsymbol{x}}_{\boldsymbol{k}}=\sum_k m_k\boldsymbol{x_k}\times(\boldsymbol{\omega}\times\boldsymbol{x_k})=\sum_k m_k\left(\boldsymbol{x_k^2\omega}-(\boldsymbol{x_k}\cdot\boldsymbol{\omega})\boldsymbol{x_k}\right)=I(\boldsymbol{\omega}).$$

In the last step we have used the formula $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$ for the double cross product.

4.3.- Kinetic energy.

The kinetic energy relative to O is given by the formula

$$\boldsymbol{T}=\frac{1}{2}\boldsymbol{\omega}\cdot I\boldsymbol{\omega}=\frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}\cdot$$

The proof is again a short computation: $T=\frac{1}{2}\sum_k m_k\dot{\boldsymbol{x}}_k^2=\frac{1}{2}\sum_k m_k(\boldsymbol{\omega}\times\boldsymbol{x_k})^2$.

But $(\boldsymbol{\omega}\times\boldsymbol{x}_k)^2=\boldsymbol{\omega}^2\boldsymbol{x}_k^2-(\boldsymbol{\omega}\cdot\boldsymbol{x}_k)^2=(\boldsymbol{x}_k^2\boldsymbol{\omega}-(\boldsymbol{x_k}\cdot\boldsymbol{\omega})\boldsymbol{x_k})\cdot\boldsymbol{\omega}$ and hence

$$T=\frac{1}{2}\sum_k m_k(x_k^2\boldsymbol{\omega}-(x_k\cdot\boldsymbol{\omega})x_k)\cdot\boldsymbol{\omega}=\frac{1}{2}\boldsymbol{\omega}\cdot\left(\sum_k m_k(x_k^2\boldsymbol{\omega}-(x_k\cdot\boldsymbol{\omega})x_k)\right),$$

which establishes the claim because $\sum_k m_k(\boldsymbol{x}_k^2\boldsymbol{\omega}-(x_k\cdot\boldsymbol{\omega})x_k)=I(\omega)$.

Remark. If we let d_k be the distance of m_k to the rotation axis $0+\langle\boldsymbol{\omega}\rangle$, then $(\boldsymbol{\omega}\times\boldsymbol{x}_k)^2=\boldsymbol{\omega}^2 d_k^2$. Indeed, $(\boldsymbol{\omega}\times\boldsymbol{x}_k)^2=\boldsymbol{\omega}^2\boldsymbol{x}_k^2-(\boldsymbol{\omega}\cdot\boldsymbol{x}_k)^2=\boldsymbol{\omega}^2\left(\boldsymbol{x}_k^2-(\boldsymbol{x}_k\cdot\boldsymbol{u})^2\right)$, with $\boldsymbol{u}=\boldsymbol{\omega}/\omega$, and the claim follows from Pythagoras' theorem, as $\boldsymbol{x_k}\cdot\boldsymbol{u}$ is the orthogonal projection of $\boldsymbol{x}_k$ to the rotation axis. Note that this shows that T is indeed the rotation kinetic energy of the solid.

The kinetic energy with respect to an observer L for which $O=L+\boldsymbol{r_0}$ is, according to the second formula in §3.1 (with $L=O-\boldsymbol{r_0}$ playing the role of O')

$$\frac{1}{2}\boldsymbol{\omega}\cdot I\boldsymbol{\omega}+\frac{1}{2}mv_0^2+\boldsymbol{P}\cdot\dot{\boldsymbol{x}}_G,$$

where $\boldsymbol{v_0} = \dot{\boldsymbol{r}}_0$, $\boldsymbol{P} = m\boldsymbol{v_0}$ (the linear momentum of a point mass m moving with O), and $\dot{\boldsymbol{x}}_G$ is the velocity of G with respect to O. This formula can also be established directly, for the kinetic energy in question is

$$\frac{1}{2}\sum_k m_k \dot{\boldsymbol{r}}_k^2 = \frac{1}{2}\sum_k m_k(\boldsymbol{v_0} + \dot{\boldsymbol{x}}_k)^2 = \frac{1}{2}\sum_k m_k \dot{\boldsymbol{x}}_k^2 + \frac{1}{2} m v_0^2 + \sum_k m_k \boldsymbol{v_0} \cdot \dot{\boldsymbol{x}}_k,$$

and the first term of last expression is the kinetic energy relative to O and the third is $m\boldsymbol{v_0} \cdot \dot{\boldsymbol{x}}_G$.

