Discrete Charms of Kähler Geometry

Sebastian Xambó-Descamps
FME & IMTech
sebastia.xambo@upc.edu

Abstract

The aim of this paper is to revisit Grothendieck's Standard Conjectures from two main perspectives: firstly, the contexts in which they arose and their significance in algebraic geometry (particularly in intersection theory), and secondly, their ramifications up till now, including the breakthroughs in combinatorics by June Huh and his collaborators, and their incarnation in arithmetic algebraic geometry.

Prelude

Once upon a time Kähler geometry was at the cutting edge of algebraic geometry, driven by the likes of S. Lefschetz (1884-1972), W. V. D. Hodge (1903-1975), and K. Kodaira (1915-1997). At this stage, the discrete side appeared through the homology and cohomology pioneered by H. Poincaré (1854-1912) and G. de Rham (1903-1990), particularly in the form of their dimensions as vector spaces (Betti and Hodge numbers, for example) and their combinatorial relations (like the Hodge staircases). Then another discrete aspect burst out of that realm with the birth of abstract algebraic geometry over any field, mainly by work of O. Zariski (1899-1986), L. van der Waerden (1903-1996), A. Weil (1906-1998), and then by the generalizations brought about by J.-P. Serre (1926–) and A. Grothendieck (1928-2014).

A fundamental development occurred when Weil came up with his celebrated conjectures about the zeta function of a non-singular projective variety over a finite field. This led Grothendieck to state his "standard conjectures" for general smooth projective varieties as an avenue to prove Weil's conjectures. The plan succeeded only partially, as the most difficult part (the "Riemann hypothesis" for the zeta function) was proved by P. Deligne using different ideas. Grothendieck's standard conjectures remain conjectures to this day (a major challenge in intersection theory), but amazingly they have been adapted by June Huh (Fields Medal 2022) and his coworkers into what they call the "Kähler package", a well tuned framework with which they have proved a number of important long-standing conjectures in Combinatorics.

The goal of this paper is to inspect the "Kähler package" ingredients, with emphasis in its roots in Kähler geometry and Grothendieck's intersection theory;

to review some of its discrete instances, and how they have yielded proofs of significant combinatorial conjectures; to analyze advances in some recent works in Algebraic Geometry and Arithmetic Algebraic Geometry; and to consider its potential for approaching old and new problems in various areas.

Notations. In addition to the standard symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, we let $\mathbb{N}$ denote the non-negative integers, $\mathbb{B} = \{0, 1\}$ (binary digits), and $\mathbb{F} = \mathbb{F}_q$ a finite field of q elements, with $\bar{\mathbb{F}} = \bar{\mathbb{F}}_q$ its algebraic closure.

Given integers a, b, the range $\{j \in \mathbb{Z} \,|\, a \leqslant j \leqslant b\}$ is denoted by $[a, b]$. For a positive integer n, $[n]$ is an abreviation of $[1, n]$, and $n /\!\!/ 2$ (quotient of the integer division of n by 2) is an alternative notation for $\lfloor \frac{n}{2} \rfloor$ (the integer part of $n/2$).

Acknowledgments. The author is thankful to José Ignacio Cogolludo for his invitation to participate in the 2023 conference on *Algebraic and topological interplay of algebraic varieties* held in Jaca (Spain) in homage of professors Enrique Artal and Alejandro Melle. This paper is a write up of the author's talk there, *The discrete charms of Kähler Geometry (a view of June Huh's Kähler package)*, supplemented with a section devoted to more recent developments, and it is heartily dedicated to the conference honorees.

I am also grateful to the organizers of the 2023 Barcelona Seminar on Algebraic Geometry, Josep Álvarez Montaner and Simone Marchesi, for their invitation (17 February 2023) to present an earlier version of my conference talk (its title was *The discrete charm of algebraic geometry—after June Huh*). All that toil was eased by Anna de Mier's teachings on graphs and matroids, and it is a most pleasant duty to express here my gratitude to her.

The pointer to the arithmetic Kähler package included at the end of §6.46 owes much to Roberto Gualdi's lecture, *On the arithmetic Kähler package*, presented to the Barcelona Algebraic Geometry Seminar on 23 February 2024. I also thank him for enlightening discussions over this topic.

The laborious task of collecting reference materials for this work has been facilitated by the staff librarians of the "Universitat Politècnica de Catalunya", and especially those working a the "Facultat de Matemàtiques i Estadística". My thanks to all of them, and particularly to the diligent director of the service, Gemma Flaquer.

This list would be flagrantly defective if it omitted June Huh, whose brilliant ICM-2002 Fields lecture switched on for many a much yearned living link of the two souls, 'continuous'↔'discrete', that to a different degree simply coexist in any mathematician. For many, as for this author, the mathematical panorama shines brighter and more meaningful ever since. Let this words of thanks be extensive to Huh's collaborators and to all that share this epiphany.

1 Manifolds and their cohomologies

Up to variations in notation, the text [1] is an excellent reference for most of the notions and properties considered in this section. In particular we refer to it for the following notions and their general properties: (real) *smooth manifolds* (possibly with boundary) and *smooth maps* between manifolds; *differential forms* on a manifold; *oriented manifolds*, *integration of forms on an oriented manifold*, and *Stokes theorem*; and *de Rham cohomology* and the *de Rham theorem*. Additional references will be provided whenever necessary.

In this section, X denotes a n-dimensional manifold and ∂X its boundary (a manifold of dimension $n-1$ or, when X has no boundary, the empty set).

1.1. $A^*(X) = \bigoplus_{k=0}^n A^k(X)$: *graded algebra of smooth exterior forms on* X. Its product is the *wedge product*, $\wedge$. It is associative and grade-commutative, which means that $\alpha \wedge \alpha' = (-1)^{kk'}\alpha' \wedge \alpha$ when $\alpha \in A^k(X)$ and $\alpha' \in A^{k'}(X)$. We also set $A_c^k(X)$ to denote the subspace of $A^k(X)$ of k-forms with compact support.

1.2. $C^*(X) = \bigoplus_{k=0}^n C^k(X) \subseteq A^*(X)$: graded subalgebra of *closed forms*. Thus $\alpha \in C^*(X)$ if and only if $d\alpha = 0$, where $d : A^*(X) \to A^{*+1}(X)$ is the *exterior derivative*. Note that the formula $d(\alpha \wedge \alpha') = (d\alpha) \wedge \alpha' + (-1)^k \alpha \wedge (d\alpha')$ implies that $\alpha \wedge \alpha' \in C^{k+k'}(X)$ whenever $\alpha \in C^k(X)$ and $\alpha' \in C^{k'}(X)$.

1.3. $E^*(X) = \bigoplus_{k=0}^n E^k(X) \subseteq C^*(X)$: graded C^*-ideal of *exact forms*. In other words, $E^*(X) = dA^*(X)$. The relation $E^*(X) \subseteq C^*(X)$ holds because $d^2 = 0$ and hence exact forms are closed. Moreover, if $\alpha \in C^k(X)$ and $\alpha' = d\beta \in E^{k'}(X)$, then $\alpha \wedge \alpha' \in E^{k+k'}(X)$, because $\alpha \wedge \alpha' = \alpha \wedge (d\beta) = d((-1)^k \alpha \wedge \beta)$, and similarly $\alpha' \wedge \alpha \in E^{k+k'}(X)$.

1.4. $H^*_{\mathrm{dR}}(X) = C^*(X)/E^*(X)$: The *de Rham graded cohomology algebra of* X. Its product, which is induced by the wedge product, is denoted with the same symbol $\wedge$. In detail, if we let $[\alpha]$ denote the class of $\alpha \in C^*(X)$ modulo $E^*(X)$, then $[\alpha] \wedge [\alpha'] = [\alpha \wedge \alpha']$. If the spaces $H^*_{\mathrm{dR}}(X)$ are finite-dimensional (which happens, for instance, when X is compact), then $\mathsf{b}_k(X) = \dim H^k_{\mathrm{dR}}(X)$ are the *Betti numbers* of X, while $\chi(X) = \sum_k (-1)^k \mathsf{b}_k(X)$ is its *Euler characteristic*. H^*_{dR} is contravariant functor: note that if $f : X \to X'$ is smooth map, the map $f^* : A^k(X') \to A^k(X)$ induces maps

$$f^* : C^k(X') \to C^k(X) \quad \text{and} \quad f^* : E^k(X') \to E^k(X)$$

(because $f^* \circ d = d \circ f^*$), and hence a linear map $f^* : H_{\mathrm{dR}}(X') \to H^k_{\mathrm{dR}}(X)$.

Since $C^0(X) = \{f \in A^0(X) \,|\, df = 0\}$, and this implies that f is constant on any (parameterized) smooth curve on X, we have $C^0(X) \simeq \mathbb{R}^s$, where s is the number of connected components of X, and hence $H^0_{\mathrm{dR}}(X) \simeq \mathbb{R}^s$, as $E^0(X) = dA^{-1}(X) = 0$. In particular $H^0_{\mathrm{dR}}(X) \simeq \mathbb{R}$ if X is connected.

1.5. *Stokes theorems*. If X is oriented and compact, then for any $\alpha \in A^{n-1}(X)$ we have $\int_X d\alpha = \int_{\partial X} \alpha$. In particular $\int_{\partial X} \alpha = 0$ if α is closed and $\int_X d\alpha = 0$ if X has no boundary. If X is oriented but not compact, then the Stokes relation holds for any $\alpha \in A_c^{n-1}(X)$. If we now apply Stokes to an oriented compact k-dimentional submanifold Y of X and to the restriction $\bar{\alpha} = \alpha|Y$ of $\alpha \in A^k(X)$ to Y, we get $\int_Y d\bar{\alpha} = \int_{\partial Y} \bar{\alpha}$, which we can abridge to $\int_Y d\alpha = \int_{\partial Y} \alpha$ because $d\bar{\alpha} = (d\alpha)|Y$ (to note that Y need not be compact if $\alpha \in A_c^k(X)$). This last version of the Stokes relation also holds (borrowing the terminology of singular homology) if $Y \in \mathcal{C}_k(X)$, the group of singular k-chains of X: for any

$\alpha \in A^k(X)$, $\int_Y d\alpha = \int_{\partial Y} \alpha$, where now ∂Y is the boundary of Y, which belongs, as $\partial(\partial Y) = 0$, to $\mathcal{Z}_{k-1}(X) \subseteq \mathcal{C}_{k-1}(X)$, the group of singular $(k-1)$-cycles of X. The integral $\int_Y \alpha$ is defined, if $Y = \sum n_j Y_j$ (Y_j a singular simplexs, n_j integers) as $\sum_j n_j \int_{Y_j} \alpha$. Instead of singular chains, we could also use (see [2, Chapter 23]) *cubical chains*. Let us just recall that the groups $H_k(X) = \mathcal{Z}_k(X)/\partial\mathcal{C}_{k+1}(X)$ are the *homology groups* of X and $H^k(X) = \mathsf{Hom}(H_k(X), \mathbb{R})$ its *cohomology groups* (actually real vector spaces). Since H_k are covariant functors from the category of manifolds $\mathcal{X}$ to the category of abelian groups, H^k is a contravariant functor from $\mathcal{X}$ to category of real vector spaces.

1.6. *The de Rham Theorem.* Given $\alpha \in C^k(X)$, X oriented and compact, it has an associated homomorphism $\bar{\alpha} : \mathcal{Z}_k(X) \to \mathbb{R}$ defined by $\bar{\alpha}(z) = \int_z \alpha$. This homomorphism vanishes on boundaries $\partial w \in \mathcal{Z}_k(X)$, because by Stokes theorem we have $\bar{\alpha}(\partial w) = \int_{\partial w} \alpha = \int_X d\alpha = 0$. Therefore we get an induced homomorphism $\bar{\alpha} : \mathcal{Z}_k(X)/\partial\mathcal{C}_{k+1}(X) = H_k(X) \to \mathbb{R}$ defined by $[z] \mapsto \bar{\alpha}(z) = \int_z \alpha$. This homomorphism vanishes when α is exact, for if $\alpha \in dA^{k-1}(X)$, say $\alpha = d\beta$, then $\bar{\alpha}([z]) = \int_z d\beta = \int_{\partial z} \beta = \int_0 \beta = 0$, again by Stokes theorem. So we can actually regard $\bar{\alpha}$ as a homomorphism $H_k(X) \to \mathbb{R}$ associated to $[\alpha] \in H^k_{\mathrm{dR}}(X)$. In sum, we have a natural homomorphism $h : H^k_{\mathrm{dR}}(X) \to \mathsf{Hom}(H_k(X), \mathbb{R}) = H^k(X)$. The de Rham theorem asserts that *this homomorphism is an isomorphism.* This is a deep theorem, as the right hand side is defined in terms of the topological structure, and it is a homotopy-type invariant, while the left hand side is defined in terms of the differential structure of X. In particular we get that the cohomology spaces $H^k(X)$ are finite dimensional and that $\dim H^k(X) = \mathsf{b}_k(X)$. For example, $\mathsf{b}_0(X)$ is the number of connected components of X (see §1.4).

1.7. The previous result is exactly the same if instead of $H_k(X)$ we proceed with the real vector space $H_k(X, \mathbb{R})$, which by definition is equal to

$$\mathcal{Z}_k(X, \mathbb{R})/\partial\mathcal{C}_{k+1}(X, \mathbb{R}),$$

with $\mathcal{Z}_k(X, \mathbb{R})$ the real vector space of (singular) k-cycles with real coefficients and $\mathcal{C}_{k+1}(X, \mathbb{R})$ the real vector space of (singular) $(k+1)$-chains on X with real coefficients. Indeed, this follows from the tautological isomorphism

$$\mathsf{Hom}(H_k(X), \mathbb{R}) \simeq \mathsf{Hom}_{\mathbb{R}}(H_k(X, \mathbb{R}), \mathbb{R}).$$

It is also important to note that the same result is obtained if we restrict our attention to smooth singular (or cubical) k-chains, $C_k^\infty(X)$, and smooth singular (or cubical) k-cycles, $\mathcal{Z}_k^\infty(X) \subset \mathcal{C}_k^\infty(X)$, thus getting a homology group

$$H_k^\infty(X) = \mathcal{Z}_k^\infty(X)/\partial\mathcal{C}_{k+1}^\infty(X)$$

which is canonically isomorphic to $H_k(X)$ (see [1, Theorem 18.7]).

Instead of real coefficients, we can also use complex coefficients, thus getting the complex spaces $H^*_{\mathrm{dR}}(X, \mathbb{C})$, $H_*(X, \mathbb{C})$, $H^*(X, \mathbb{C})$, and the corresponding de Rham isomorphism $H^*_{\mathrm{dR}}(X, \mathbb{C}) \simeq H^*(X, \mathbb{C})$.

1.8. *Poincaré duality.* If X is compact and oriented, there is a natural isomorphism $H^k_{\mathrm{dR}}(X) \simeq H^{n-k}_{\mathrm{dR}}(X)^*$ for all k.

The natural linear map $H^k_{\mathrm{dR}}(X) \to H^{n-k}_{\mathrm{dR}}(X)^*$ providing this isomorphism can be established as follows. Given $\alpha \in C^k(X)$, define $\hat{\alpha} : C^{n-k}(X) \to \mathbb{R}$ by the formula $\hat{\alpha}(\beta) = \int_X \alpha \wedge \beta$. Thus we have a linear map $C^k(X) \to C^{n-k}(X)^*$ given by $\alpha \mapsto \hat{\alpha}$. The image of this map is contained in $H^{n-k}_{\mathrm{dR}}(X)^* \subset C^{n-k}(X)^*$ (this inclusion is the dual of the quotient map $C^{n-k}(X) \to C^{n-k}(X)/dA^{n-k-1}(X) = H^{n-k}_{\mathrm{dR}}(X)$). Indeed, for any $d\beta \in dA^{n-k-1}(X)$, we have $\hat{\alpha}(d\beta) = \int_X \alpha \wedge d\beta = (-1)^k \int_X d(\alpha \wedge \beta) = 0$ (see §1.6). We also have that $\hat{\alpha} = 0$ when $\alpha = d\alpha' \in dA^{k-1}(X)$, as this implies that $\hat{\alpha}(\beta) = \int_X d\alpha' \wedge \beta = \int_X d(\alpha' \wedge \beta) = 0$ for any $\beta \in C^{n-k}(X)$. These two facts provide a linear map $H^k_{\mathrm{dR}}(X) \to H^{n-k}_{\mathrm{dR}}(X)^*$, which turns out to be an isomorphism for all k. In particular we have, if X is connected, $H^n_{\mathrm{dR}}(X) \simeq H^0_{\mathrm{dR}}(X)^* \simeq \mathbb{R}$. Alternatively, $\mathbb{R} \simeq H^0_{\mathrm{dR}}(X) \simeq H^n_{\mathrm{dR}}(X)^*$, and $1 \in \mathbb{R}$ is mapped to $\int_X : H^n_{\mathrm{dR}}(X) \to \mathbb{R}$.

