2019 Interdisciplinary Colloquium
in Topology and its Applications

UNIVERSIDAD DE VIGO

A Eusebio Corbacho Rosas y Elena Martín Peinador

A Light Dream

S. Xambó-Descamps

UPC

19-22/06/2019

Jornadas Sage y Python en Jarandilla de la Vera, 31-05-2014.

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Jornada de homenaje a José María Montesinos Amilibia en la FM de la UCM, 8 de septiembre de 2015.

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- The science of light and its giants
- Maxwell's electromagnetism (EM)
- Synopsis of geometric algebra (GA)
- EM with GA
- The Dirac equation after Hestenes
- Backpack for a voyage

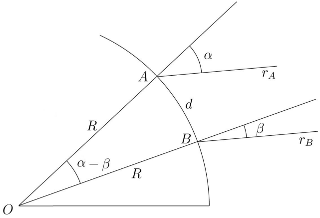
Light and its giants

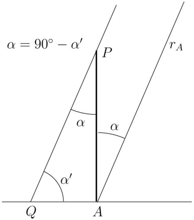
EUCLID of Alexandria (−325, −265). The *Elements*. Ray optics (or geometric optics).

ARCHIMEDES of Syracuse (−287,−212) Approximations of π. Area of a circle. Area and volume of a sphere. Hydrostatics. Law of the lever. Theory and applications of mirrors.

ERATOSTHENES of Cyrene (−276, −195). Measured the size of the Earth using geometry and ray optics and the distance Earth-Sun.

PTOLEMY, Claudius (85 – 165) Treatise on *Optics* (on reflection, refraction and color).

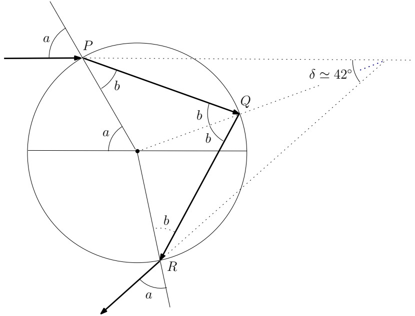


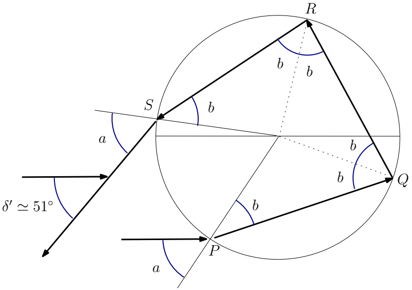


SAHL, Ibn (940 – 1000). Discovered the law of refraction in 984.

ALBERTUS MAGNUS (1193 – 1280). The first to propose that the rainbow was produced by the interaction of light with the spherical rain drops.

THEODORIC of Freiberg (1250 – 1310). First to provide a correct explanation of the rainbow by ray geometry.



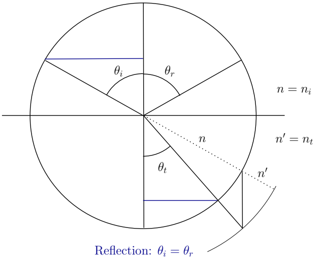


GALILEO Galilei (1564 – 1642). Developed the refraction telescope.

KEPLER, Johannes (1571 – 1630). Improved refracting telescope. Observer of the 1604 supernova and analyzed its scientific significance.