As a corollary we get, taking $O = G$, that the kinetic energy with respect to L is, setting $\boldsymbol{V} = \dot{\boldsymbol{r}}_G$,

$$\frac{1}{2}\boldsymbol{\omega} \cdot I\boldsymbol{\omega} + \frac{1}{2} m V^2.$$

In other words, *the kinetic energy of a rigid body is*, for any observer, *the sum of the kinetic energy* $1/2\,mV^2$ *of a point particle of mass* m *moving as the bray-center of the body, and the rotation energy* $T_{rot} = 1/2\,\boldsymbol{\omega} \cdot I\boldsymbol{\omega}$ *of the rotation about the axis* $G + \langle\boldsymbol{\omega}\rangle$ *with angular velocity* ω.

4.4.- Moments of inertia.

Let O be an observer that is stationary with respect to the solid Σ. Let $\boldsymbol{u}$ be a unit vector. If we let Σ turn with angular velocity $\boldsymbol{\omega} = w\boldsymbol{u}$ about the axis $0 + \langle\boldsymbol{u}\rangle$, the rotation kinetic energy is

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot I\boldsymbol{\omega} = \frac{1}{2}(w\boldsymbol{u}) \cdot I(w\boldsymbol{u}) = \frac{1}{2}(\boldsymbol{u} \cdot I\boldsymbol{u})w^2 = \frac{1}{2}\mu_{O,\boldsymbol{u}} w^2,$$

where $\mu_{O,\boldsymbol{u}} = \boldsymbol{u} \cdot I\boldsymbol{u}$ is called the *moment of inertia* with respect to the axis $0 + \langle\boldsymbol{u}\rangle$.

4.5.- Inertia axes.

The inertia tensor I is symmetric (i.e., its matrix with respect to a rectangular basis is symmetric), and hence there is an Cartesian basis $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ with respect to which I has a diagonal matrix, say

$$I \equiv \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{31} \end{pmatrix}.$$

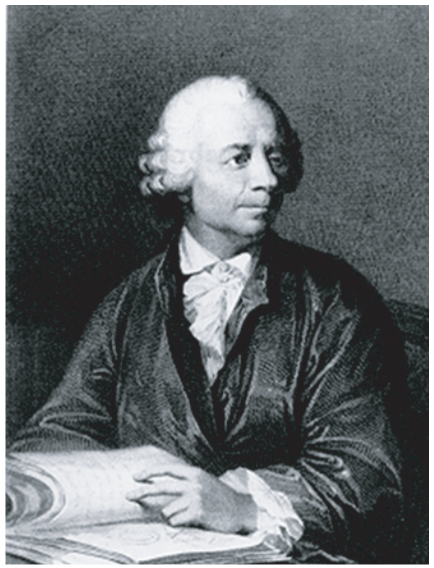
The axes $0 + \langle\boldsymbol{u}_j\rangle$ are then called *principal axes* (*of inertia*) relative to O and the quantities I_j, *principal moments of inertia* (note that I_j is the moment of inertia with respect to the corresponding principal axis). The axes are uniquely determined if the principal moments of inertia are distinct. In case two are equal, but the third is different, say $I_1 = I_2$ and $I_3 \neq I_1$, then the I_3 axis is uniquely determined but the other two may be any pair of axis through O that are orthogonal and orthogonal to $\boldsymbol{u_3}$. In this case we say that the solid is a gyroscope with axis $0 + \langle\boldsymbol{u_3}\rangle$. Finally, if $I_1 = I_2 = I_3$, then any orthonormal basis gives principal axes through O and we say that Σ is a *spherical gyroscope*.