Poincaré duality can also be phrased by saying that the bilinear map

$$C^k(X) \times C^{n-k}(X) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_X \alpha \wedge \beta$$

induces a bilinear map $H^k_{\mathrm{dR}}(X) \times H^{n-k}_{\mathrm{dR}}(X) \to \mathbb{R}$ and that this bilinear map is non-degenerate.

1.9. *The cohomology class of a submanifold or cycle.* Assume X compact and oriented, and let $Y \subset X$ be either an oriented closed k-dimensional submanifold without boundary or a k-cycle (singular or cubical). Then any $\alpha \in A^k(X)$ can be integrated along Y (see §1.5), thus providing a linear map $\bar{Y} : A^k(X) \to \mathbb{R}$, $\alpha \mapsto \int_Y \alpha$. This map vanishes on exact forms $d\beta \in dA^{k-1}(X)$, for $\int_Y d\beta = \int_{\partial Y} \beta$ and either $\partial Y = \emptyset$ or $\partial Y = 0$. Therefore we get a linear map

$$H^k_{\mathrm{dR}}(X) = C^k(X)/dC^{k-1}(X) \to \mathbb{R}, \quad [\alpha] \mapsto \int_Y \alpha.$$

This means that $\bar{Y} \in H^k_{\mathrm{dR}}(X)^*$, which by Poincaré duality can be regarded as $\bar{Y} \in H^{n-k}_{\mathrm{dR}}(X)$. This cohomology class will be denoted by $\mathsf{cl}(Y)$. By the de Rham theorem, it can also be seen as $\mathsf{cl}(Y) \in H^{n-k}(X)$. If Y is a boundary, then $\mathsf{cl}(Y) = 0$, which implies that we have a linear map $\mathsf{cl} : H_k(X, \mathbb{R}) \to H^{n-k}(X)$.

1.10. *The cup and cap products.* Through the de Rham isomorphism and Poincaré duality we can define a product

$$H^k(X) \times H^{k'}(X) \to H^{k+k'}(X), \quad (h([\alpha]), h([\alpha'])) \mapsto h([\alpha] \wedge [\alpha']),$$

where $\alpha \in C^k(X)$, $\alpha' \in C^{k'}(X)$. This map was originally established combinatorially in singular cohomology as the cup product (denoted by $\smile$). Thus we have $h([\alpha]) \smile h([\alpha']) = h([\alpha] \wedge [\beta])$.

On the other hand the cup product, via $\mathsf{cl} : H_k(X, R) \to H^{n-k}(X)$, induces a product

$$H_k(X, \mathbb{R}) \times H_{k'}(X, \mathbb{R}) \to H_{k+k'-n}, \quad ([z], [z']) \mapsto \mathsf{cl}^{-1}(\mathsf{cl}([z]) \smile \mathsf{cl}([z'])),$$

where $z \in \mathcal{Z}_k(X)$, $z' \in \mathcal{Z}_{k'}(X)$. This product was first introduced by Poincaré as an intersection product of homology classes, which itself was induced by the oriented intersection of cycles that meet transversally, and we will denoted by $[z] \cdot [z']$. It is often called the cap product and denoted by $\cap$.

1.11. The appeal to singular (or cubical) real homology can be bypassed by defining $H_k(X, \mathbb{R}) = \mathcal{Z}_k(X, \mathbb{R})/\partial\mathcal{B}_{k+1}(X, \mathbb{R})$, where $\mathcal{Z}_k(X, \mathbb{R})$ and $\mathcal{B}_{k+1}(X, \mathbb{R})$ are the real vector spaces with basis the k-dimensional closed oriented submanifolds $Z \subset X$ with no boundary and the $(k+1)$-dimensional closed oriented submanifolds with boundary, respectively (this idea is suggested in [3]). This definition has advantages as compared to the singular theory: it neatly sidesteps the technicalities involved in dealing with (regular) singular chains and cycles; the integral $\int_Z \alpha$ $(\alpha \in C^k(X))$ confers a clearer geometric meaning to the duality $H_k(X, \mathbb{R}) \simeq H^k_{\mathrm{dR}}(X)^* \simeq H^{n-k}_{\mathrm{dR}}(X)$ and hence to the class $\mathsf{cl}(Z) \in H^{n-k}_{\mathrm{dR}}(X)$ for $Z \in \mathcal{Z}_k(X, \mathbb{R})$; the intersection product $H_k(X, \mathbb{R}) \times H_{n-k}(X, \mathbb{R}) \to \mathbb{R}$, $([Z], [Z']) \mapsto \deg(Z \cdot Z')$, is derived from a geometric oriented intersection (here we rely on the fact that there is a 'moving lemma', i.e. the intersection $Z \cap Z'$ of two submanifolds can be assumed to be transversal without modifying the class $[Z']$, and then $\deg(Z \cdot Z')$ counts each point in $Z \cap Z'$ with the oriented intersecton sign); the bypass connects with Thom's cobordism;[1]; and it is a close relative of the Chow groups of algebraic varieties (to be presented later on).

2 Riemannian manifolds

2.12. Let (X, g) be an oriented riemannian manifold. The metric g can be extended, by standard linear algebra techniques, to a symmetric bilinear map

$$g : A^k(X) \times A^k(X) \to A^0(X)$$

and $A^k(X)$ inherits the symmetric positve-definite bilinear form

$$(\alpha, \beta) = \int_X g(\alpha, \beta)\boldsymbol{\omega},$$

where $\boldsymbol{\omega} \in A^n(X)$ is the *volume form*, namely the unique orientation form such that $\boldsymbol{\omega}_x(u_1, \ldots, u_n) = 1$ for any $x \in X$ and any positively oriented orthonormal basis $u_1, \ldots, u_n \in T_xX)$.

We also know, again from linear algebra, that there is a unique linear isomorphism $* : A^k(X) \to A^{n-k}(X)$ (called the *Hodge $*$-operator*) such that $\alpha \wedge *\beta = g(\alpha, \beta)\boldsymbol{\omega}$, and hence $(\alpha, \beta) = \int_X \alpha \wedge *\beta$. It satisfies $(\alpha, \beta \in A^k(X))$:

(1) $\alpha \wedge *\alpha = 0 \Leftrightarrow g(\alpha, \alpha) = 0 \Leftrightarrow (\alpha, \alpha) = 0 \Leftrightarrow \alpha = 0$;

(2) $* * \alpha = (-1)^{k(n-k)}\alpha$; and

(3) $(*\alpha, *\beta) = (\alpha, \beta)$.

The *codiferential* $\delta : A^k(X) \to A^{k-1}(X)$ is defined by $\delta = (-1)^{n(k+1)+1} * d *$. It is immediate to check that $\delta \circ \delta = 0$. It also satisfies $(d\alpha, \beta) = (\alpha, \delta\beta)$ for

[1] see `https://ncatlab.org/nlab/show/bordism+ring` for a summary and references

$\alpha \in A^{k-1}(X)$, $\beta \in A^k(X)$ (so δ, which is often denoted by d^*, is adjoint of d). To prove this, it is enough to notice that

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta - (-1)^k \alpha \wedge d * \beta = d\alpha \wedge *\beta - \alpha \wedge *\delta\beta$$

(check that $(-1)^k d * \beta = *\delta\beta$). Taking the integral over X, the left hand side vanishes, as it is an exact form, and consequently

$$(d\alpha, \beta) = \int_X d\alpha \wedge *\beta = \int_X \alpha \wedge *\delta\beta = (\alpha, \delta\beta).$$

2.13. *The Laplacian.* The *Laplacian* (also called the *Laplace-Beltrami operator*) of an oriented riemannian manifold (X, g) is the operator defined by

$$\Delta = \Delta_d = d\delta + \delta d : A^k(X) \to A^k(X).$$

It is immediate that Δ is selfadjoint, i.e. $(\Delta\alpha, \beta) = (\alpha, \Delta\beta)$, and it is not hard to check that that $*\Delta = \Delta*$ (manage carefully the various signs appearing in the computation).

A form $\alpha \in A^k(X)$ is said to be *harmonic* if $\Delta\alpha = 0$. It is clear that α is harmonic if $d\alpha = \delta\alpha = 0$. The converse is also true: from

$$(\Delta\alpha, \alpha) = (d\delta\alpha, \alpha) + (\delta d\alpha, \alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha)$$

it follows that if $\Delta\alpha = 0$, then $\delta\alpha = d\alpha = 0$.

2.14. *Hodge decomposition theorem.* Let

$$\mathcal{H}^k(X) = \mathcal{H}^k_\Delta(X) = \{\alpha \in A^k(X) \,|\, \Delta(\alpha) = 0\}$$

(the space of *harmonic* k-forms). Then [4, § 6.8]:

$$A^k(X) = dA^{k-1}(X) \oplus \delta A^{k+1}(X) \oplus \mathcal{H}^k(X).$$

It follows that $C^k(X) = dA^{k-1}(X) \oplus \mathcal{H}^k(X)$ and $\mathcal{H}^k(X) \simeq H^k_{\mathrm{dR}}(X)$. In particular, $\mathcal{H}^k(X)$ is finite dimensional and its dimension is $\mathsf{b}_k(X)$. Notice that $C^k(X) \cap \delta A^{k+1}(X) = \{0\}$, as forms in this intersection are harmonic. Moreover, the decomposition is orthogonal. For example, if $\alpha \in A^{k-1}(X)$ and $\beta \in \mathcal{H}^k(X)$, then $(d\alpha, \beta) = (\alpha, \delta\beta) = 0$.

3 Kähler manifolds

The references [5, 6], [7], and [8] are excellent resources for filling in the details of the overview provided in this section. Other convenient references: [9], [10, 11] and [12]. Whenever deemed useful, we will provide more specific pointers according to the context.

To avoid confusion with i used as an index, the imaginary unit in $\mathbb{C}$ is denoted by i.

3.15. *Preliminairies.* A *hermitian metric* of a complex vector space V is a map $h : V \times V \to \mathbb{C}$ that is $\mathbb{C}$-linear in the first component, that satisfies $h(v', v) = \overline{h(v, v')}$ (so that $h(v, v)$ is real for any $v \in V$), and which is *positive definite*, namely $h(v, v) > 0$ for any non-zero $v \in V$. If we split h in real and imaginary parts, say $h = g + \mathrm{i}\omega$, then $g(v', v) + \mathrm{i}\omega(v', v) = g(v, v') - \mathrm{i}\omega(v, v')$ for all $v, v' \in V$. It follows the g is a euclidean metric on $V_{\mathbb{R}}$ (V considered as real vector space), as $g(v, v) = h(v, v) > 0$ for any nonzero $v \in V$, and that $\omega \in \wedge^2 V_{\mathbb{R}}^*$ (space of skew-symmetric bilinear forms on $V_{\mathbb{R}}$). The forms g and ω are related on account of the linearity of h with respect to v, which yields $\omega(v, v') = -g(\mathrm{i}v, v')$ or, equivalently, $g(v, v') = \omega(\mathrm{i}v, v')$. Moreover, since $h(\mathrm{i}v, \mathrm{i}v') = h(v, v')$, we also have $g(\mathrm{i}v, \mathrm{i}v') = g(v, v')$ and $\omega(\mathrm{i}v, \mathrm{i}v') = \omega(v, v')$. The following converse statement is also true: if g is an euclidean metric on $V_{\mathbb{R}}$ such that $g(\mathrm{i}v, \mathrm{i}v') = g(v, v')$ and we set $\omega(v, v') = -g(\mathrm{i}v, v')$, then $\omega \in \wedge^2 V_{\mathbb{R}}^*$ and $h = g + \mathrm{i}\omega$ is a hermitian metric on V.

3.16. *Notations.* In this section X denotes a complex manifold of complex dimension n. Locally it has holomorphic coordinates $z_1, \ldots, z_n$ and changes of coordinates are expressed by biholomorphic functions. The tangent space $T_x X$ at $x \in X$, and its $\mathbb{C}$-dual $T_x^* X$, are complex vector spaces with basis (over $\mathbb{C}$) $\partial_{z_1}, \ldots, \partial_{z_n}$ and $dz_1, \ldots, .dz_n$, respectively. These two basis are dual: $dz_j(\partial_{z_k}) = \delta_{jk}$.

The decomposition $z_j = x_j + \mathrm{i}y_j$ provides local coordinates $x_1, y_1, \ldots, x_n, y_n$ of $X_{\mathbb{R}}$ (X considered as a real manifold). In particular $\dim_{\mathbb{R}} X_{\mathbb{R}} = 2n$. The vectors $\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_n}, \partial_{x_n}$ form a basis $T_x X_{\mathbb{R}} = (T_x X)_{\mathbb{R}}$, the real tangent space, and $dx_1, dy_1, \ldots, dx_n, dy_n$ a basis of $T_x^* X_{\mathbb{R}}$, the $\mathbb{R}$-dual $T_x X_{\mathbb{R}}$. Actually $X_{\mathbb{R}}$ is endowed with the natural orientation given by the bases of the form $\partial_{x_1}, \partial_{y_1}, \ldots, \partial_{x_n}, \partial_{y_n}$, and $dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is a local orientation form.

Properly expressing the relation between the real bases and the complex bases of the preceding two paragraphs requires going up to the complexified spaces $T_x^* X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $T_x X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, which agree with $T_x^* X \otimes_{\mathbb{R}} \mathbb{C}$ and $T_x X \otimes_{\mathbb{R}} \mathbb{C}$. Indeed, the relations $dz_j = dx_j + \mathrm{i}dy_j$ and $d\bar{z}_j = dx_j - \mathrm{i}dy_j$ make sense in $T_x^* X \otimes_{\mathbb{R}} \mathbb{C}$, and $\partial_{z_j} = \partial_{x_j} - \mathrm{i}\partial_{y_j}$ and $\partial_{\bar{z}_j} = \partial_{x_j} + \mathrm{i}\partial_{y_j}$ in $T_x X \otimes_{\mathbb{R}} \mathbb{C}$. Note that $(dx_j + \mathrm{i}dy_j)(\partial_{x_k} - \mathrm{i}\partial_{y_k}) = \delta_{jk}$.

3.17. *The spaces $A^{p,q}(X, \mathbb{C})$ and the operators ∂ and $\bar{\partial}$.* We have a decomposition

$$A^k(X, \mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X, \mathbb{C}).$$

Locally, with respect to the coordinates $z_1, \ldots, z_n$, the forms in $A^{p,q}(X, \mathbb{C})$ have the form $\sum a_{\boldsymbol{i},\boldsymbol{j}} dz_{\boldsymbol{i}} \wedge d\bar{z}_{\boldsymbol{j}}$, where $\boldsymbol{i}$ and $\boldsymbol{j}$ are increasing sequences of integers in $[n]$ with $|\boldsymbol{i}| = p$, $|\boldsymbol{j}| = q$, $dz_{\boldsymbol{i}} = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, $d\bar{z}_{\boldsymbol{j}} = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, and $a_{\boldsymbol{i},\boldsymbol{j}} \in A^0(X_{\mathbb{R}}, \mathbb{C})$. The forms in $A^{p,q}(X, \mathbb{C})$ are said to be of *type* (or *bidegree*) (p, q). Note that $\overline{A^{p,q}}(X, \mathbb{C}) = A^{q,p}(X, \mathbb{C})$, where the overline denotes complex conjugation.

If $f \in A^0(X, \mathbb{C})$, let $\partial f = \sum (\partial_j f) dz_j$ and $\bar{\partial} f = \sum (\bar{\partial}_j f) d\bar{z}_j$, where we set $\partial_j = \partial_{z_j}$ and $\bar{\partial}_j = \partial_{\bar{z}_j}$. Thus $df = \partial f + \bar{\partial} f$. This decomposition implies that on $A^{p,q}(X, \mathbb{C})$ we have $d = \partial + \bar{\partial}$, where $\partial : A^{p,q}(X, \mathbb{C}) \to A^{p+1,q}(X, \mathbb{C})$ and

$\bar\partial : A^{p,q}(X,\mathbb{C}) \to A^{p,q+1}(X,\mathbb{C})$.

3.18. *Kähler manifolds.* A *hermitian manifold* is a complex manifold X endowed with a *hermitian metric* h. This means that we have a hermitian metric h_x of the tangent space T_xX for any point $x \in X$, and that h_x depends smoothly on x. In local holomorphic coordinates $z_1, \ldots, z_n$, the expression of h has the form $\sum_{i,j=1}^n h_{ij} dz_i d\bar z_j$, where the matrix $(h_{ij}(x))$ is hermitian and positive definite (this follows on imposing that $h(v,v)$ is real and positive for $v \neq 0$). The imaginary part of h_x is a 2-form $\omega_x \in \wedge^2(T_x^*X,\mathbb{C})$, and so we get a 2-form $\omega \in A^2(X,\mathbb{C})$. From the local expression in h with respect to the coordinates $z_1, \ldots, z_n$, we find that the corresponding expression of ω is $\frac{1}{2\mathrm{i}} \sum_{i,j=1}^n h_{ij} dz_i \wedge d\bar z_j$ and hence $\omega \in A^{1,1}(X,\mathbb{C})$. Notice that $\bar h = \sum \bar h_{ij} d\bar z_i dz_j = \sum h_{ji} d\bar z_i dz_j = \sum h_{ij} d\bar z_j dz_i$.