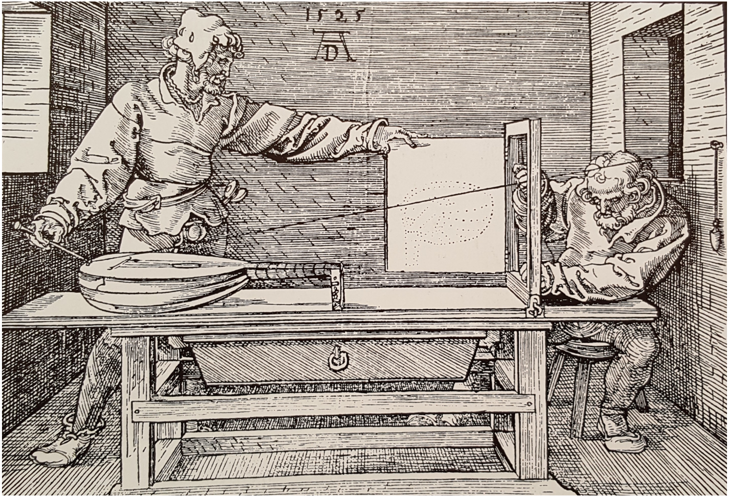
SNELLIUS, Willebrord (1580 – 1626). Born Willebrord SNEL VAN ROYEN, known as SNELL in the English-speaking world. Experimental studies of the reflection and refraction laws. (1621, unpublished; cf. SAHL).

Snell's laws



Reflection: $\theta_i = \theta_r$
Refraction: $\sin\theta_t = n\sin(\theta_i)/n'$

DESARGUES, Girard (1591, 1661). Founder of projective geometry (the geometry derived of the visual intuition of space, in which perspective plays a fundamental role).



DESCARTES, René (1596,1650). *Discours de la méthode* (1637). (M) Les Météores, (D) La Dioptrique, and (G) La Géométrie. D is Descartes' greatest contribution to optics, and contains the first publication of the law of refraction.

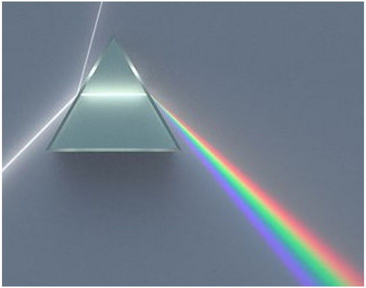
FERMAT, Pierre de (1607–1665). Optics (Fermat's principle of propagation in least time, or least optical path).

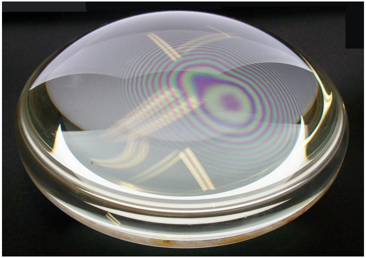
Remark A homogeneous optical medium is characterized by its *refractive index* $n \geq 1$. It is related to the speed of light c in the medium by the formula $n = c_0/c$, where c_0 is the speed of light in free space. Thus the time taken by a light ray to travel a distance d in the medium is $d/c = nd/c_0$. The quantity nd is called the *optical path length* of d. In general, the optical length of path γ is $\int_\gamma nds$.

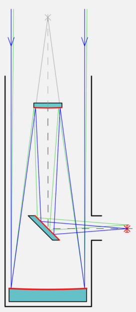
HUYGENS, Christiaan (1629, 1695). Wave theory of light. Improved telescope.

NEWTON, Isaac (1642 – 1727). Reflecting telescope, spectrum of white light, light interference (Newton's rings). Corpuscular theory of light.

RØMER, Ole Christensen (1644–1710). First measurement of the speed of light (1676). N



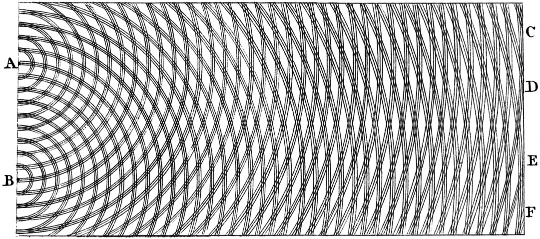


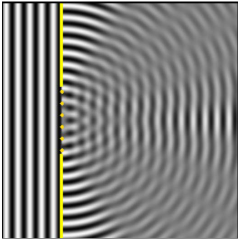


EULER, Leonhard (1707 – 1783). Author of *Opticks*, in which he disagrees with Newton's corpuscular theory and favored Huygens wave theory.