Remark that if $\boldsymbol{\omega} \equiv (w_1, w_2, w_3)$ in the principal axis, then

$$\boldsymbol{L} = I\boldsymbol{\omega} \equiv (I_1 w_1, I_2 w_2, I_3 w_3),$$

$$T_{rot} = \frac{1}{2}(I_1 w_1^2 + I_2 w_2^2 + I_3 w_3^2) = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}.$$

5.- Dynamics of rigid bodies.



5.1.- We have established the fundamental equations that rule the dynamics of a rigid body Σ for any observer O: the momentum principle and the angular momentum principle. If $\boldsymbol{F}$ and $\boldsymbol{N}$ are the total external force and total external moment of Σ relative to O, and $\boldsymbol{P}$ and $\boldsymbol{L}$ are the linear and angular moments of Σ relative to O, those principles state that

Momentum principle (§1.6)

$$\dot{\boldsymbol{P}} = \boldsymbol{F}$$

Angular momentum principle (§2.4)

$$\dot{\boldsymbol{L}} = \boldsymbol{N}$$

We also have proved that

$$\dot{E} = \Pi,$$

where E and Π are the energy of Σ and the power of the external forces acting of Σ. We may call this the "energy principle".

5.2.- Euler's equation.

Let O be an observer that is at rest with respect the solid Σ. Let I and $\boldsymbol{N}$ be the inertia tensor and the total external moment of Σ relative to O. Then we have

$$[*] \qquad \boldsymbol{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega}.$$

We know that $\boldsymbol{N} = \dot{\boldsymbol{L}}$ (§5.1) and $\boldsymbol{L} = I\boldsymbol{\omega}$ (§4.2). Thus we have

$$\boldsymbol{N} = \dot{\boldsymbol{L}} = \boldsymbol{L}' + \boldsymbol{\omega} \times \boldsymbol{L} = (I\boldsymbol{\omega})' + \boldsymbol{\omega} \times I\boldsymbol{\omega}$$

Since I is independent of the motion, $(I\boldsymbol{\omega})' = I\boldsymbol{\omega}' = I\dot{\boldsymbol{\omega}}$ (note that $\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}'$, because $\dot{\boldsymbol{\omega}} = \boldsymbol{\omega}' + \boldsymbol{\omega} \times \boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$), and this completes the proof.

5.3.- As a corollary we have that *in the absence of external forces the angular velocity can be constant only if it is parallel to a principal axis*. Indeed, if ω is constant and there are no external forces, then $\boldsymbol{\omega} \times I\boldsymbol{\omega} = 0$, and this relation is equivalent to say that $\boldsymbol{\omega}$ is an eigenvector of I.

As another corollary we obtain that

$$\dot{T}_{rot} = \boldsymbol{N} \cdot \boldsymbol{\omega},$$

for $\dot{T}_{rot} = \frac{1}{2} \boldsymbol{\omega} \dot{\cdot} I\boldsymbol{\omega} = \boldsymbol{\omega} \cdot I\dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \cdot (\boldsymbol{N} - \boldsymbol{\omega} \times I\boldsymbol{\omega}) = \boldsymbol{\omega} \cdot \boldsymbol{N}$

(in the second step we have used that I is symmetric).

5.4.- Euler's equations.

Writing the equation [*] in the principal axes through O we get the equations

$$[**] \qquad \begin{aligned} N_1 &= (I_3 - I_2)\omega_2\omega_3 + I_1\dot{\omega}_1 \\ N_2 &= (I_1 - I_3)\omega_3\omega_1 + I_2\dot{\omega}_2 \\ N_3 &= (I_2 - I_1)\omega_1\omega_2 + I_3\dot{\omega}_3 \end{aligned}$$

6.- Continuous systems.

Let us indicate how one can proceed to extend the theory to continuous systems. Instead of a finite number of masses located at some points, we consider a *mass distribution* $\boldsymbol{\mu}$ on a region $\boldsymbol{R}$ in $\boldsymbol{E}_3$, that is, a positive continuous function $\boldsymbol{\mu} : \boldsymbol{R} \to \Re$. We will say that $\boldsymbol{\Sigma} = (\boldsymbol{R}, \boldsymbol{\mu})$ is a *material system* (or a *material body*).