A *Kähler manifold* is a complex manifold X equipped with a Hermitian metric (called the *Kähler metric*) whose imaginary part ω is *closed*. This $(1,1)$-form is called the *Kähler form* of the Kähler metric.

Submanifolds of a Kähler manifold are Kähler, as the restriction of a closed form to a submanifold Y is a closed form of Y.

3.19. *Hodge decomposition of cohomology.* A Kähler manifold is in particular a riemannian manifold of dimension $2n$ and it turns out that the (p,q) components of an harmonic k-form are harmonic. This and the Hodge theorem imply a *Hodge decomposition* of cohomology:

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C}),\ k = 0, 1, \ldots, 2n.$$

Thus $\big(H^*(X,\mathbb{C}), \wedge\big)$ is a bigraded algebra.

Since $\overline{H}^{p,q}(X) = H^{q,p}(X)$, the *Hodge numbers*, i.e. $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C})$, satisfy $h^{p,q} = h^{q,p}$ (see Figure 3.1).

3.20. *Complex projective varieties.* The restriction to S^{2N+1} of the Fubini-Study hermitian metric $ds^2 = \sum_{j=0}^N dz_j \otimes d\bar z_j$ on $\mathbb{C}^{N+1}$ is invariant by the action of S^1 and hence it induces a hermitian metric on $S^{2N+1}/S^1 = \mathbb{P}^N_{\mathbb{C}}$. Setting $z_j = x_j + \mathrm{i} y_j$, the imaginary part of ds^2 is $\omega = \sum_j dx_j \wedge dy_j$. This form has type $(1,1)$ and is closed. Therefore it induces a Kähler structure ω on $\mathbb{P}^N_{\mathbb{C}}$. The class $[\omega] \in H^{1,1}(\mathbb{P}^N_{\mathbb{C}})$ coincides with the class $\mathsf{cl}(H)$ of a hyperplane $H \subset \mathbb{P}^N_{\mathbb{C}}$.

Closed complex submanifolds of the complex projective space are Kähler, and they are projective subvarieties by Chow's theorem [13]. If X is such a submanifold and $n = \dim(X)$, the class $[\omega|X] \in H^{1,1}(X,\mathbb{C}) \subset H^2(X,\mathbb{C})$ coincides with the cohomology class $\mathsf{cl}(H \cap X)$ of a general hyperplane section of X. Recall that $\mathsf{cl}(H \cap X)$ is defined as the Poincaré dual of the $(n-2)$-class $[H \cap X] \in H_{n-2}(X)$.

3.21. *The Kodaira embedding theorem* [14]. A compact complex manifold admits a holomorphic embedding into complex projective space (and hence it is a smooth algebraic variety, by Chow's theorem) if and only if it admits a Kähler metric whose Kähler form is a rational class (i.e. it belongs to the image of

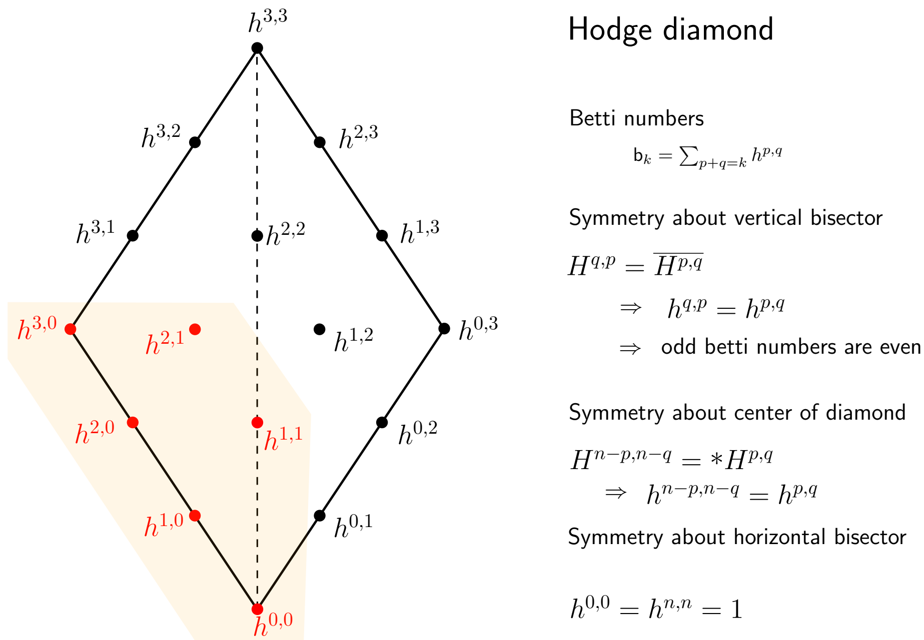


Figure 3.1: Properties of the Hodge numbers $h^{p,q}$ of a compact Kähler manifold. The symmetries about the vertical and horizontal bisectors of the diamond imply that any number in the diamond is equal to one of the numbers distinguished in read. The symmetry about the horizontal bisector is the composition of the symmetry about the center of the diamond with the symmetry about the vertical bisector.

$H^2(X, \mathbb{Q})$ in $H^2(X, \mathbb{C})$). Kodaira, who was awarded the Fields Medal in 1954 in part for this work, qualified such Kähler manifolds as of *restricted type*.

3.22. *Weak and Hard Lefschetz theorems.* Let (X, ω) be a compact Kähler manifold. The *Lefschetz operator* $L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$ is defined by $[\alpha] \mapsto [\omega \wedge \alpha]$. In the Hodge diamond of X, L connects each node with the node one vertical step above it. The *weak Lefschetz theorem* asserts that for $k < n$, L is injective. In consequence, for $k < n$, $\mathsf{b}_k(X) \leqslant \mathsf{b}_{k+2}(X)$ and $h^{k-i,i} \leqslant h^{k-i+1,i+1}$ $(i \in [k])$. By Poincaré duality, we also get $\mathsf{b}_{n-k}(X) \leqslant \mathsf{b}_{n-k-2}(X)$ and $h^{n-i,n-k+i} \leqslant h^{n-i-1,n-k+i-1}$ $(i \in [k])$. These inequalities are dubbed *Hodge staircases*: the Hodge numbers on a vertical line of the Hodge diamond are non-decreasing in the bottom half and non-increasing in the top half; and the even (respectively odd) Betti numbers have the same property.

Note that $L : H^{n-1}(X, \mathbb{C}) \to H^{n+1}(X, \mathbb{C})$ is an isomorphism, as it is injective and both spaces have the same dimension. This is a special case of the *hard Lefschetz theorem*: $L^j : H^{n-j}(X) \to H^{n+j}(X)$ is an isomorphism for all $j \geqslant 0$.

If $H^{p,q}$ is a Hodge component of $H^{n-j}(X)$, so $p+q = n-j$, then L^j maps it isomorphically to $H^{p+j,q+j} = H^{n-q,n-p}$. We get again that the Hodge diamond is symmetric about the horizontal diagonal; see Fig. 3.1).

3.23. *Lefschetz Decomposition Theorem.* For $k \leqslant n$, the *primitive subspace* of $H^k(X, \mathbb{C})$ is defined as the kernel of $L^{n-k+1} : H^k(X) \to H^{2n-k+2}$, and is denoted by $H_0^k(X)$. For $k = 0, 1$, $H_0^k(X) = H^k(X)$. Let $q_k = \lfloor k/2 \rfloor = k /\!\!/ 2$. Then we have:

- $H^k(X, \mathbb{C}) = \bigoplus_{j \geqslant (k-n)^+} L^j H_0^{k-2j}(X)$, where $(k-n)^+ = \max(k-n, 0)$;
- For $k \leqslant n$, $H^k = H_0^k \oplus LH_0^{k-2} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k} = H_0^k \oplus LH^{k-2}$;
- For $k = n + k'$, $1 \leqslant k' \leqslant n$, $H^k = L^{k'} H_0^{k-2k'} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k}$.

3.24. *The Hodge-Riemann relations.* Define $Q : H^k(X, \mathbb{C}) \times H^k(X, \mathbb{C}) \to \mathbb{C}$, by $Q(\alpha, \alpha') = (-1)^{k /\!\!/ 2} \int_X \alpha \wedge \alpha' \wedge \omega^{n-k}$. This is called the *Hodge-Riemann pairing*. With respect to the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$, it has the following properties:

(1) $Q(H^{p,q}, H^{p',q'}) = 0$ if $(p', q') \neq (q, p)$, and
(2) $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for $0 \neq \alpha \in H_0^{p,q}(X)$.

Note that on $H^{p,p}$ we have $Q(\alpha, \alpha') = (-1)^p \int_X \alpha \wedge \alpha' \wedge \omega^{n-2p}$ and that $Q(\alpha, \bar{\alpha}) > 0$ for $0 \neq \alpha \in H_0^{p,p}(X)$.

3.25. *Hodge conjecture.* Let X is a compact Kähler manifold and Z a submanifold of codimention k. Then $\mathsf{cl}(Z) \in H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X) = H^{k,k}(X, \mathbb{Q})$ (it is easy to see that the harmonic form representing $\mathsf{cl}(Z)$ cannot have components $(p, q) \neq (k, k)$). The same is true if $Z \in \mathcal{Z}_{\mathbb{Q}}^k$, the group of rational linear combinations of submanifolds of codimension k (rational cycles of codimension k).

At present, the *Hodge conjecture* [15] is stated as follows: if X is a smooth complex projective variety (or, equivalently, a compact Kähler manifold of restricted type), then $\mathsf{cl} : \mathcal{Z}_{\mathbb{Q}}^k \to H^{k,k}(X, \mathbb{Q})$ is surjective.

This conjecture and its status were presented by Pierre Deligne [16] on the occasion of the selection, in the year 2000, of the *Millennium Prize Problems* [17] by the Clay Mathematics Institute. The two examples the author presents of page 4 point at core difficulties concerning the conjecture. To remark also the contributions of Claire Voisin in her treatises [10, 11] and in her paper [8]. See also [18].

Let us summarize the cases in which the conjecture is known and some of the generalizations that have turned out to be false. To account for variations about the conjecture, the label $\mathrm{HC}_R^k(\mathcal{C})$ stands for the conjecture concerning cohomology classes over the ring R ($\mathbb{Z}$ or $\mathbb{Q}$) of k-cycles of objects from a class $\mathcal{C}$ of manifolds, where $\mathcal{C} = \mathcal{X}, \mathcal{K}, \ldots$, with $\mathcal{X}$ standing of smooth complex projective varieties (or equivalently, by Kodaira's theorem, compact Kähler manifolds of restricted type), $\mathcal{K}$ for general compact Kähler manifolds,... If k is not quoted, it is to be understood that the conjecture refers to all possible codimensions. Thus, for example, the original conjecture is labeled $\mathrm{HC}_{\mathbb{Q}}(\mathcal{X})$.

- $\mathrm{HC}_{\mathbb{Z}}^1(\mathcal{X})$ is true. This is originally due to Lefschetz, but [19] offers a well known sheaf theoretic proof. This and the hard Lefschetz theorem imply that $\mathrm{HC}_{\mathbb{Z}}^{n-1}(\mathcal{X})$ is also true. On the other hand, $\mathrm{HC}_{\mathbb{Z}}(\mathcal{X})$ is false [3].
- The paper [20] shows that $\mathrm{HC}_{\mathbb{Q}}(X_3^4)$ is true, where X_3^4 is a cubic fourfold in

$\mathbb{P}^5$ and also that $\mathrm{HC}^n_{\mathbb{Q}}(\mathcal{K}_{2n})$ is false ($\mathcal{K}_{2n}$: compact Kähler manifolds of complex dimension $2n$). The counterexamples (in Appendix B of that paper) are complex tori T of dimension $2n$ that have no analytic submanifolds of dimension n and yet are such that $H^{n,n}(T,\mathbb{Z})$ has rank 2.

• $\mathrm{HC}^2_{\mathbb{Q}}(\mathcal{U}^4)$ ($\mathcal{U}^4$ fourfolds admitting an algebraic family of rational curves covering it, as for example hypersurfaces in $\mathbb{P}^5_{\mathbb{C}}$ of degree 5): [21].

• $\mathrm{HC}_{\mathbb{Q}}(\mathcal{P})$ ($\mathcal{P}$ Prym varieties) is true: [22].

3.26. *Interlude.* The objects we have been considering up till now (manifolds of various kinds) have an underlying 'continuous' nature, and their relation to the 'discrete' (Betti and Hodge numbers, for example) has been mediated by algebraic structures (homology and cohomology groups and algebras).

A further appearance of discreetness in algebraic geometry, again mediated by algebra, was the introduction of (abstract) algebraic varieties defined over an arbitrary field, and in particular fields of characteristic $p > 0$, e.g. finite fields. This move, pioneered mainly by L. van der Waerden, A. Weil and O. Zariski, was soon continued with further generalizations by J.-P. Serre, A. Grothendieck, and many others. In this era the objects themselves, the assortment of associated cohomologies, and their deep relations to number theory, had a much stronger discrete character than in the preceding epoch. The Fields medals to J.-P. Serre (1954), A. Grothendieck (1966), H. Hironaka (1970), D. Mumbord (1974), P. Deligne (1978), G. Faltings (1986), S. Mori (1990), M. Kontsevich (1998), T. Tao (2006), C. Birkar (2018), P. Scholze (2018) are successive highlights of a most fruitful era in algebraic geometry.

Yet a strong early archetype of the 'discrete' march in algebraic geometry was born when A. Weil focused on counting the number of points on algebraic varieties defined over a finite field. This led him to state his celebrated conjectures, a brilliant moment of when the discrete charmed algebraic geometry. Then A. Grothendieck, in an attempt to prove those conjectures, introduced his 'standard' conjectures in intersection theory. These avenues, which are summarized in next section, are harbingers of the revolution in combinatorics represented by the breakthroughs of J. Huh and his collaborators, which somehow culminate the efforts initiated by the likes of R. P. Stanley visible in his treatises [23, 24]. The highlights of this revolution, including the Fields Medal to J. Huh (2022), will be considered in the sections after next one.

4 Weil's conjectures and Grothendieck's standard conjectures

In [25, p 489], the author says: "This will contain nothing new, except perhaps in the mode of presentation of the final results, which will lead to the *statement of some conjectures concerning the numbers of solutions of equations over finite fields, and their relation to the topological properties of the varieties defined by the corresponding equations over the field of complex numbers*"; and later (p 507): “This, and other examples which we cannot discuss here, seem to lend some support to the following conjectural statements, which are known to be *true for curves*, but which I have not so far been able to prove for varieties of higher dimension” (emphases added).

Even though the bibliography about the Weil conjectures is very extensive, for the purposes of this paper the following titles may suffice: [26, Appendix C, §1]; [27], a recent assessment of the major impacts of [25]; and [28], a selection of MRs related to [25]. Stemming from interest in the applications of algebraic geometry to error-correcting coding, the note [29] presents a fast algorithm for computing $N_r(X)$ when X is a curve.

4.27. *The Hasse-Weil zeta function.* Let $X/\mathbb{F}_q$ be a non-singular projective algebraic variety over $\mathbb{F}_q$, and $n = \dim(X)$. Define $N_r = \#X(\mathbb{F}_{q^r})$, where $X(\mathbb{F}_{q^r})$ denotes the set of $\mathbb{F}_{q^r}$-points of X, and

$$Z_X(t) = \exp(\textstyle\sum_{r=1}^{\infty} \frac{N_r}{r} t^r)$$

(the *zeta function* of X). This is a *generating function* of the N_r, in the sense that

$$N_r = \frac{1}{(r-1)!}\frac{d^r}{dt^r}\log Z(t)|_{t=0}$$

(this follows easily from the definition of $Z(t)$).

4.28. *Examples.* (1) $Z_{\mathbb{A}^n}(t) = \frac{1}{1-q^n t}$. Indeed: $N_r = q^{rn}$, and hence

$$\exp(\textstyle\sum_{r=1}^{\infty} N_r \frac{t^r}{r}) = \exp(\sum_{r=1}^{\infty} \frac{(q^n t)^r}{r}) = \frac{1}{1-q^n t}.$$

In the last identity we use the formal relation $\sum_{r=1}^{\infty} \frac{x^r}{r} = \ln \frac{1}{1-x}$.

(2) If $Y \subset X$ is open, $Z_X(t) = Z_Y(t) Z_{X-Y}(t)$. This relation follows from the equality $N_r(X) = N_r(Y) + N_r(X-Y)$.

(3) $Z_{\mathbb{P}^n}(t) = \frac{1}{1-q^n t} Z_{\mathbb{P}^{n-1}}(t) = \prod_{j=0}^{n} \frac{1}{1-q^j t}$. Apply (2) to $\mathbb{P}^n = \mathbb{A}^n \sqcup \mathbb{P}^{n-1}$, then (1) to $\mathbb{A}^n$, and finally induction to $Z_{\mathbb{P}^{n-1}}(t)$.

4.29. *Statement of Weil's conjectures*

W1 (Rationality). $Z(t) = \frac{P_1(t)\cdots P_{2n-1}(t)}{P_0(t)\cdots P_{2n}(t)}$, where $P_j(t) \in \mathbb{Z}[t]$ with $P_0(t) = 1-t$, $P_{2n}(t) = 1 - q^n t$.