FOURIER, Jean-Baptiste Joseph (1768–1830).*Fourier series*. Applications to heat transfer and vibrations (sound and music, for example). Gave rise to *Fourier optics*.

YOUNG, Thomas (1773–1829). In optics, he contributed major steps to establish the wave nature of light. Double-slit experiments.





FRESNEL, Augustin-Jean (1788–1827). His researches upheld and greatly advanced the theory of light as a *transverse wave* (in Young's account it was assumed to be a longitudinal wave). He provided quantitative explanations for the rectilinear propagation of light, its diffraction by straight edges, the significance of wave phase, the nature of polarization, the transmission and reflection coefficients at the interface between two transparent isotropic media, et cetera.

$$r_\perp = \frac{n_i \cos(\theta_i) - n_t \cos(\theta_t)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \quad r_\parallel = \frac{n_i \cos(\theta_t) - n_t \cos(\theta_i)}{n_i \cos(\theta_t) + n_t \cos(\theta_i)}$$

$$t_\perp = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_i) + n_t \cos(\theta_t)} \quad t_\parallel = \frac{2n_i \cos(\theta_i)}{n_i \cos(\theta_t) + n_t \cos(\theta_i)}$$

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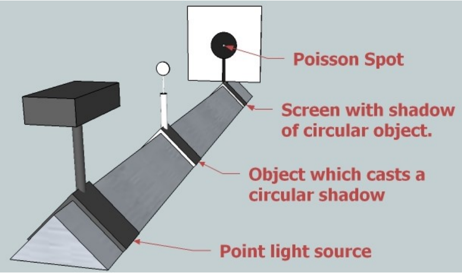
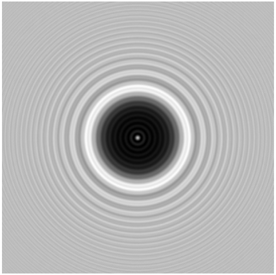
WOLLASTON, William Hyde (1766–1828). Initiator of *spectroscopy*: Earliest observations of the dark lines in the solar spectrum (1802).

ARAGO, François (1786–1853). He demonstrated experimentally the reality of *Poisson's spot* (also known as *Arago spot* or *Fresnel bright spot*). This was a major support of the wave theory of light. Arago discovered rotary polarization, invented the first polarization filter (1812), and advanced the ideas used by FIZEAU and FOUCAULT to determine the speed of light by terrestrial means only.





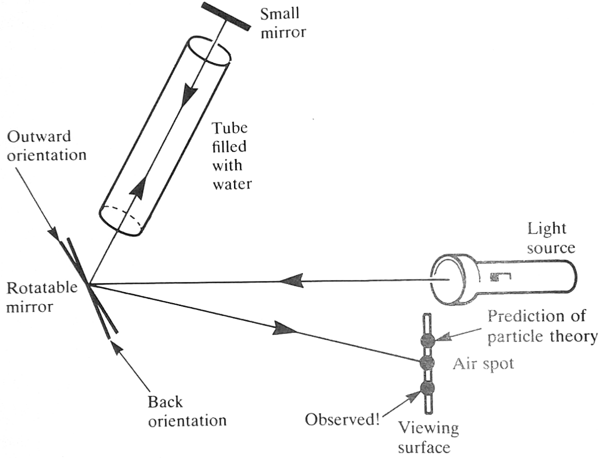
FRAUNHOFER, Joseph (1787–1826): double line of sodium; first wavelength determinations using diffraction gratings.

FARADAY, Michael (1791 – 1867). In optics, he showed that magnetism and light were related phenomena.