6.1.- The *total mass* of Σ is $m = \int_R \mu d\omega$, where $d\omega$ is the volume element of $\boldsymbol{E}_2$. More generally, if $R' \subseteq R$ is a subregion of R (usually called a *part* of R) we say that $m(R') = \int_{R'} \mu d\omega$ is the *mass of*, or contained in, R'.

The intuition behind this model is that $\mu d\omega$ represents the (infinitesimal) mass contained in the volume element $d\omega$ and so the mass contained in R' is the "sum" of all the $\mu d\omega$. But the "sum" of infinitesimal terms is just the integral.

6.2.- The *center of mass* G *of* Σ is the point $O + \boldsymbol{r}_G$ with

$$\boldsymbol{r}_G = \frac{1}{m} \int_R \mu(\boldsymbol{r})\boldsymbol{r} d\omega.$$

The point G *does not depend on the observer* O used to calculate it. The proof is similar to the discrete case.

6.3.- The instantaneous motion of the material system is represented by a vector field $\boldsymbol{u}$ defined on R. We will say that $\boldsymbol{u}$ is the *velocity field* of the system, and that the velocity of the mass element $\mu(\boldsymbol{r})d\omega$ is $\boldsymbol{u}(\boldsymbol{r})$. In general, both R and $\boldsymbol{u}$ are dependent on time.

6.4.- The *momentum* of $\mu d\omega$ is $\mu \boldsymbol{u} d\omega$ and the *momentum* of the region R' is

$$\boldsymbol{P}(R') = \int_{R'} \mu \boldsymbol{u} d\omega.$$

The *momentum principle* states that the instantaneous variation with time of $\boldsymbol{P}(R')$, for any part R', is equal to the external force $\boldsymbol{F}(R')$ acting on R'. The external forces include those that the exterior of R' in R exert on R' along the boundary of R'.

6.5.- The *angular momentum* of $\mu(\boldsymbol{r})d\omega$ is $\mu(\boldsymbol{r})\boldsymbol{r}\times\boldsymbol{u}(\boldsymbol{r})d\omega$ and the *momentum* of the region R' is

$$\boldsymbol{L}(R') = \int_{R'} \mu(\boldsymbol{r})\boldsymbol{r}\times\boldsymbol{u}(\boldsymbol{r})d\omega .$$

The *angular momentum principle* states that the instantaneous variation with time of $\boldsymbol{L}(R')$, for any part R', is equal to the external torque $\boldsymbol{N}(R')$ acting on R'. The external torque includes that produced by the exterior of R' in R along the boundary of R'.

6.6.- The inertia tensor of a rigid continuous body R with respect to the observer O is defined as

$$I \equiv \int_R \mu(\boldsymbol{x})(\boldsymbol{x}^2 Id - \hat{\boldsymbol{x}}\otimes\boldsymbol{x})d\omega$$

$$\equiv \int_R \mu(x_1,x_2,x_3)\begin{pmatrix} x_2^2+x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & x_1^2+x_3^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & x_1^2+x_2^2 \end{pmatrix} dx_1dx_2dx_3 .$$

Here $\boldsymbol{x}$ is the position vector of a point on the body relative to O, $\hat{\boldsymbol{x}}\otimes\boldsymbol{x}$ denotes the linear map $\boldsymbol{y}\mapsto(\boldsymbol{x}\cdot\boldsymbol{y})\boldsymbol{x}$ and the coordinates x_1,x_2,x_3 are Cartesian coordinates with origin O.

6.7.- Angular velocity is defined as in the discrete case, so that we still have, if the principles of momentum and angular momentum are true, all the relations that were established for discrete rigid bodies. In particular we have Euler's equation

$$\boldsymbol{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times I\boldsymbol{\omega} .$$

7.- Historical notes.

7.1.- Mechanica.