For $j = 1, \ldots, 2n-1$, let $P_j(t) = \prod_k (1 - \alpha_{jk} t)$, $\alpha_{jk} \in \mathbb{C}$.

W2 (Functional equation). $Z_X(\frac{1}{q^n t}) = \pm q^{n\chi/2} t^{\chi} Z_X(t)$, where χ is the Euler characteristic of X.

W3 (Riemann hypothesis). $|\alpha_{jk}| = q^{j/2}$ ($j = 1, \ldots, 2n-1$, all k). This means, with the change of variable $t = q^{-s}$, that the roots $1/\alpha_{jk}$ of P_j, lie on the line $\mathsf{re}(s) = j/2$.

W4 (Betti numbers). If X' is a non-singular projective variety defined over a number field embedded in $\mathbb{C}$ (e.g. $\mathbb{Q}$) and it has good reduction mod p to $X/\mathbb{F}_p$, then $\deg P_j = \mathsf{b}_j(\mathbb{C}(X'))$.

Notes. The first proofs of **W1** and **W2** are due to B. Dwork [30]. **W4** was proved by Grothendieck in 1964/65, in collaboration with M. Artin and J.-L. Verdier, by means of étale cohomology (see, for example, [31]), and the technique also gave new proofs of **W1** and **W2**. For more details, see for example [26, Appendix C].

The first proof of **W3** was produced by P. Deligne in 1974 [32]. See also [33], which was announced in [32] with these words: "Dans un article faisant suite à celui-ci, je donnerai divers raffinements des résultats intermédiaires, et des applications, parmi lesquelles le théorème de Lefschetz ≪difficile≫ (sur les cup-produits itérés par la classe de cohomologie d'une section hyperplane)".

4.30. *Kählerian analogue of Weil's Riemann hypothesis.* Let $X/\mathbb{C}$ smooth irreducible projective variety, $Y \subset X$ a hyperplane section, and $f : X \to X$ an endomorphism. In [34, Th. 1], J.-P. Serre proves the following result: If $f^*(Y) \sim_{\rm alg} qY$ for some positive integer q, then the modulus of the eigenvalues of $f_j^* : H^j(X,\mathbb{C}) \to H^j(X,\mathbb{C})$ are all equal to $q^{j/2}$.

We see the analogy on replacing $\mathbb{C}$ by $\mathbb{F}_q$ and letting f be the Frobenius endomorphism, which satisfies $f^*(Y) \sim qY$.

4.31. *Intersection theory.* For compact oriented manifolds, we have seen a glimpse of 'intersection theory' in §1.11. For smooth quasi-projective varieties X over an algebraically closed field F, the notion analogous to the k-cycles on an oriented manifold is the free abelian group $\mathcal{Z}_k(X)$ generated by closed irreducible k-dimensional subvarieties of X (its elements are also called k-cycles of X); the notion of homologous cycles is replaced by the notion of *rationally equivalent* cycles, $Z \sim_{\rm rat} Z'$; and the homology group is replaced by the *Chow group* $A_k(X) = \mathcal{Z}_k(X)/\mathcal{R}_k(X)$, where $\mathcal{R}_k(X)$ is the subgroup of cycles rationally equivalent to 0 (for this, and what follows in this §, we refer to [26, Appendix A] for further details).

In the case of complete irreducible smooth algebraic varieties X of dimension n, let $A^k(X) = A_{n-k}(X)$, the Chow group of codimension k, and $A(X) = \bigoplus_k A^k(X)$. Then $A(X)$ is an associative commutative ring when endowed with the *intersection product* $[Z] \cdot [Z']$ of rational classes. In the case of two irreducible cycles $Z \in \mathcal{Z}^k(X)$ and $Z' \in \mathcal{Z}^{k'}(X)$ intersecting properly (meaning that all components C of $Z \cap Z'$ have codimension $k+k'$), we have $[Z]\cdot[Z'] = [Z\cdot Z']$, where $Z \cdot Z' = \sum_C i_C(Z,Z')C$, with $i_C(Z,Z')$ the multiplicity of C in the intersection $Z \cap Z'$. The general case is reduced to the proper case by means of a 'moving lemma' for rational equivalence. The *Chow ring* $A(X)$ has many remarkable features, like the formal properties A1-A8 (*loc. cit.*), and for providing a natural framework for the the theory of Chern classes of coherent sheaves, the Riemann-Roch theorem of Hirzebruch, and the Riemann-Roch theorem of Grothendieck.

Besides rational equivalence, there are other relevant relations. *Algebraic equivalence*, $Z \sim_{\rm alg} Z'$, works like the rational equivalence but replacing $\mathbb{P}^1$ by an arbitrary parameter variety S. In the *numerical equivalence*, $Z \sim_{\rm num} Z'$, the criterion is that $\deg(W \cdot (Z' - Z)) = 0$ for any irreducible closed subvariety W of dimension $n-k$. There is also the *homological equivalence*, $Z \sim_{\rm hom} Z'$, that we will consider in §4.33. It is clear that rational equivalence implies algebraic equivalence, and Severi's *principle of conservation of number* asserts that algebraic equivalence implies numerical equivalence (cf [35]). The numerical equivalence has been at the root of *enumerative geometry* since its beginnings, and most definitely since Schubert's foundational treatise [36]. The treatise [37], a modern version of intersection theory, has many pointers to enumerative geometry. Cf. also [38], now accessible as a pdf online.

4.32. *Origin of Grothendieck's standard conjectures.* According to the first paragraph of [39], the standard conjectures "arose from an attempt at understanding the conjectures of Weil on the ζ-functions of algebraic varieties ... and they were worked out about three years ago independently by Bombieri and myself." And in the paper's Conclusions we read: "The proof of the two standard conjectures would yield results going considerably further than Weil's conjectures. They would form the basis of the so-called 'theory of motives' which is a systematic theory of 'arithmetic properties' of algebraic varieties as embodied in their groups of classes of cycles for numerical equivalence. ... Alongside the problem of resolution of singularities,[2] the proof of the standard conjectures seems to me to be the most urgent task in algebraic geometry."

As we have seen in §4.29 *Notes*, the core issue was producing a proof of **W3**. As said, this was settled by Deligne following a different approach—which, by the way, also allowed him to prove the Ramanujan conjecture on the τ function [32, 33]. Nevertheless, the fact is that the standard conjectures remain conjectures, and it is to be hoped that the fertile work of June Huh and collaborators (which we will explore in §5, §6, and §7.50 following the steps outlined §4.35), may rekindle the interest in them.

4.33. *Weil cohomologies.* Consider smooth irreducible projective algebraic varieties X over an algebraically closed field $\boldsymbol{\kappa}$ (henceforth *varieties*), and set $n = n_X = \dim(X)$. The standard conjectures (see §4.34) are phrased in terms of a *Weil cohomology* $H^*(X)$ with *coefficient field* K (a field of characteristic 0), which means that $X \mapsto H^*(X)$ is a contravariant functor from varieties to finite-dimensional graded K-algebras such that $H^k(X) = 0$ for $k < 0$ and $k > 2n$. It is further assumed that $H^{2n}(X)$ is endowed with an isomorphism $\int_X : H^{2n}(X) \simeq K$ (also denoted deg) and with a functorial *class homomorphism* $\mathsf{cl} = \mathsf{cl}_X : \mathcal{Z}^k(X) \to H^{2k}(X)$, so that the following properties are satisfied:

1. Poincaré duality: The map $H^k(X) \times H^{2n-k}(X) \to K$, $(\alpha, \alpha') \mapsto \int_X \alpha \cdot \alpha'$ is non-degenerate for all k. In particular, $H^{2n-k}(X) \simeq H^k(X)^*$, or, defining $H_k(X) = H^k(X)^*$, $H^{2n-k}(X) \simeq H_k(X)$ (the duals are as K-vector spaces). If $f : X \to X'$ be a morphism of varieties, the adjoint of $f^* : H^*(X') \to H^*(X)$ by Poincaré duality is a K-linear map $f_* : H^*(X) \to H^{*+r}(X')$, where $r = n_{X'} - n_X$ (dubbed the *codimension* of f). These two maps are related through the *projection formula*: $f_*(\alpha \cdot f^*\alpha') = f_*\alpha \cdot \alpha'$ ($\alpha \in H^*(X)$, $\alpha' \in H^*(X')$).

[2]Let us just recall here that the problem of the resolution of singularities was solved by H. Hironaka in [40] (in characteristic zero; for positive characteristic, see [41]).

2. Künneth formula: The natural map $H^*(X) \otimes H^*(X') \to H^*(X \times X')$, $\alpha \otimes \alpha' \mapsto \pi_X^*(\alpha) \cdot \pi_{X'}(\alpha')$, is an isomorphism for any varieties X and X' (π_X and $\pi_{X'}$ are the projection maps of $X \times X'$ onto X and X', respectively).

3. Class map: Aside from its functoriality, which means that $\mathsf{cl}_X(f^*Z') = f^*(\mathsf{cl}_{X'}Z')$ for any morphism of varieties $f : X \to X'$ and any $Z' \in \mathcal{Z}^k(X')$, it is also required that it be Künneth-compatible (by this we undestand that $\mathsf{cl}_{X\times X'}(Z \times Z') = \mathsf{cl}_X(Z) \otimes \mathsf{cl}_{X'}(Z')$ for any varieties X, X' and any cycles $Z \in \mathcal{Z}^*(X), Z' \in \mathcal{Z}^*(X')$) and that $\int_X \mathsf{cl}(Z) = \deg(Z)$ for any $Z \in \mathcal{Z}^n(X)$ (here $\deg(Z) = \sum n_i$ if $Z = \sum_i n_i P_i,\ P_i \in X$).

The elements of $H^*(X)$ are called *cohomology classes* of X and the classes in the subring $\mathcal{A}^*(X) = \mathsf{cl}(\mathcal{Z}^*(X)) \subseteq H^{2*}(X)$ are said to be *algebraic*. Two algebraic cycles are said to be *homologically equivalent* when they define the same algebraic class. To note that $\int_X \alpha \cdot \alpha' \in \mathbb{Z}$ for any $\alpha, \alpha' \in \mathcal{A}^*(X)$ (with the convention that $\int_X \xi = \int_X \xi_n$ for any $\xi = \xi_0 + \xi_1 + \cdots + \xi_n \in \mathcal{A}^*(X)$.

One instance of Weil cohomology is Grothendieck's *ℓ-adic cohomology*. In this theory K is the field $\mathbb{Q}_\ell$ of ℓ-adic rational numbers, with ℓ different from the characteristic of the ground field $\boldsymbol{\kappa}$. See, for instance, [31]. Weil cohomologies are presented in many references, as for example [42].

4.34. *Statement of the standard conjectures.* We extract from Grothendieck's original paper [39] just the conjectural statements that play a role in the description of the Kähler package in §6, namely the *hard Lefschetz property* and the *Hodge–Riemann relations*, which themselves mimic for algebraic varieties the homonymous results for Kähler manifolds. For a rather meticulous discussion of Grothendieck's standard conjectures, see [42] and [43]. See also §7.52 for references about their current status.

Let Y by a hyperplane section of X, set $y = \mathsf{cl}(Y) \in H^2(X)$, and define $L : H^k(X) \in H^{k+2}(X)$ to be the multiplication by y. Clearly, we also have that $L^j : H^k(X) \to H^{k+2j}(X)$ is multiplication by y^j.

1. Hard Lefschetz property. It states that $L^j : H^{n-j}(X) \to H^{n+j}(X)$ is an isomorphism for all $j \in [n]$ (equivalently, $L^{n-k} : H^k(X) \to H^{2n-k}(X)$ is an isomorphism for all $k \in [n]$). A corollary is that $L^j : H^k(X) \to H^{k+2j}(X)$ is injective for $j \leqslant n-k$ and surjective for $j \geqslant n-k$. Indeed, in the first case, composing $L^j : H^k(X) \to H^{k+2j}(X)$ with $L^{n-k-j} : H^{k+2j}(X) \to H^{2n-k}(X)$ is $L^{n-k} : H^k(X) \to H^{2n-k}(X)$, which is an isomorphism; so the first map must be injective. In the second case, the map $L^{k-(n-j)} : H^{n-(k+2j-n)}(X) \to H^{2k}(X)$ followed by $L^j : H^k(X) \to H^{k+2j}(X) = H^{n+(k+2j-n)}$ produces the map $L^{j+(j-(n-k))} : H^{n-(k+2j-n)}(X) \to H^{n+(k+2j-n)}(X)$, which is an isomorphism. Therefore the second map must be surjective.

2. Hodge–Riemann relations. For $j \leqslant n/2$ let $\mathcal{A}_0^j(X) = \{\alpha \in \mathcal{A}^j \mid L^{n-2j+1}\alpha = 0\}$ (the *primitive part* of $\mathcal{A}^j(X)$; note that $\mathcal{A}_0^0(X) = \mathcal{A}^0(X)$). Then the intersection pairing $\mathcal{A}_0^j(X) \times \mathcal{A}_0^j(X) \to \mathbb{Z}$ given by $(-1)^j \int_X L^{n-2j}(\alpha \cdot \beta)$ is positive definite (cf. §4.33, 3, Class map). As we have seen in §3.24, this statement holds for complex algebraic varieties (Hodge theory).

4.35. *Interlude.* The nomination of June Huh for the Fields Medal was disclosed on 5 July 2022: "For bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron-Rota-Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture." Gil Kalai was in charge of the *Laudatio* [44], the Fields Medal Lecture [45] was delivered on 6 July 2022 with the title *Combinatorics and Hodge Theory*, and the corresponding paper in the ICM-2022 Proceedings appeared as [46]. Those dazzling events revealed to workers in various areas a wealth of fresh and beautiful interconnections between mathematical fields, feeling thereby inspired to pursue the greater understanding that those accomplishments connoted.

The results obtained by J. Huh, often with collaborators, are outlined in the next two sections: §5 is devoted to Combinatorics and §6 to the Kähler Package and Lorentzian polynomials, including examples of how this machinery works for solving conjectures in combinatorics "that had hitherto been unreachable by other means". In §7 we overview recent productions, either rooted in the topics previously outlined or closely connected to them.

5 Combinatorics

In this section we collect a small parade of basic notions, conjectures, results, and references about a facet of the discrete side of mathematics. In the main, it is no more than a handy passage to a more attentive perusal of J. Huh's contributions and influence, which, except for a few bibliographical pointers, is deferred to §6.

5.36. *The birth of modern graph theory.* In a communication at Harvard dated January 14, 1931, H. Whitney (1907-1989) stated this (emphasis added): "We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a *combinatorial character*; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper" (namely, [47] in our references). Whitney, who was one of the founders of singularity theory, did groundbreaking research on manifolds, embeddings and immersions, characteristic classes, and geometric integration theory.

Graph theory has evolved into a very large industry. Here are some basic references, ordered by year of publication: [48], [49], [50], [51], [52]. Let us just recall here that an a graph with no cycles (acyclic) is also called a *forest*, and that connected forests are called *trees*. Consequently, the connected components of a forest are trees. A *spanning tree* of a connected graph G is a subgraph T of G such that T is a tree and $V(T) = V(G)$. Note that $|E(T)| = |V(T)| - 1$ holds for any tree and hence $|E(T)| = |V(G)| - 1$ for any spanning tree of a connected graph G.

5.37. *The birth of the matroid.* It happened in 1935 and its birth certificate is Whitney's paper [53]. It is worth reproducing its first few lines:

Let $C_1, C_2, \dots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid". The present paper is devoted to a study of the elementary properties of matroids.

5.38. *Ubiquity of the matroid.* The following quotations are from [54]:

> As the word suggests, Whitney conceived a matroid as an abstract generalization of a matrix, and much of the language of the theory is based on that of linear algebra. However, Whitney's approach was also motivated by his work in graph theory and as a result some of the matroid terminology has a distinct graphical flavor. Some time later Van der Waerden also used the concept of abstract dependence in his *Moderne Algebra*.
>
> Apart from [several] isolated papers [up to 1949] ... the subject lay virtually dormant until the late fifties when W. T. Tutte (1958,1959) published his fundamental papers on matroids and graphs [[55], [56] in our references] and Rado (1957) studied the representability problem for matroids. Since then interest in matroids and their application in combinatorial theory has accelerated rapidly. Indeed it was realized that matroids have important applications in the field of *combinatorial optimization* and also that they *unify and simplify* apparently diverse areas of pure combinatorics".

In the last thirty years, matroid theory has kept its unfolding into a huge field. The following are excellent references: [57] (a comprehensive treatise), [58] (an earlier and a bit slimmer book, republished by Dover in 2010), and [59] (an useful short summary, updated in 2014).

5.39. *Matroids.* For convenience of readers predominantly trained in the 'continuous' side of mathematics but wishing to explore connections with the 'discrete' side, we briefly recall various definitions of matroids and how they relate to each other. In what follows E will denote a finite set, and 2^E (or better $\mathbb{B}^E$, $\mathbb{B} = \{0,1\} \subset \mathbb{N} = \mathbb{Z}_{\geqslant 0}$) the set of all subsets of E.