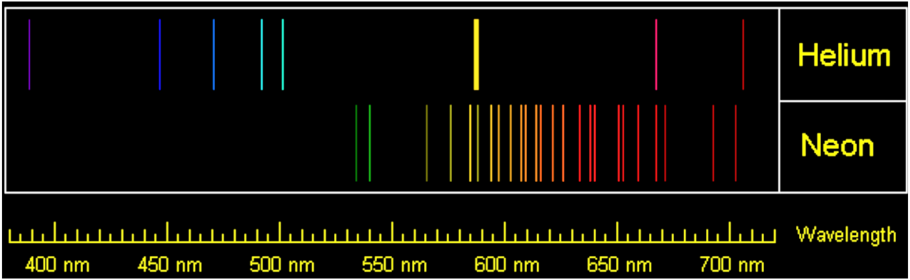
HAMILTON, Willian Rowan (1805 – 1865). His approch to classical mechanics, including his *principle of least action*, has been very influencial in physics and mathematics. Similarly with his views of optics, that meant, through his "principal function" (later called the Hamilton-Jacobi function), a unified treatment of optics and mechanics. Discoverer of the quaternion algebra and its significance for geometry.

FOUCAULT, Jean Bernard Léon (1819 – 1868). Made precise measurements of the speed of light in air and in transparent media.

FIZEAU, Armand Hippolyte Louis (1819–1896). Contributed significantly to measure the speed of light.



Kirchhoff, Gustav Robert (1824–1887) and Bunsen, Robert Wilhelm (1811–1899): each kind of atom has its own signature in a characteristic array of spectral lines.



Thomson, William (1824-1907), Lord Kelvin. Discovered the electron (1897). Could atoms be vortex knots of ether? *Molecular dynamics and the wave theory of light* (1884 Master class at Johns Hopkins, with Michelson and Morley as students).

MAXWELL, James Clerck (1831–1879). His theory of electromagnetism (1865), summarized by his celebrated equations, unified electricity, magnetism and optics. He predicted electromagnetic waves (electromagnetic spectrum), described their physical nature, and obtained that they propagated at the speed of light. He invented and popularized the color wheels. Einstein described his work as "the most profound and most fruitful that physics has experienced since the time of Newton" (1931).

CLIFFORD, William Kingdom (1845-1879). Developed *geometric algebra* as a synthesis of Hamilton's quaternions and Grassmann's extension theory (exterior algebra).

RÖNTGEN, Wilhelm Conrad (1845-1923). Discovered *X-rays* (1895).

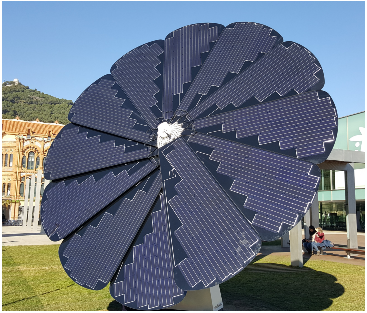
MICHELSON, Albert Abraham (1852 – 1931). Measured the speed of light with unprecedented precision (by interpherometry) and lead the crucial Michelson-Morley experiment showing that the movement of the Earth has no observable effect on the speed of light.

LORENTZ, Hendrik Antoon (1853–1928). The *Lorentz transformations* (thus named by Poincaré) played a fundamental role in the special theory of relativity. Lorentz provided an explanation of the *Zeeman effect* (splitting of the spectral lines in several components caused by a magnetic field). This effect has many theoretical and practical applications.

POINCARÉ, Jules Henri (1854–1912). Was the closest precursor of relativity theory. *Leçons sur la théorie mathématique de la lumière* (I, 1899; II, 1892) The *Poincaré sphere* parameterizes the polarization states of monochromatic light.

HERTZ, Heinrich Rudolf (1857-1894). *Experimental confirmation of Maxwell's electro-magnetic waves*.

PLANCK, Max (1858–1947). $E = h\nu = hc/\lambda$ giving the "quantum" of energy carried by an electromagnetic radiation of frequency ν (particle-like nature of radiation), which was the basis for his derivation of the black-body radiation law.