Euler published his treatise on Mechanics in 1736, in two volumes. They can be found through *http://math.dartmouth.edu/~euler/*, both in the original Latin and in English (translation by Ian Bruce). For our purposes, it is appealing to quote a few striking sentences of the translators preface to the English version:

> *"while Newton's Principia was fundamental in giving us our understanding of at least a part of mechanics, it yet lacked in analytical sophistication, so that the mathematics required to explain the physics lagged behind and was hidden or obscure, while with the emergence of Euler's Mechanica a huge leap forwards was made to the extend that the physics that could now be understood lagged behind the mathematical apparatus available. A short description is set out by Euler of his plans for the future, which proved to be too optimistic. However, Euler was the person with the key into the magic garden of modern mathematics, and one can savour a little of his enthusiasm for the tasks that lay ahead : no one had ever been so well equipped for such an undertaking. Although the subject is mechanics, the methods employed are highly mathematical and full of new ideas."*

It is also interesting to reproduce, in Euler's own words, what "his plans for the future" in the field of Mechanics were (vid. §98 of EULER (1936)):

> *"The different kinds of bodies will therefore supply the primary division of our work. First … we will consider infinitely small bodies... Then we will approach those bodies of finite magnitude which are rigid… Thirdly, we will consider flexible bodies. Fourthly, … those which allow extension and contraction. Fifthly, we will examine the motions of several separated bodies, some of which hinder each other from their own motions… Sixthly, at last, the motion of fluids…"*

These words are preceded, however, by the acknowledgement that

> *"this hitherto … has not been possible … on account of the insufficiency of principles …"*

7.2.- Motion of rigid bodies.

Daniel Bernoulli states in a letter to Euler dated 12 December 1745 that the motion of a rigid bodies is "and extremely difficult problem that will not be easily solved by anybody…" (quotation extracted from TRUESDELL (1975)).

The prediction was not very sharp, as fifteen years later Euler had completed the masterpiece EULER (1965*a*) that collects a systematic account of his findings in the intervening years and which amount to a complete and detailed solution of the problem.

Euler's earliest breakthrough was his landmark paper EULER (1752), which "has dominated the mechanics of extended bodies ever since". This quotation, which is based in TRUESDELL (1954), is from the introduction to E177 in [1] and it is fitting that we reproduce it here:

> *"In this paper, Euler begins work on the general motion of a general rigid body. Among other things, he finds necessary and sufficient conditions for permanent rotation, though he does not look for a solution. He also argues that a body cannot rotate freely unless the products of the inertias vanish. As a result of his researches in hydraulics during the 1740s, Euler is able, in this paper, to present a fundamentally different approach to mechanics, and this paper has dominated the mechanics of extended bodies ever since. It is in this paper that the so-called Newton's equations* $\boldsymbol{f} = m\boldsymbol{a}$ *in rectangular coordinates appear, marking the first appearance of these equations in a general form since when they are expressed in terms of volume elements, they can be used for any type of body. Moreover, Euler discusses how to use this equation to solve the problem of finding differential equations for the general motion of a rigid body (in particular, three-dimensional rigid bodies). For this application, he assumes that any internal forces that may be within the body can be ignored in the determination of torque since such forces cannot change the shape of the body. Thus, Euler arrives at "the Euler equations" of rigid dynamics, with the angular velocity vector and the tensor of inertia appearing as necessary incidentals."*

For example, on p. 213 (of the original version, or p. 104 in OO II **1**) we find the equations

$$\frac{Pa}{2M} = \frac{ffd\lambda}{dt} - \frac{nnd\mu}{dt} - \frac{mmd\mu}{dt} + \lambda\nu nn - \lambda\mu nn - (\mu\mu - \nu\nu)ll + \mu\nu(hh - gg)$$

$$\frac{Qa}{2M} = \frac{ggd\lambda}{dt} - \frac{lld\mu}{dt} - \frac{nnd\mu}{dt} + \lambda\mu ll - \mu\nu nn - (\nu\nu - \lambda\lambda)mm + \mu\nu(ff - hh)$$

$$\frac{Ra}{2M} = \frac{hhd\lambda}{dt} - \frac{mmd\mu}{dt} - \frac{lld\mu}{dt} + \mu\nu mm - \lambda\nu ll - (\lambda\lambda - \mu\mu)nn + \mu\nu(gg - ff)$$