▪ Definition by independence systems. A *matroid* on E is a pair $M = (E, \mathcal{I})$, where $\mathcal{I}$ is a family of subsets of E (called *independent sets*) that satisfy:

(**i1**) the empty subset is independent;
(**i2**) any subset of an independent set is independent; and
(**i3**) if X, X' are independent and $|X| > |X'|$, there exists $x \in X - X'$ such that $X' \cup x$ is independent (as usual in the literature, we let $X' \cup x$ be a shorthand for $X' \cup \{x\}$).

This definition is an abstraction of the notion of linearly independent sets of a finite set of vectors in a K-vector space (such matroids are said to be *representable* over the field K). It is also important to note that a graph $G = (V, E)$ gives rise to a matroid by declaring that a subset of E (edges of G) is independent if it contains no cycles of G. Actually, most of the terminology about matroids has its roots in these two sources (vector spaces and graphs).

▪ By bases. Given a matroid $M = (E, \mathcal{I})$, let us denote by $\mathcal{B}$ the family of *maximal independent sets*, which are called *bases* of M. It is clear that $\mathcal{B}$ determines $\mathcal{I}$ as the family of subsets X of E such that $X \subseteq B$ for some $B \in \mathcal{B}$. The family $\mathcal{B}$ satisfies:
(**b1**) $\mathcal{B}$ in non-empty;
(**b2**) if $B, B' \in \mathcal{B}$ and $b \in B - B'$, there exists $b' \in B'$ such that $(B - b) \cup b' \in \mathcal{B}$ ($B - b$ is shorthand for $B - \{b\}$). This is the *base exchange property* and corresponds to the property **i3** of $\mathcal{I}$. Now the point is that a matroid can be defined as a pair $M = (E, \mathcal{B})$ with B satisfying **b1** and **b2**.

▪ By circuits. A *circuit* of a matroid $M = (E, \mathcal{I})$ is a *minimal dependent set*, that is, a non-independent set whose proper subsets are independent. The set $\mathcal{C}$ of circuits has the following properties:
(**c1**) $\emptyset \notin \mathcal{C}$;
(**c2**) if $C, C' \in \mathcal{C}$ and $C \subseteq C'$, then $C = C'$; and
(**c3**) it satisfies the *circuit elimination axiom*, namely that if $C, C' \in \mathcal{C}$ are distinct and $e \in C \cap C'$, then there exists $C'' \in \mathcal{C}$ such that $C'' \subseteq (C \cup C') - e$.

Again a matroid can be defined as a pair $(E, \mathcal{C})$ where $\mathcal{C}$ satisfies **c1**–**c3**. Note that $\mathcal{I}$ is the family of subsets of E that contain no circuit.

▪ By rank. Given a matroid $(E, \mathcal{I})$, we have the rank function: $r : \mathbb{B}^E \to \mathbb{N}$, where $r(X)$ is the maximum cardinal $|I|$ for independent sets $I \subseteq X$. It has the following properties:
(**r1**) $r(X) \leqslant |X|$;
(**r2**) if $X \subseteq X'$, then $r(X) \leqslant r(X')$; and
(**r3**) $r(X \cup X') + r(X \cap X') \leqslant r(X) + r(X')$ (*submodular property*).

Then a matroid on E can be defined by a function $r : \mathbb{B}^E \to \mathbb{N}$ satisfying those properties **r1**–**r3**. Note that the independent sets are the $I \subset E$ such that $r(I) = |I|$. The rank of E is also called the rank of M and denoted by $r(M)$.

▪ By a closure operator. Given $M = (E, \mathcal{I})$, set $\bar{X} = \{x \in E \,|\, r(X \cup x) = r(X)\}$ for any $X \subseteq E$. This defines the operator $\mathbb{B}^E \to \mathbb{B}^E$, $X \mapsto \bar{X}$. This operator has the following properties:
(**d1**) $X \subseteq \bar{X}$ for any $X \subseteq E$;
(**d2**) if $X \subseteq Y \subseteq E$, $\bar{X} \subseteq \bar{Y}$;
(**d3**) $\bar{\bar{X}} = \bar{X}$ for all $X \subset E$; and

(**d4**) If $X \subset E$, $x \in E$, and $x' \in \overline{X \cup x} - \bar{X}$, then $x \in \overline{X \cup x'}$.

It turns out that a matroid on E can be specified by giving a *closure operator* $\mathbb{B}^E \to \mathbb{B}^E$, $X \mapsto \bar{X}$, by which we understand that it satisfies **d1**–**d4**. One important point to note is how independent sets are determined by means of a closure operator: $I \subseteq E$ is independent if and only if $x \notin \overline{I - x}$ for all $x \in I$.

▪ By flats. Given a matroid, its *flats* are the subsets $F \subseteq E$ such that $\bar{F} = F$ (*closed* subsets). The collection of flats, $\mathcal{L}$, has the following properties:
(**f1**) $\emptyset$ and E are flats;
(**f2**) the intersection of any two flats is a flat; and
(**f3**) for any flat F and any $e \in E - F$, there is a unique flat that is minimal among the flats containing $F \cup e$.

Moreover, the family $\mathcal{L}$ of flats is a lattice with the partial order given by inclusion. The lattice operations $\wedge$ and $\vee$ are defined by $F \wedge F' = F \cap F'$ and $F \vee F' = \overline{F \cup F'}$.

Once more, a matroid can be defined as a pair $M = (E, \mathcal{L})$, where $\mathcal{L}$ is a family of subsets of E satisfying **f1**–**f3**. In terms of flats, the bases are the $B \subseteq E$ such that $\bar{B} = E$ but $\overline{B - b} < E$ for all $b \in B$.

▪ In §5.45(2) we recall another definition due to J. Huh.

5.40. *Four useful properties of finite numeric sequences.* Let $a_0, \ldots, a_m$ be a sequence of non-negative real numbers. It is said to be:

(1) *Unimodal*: if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$ for some $j \in [m]$. For instance, the sequences of Betti numbers $\mathsf{b}_0, \mathsf{b}_2, \ldots, \mathsf{b}_{2n}$ and $\mathsf{b}_1, \mathsf{b}_3, \ldots, \mathsf{b}_{2n-1}$ of a compact Kähler manifold are unimodal. They also happen to be *symmetric*. These two properties would still hold for smooth projective algebraic varieties over any field $\boldsymbol{\kappa}$ if the hard Lefschetz property turns out to be true (cf. §4.34).

(2) *Log-concave*: if $a_j^2 \geqslant a_{j-1}a_{j+1}$ for all $j \in [m-1]$. A log-concave sequence of *positive* terms is unimodal. For example, the symmetric sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is log-concave, hence also unimodal (which in this case is clear from the properties of binomial numbers). Let the following quote from [60] be a motivation for continuing this journey in next section: "We believe that behind any log-concave sequence that appears in nature there is … a 'Hodge structure' responsible for the log-concavity".

(3) *Ultra-log-concave*: If $a_j/\binom{m}{j}$, $j \in [m]$, is log-concave.

(4) *Top-heavy*: if $a_j \leqslant a_{m-j}$ for $j \in [0, m/\!\!/2]$.

For the ubiquity of these notions in algebra, combinatorics and geometry, see the surveys [61] and [62]. For specific occurrences in the theory of projective hypersurface singularities, see [63].

Example 1 (I. Newton). Let $\sum_{j=0}^n b_j x^j = \sum_{j=0}^n \binom{n}{j} a_j x^j$ be a real polynomial with *real roots*. Then $b_0, b_1, \ldots, b_n$ is ultra-log-concave ($\Leftrightarrow a_0, a_1, \ldots, a_n$ is log-concave). Moreover, if $b_j \geqslant 0$, then $b_0, b_1, \ldots, b_n$ has no internal zeros. See [64, Theorem 5.12].

Example 2 (J. Huh, [65, 66]). Let $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. Then $H_{2k} = H_{2k}(\mathbb{P}^m \times \mathbb{P}^n, \mathbb{Z})$ is free with basis the classes $[\mathbb{P}^{k-j} \times \mathbb{P}^j]$, where $j \in I_k = [\max(0, k-m), \min(k, n)]$. Thus any $\xi \in H_{2k}$ can be (uniquely) expressed as $\xi = \sum_{j \in I_k} c_j [\mathbb{P}^{k-j} \times \mathbb{P}^j]$, $c_j \in \mathbb{Z}$. When can some integer multiple of ξ be representable, i.e. be equal to the homology class $[Z]$ of a subvariety Z? The answer is as follows in [66, Theorem 1]: if ξ is an integer r times either $[\mathbb{P}^m \times \mathbb{P}^n]$, $[\mathbb{P}^m \times \mathbb{P}^0]$, $[\mathbb{P}^0 \times \mathbb{P}^n]$, or $[\mathbb{P}^0 \times \mathbb{P}^0]$, then ξ is representable if and only if $r = 1$; otherwise, some positive multiple of ξ is representable if and only if $\{c_j\}_{j \in I_k}$ is a nonzero log-concave sequence of nonnegative integers with no internal zeros.

Intersection cohomology staircases. If X is an irreducible complex projective variety of dimension n, possibly with singularities, Goresky and MacPherson [67, 68] introduced the *intersecction cohomology* of X,

$$IH^*(X) = IH^0(X) \oplus IH^1(X) \oplus \cdots \oplus IH^{2n}(X).$$

Let $\beta_j = \dim IH^j(X)$ (*IH* Betti numbers). Then the sequences $\beta_0, \beta_2, \ldots, \beta_{2n}$ and $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ are symmetric and unimodal. For a detailed overview of the development of *IH*, see [69].

5.41. *The Read-Hoggar conjecture.* Given a graph $G = (V, E)$, and a positive integer q, a *proper coloring* of G with q colors is a map $c : V \to [q]$ such that $c(a) \neq c(b)$ when $(a, b) \in E$.

The number of proper colorings of G with q colors turns out to be a polynomial in q (the *chromatic polynomial* of G) of the form

$$\chi_G(q) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^{n-1} a_1 q,$$

where $n = |V|$ and $a_j \geqslant 0$ for $j = 1, \ldots, n$.

The *Read-Hoggar conjecture* (1968, 1974) says that $a_1, \ldots, a_n$ is *log-concave.* It was proved by J. Huh in his PhD research (see [63] and [65]). The sequence is also *unimodal* (this was Read's conjecture). This turns out to be a special case of the conjecture considered next.

5.42. *Heron–Rota–Welsh conjecture.* The *characteristic polynomial*, $\chi_M(q)$, of a matroid $M = (E, \mathcal{I})$ is defined as

$$\chi_M(q) = \sum_{X \subseteq E} (-1)^{|X|} q^{r(E)-r(X)} = \sum_{j=0}^{r(E)} (-1)^j w_j q^{d-j},$$

where the coefficients w_j are called *Whitney numbers* (of the first kind).

The characteristic polynomial *generalizes the notion of chromatic polynomial of a graph* (see [57, p. 588]): if (G, E) is a connected graph and $(M, \mathcal{I})$ its associated matroid, then $\chi_G(q) = q\chi_M(q)$. This is why the characteristic polynomial of a matroid is also called chromatic polynomial.

The Heron-Rota-Welsh conjecture asserts that $w_0, w_1, \ldots, w_{r(E)}$ is log-concave, and it was proved in [60].

5.43. *The Dowling-Wilson conjecture.* Let $\mathcal{L}$ be a finite lattice, $r : \mathcal{L} \to \mathbb{N}$ its *rank* function, $\mathcal{L}^k = \{x \in \mathcal{L} : r(x) = k\}$, and $d = \operatorname{rank}(\mathcal{L})$ (the rank of its maximum element). $\mathcal{L}$ is said to be *geometric* if it is generated by $\mathcal{L}^1$ (the *atoms* of $\mathcal{L}$) and r satisfies the *submodular* property, namely $r(x) + r(x') \geqslant r(x \vee x') + r(x \wedge x')$ for all $x, x' \in \mathcal{L}$.

The Dowling-Wilson *top-heavy* conjecture (1974) asserts that

$$|\mathcal{L}^k| \leqslant |\mathcal{L}^{d-k}| \text{ for all } k \leqslant d/2. \qquad (*)$$

Actually the conjecture was phrased for the lattice $\mathcal{L}(M)$ of flats of a matroid $M = (E, \mathcal{I})$ (recall that a flat is a subset of E that is maximal for its rank) and it was proved in [70] (see also [71] for further enhancements). But this is not a less general statement than Eq. $(*)$, as the class of geometric lattices agrees with the class of lattices of flats of matroids.

The maximal elements of $\mathcal{L}^k$ will be called k-flats of $\mathcal{L}$ (points for $k = 1$, lines for $k = 2$ and planes for $k = 3$). See Fig. 6.2 for two simple illustrations.

5.44. *Mason's conjecture.* Let $i_k = i_k(M)$ be the number of independent sets of cardinal k of a finite matroid $M = (E, \mathcal{I})$. *Mason's ultra-strong conjecture* says that the i_k form an ultra log-concave sequence, i.e.

$$i_k^2 \geqslant (1 + \frac{1}{k})(1 + \frac{1}{n-k}) i_{k-1} i_{k+1},\ n = |E|,$$

and it was proved in [72]. Notice that if we set $i'_k = i_k / \binom{n}{k}$, the ultrastrong condition is equivalent, by definition, to $i'_k \geqslant i'_{k-1} i'_{k+1}$. As explained in the footnote 2 of [46], it was independently proved in [73]. In this paper the authors introduce the notion of *completely log-concave polynomials* and proved, for any finite matroid, that the (homogenization of the) generating polynomial of its independent sets has this property. We will return to this point in next section.

5.45. *Interlude.* It is a good place to recall, following [45] and [46], together with other references to be specified, a few additional combinatorial concepts and notations that will be serviceable in next section.

(1) A *generalized permutohedron* is a polytope in $\mathbb{R}^E$ (E a finite set) all of whose edges are in the direction $e_i - e_j$ for some $i, j \in E$ ($\{e_j\}_{j \in E}$ denotes the standard basis of $\mathbb{R}^E$), and it is said to be *integral* if its vertices belong to the lattice $\mathbb{Z}^E$. Examples: the *standard permutohedron* $P(1, 2, \ldots, n)$, which is the convex hull of all the permutations of $(1, 2, \ldots, n)$, and the *hyperoctahedron* $P(\pm 1, 0, \ldots, 0)$, which is the convex hull of all the permutations of $(\pm 1, 0, \ldots, 0)$ (see Fig. 5.1), are integral generalized permutohedra. According to [74], *all generalized permutahedra in $\mathbb{R}^n$ are obtained from the standard permutohedron by moving its vertices so that all the edge directions are preserved.* See also [75].

(2) We can identify the subsets of a finite set E with $\mathbb{B}^E$ (also called zero-one vectors). A subset J of the lattice $\mathbb{Z}^E$ is said to be *M-convex* if it is the set of lattice points of an integral generalized permutohedron. A finite set $J \subset \mathbb{N}^E$ is M-convex when it satisfies the *symmetric basis exchange property*: For any $\alpha, \beta \in J$ and an index i such that $\alpha_i > \beta_i$, there is an index j such that $\alpha_j < \beta_j$ and $\alpha - e_i + e_j, \beta + e_i - e_j \in J$. Examples: given a positive integer d, the set of

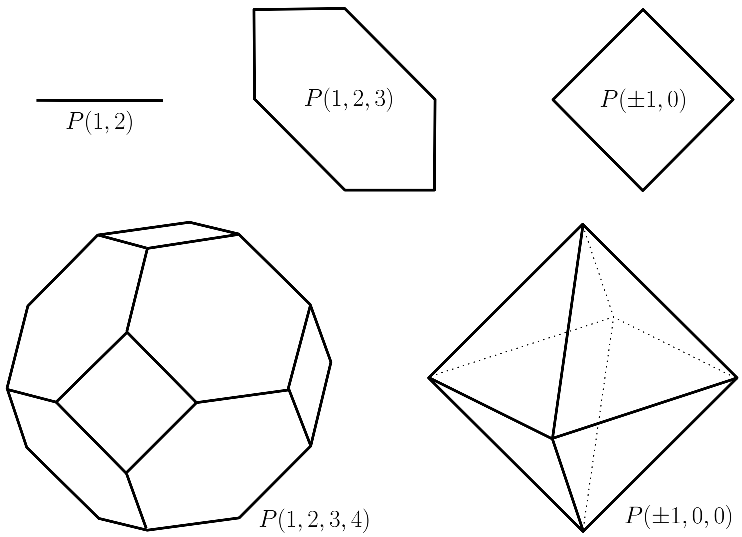


Figure 5.1: $P(1,2)$, $P(1,2,3)$ and $P(1,2,2,4)$ are the standard permutohedra in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. Note that $P(1,2,\ldots,n)$ is $(n-1)$-dimensional, as it lies in the hyperplane $x_1+\cdots+x_n = \binom{n+1}{2}$ of $\mathbb{R}^n$. $P(\pm 1,0) \subset \mathbb{R}^2$ (a square) and $P(\pm 1,0,0) \subset \mathbb{R}^3$ (an octahedron) are examples of hyperoctahedra. Note also that $P(\pm 1,0,0)$ is an edge direction preserving deformation of $P(1,2,3,4)$ –shrink each of the square faces to a vertex while preserving the directions of all the edges).

vectors in $\alpha \in \mathbb{N}^E$ (or $\alpha \in \mathbb{B}^E$) such that $\sum \alpha_i = d$. Importantly, matroids on the set E can be defined as M-convex subsets of $\mathbb{B}^E$ (see [46]).
(3) Let $G = (V, E)$ be a connected graph and $J_G \subseteq \mathbb{B}^E$ be the set of spanning trees of G (see Fig. 5.2 for an example). For any G, J_G is a matroid (in other words, it is the set of lattice points of an integral generalized permutohedron). Such matroids are said to be *graphic*.