MINKOWSKI, Hermann (1864 – 1909). His approach to Einstein's special relativity initiated the study of *spacetime geometry* (Minkowski spacetime).

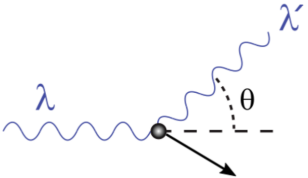
EINSTEIN, Albert (1879–1955). Special and general relativity theories (1905 and 1016). Quantum physics (idea of the photon, although not the name; explanation of the photoelectric effect). Founder of modern physical cosmology. Predicted many effects that have been observed hitherto, including $E = mc^2$, gravitational waves, black holes, the atomic stimulated emission of light (basis of masers and lasers).

BOHR, Niels Henrik David (1885–1962): explanation of atomic spectra (role of outermost electrons).

DE BROGLIE, Louis (1892-1987). Wavy character of particles:

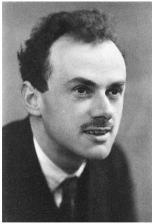
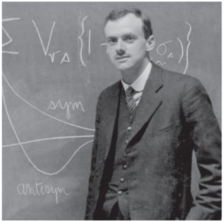
$$\lambda = h/p = h/mv = h/\gamma m_0 v, \quad \gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}.$$

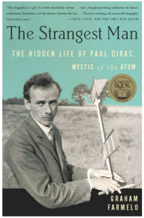
COMPTON, Arthur (1892–1962). Discoverer of the *Compton effect*, an experimental demonstration of the corpuscular aspect of electromagnetic radiation predicted by Einstein. This provided compelling evidence for the wave-particle duality of electromagnetic fields.



$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$

DIRAC, Paul (1902–1984). With *Dirac's equation*, he is an early founder of quantum electrodynamics. Predicted the existence of antimatter.



$i\gamma \cdot \partial\psi = m\psi$ $\qquad$ $ge = n\hbar$ (n integer)

FEYNMAN, Richard (1918–1988). One of the founders of quantum electrodynamics, QED (Feynman diagrams). Author of the popular *Feynman lectures on physics*.

$$e^- + e^+ \leftrightarrow \gamma + \gamma, \qquad \gamma \to \mathrm{e}^- + \mathrm{e}^+$$

Maxwell's electromagnetism

ε_0 *electric constant* 8.854 187 8128(13)$\times 10^{-12}$ (F/m)

μ_0 magnetic constant $4\pi \times 1.00000000082(20)\,10^{-7}$ (H/m)

ρ electric charge density (C/m^3)

$\boldsymbol{J} = (J_x, J_y, J_z)$ electric current density (A/m^2)

$\boldsymbol{E} = (E_x, E_y, E_z)$ electric field (V/m)

$\boldsymbol{B} = (B_x, B_y, B_z)$ magnetic induction (T)

$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B},$

$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\varepsilon_0\partial_t \boldsymbol{E}$

$\partial_t\rho + \nabla \cdot \boldsymbol{J} = 0$ (continuity equation)

$\nabla^2 \boldsymbol{E} - \mu_0\varepsilon_0\partial_t^2 \boldsymbol{E} = 0, \qquad \nabla^2 \boldsymbol{B} - \mu_0\varepsilon_0\partial_t^2 \boldsymbol{B} = 0$ (wave equations)

Wave velocity: $1/\sqrt{\mu_0\varepsilon_0} = c$ (speed of light!)

Combine $\boldsymbol{E}$ and $\boldsymbol{B}$ into the *electromagnetic field* $F = \widehat{\boldsymbol{E}}\,dt + \widetilde{\boldsymbol{B}}$, where

$$\widehat{\boldsymbol{E}} = E_x dx + E_y dy + E_z dz$$
$$\widetilde{\boldsymbol{B}} = B_x dydz + B_y dzdx + B_z dxdy$$

and combine ρ and $\boldsymbol{J}$ into the *electromagnetic current* $J = -\rho dt + \widehat{\boldsymbol{J}}$.