They can be decoded in terms of our presentation as follows:

$$(w_1, w_2, w_3) = (\lambda, \mu, \nu)$$

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix} = \begin{pmatrix} ff & -mm & -nn \\ -mm & gg & -ll \\ -nn & -ll & hh \end{pmatrix}$$

$$(N_1, N_2, N_3) = \left(\frac{Pa}{2M}, \frac{Qa}{2M}, \frac{Ra}{2M}\right),$$

and in this way we get equations that are equivalent to Euler's equation (§5.2)

$$\boldsymbol{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \wedge I\boldsymbol{\omega}$$

written in a reference tied to the body, but with general axis.

It will also be informative to the reader to summarize here the introductory sections of E177. Rigid bodies are defined in §1, and the problems of their kinematics and dynamics are compared with those of fluid dynamics and of elasticity. Then in §2 and §3 the two basic sorts of movements of a solid (translations and rotations) are explained. The "mixed" movements are also mentioned, with the Earth movement as an example. The main problem to which the memoir is devoted is introduced in §4: up to that time, only rotation axes fixed in direction had been considered, "faute de principles suffisants", and Euler suggests that this should be overcome. Then it is stated (§5) that any movement of a rigid body can be understood as the composition of a translation and a rotation. The role of the barycenter is also stressed, and the fact that the translation movement plays no role in the solution of the rotation movement. The momentum principle is introduced in §6. It is used to split the problem in two separate problems:

> *"… on commencera par considérer … comme si toute la masse étoit réunie dans son centre de gravité, et alors on déterminera par les principes connues de la Mécanique le mouvement de ce point produit par les forces sollicitantes; ce sera le mouvement progressif du corps. Après cela on mettra ce mouvement … à part, et on considérera ce même corps, comme si le centre de gravité étoit immobile, pour déterminer le mouvement de rotation …"*

The determination of the rotation movement for a rigid body with a fixed barycenter is outlined in §7. In particular, the instantaneous rotation axis is introduced and its key role explained

> *"… quel que soit le mouvement d'un tel corps, ce sera pour chaque instant non seulement le centre de gravité qui demeure en repos, mais il y aura aussi toujours une infinité de points situés dans une ligne droite, qui passe par le centre de gravité, dont tous ce trouveront également sans mouvement. C'est à dire, quel que soit le mouvement du corps, il y aura en chaque instant un mouvement de rotation, qui se fait autour d'un axe, qui passe par le centre de gravité, et toute la diversité qui pourra avoir lieu dans ce mouvement, dépendra, outre la diversité de la vitesse, de la variabilité de cet axe …"*

In §8 the main goal of the memoir is explained in detail:

> *"… je remarque que les principes de la Mécanique, qui ont été établis jusqu'à présent, ne sont suffisants, que pour le cas, où le mouvement de rotation se fait continuellement autour du même axe. … Or dès que l'axe de rotation ne demeure plus le même, … alors les principes de Mécanique connues jusqu'ici ne sont plus suffisants à déterminer ce mouvement. Il s'agit donc de trouver et d'établir de nouveaux principes, qui soient propres à ce dessin ; et cette recherche sera le sujet de ce Mémoire, dont je suis venu à bout après plusieurs essais inutiles, que j'ai fait depuis longtemps."*

The principle that is missing, and which is established in E177, is the angular momentum principle, and with it he can finally arrive at Euler's equations that give the relation between the instantaneous variation of the angular velocity and the torque of the external forces. And with regard to the sustained efforts toward the solution of the problem, there is a case much in point, namely, the investigations that led to the two volumes of *Scientia navalis* (published in 1749 in San Petersburg) and in which some special problems of the dynamics of rigid bodies are solved.

The discovery of the principal axes, that brought much simplification to the equations, was published in EULER (1765*b*).