Figure 5.2: Set J_G of spanning trees of the full 4-vertex graph G.

Representable (or realizable) matroids over a field $\mathbb{F}$ can be defined by a similar procedure. Let W be a vector space over $\mathbb{F}$, $\varphi : E \to W$ a map, and denote by $J_\varphi \subseteq \mathbb{B}^E$ be the set of subsets B of E such that $\varphi(B)$ is a basis of W. Then J_φ is a matroid (the set of lattice points of an integral generalized permutohedron), and such matroids are said to be representable over $\mathbb{F}$. For example, if $v_1, v_2, v_3, v_4 \in \mathbb{R}^2$ are general vectors, we get six bases $\{v_i, v_j\}_{1 \leqslant i < j \leqslant 4}$ which form, regarded as elements of $\mathbb{B}^4$, the octahedron $\{1100, 1010, 1001, 0110, 0101, 0011\}$. Graphic matroids are representable over any field [57, §5.1].

(4) Let $\mathcal{L}$ be a geometric lattice of rank d (see §5.43) and $E = \mathcal{L}^1$ the set of points (or atoms) of $\mathcal{L}$ (elements of rank 1). Let $J_{\mathcal{L}} \subset \mathbb{B}^E$ be the set of bases

of $\mathcal{L}$ (recall that a basis is a set of d points whose join has rank d). Then $J_{\mathcal{L}}$ is the set of lattice points of an integral generalized permutohedron.

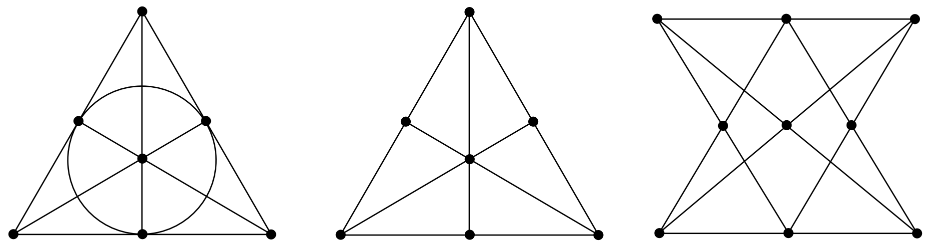


Figure 5.3: Three graphical geometries which are not graphic. The first, on the left, is the lattice of linear subspaces of $\mathbb{Z}_2^3$ (often called the Fano plane). It has 7 points, 7 lines and $\binom{7}{3} - 7 = 28$ bases and it is realizable over $\mathbb{F}$ if and only if the characteristic of $\mathbb{F}$ is 2. The second, in the middle, has 7 points, 6 lines, and 29 bases, and is realizable over $\mathbb{F}$ if and only the characteristic of $\mathbb{F}$ is not 2. The third has 9 points, 8 lines, and 84 bases, and it is not realizable over any field. See [57, Prop. 6.4.8] and [76].

6 The Kähler package

Honoring Kähler, for he "first emphasized the importance of the respective objects in topology and geometry", a *Kähler package* has, as presented by J. Huh in [46], and previously in [77], has three ingredients and three postulates. Although in the latter two references it is phrased in a 'mixed' approach, we follow the 'unmixed' formalism, so that we will have powers L^j of an operator L instead of products of operators $L_1, \ldots, L_j \in K$ (see ingredient (2) below). As mentioned in the footnote 1 to [46], this stance does not incur a loss of generality, and in any case it is sufficient for our purposes.

Ingredients

(1) A graded real vector space $A = \bigoplus_{j=0}^{d} A^j$;

(2) A convex cone K of graded linear maps $L : A^\star \to A^{\star+1}$; and

(3) A bilinear pairing $P : A^\star \times A^{d-\star} \to \mathbf{R}$ that is symmetric, $P(x, y) = P(y, x)$, and satisfies $P(x, Ly) = P(Lx, y)$ for all x, y and all $L \in K$.

Postulates. For any $j \leqslant d/2$,

Poincaré Duality: $P_j : A^j \to (A^{d-j})^*$ is an isomorphism;

Hard Lefschetz Propery: For any $L \in K$, $L^{d-2j} : A^j \to A^{d-j}$ is an isomorphism;

Hodge-Riemann Relations: The pairing (henceforth labeled HR_j)

$$A^j \times A^j \to \mathbf{R}, \quad (x, y) \mapsto (-1)^j P(x, L^{d-2j} y) = (-1)^j \langle x, y \rangle,$$

is positive definite on the kernel $A_0^j \subseteq A^j$ of L^{d-2j+1} (the *primitive part* of A^j, to borrow the name from Lefschetz theory).

We will further assume that there is a distinguished nonzero element $\mathbf{1} \in A^0$, as this will be the case in all instances we shall consider.

It is clear that $A^0_0 = A^0$, for $A^{d+1} = 0$, and HR_0 just says that $\langle x, x\rangle > 0$ for any nonzero $x \in A^0$. We define $\deg : A^d \to \mathbb{R}$ by the formula $\deg(\alpha) = P(\mathbf{1}, \alpha)$. In particular, $\deg(L^d\mathbf{1}) = P(\mathbf{1}, L^d\mathbf{1}) > 0$ by HR_0. Now $A^1_0 = \{x \in A^1 \mid L^{d-1}x = 0\}$ and for any such x we have $\langle L\mathbf{1}, x\rangle = P(L\mathbf{1}, L^{d-2}x) = P(\mathbf{1}, L^{d-1}x) = 0$, which implies that $A^1 = \langle L\mathbf{1}\rangle \perp A^1_0$ with respect to the bilinear form $\langle x, y\rangle$. For any nonzero $x \in A^1_0$, we have $\langle x, x\rangle = P(x, L^{d-2}x) < 0$, by HR_1, while $\langle L\mathbf{1}, L\mathbf{1}\rangle = P(L\mathbf{1}, L^{d-2}L\mathbf{1}) = P(\mathbf{1}, L^d\mathbf{1}) > 0$. In other words, $\langle\,,\rangle$ has Lorentzian signature $(+-\cdots-)$.

6.46. *Examples.* In a Kähler package, $A = A(X)$ depends on the objects X of some species. The examples below are described in [46]:

(1) The content of Grothendieck's standard conjectures [39] can be phrased by saying, for smooth projective algebraic varieties X over any field $\boldsymbol{\kappa}$, that the ring $\mathcal{A}^*(X)$ of algebraic classes in the (even) ℓ-adic cohomology ring $H^{2*}_\ell(X)$ is a Kähler package. These conjectures are still open, but see §7.52 for a glimpse on some positive results.

(2) X is a convex polytope and $A(X)$ its combinatorial cohomology. See [78].

(3) X a matroid and $A(X)$ one of the following:
- (a) The Chow ring of X [60];
- (b) The conormal Chow ring of X [79];
- (c) The intersection cohomology of X [71].

(4) X is an element of a Coxeter group and $A(X)$ its Soergel bimodule [80]. Additional references: [81] (a gentle introduction) and the treatise [82].

Another remarkable example is the arithmetic Kähler package, which was introduced in [83] using the arithmetic intersection theory developped in [84]. A convenient entry to this topic is the overview [85].

6.47. *Lorentzian polynomials.* The main reference for this section is [72], helpfully framed by [45] and [46]. The following quotation from [46, page 4] reveals interesting aspects of its author research temper (emphasis not in the source):

> The known proofs of the Poincaré duality, the hard Lefschetz property, and the Hodge–Riemann relations for the objects listed above [§6.46] have certain structural similarities, but there is no known way of deducing one from the others. *Could there be a Hodge-theoretic framework general enough to explain this miraculous coincidence?*
>
> A related goal is to produce a flexible analytic theory that would reflect certain basic features of the unified theory: If one postulates the existence of the satisfactory cohomology $A(X)$, what can we say about X at an elementary and numerical level? *This is a worthwhile question because, depending on X, the construction and the study of $A(X)$ might be beyond the reach of our current understanding.* A step in this direction is taken in a joint work with Petter Brändén

[namely [72]], where the difficult goal of finding $A(X)$ is replaced by an easier goal of producing a Lorentzian polynomial from X. *Such a Lorentzian polynomial can be used to settle and generate conjectures on various* X (Section 2) and, *sometimes, leads to a satisfactory theory of* $A(X)$ (Section 3).

Let H_n^d be the space of real homogeneous polynomials of degree $d \geqslant 2$ in n variables $x_1, \dots, x_n$. The set of *Lorentzian polynomials* $L_n^d \subset H_n^d$ is defined by induction on d follows. The elements of L_n^2 are specified by two conditions:

(a_2) their coefficients are non-negative; and
(b_2) their signature has at most one positive sign.

For degrees $d > 2$, the set L_n^d is defined recursively by the following conditions:

(a_d) $\partial_j f \in L_n^{d-1}$ for all $j \in [n]$, where $\partial_j = \partial/\partial_{x_j}$; and
(b_d) the set of (exponents of) monomials of f (the *support* of f) is the set of lattice points of an *integral generalized permutohedron* (cf. §5.45(1)).

One of the crucial results in [72] is that L_n^d is the closure of $\mathring{L}_n^d$, which is the set of $f \in L_n^d$ satisfying the following conditions: for $d = 2$,
($\mathring{a}_2$) the coefficients of f are *positive* real numbers; and
($\mathring{b}_2$) the signature of f has *exactly* one positive sign; and for $d > 2$,
($\mathring{a}_d$) $\partial_j f \in \mathring{L}_n^{d-1}$ for all $j \in [n]$.

In [72] it is also established that the compact set $\mathbb{P}L_n^d \subset \mathbb{P}H_n^d$ is contractible, with contractible interior $\mathbb{P}\mathring{L}_n^d$ (Theorem 2.28), and conjectured that $\mathbb{P}L_n^d$ is homeomorphic to a closed Euclidean ball, a fact that was proved by Brändén in [86]. See Figure 6.1.

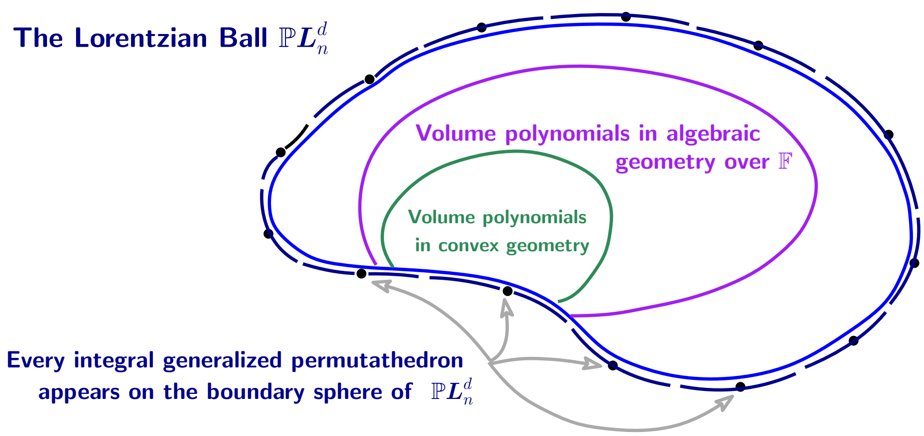


Figure 6.1: This image is a rough reproduction of the image on the slide #13 of J. Huh's Fields lecture at the ICM-22. Note the statement about the boundary sphere. Concerning volume polynomials, see §6.48 for an example in algebraic geometry, and we refer to [78] for examples in convex geometry.

6.48. *Example.* Let $D = D_1, \dots, D_n$ be nef Cartier divisors on d-dimensional irreducible projective variety X over an algebraically closed field. Consider the polynomial function

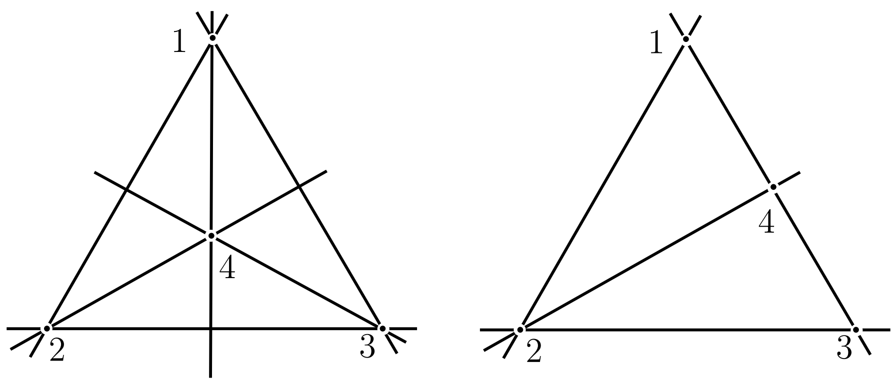
$$\mathrm{vol}_D : \mathbb{R}_+^n \to \mathbb{R}, \quad w \mapsto \frac{1}{d!}\deg(w_1 D_1 + \cdot + w_n D_n)^d,$$

where $\deg(D_1^{r_1} \cdots D_n^{r_n})$ $(r_1 + \cdots + r_n = d)$ is the degree (or $\int_X$) in the sense of [37, Definition 2.4.2]. If X admits a resolution of singularities Y and the Hodge-Riemann relations hold in degree $\leqslant 1$ (so HR_j for $j = 0, 1$) for the ring of algebraic cycles $A(Y)$, then $\mathrm{vol}_D(w)$ is Lorentzian [46, Example 7].

We defer to Section 7 the discussion of other manifestations of Lorentzian polynomials, especially significant in algebraic geometry and intersection theory.

6.49. *On the proof of the Dowling-Wilson conjecture.* Let us end this section by describing, after J. Huh's Fields Lecture [45], how the Dowling–Wilson conjecture was solved.

Given a geometric lattice $\mathcal{L}$ of rank d, consider the set $\mathcal{B}$ of its *bases*, that is, subsets of size d of $E = \mathcal{L}^1$ (the set of atoms) whose join has rank d. Then $\mathcal{B}$ is the set of *lattice points of an integral generalized permutohedron* (cf. §5.45 (2)), and the basis generating function $g = \sum_{\nu \in \mathcal{B}} w^\nu$ is a Lorentzian polynomial (cf. Fig. 6.2 for examples).



$g = w_1w_2w_3 + w_1w_2w_4 + w_1w_3w_4 + w_2w_3w_4$ $\qquad$ $g = w_1w_2w_3 + w_1w_2w_4 + w_2w_3w_4$

Figure 6.2: In these examples, the proper non-trivial flats are the points and the lines. Bases are minimal sets of points that span the lattice: $\{1,2,3\}$, $\{1,2,4\}$,{1,3,4}, $\{2,3,4\}$ for the left-hand lattice and $\{1,2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$ for the right-hand one. In each case, the corresponding generating function is displayed.

Now define $\mathbf{H}(\mathcal{L}) = \{f : \mathcal{L} \to \mathbb{Q}\} = \bigoplus_{F \in \mathcal{L}} \mathbb{Q}\delta_F$ and make it a graded $\mathbb{Q}$-algebra (the *Möbius algebra* of $\mathcal{L}$, [70, §2]) with the multiplication determined by

$$\delta_F \cdot \delta_{F'} = \begin{cases} \delta_{F \vee F'} & \text{if } r(F \vee F') = r(F) + r(F') \\ 0 & \text{otherwise.} \end{cases}$$

The *bases generating function* of $\mathcal{L}$ is $\frac{1}{d!}(\sum_{j \in E} w_j \delta_j)^d$. This suggests taking $A(\mathcal{L}) = \mathbf{H}(\mathcal{L})$; $K(\mathcal{L})$, the set of multiplications by positive linear combinations of the δ_j; and $P(\mathcal{L})$, multiplication in $\mathbf{H}(\mathcal{L})$ composed with $\mathbf{H}^d(\mathcal{L}) \simeq \mathbb{Q}$. But $\mathbf{H}(\mathcal{L})$ already fails to satisfy Poincaré duality, for $\dim \mathbf{H}^j(\mathcal{L}) = |\mathcal{L}^j|$ and in general $|\mathcal{L}^j| \neq |\mathcal{L}^{d-j}|$.

As shown in [71], the rescue from this failure came from the *intersection cohomology* of $\mathcal{L}$, $\mathbf{IH}(\mathcal{L})$, which is an indecomposable graded $\mathbf{H}(\mathcal{L})$-module endowed with a map $P : \mathbf{IH}(\mathcal{L}) \to \mathbf{IH}(\mathcal{L})^*[-d]$ that satisfies the following properties for every $j \leqslant d/2$ and every $L \in K(\mathcal{L})$:

Poincaré duality. $P : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})^*$ is an isomorphism;
Hard Lefschetz: $L^{d-2j} : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})$ is an isomorphism; and
Hodge-Riemann relations: The pairing

$$\mathbf{IH}^j(\mathcal{L}) \times \mathbf{IH}^j(\mathcal{L}) \to \mathbb{Q}, \quad (x, y) \mapsto (-1)^j P(x, L^{d-2j} y),$$

is positive definite on the kernel of L^{d-2j+1}. In addition, $\mathbf{IH}^0(\mathcal{L})$ generates a submodule isomorphic to $\mathbf{H}(\mathcal{L})$.

The construction relies on the resolution of singularities of algebraic varieties, and in particular on the 'wonderful models' in [87] (a really wonderful book).

Since the composition of $\mathbf{H}^j(\mathcal{L}) \hookrightarrow \mathbf{IH}^j(\mathcal{L})$ with the Hard-Lefschetz isomorphism $\mathbf{IH}^j(\mathcal{L}) \simeq \mathbf{IH}^{d-j}(\mathcal{L})$ is injective, $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ composed with $\mathbf{H}^{d-j} \to \mathbf{IH}^{d-j}(\mathcal{L})$ is injective (see diagram below) and consequently $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ is injective, which proves that $|\mathcal{L}^j| \leqslant |\mathcal{L}^{d-j}|$.

$$\begin{array}{ccc} \mathbf{H}^j(\mathcal{L}) & \hookrightarrow & \mathbf{IH}^j(\mathcal{L}) \\ L^{n-2j} \downarrow & & \downarrow L^{n-2j} \\ \mathbf{H}^{d-j}(\mathcal{L}) & \to & \mathbf{IH}^{d-j}(\mathcal{L}) \end{array}$$

7 Postfaces

7.50. *Additional notes on June Huh's main works.* In the preceding sections we have cited the following papers of J. Huh, often with collaborators: [63] (2012), [65] (2014, PhD thesis), [66] (2015), [70]* (2017), [60] (2018), [77]* (2018), [71]* (2020), [72] (2020), [46] (2022), [79] (2022). The items distinguished with an asterisk are cited in [88], to which we refer not only for his insightful comments on them, but also for his introduction (Appendix C) to basic *tropical geometry* notions and references for them.

The early paper [89] (2012) extends to all realizable matroids the result obtained in [63] asserting the log-concavity of the coefficients of the characteristic polynomial of matroids realizable over a field of characteristic zero. This step was a clear progress toward the proof of the Rota-Heron-Welsh conjecture. On the other hand, in [90] the authors first define morphism of matroids, then introduce the notion of *bases* of such a morphism and show that the generating function for such bases is strongly log-concave.

Okounkov's log-concavity conjecture for Littlewood–Richardson coefficients is proved for the case of Kostka numbers in the paper [91], and the fact that the Chow ring of a polymatroid yields a Kähler package is established in [92]. The paper [93] is a significant continuation of [79], while in [94] the authors introduce three equivalence relations (valuative, homological, and numerical; definitions 1.1, 1.2, 1-3) on the free abelian group $\mathrm{Mat}_r(E)$ generated by rank r matroids on E, and the main result of the paper (Theorem 1.4) is to prove that they are equivalent. The techniques used rely on the combinatorics and algebraic geometry of the *stellahedron* Π_E of E, for which two definitions are provided. One is derived from the standard permutohedron and the other is expressed as the Minkowski sum of the independence polytopes of the uniform matroids $U_{r,E}$ for $r = 0, \ldots, n$ (the bases of $U_{r,E}$ are all size r subsets of E).

Finally two very recent works that offer perspectives on J. Huh's track: [95], about the Hodge theory of matroids, and the bachelor thesis [96], a study of log-concavity in combinatorics, with new results, particularly on posets and matroids.

7.51. *Some recent references on Lorentzian polynomials and their applications.* [97] introduces the notion of dually Lorentzian polynomials and establishes that "any theory that admits a mixed Alexandrov–Fenchel inequality also admits a generalized Alexandrov–Fenchel inequality involving dually Lorentzian polynomials", and from this the authors derive such inequalities in various settings, such as for mixed discriminants, for mixed volumes, and for integrals of Kähler classes.

The paper [98] develops a theory of Lorentzian polynomials on cones and provides several characterizations of them. It introduces the notion of hereditary (multivariate) polynomials, and offers a sharp characterization of hereditary Lorentzian polynomials.

The set of real homogeneous polynomials of degree d in n variables that can be represented as the volume polynomial of n convex bodies in the Euclidean space $\mathbb{R}^n$ is considered in [99] and shown to be a subset of the Lorentzian polynomials. Moreover, a classification of the cases when the two sets are equal is provided.

The work [100] "explains connections among several, a priori unrelated, areas of mathematics: combinatorics, algebraic statistics, topology, and enumerative algebraic geometry". Such connections are mediated by the theory of Lorentzian polynomials. A surprising result is the interpretation of the number of hyperquadrics in $\mathbb{P}^n_{\mathbb{C}}$ that pass through r general points and are tangent to $\binom{n+1}{2} - r - 1$ general hyperplanes as the degree $\phi(n, r)$ of the statistical "general linear concentration model" (compactly introduced on the left column of page 4).

One important feature of [101] is to provide "the first series of examples of hard Lefschetz classes of dimension two both in algebraic geometry and analytic geometry". A key result is a "local Hodge index inequality for Lorentzian polynomials, which is the algebraic analogue of the local Alexandrov-Fenchel inequality obtained by Shenfeld-van Handel for convex polytopes". In [102], the same authors seek further applications of the theory of Lorentzian polynomials to algebraic geometry, analytic geometry and convex geometry. A guiding thread of their approach is the notion of 'reverse Khovanskii–Teissier' inequality (rKT in their shorthand), which has multiple connections with other themes. For example, the result that "any theory that admits a mixed Alexandrov-Fenchel inequality admits a rKT property" has a clear parallelism with the result in [97] (commented above), with rKT in place of generalized Alexandrov–Fenchel inequality involving dually Lorentzian polynomials.

The chromatic symmetric function of a graph was introduced and studied by Stanley in 1995. Among many other details, the paper [103] conjectures that such functions are Lorentzian. The evidence they provide is the proof of a special case (abelian Dick paths, which are presented in section 2.3).

An interesting application of the 'Lorentzian approach' is found in [104]: the authors show that the coefficients of $\Delta_L(-t)$ (where $\Delta_L(t)$ is the Alexander polynomial of a *special* alternating link L) form a log-concave sequence with no internal zeros, hence it is also a unimodal sequence. This settles (in a stronger form) a case of a long standing conjecture of Fox, which claims the said unimodality for arbitrary alternating links. Here it seems reasonable to expect that log-concavity holds also for arbitrary alternating links, a point made by Stoimenow (2005) and stressed by J. Huh in his lecture at the ICM-2018 (Rio de Janeiro).

We end this point with a few words about [105], which extends previous works of the author in several directions that bridge over to the Lorentzian realm and beacons a message that deserves a careful study by algebraic geometers interested in such discrete connections to intesection theory. The focus of its first part is on *covolume polynomials*, which are defined as "limits of positive multiples of polynomials whose coefficients are multidegrees of irreducible algebraic subvarieties of products of projective spaces". This appears to be a notion dual of volume polynomials studied in [72]. Indeed, after their normalization by the operator N introduced in [72], covolume polynomials turn out to be dual Lorentzian polynomials (in the sense of [97]), while volume polynomials of projective varieties are Lorentzian (cf. §6.48). In any case, covolume polynomials are 'sectional long-concave', a notion that plays for such polynomials the role of 'sectional ultra-log-concave' enjoyed by Lorentzian polynomials. The second part of the paper is devoted to the *Segre zeta function* $\zeta_I(t_1, \ldots, t_\ell)$, a power series in the variables $t_1, \ldots, t_\ell$ that depends on a list of multihomogeneous polynomials I of multidegree $(n_1, \ldots, n_\ell)$. This zeta function turns out to be a rational function (Theorem 3.2 of the paper) and encodes information on the Segre classes of $Z(I) \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_\ell}$, where $Z(I)$ is the subscheme defined by I. The main result is that the homogenization of the numerator of $1-\zeta_I$ (call it h_I) is a covolume polynomial, and that the homogenization of the numerator of ζ_I is also a covolume polynomial provided that the projective normal cone of $Z(I)$ is irreducible. Actually the author conjectures, on experimental evidence, that h_I is Lorentzian. The third and last part of the paper is devoted to applications to adjoint polynomials of convex polytopes and the main result is that they are covolume polynomials in the case of convex polyhedral cones contained in the non-negative orthant and sharing a face with it. As expressed by the author, it is conceivable that the result is valid without the face condition, and that the said covolume polynomials are Lorentzian afte normalization.

7.52. *Notes on the standard conjectures.* While Grothendieck's standard conjectures (cf. §4.34) provided elegant proofs of three of the Weil conjectures, they have hitherto not delivered a proof of the Riemann-Weil conjecture, which yielded, as recalled in §4.32, to a different approach by Deligne.

After presenting an enlightening historical synopsis, the paper [106] proves the standard conjecture of Hodge type for abelian fourfolds in characteristic p. The author relies on p-adic Hodge theory. The paper also proves that numerical equivalence for such fourfolds agrees with ℓ-adic cohomological equivalence for infinitely many primes ℓ, which falls short of getting the agreement for all primes $\ell \neq p$ (ℓ-independence) required by the conjecture.

A systematic approach to what is known about the standard conjectures and their relations to other conjectures, is offered in the papers [107] and [108]. All in all, they are technically quite demanding, as they depend on many previous works, and for the most part fall outside the scope of our paper. The first deals with consequences of the Lefschetz type conjecture for irreducible smooth projective varieties X over $\bar{\mathbb{F}}_p$ (p a prime number). For instance, the standard conjecture of Hodge type for *abelian varieties* in characteristic p follows from the Lefschetz conjecture for all X [107, Theorem 2], and also that the standard conjectures for X are a consequence of the *full Tate conjecture*. This conjecture, which predicts that *ℓ-adic Tate classes* are algebraic, is studied in the second paper. In particular it is proved [108, Theorem 2.8] that if the Tate and standard conjectures are true for X and one $\ell \neq p$, then they are true for X and all $\ell \neq p$. Obviously, this conditional statement does not reach to guarantee the independence of ℓ in Ancona's (unconditional) theorem.

Since the standard conjectures were tied to the theory of motives from the very beginning, we just provide a few basic references on this topic: [109], [110], [111], [112], and [113].

7.53. *Extras.* It is no secret that combinatorics has rendered many services to other branches of mathematics, as for example probability theory and theoretical physics, often through a background of probability concepts. Ready examples are found in statistical physics, for example. The book [114] is an excellent recent compendium of this topic (for a recent review of the comprehensive five-volume collection of these authors, to wich the cited volume belongs, see [115]). The Feynman graphs (or diagrams) in quantum electrodynamics, used for probability computations, is another illustration.

Now it is rather alluring to find that pure combinatorics can go a long way to provide a solid foundation for quantum field theory. This perspective is epitomized by the rather brief text [116]. Its combinatorics basis is very explicit and forceful (cf. for example [116, §5.2], on "combinatorial physical theories"). It is interesting to realize that all along you encounter topics and problems for further research.

For people interested in mathematical foundations of quantum field theory, combinatorial or otherwise, perhaps also in research problems stemming from those foundations, the great treatise [117] should be very welcome. Its author was awarded the 2024 Abel Prize "for his groundbreaking contributions to probability theory and functional analysis, with outstanding applications in mathematical physics and statistics" and the following quote summarizes its aims and scope (our emphasis):

> QFT is one of the great achievements of physics, of *profound interest to mathematicians*. Most pedagogical texts on QFT are geared toward budding professional physicists, however, whereas mathematical accounts are abstract and difficult to relate to the physics. *This book bridges the gap*. While the treatment is rigorous whenever possible, the accent is not on formality but on *explaining what the physicists do and why, using precise mathematical language*. In particular, it *covers in detail the mysterious procedure of renormalization*. Written for readers with a mathematical background but *no previous knowledge of physics* and largely self-contained, it *presents both basic physical ideas from special relativity and quantum mechanics* and *advanced mathematical concepts in complete detail*. It will be of interest to mathematicians wanting to learn about QFT and, with nearly 300 exercises, also to physics students seeking greater rigor than they typically find in their courses."

As final note, let us mention a return to the continuous from the discrete in the form of combinatorial approaches to algorithmic learning, like *graph learning* in general and *manifold learning* in particular. There is a growing number of contributions to these topics in recent times, as for example [118] (and the references mentioned there) and the surveys [119] and [120]. Connected to this, let me point to the announcement [121].

References

[1] J. M. Lee, *Smooth manifolds*, vol. 218 of *Graduate Texts in Mathematics*. Springer, 2013.

[2] W. Fulton, *Algebraic topology: a first course*, vol. 153 of *Graduate Texts in Mathematics*. Springer, 2013.

[3] M. F. Atiyah and F. Hirzebruch, "Analytic cycles on complex manifolds," *Topology*, vol. 1, pp. 25–46, 1961.

[4] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, vol. 94 of *Graduate Texts in Mathematics*. Springer, 1983.

[5] A. Weil, "Sur la théorie des formes différentielles attachées à une variété analytique complexe," *Comment. Math. Helv*, vol. 20, pp. 110–116, 1947. pdf.

[6] A. Weil, *Introduction à l'étude des variétés kählériennes*. Hermann, 1958. Nouvelle édition corrigée.

[7] E. Cattani, "Introduction to kähler manifolds," 2010. Abdus Salam ICTP, Summer School and Conference on Hodge Theory and Related Topics, 14.06–2.07, 2010.

[8] C. Voisin, "Algebraic geometry versus Kähler geometry," *Milan journal of mathematics*, vol. 78, pp. 85–116, 2010. pdf.

[9] P. Griffiths and J. Harris, *Principles of algebraic geometry.* John Wiley & Sons, 1978.

[10] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, vol. 76 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 2002.

[11] C. Voisin, *Hodge Theory and Complex Algebraic Geometry II*, vol. 77 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 2003.

[12] D. Huybrechts, *Complex geometry: an introduction.* Universitext, Springer, 2005.

[13] W.-L. Chow, "On compact complex analytic varieties," *American Journal of Mathematics*, vol. 71, no. 4, pp. 893–914, 1949.

[14] K. Kodaira, "On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)," *Ann. of Math.(2)*, vol. 60, no. 1, pp. 28–48, 1954.

[15] W. V. D. Hodge, "The topological invariants of algebraic varieties," in *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., Aug. 30-Sept. 6, 1950*, vol. 1, pp. 182–192, American Mathematical Society, 1952.

[16] P. Deligne, "The Hodge conjecture," in *The millennium prize problems*, pp. 45–53, American Mathematical Society, 2006. Claymath pdf.

[17] J. A. Carlson, A. Jaffe, and A. Wiles, *The millennium prize problems.* American Mathematical Soc., 2006. Claymath pdf.

[18] B. Totaro, "Why believe the Hodge Conjecture?," 2012. Post 18.3.2012.

[19] K. Kodaira and D. C. Spencer, "Divisor class groups on algebraic varieties," *Proceedings of the National Academy of Sciences*, vol. 39, no. 8, pp. 872–877, 1953.

[20] S. Zucker, "The Hodge conjecture for cubic fourfolds," *Compositio Mathematica*, vol. 34, no. 2, pp. 199–209, 1977. Numdam pdf.

[21] A. Conte and J. P. Murre, "The Hodge conjecture for fourfolds admitting a covering by rational curves," *Mathematische Annalen*, vol. 238, pp. 79–88, 1978.

[22] I. Biswas and K. H. Paranjape, "The Hodge conjecture for general Prym varieties," *Journal of Algebraic Geometry*, vol. 11, no. 1, 2002. arXiv pdf.

[23] R. P. Stanley, *Enumerative Combinatorics* (Volume 1, second edition; xiv+626p). Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2012. First edition published by Wadsworth, 1986; first Cambridge edition, 1997.

[24] R. P. Stanley, *Enumerative Combinatorics* (Volume 2), vol. 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1999. xii+581p.

[25] A. Weil, "Numbers of solutions of equations in finite fields," *Bulletin AMS*, vol. 55, no. 5, pp. 497–508, 1949.

[26] R. Hartshorne, *Algebraic geometry*, vol. 52 of *Graduate Texts in Mathematics.* Springer-Verlag, 1977. Corrected third printing 1983.

[27] M. Goresky, "Commentary on "Numbers of solutions of equations in finite fields" by André Weil," *Bulletin of the AMS*, vol. 55, no. 3, 2018. BAMS pdf.

[28] M. Deuring *et al.*, "Selected mathematical reviews related to the paper [25]," *Bulletin of the AMS*, vol. 55, no. 3, pp. 381–397, 2018. Reviews by M. Deuring, N. M. Katz, S. Bloch, O. F. G. Schilling (2), H. Popp, T. Oda, and E. M. Friedlander of works by B. Dwork, P. Deligne (2), A. Weil (2), S. L. Kleiman, A. Grothendieck, and A. Suslin & V. Voevodsky. AMS pdf.