Of the final treatise EULER (1965*a*), it is worth reproducing the short assessment at the beginning of BLANC (1946):

> *"… est un traité de dynamique du solide; il s'agit d'un ouvrage complet, de caractère didactique, exposant d'une façon systématique ce que l'auteur avait, dans les années 1740 à 1760, publié dans divers mémoires. L'établissement des équations différentielles du mouvement d'un solide (celles que l'on appelle aujourd'hui les équations d'Euler) en constitue l'objet essentiel."*

Let us also say that the words in TRUESDELL (1954) concerning Euler's works in fluid mechanics are also fitting for the case of the rigid body and for Mechanics in general. The results of Euler are "not forged by a brief and isolated intuition" and

> *"… we shall learn how the most creative of all mathematicians searched, winnowed, and organized the works of his predecessors and contemporaries; shaping, polishing, and simplifying his ideas anew after repeated successes which any other geometer would have let stand as complete; ever seeking first principles, generality, order, and, above all, clarity."*

The works of Euler on Mechanics, and on rigid bodies in particular, have been the source of much of the subsequent texts, like the "classics" GOLDSTEIN (1950) and LANDAU; LIFCHITZ (1966), or in the recent GREGORY (2006). The latter, however, is (rightly) critical about the significance of Euler's equations and points out two "deficiencies" (p. 548): The knowledge of the time variation of $\boldsymbol{\omega}$ does not give the position of the body, and the knowledge of N^e does not yield its principal components, as the orientation of the body is not known.

8.- References.

BLANC, Ch. (1946) "Préface de l'Éditeur" to Euler's *Opera Omnia* (II **3, 4**).

BLANC, Ch. (1966) "Préface" to Euler's *Opera Omnia* (II **8, 9**).

EULER, L. (1736) *Mechanica sive motus scientia analytice exposita, I, II.* E015 and E016 in Euler's *Opera Omnia* (II **1, 2**). Originally published by the Acad. Scient. Imper. Petropoli.

EULER, L. (1752) "Découverte d'un nouveau principe de la mécanique". E177 in Euler's *Opera Omnia* (II **5**, 81-108). Presented to the Berlin Academy in 3 Sep 1950 and originally published in *Mémoires de l'Académie des Sciences de Berlin* **6**, 1752, 185-217.

EULER, L. (1758) "Recherches sur la connoisance mechanique des corps". E291 in Euler's *Opera Omnia* (II **8,** 178-199). Originally published in *Mémoires de l'Académie des Sciences de Berlin* **14**, 1765, 131-153.

EULER, L. (1765*a*) *Theoria motus corporum solidorum seu rigidorum*. E289 in EULER'S *Opera Omnia* (II **3**). Originally published as a book (finished in 1760).

EULER, L. (1765*b*) "Du mouvement de rotation des corps solides autour d'un axe variable". E292 in Euler's *Opera Omnia* (II **8**, 200-235). Originally published in *Mémoires de l'Académie des Sciences de Berlin* **14**, 1765, 154-193.

EULER, L. (1776) "Nova methodus motum corporum rigidorum degerminandi". E479 in Euler's *Opera Omnia* (II **9**, 99-125). Originally published in *Novi Commentarii academiae scientiarum Petropolitanae* **20**, 1776, 208-238. Presented to the St. Petersburg Academy on October 16, 1775.

GOLDSTEIN, H. (1950) *Classical Mechanics,* Addison-Wesley. There is recent third edition.

GREGORY, R. D. (2006) *Classical Mechanics,* Cambridge University Press.

LANDAU, L.; LIFCHITZ, E. (1966) *Mécanique,* MIR.

TRUESDELL, C. (1954) "Introduction" to Euler's *Opera Omnia*, II **12**.

TRUESDELL, C. (1975) *Ensayos de Historia de la Mecánica,* Tecnos.

XAMBÓ, S. (2001) *Geometria* (2a edició), Edicions UPC (1a ed. 1996). This text is written in Catalan. There is a Spanish version, also published by Edicions UPC.

XAMBÓ, S. (2007) *Models Matemàtics de la Física*. Last version of the notes for the course on mathematical models for physics taught at the Facultat de Matemàtiques i Estadística of the Universitat Politècnica de Catalunya (in Catalan). Sent in pdf if requested. A web version is in preparation at *http://www-ma2.upc.edu/sxd/*.