[29] S. Molina, N. Sayols, and S. Xambó-Descamps, "A bootstrap for the number of $\mathbb{F}_{q^m}$-rational points on a curve over $\mathbb{F}_q$," 2017. `https://arxiv.org/pdf/1704.04661`.

[30] B. Dwork, "On the rationality of the zeta function of an algebraic variety," *American Journal of Mathematics*, vol. 82, no. 3, pp. 631–648, 1960.

[31] J. S. Milne, "Lectures on étale cohomology," 2013. Author's pdf.

[32] P. Deligne, "La conjecture de Weil. I," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, vol. 43, pp. 273–307, 1974. Numdam pdf.

[33] P. Deligne, "La conjecture de Weil: II," *Publications Mathématiques de l'IHÉS*, vol. 52, pp. 137–252, 1980. Numdam pdf.

[34] J.-P. Serre, "Analogues Kählériens de Certaines Conjectures de Weil," *Annals of Mathematics*, pp. 392–394, 1960. Reprinted in Collected Papers, vol. 2, paper 45, Springer, 1986.

[35] S. Xambó-Descamps, "Francesco Severi and the principle of conservation of number," *Rendiconti del Circolo Matematico di Palermo, Serie II*, vol. 36, pp. 255–277, 1994. This issue of the Rendiconti collects papers presented at the conference *Algebra e Geometria* (1860-1940): *Il contibuto italiano* held in Cortona, 4-8 May, 1992.

[36] H. Schubert, *Kalkül der abzählenden Geometrie*. Springer-Verlag, 1979. Reprint, with an introduction by S. L. Kleiman, from the original Teubner edition of 1879.

[37] W. Fulton, *Intersection theory*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge*. Springer-Verlag, 1984. Second edition: Springer, 1998.

[38] S. Xambó-Descamps, *Using intersection theory*. No. 7 in Aportaciones Matemáticas, Nivel Avanzado, Sociedad Matemática Mexicana, 1996. Now accessible as an UPC-Commons pdf.

[39] A. Grothendieck, "Standard conjectures on algebraic cycles," 1969. In [122], 193-199; in the pdf, 203-211.

[40] H. Hironaka, "Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II," *Annals of Mathematics*, vol. 79, pp. 109–326, 1964.

[41] H. Hironaka, "Resolution of singularities in positive characteristics," 2017. Harvard pdf.

[42] S. L. Kleiman, "Algebraic cycles and the Weil conjectures," in *Dix exposés sur la cohomologie des schémas*, vol. 3 of *Adv. Stud. Pure Math.*, pp. 359–386, North Holland, Amsterdam, 1968.

[43] S. L. Kleiman, "The standard conjectures," in *Proceedings of the Summer Research Conference on Motives* (Seattle, Washington, July 20 – August 2, 1991) (U. Jannsen, S. L. Kleiman, and J.-P. Serre, eds.), vol. 55 [I and II] of *Proceedings of Sumposia in Pure Mathematics*, pp. I.3–20, American Mathematical Society, 1994. This paper introduces the formalism of Grothendieck's two "standard conjectures" [from the Abstract].

[44] G. Kalai, "The work of June Huh," in *Proceedings of the ICM 2022*, IMU, 2022. pdf.

[45] J. Huh, "Combinatorics and Hodge theory," 2022. ICM Fields Lecture, 6 July 2022 Video (2:34–3:37).

[46] J. Huh, "Combinatorics and Hodge theory," in *Proceedings of the ICM 2022*, IMU, 2022. pdf.

[47] H. Whitney, "Non-separable and planar graphs," *Transactions of the AMS*, vol. 34, pp. 339–362, 1932.

[48] W. T. Tutte, *Graph theory*, vol. 21. Cambridge University Press, 1984. xxii+333p. In 2001 it was published in softcover.

[49] B. Bollobás, *Graph theory: an introductory course*, vol. 63 of *Graduate Texts in Mathematics*. Springer, 1998. xiv+394p.

[50] J. M. Harris, J. L. Hirst, and M. J. Mossinghoff, *Combinatorics and graph theory* (2nd). UTM, Springer, 2008. xvi+381p.

[51] J. A. Bondy and U. S. R. Murty, *Graph theory*, vol. 244 of *Graduate Texts in Mathematics*. Springer, 2008. xii+651 p.

[52] S. K. Yadav, *Advanced Graph Theory*. Springer, 2023.

[53] H. Whitney, "On the abstract properties of linear dependence," *American Journal of Mathematics*, pp. 509–533, 1935. Included in "Hassler Whitney Collected Papers" (2 volumes), Birkhäuser, 1992.

[54] D. J. Welsh, "Matroids: fundamental concepts," 1995. In [123], Volume 1, 481-526.

[55] W. T. Tutte, "Matroids and graphs," *Transactions of the American Mathematical Society*, vol. 90, no. 3, pp. 527–552, 1959.

[56] A. M. Hobbs and J. G. Oxley, "William T. Tutte (1917-2002)," *Notices AMS*, vol. 51, no. 3, pp. 320–330, 2004.

[57] J. G. Oxley, *Matroid theory* (2nd edition), vol. 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, 2011. xiv+684p. The 1st edition was published in 1992.

[58] D. J. Welsh, *Matroid theory*. Academic Press, 1976. xii+433p. Republished by Dover in 2010.

[59] J. Oxley, "What is a matroid?," *Cubo Mat. Educ.*, vol. 5, pp. 179–218, 2003. Revised and extended version pdf.

[60] K. Adiprasito, J. Huh, and E. Katz, "Hodge theory for combinatorial geometries," *Annals of Mathematics*, vol. 188, no. 2, pp. 381–452, 2018. arXiv pdf, v2 2018.

[61] R. P. Stanley, "Log-concave and unimodal sequences in algebra, combinatorics, and geometry," *Ann. New York Acad. Sci*, vol. 576, no. 1, pp. 500–535, 1989.

[62] F. Brenti *et al.*, "Stanley's work on unimodality," *The Mathematical Legacy of Richard P. Stanley*, pp. 119–130, 2016.

[63] J. Huh, "Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs," *Journal of the American Mathematical Society*, vol. 25, no. 3, pp. 907–927, 2012. JAMS pdf.

[64] R. P. Stanley, *Algebraic combinatorics. Walks, Trees, Tableaux, and More.* Undergraduate Texts in Mathematics, Springer, 2013. xii+223p.

[65] J. Huh, *Rota's conjecture and positivity of algebraic cycles in permutohedral varieties.* PhD thesis, University of Michigan, 2014.

[66] J. Huh, "h-Vectors of matroids and logarithmic concavity," *Advances in mathematics*, vol. 270, pp. 49–59, 2015. arXiv pdf v2 2012.

[67] M. Goresky and R. MacPherson, "Intersection homology theory," *Topology*, vol. 19, no. 2, pp. 135–162, 1980.

[68] M. Goresky and R. MacPherson, "Intersection homology II," *Inventiones Mathematicae*, vol. 71, pp. 77–129, 1983.

[69] S. L. Kleiman, "The development of intersection homology theory," *Pure Appl. Math. Q*, vol. 3, no. 1, pp. 225–282, 2007.

[70] J. Huh and B. Wang, "Enumeration of points, lines, planes, etc.," *Acta Mathematica*, vol. 218, 2017.

[71] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang, "Singular Hodge theory for combinatorial geometries," 2020. arXiv pdf.

[72] P. Brändén and J. Huh, "Lorentzian polynomials," *Annals of Mathematics*, vol. 192, no. 3, pp. 821–891, 2020.

[73] N. Anari, K. Liu, S. O. Gharan, and C. Vinzant, "Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids," 2018. arXiv pdf. Published in *Proceedings of the American Mathematical Society* **152**/05 (2024) 1969-1981.

[74] A. Postnikov, "Permutohedra, associahedra, and beyond," *International Mathematics Research Notices*, vol. 2009, no. 6, pp. 1026–1106, 2009.

[75] J. S. Doker, *Geometry of generalized permutohedra.* PhD thesis, UC Berkeley, 2011. pdf.

[76] A. W. Ingleton, "Representation of matroids," in *Combinatorial mathematics and its applications* (D. J. A. Welsh, ed.), pp. 149–167, Academic Press, 1971.

[77] J. Huh, "Combinatorial applications of the hodge–riemann relations," in *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pp. 3093–3111, World Scientific, 2018. arXiv pdf.

[78] K. Karu, "Hard Lefschetz theorem for nonrational polytopes," *Inventiones Mathematicae*, vol. 157, no. 2, pp. 419–447, 2004. arXiv pdf (2002, v4).

[79] F. Ardila, G. Denham, and J. Huh, "Lagrangian geometry of matroids," *Journal of the American Mathematical Society*, 2022. JAMS pdf.

[80] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, "The Hodge theory of Soergel bimodules," *Annals of Mathematics*, pp. 1089–1136, 2014.

[81] N. Libedinsky, "Gentle introduction to Soergel bimodules I: the basics," *Sao Paulo Journal of Mathematical Sciences*, vol. 13, pp. 499–538, 2019.

[82] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, *Introduction to Soergel bimodules*, vol. 5 of *RSME Springer Series.* Springer, 2020. xxv+588p.

[83] H. Gillet and C. Soulé, "Arithmetic analogs of the standard conjectures," in *Motives* (U. Jannsen, S. Kleiman, and J.-P. Serre, eds.), vol. 55, Part I of *Proc. Sympos. Pure Math*, pp. 141–188, American Mathematical Society, 1994. Proceeding of the conference held in the University of Washington, Seattle, WA, 1991.

[84] H. Gillet and C. Soulé, "Arithmetic intersection theory," *Publications Mathématiques de l'IHÉS*, vol. 72, pp. 93–174, 1990.

[85] A. Iyengar, "The classical and arithmetic standard conjectures," 2017. Supervised by C. Soulé: pdf.

[86] P. Brändén, "Spaces of Lorentzian and real stable polynomials are Euclidean balls," in *Forum of Mathematics, Sigma*, vol. 9, p. e73, Cambridge University Press, 2021. Σ/CUP pdf.

[87] C. De Concini and C. Procesi, "Wonderful models of subspace arrangements," *Selecta Mathematica*, vol. 1, pp. 459–494, 1995.

[88] A. Okounkov, "Combinatorial geometry takes the lead," in *Proceedings ICM-2022*, vol. 1, 2022. arXiv pdf, 43p.

[89] J. Huh and E. Katz, "Log-concavity of characteristic polynomials and the Bergman fan of matroids," *Mathematische Annalen*, vol. 354, pp. 1103–1116, 2012. arXiv pdf.

[90] C. Eur and J. Huh, "Logarithmic concavity for morphisms of matroids," *Advances in Mathematics*, vol. 367, p. 107094, 2020. arXiv pdf.

[91] J. Huh, J. Matherne, K. Mészáros, and A. St Dizier, "Logarithmic concavity of Schur and related polynomials," *Transactions of the American Mathematical Society*, vol. 375, no. 06, pp. 4411–4427, 2022. TAMS pdf.

[92] C. Crowley, J. Huh, M. Larson, C. Simpson, and B. Wang, "The Bergman fan of a polymatroid," 2022. arXiv pdf, v4 2024.

[93] F. Ardila, G. Denham, and J. Huh, "Lagrangian combinatorics of matroids," *Algebraic Combinatorics*, vol. 6, no. 2, 2023.

[94] C. Eur, J. Huh, and M. Larson, "Stellahedral geometry of matroids," in *Forum of Mathematics, Pi*, vol. 11:e24, pp. 1–48, Cambridge University Press, 2023. Cambridge pdf.

[95] C. Eur, "Essence of independence: Hodge theory of matroids since June Huh," *Bulletin of the American Mathematical Society*, vol. 61, no. 1, pp. 73–102, 2024. BAMS pdf.

[96] A. Yan, "Log-concavity in combinatorics," 2024. arXiv pdf.

[97] J. Ross, H. Süss, and T. Wannerer, "Dually Lorentzian polynomials," 2023. arXiv pdf.

[98] P. Brändén and J. Leake, "Lorentzian polynomials on cones," 2023. arXiv pdf.

[99] A. Menges, "Comparing the sets of volume polynomials and Lorentzian Polynomials," 2023. arXiv pdf.

[100] M. Michałek, "Enumerative Geometry Meets Statistics, Combinatorics, and Topology," *Notices of the AMS*, vol. 70, no. 4, 2023. NAMS pdf.

[101] J. Hu and J. Xiao, "Numerical characterization of the hard Lefschetz classes of dimension two," 2023. arXiv pdf.

[102] J. Hu and J. Xiao, "Intersection theoretic inequalities via Lorentzian polynomials," *Mathematische Annalen*, pp. 1–38, 2024. arXiv pdf.

[103] J. P. Matherne, A. H. Morales, and J. Selover, "The Newton polytope and Lorentzian property of chromatic symmetric functions," *Selecta Mathematica*, vol. 30, no. 3, pp. 1–35, 2024. arXiv pdf v3, 2023.

[104] E. S. Hafner, K. Mészáros, and A. Vidinas, "Log-concavity of the Alexander polynomial," *International Mathematics Research Notices*, p. rnae058, 2024. arXiv pdf.

[105] P. Aluffi, "Lorentzian polynomials, Segre classes, and adjoint polynomials of convex polyhedral cones," *Advances in Mathematics*, vol. 437, p. 109440, 2024. arXiv pdf.

[106] G. Ancona, "Standard conjectures for abelian fourfolds," *Inventiones mathematicae*, vol. 223, pp. 149–212, 2021.

[107] J. S. Milne, "Grothendieck's standard conjecture of Lefschetz type over finite fields (v1.1)," 2022. Author's pdf.

[108] J. S. Milne, "The tate and standard conjectures for certain abelian varieties," 2022. arXiv pdf.

[109] S. del Baño Rollin and V. Navarro, "On the motive of a quotient variety," *Collectanea Mathematica*, vol. 49, no. 2-3, pp. 203–ç–226, 1998.

[110] P. Deligne, "Hodge cycles on abelian varieties," 2003. JS Milne pdf. A TeXed copy of *Hodge cycles on abelian varieties* (the notes of most of the seminar "Périodes des Intégrales Abéliennes" given by P. Deligne at I.H.E.S., 1978–79; p. 9–100 of Deligne et al. 1982), and *Motives for absolute Hodge cycles* (§5 of "Tannakian categories", ibid. p. 196–220), somewhat revised and updated.

[111] Y. André, "Motifs de dimension finie," *Séminaire Bourbaki*, vol. 46, pp. 2003–2004, 2004. Numdam pdf.

[112] J. S. Milne, "Motives—grothendieck's dream," *Open problems and surveys of contemporary mathematics*, vol. 6, pp. 325–342, 2013. Citeseer pdf.

[113] L. Barbieri-Viale and B. Kahn, "Universal Weil cohomology," 2024. arXiv pdf.

[114] K. S. Thorne and R. D. Blandford, *Statistical Physics: Volume 1 of Modern Classical Physics*. Princeton University Press, 2021.

[115] S. Xambó Descamps, "Review of "Modern Classical Physics", by Kip. S. Thorne and Roger D. Blandford," 2023. p38 of IMTech's NL06, pdf.

[116] K. Yeats, *A combinatorial perspective on quantum field theory*. Briefs in Mathematical Physics, Springer, 2017. U. Waterloo pdf.

[117] M. Talagrand, *What is a Quantum Field Theory?: A First Introduction for Mathematicians*. Cambridge University Press, 2022.

[118] S.-T. Yau, "Manifold fitting, an invitation to machine learning –a mathematician's view," 2024. Lecture at the 2024 b-M2L seminar html, organized by Roberto Rubio. slides pdf.

[119] Z. Wu, S. Pan, F. Chen, G. Long, C. Zhang, and S. Y. Philip, "A comprehensive survey on graph neural networks," *IEEE transactions on neural networks and learning systems*, vol. 32, no. 1, pp. 4–24, 2020.

[120] F. Xia, K. Sun, S. Yu, A. Aziz, L. Wan, S. Pan, and H. Liu, "Graph learning: A survey," *IEEE Transactions on Artificial Intelligence*, vol. 2, no. 2, pp. 109–127, 2021.

[121] S. Xambó Descamps (ed), *Novel deep learning methodologies in Industrial and Applied Mathematics.* Springer, 2024. Proceedings of the ICIAM2023 Mini Symposium with the same title, to appear in the Springer ICIAM-2023 series, and featuring chapters by Marta Barroso, Taco Cohen, Silvia Franchini, Eduardo U. Moya, Steven van Vaerenbergh, Yolanda Vidal, and Sebastian Xambó (these are the corresponding authors ordered by family name).

[122] M. S. Narasimhan, ed., *Algebraic Geometry*, Tata Institute Studies in Mathematics, Oxford University Press & Tata Institute of Fundamental Research, 1969.

[123] R. L. Graham, M. Grötschel, L. Lovász, and L. Lovász, eds., *Handbook of Combinatorics* (2 volumes). Elsevier, 1995. Volume 1: xii+1018; Volume 2: xii+(1119-2198); they share the author and subject indices, 45 and 54 pages, respectively.