Then MAXWELL's equations are equivalent to the equations (see Xambo-2019 [22] for details and further references)

$$dF = 0, \quad \delta F = J,$$

where d is the exterior differential and δ the codifferential (adjoint to d).[1]

[1] To note that the equations $d\psi = \delta\psi = 0$ make sense on any Riemannian manifold for any differential form ψ, and that they characterize the harmonic forms on that manifold (HODGE). See [19].

Synopsis of GA

Ingredients

E a real vector space of dimension n.

q a *metric* on E: non-degenerate quadratic form of E;
(r, s) its signature.

$(E, q) = E_{r,s}$: *orthogonal geometry* of signature (r, s).

Examples

Euclidean space $E_n = E_{n,0}$ (signature $(n, 0)$).
E_2 (Euclidean plane), E_3 (ordinary Euclidean space).

$E_{1,3} = (E, \eta)$ (*Minkowski space*).

The metric is regarded as a bilinear non-degenerate form:

$$2q(e, e') = q(e + e') - q(e) - q(e'), \quad q(e) = q(e, e).$$

The GA of $(E,q)=E_{r,s}$, denoted $\mathcal{G}_q=\mathcal{G}_{r,s}$, can be constructed by enriching Grassmann's exterior algebra $\wedge E$ with the *geometric product* xy (Clifford). It is unital, bilinear and associative. Moreover,

- The grade involution $\hat{x}=\sum(-1)^i x_i$ of $\wedge E$ satisfies $\widehat{xy}=\hat{x}\hat{y}$.
- The reverse involution $\tilde{x}=\sum(-1)^{i/\!\!/2}x_i$ of $\wedge E$ satisfies $\widetilde{xy}=\widetilde{y}\widetilde{x}$.
- If $x\in\wedge^j E$ and $y\in\wedge^k E$, then $(xy)_i$ is 0 unless i is in the range $|j-k|, |j-k|+2,\dots,j+k-2, j+k$, with

$$(xy)_{|j-k|}=x\cdot y \text{ for } j,k>0, \quad \text{and} \quad (xy)_{j+k}=x\wedge y.$$

- For $e\in E$ and $x\in\wedge E$,

$$ex=e\cdot x+e\wedge x=(i_e+\mu_e)(x).$$

In particular, for $e'\in E$, $ee'=e\cdot e'+e\wedge e'$ (*Clifford's relations*), with $e\cdot e'=i_e(e')=q(e,e')$. Thus $ee'=-e'e$ iff $e\cdot e'=0$ and $e^2=q(e)$ (*Clifford's reduction rule*). If $q(e)\neq 0$ (*non-isotropic*, or *non-null* vector), $e^{-1}=e/q(e)$.

Let $\boldsymbol{e} = e_1, \ldots, e_n$ be a basis of E and $N = \{1, \ldots, n\}$ the *set of indexes*.

If $I = i_1, \ldots, i_k$ is a sequence of indexes, set

$$e_{\hat{I}} = e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad e_I = e_{i_1} \cdots e_{i_k}.$$

Let $\mathcal{I}$ be the set of *multiindices* and $\mathcal{I}_k$ the subset of multiindices of cardinal k.

- $\{e_{\hat{I}}\}_{I \in \mathcal{I}_k}$ is a basis of $\mathcal{G}^k = \wedge^k E$, hence $\{e_{\hat{I}}\}_{I \in \mathcal{I}}$ a basis of $\mathcal{G} = \wedge E$.
- $\{e_I\}_{I \in \mathcal{I}}$ is a basis of $\mathcal{G} = \wedge E$.
- If $\boldsymbol{e}$ is orthogonal, then $e_I = e_{\hat{I}}$, as *the geometric product of pair-wise orthogonal vectors is equal to their exterior product*.

$\mathcal{G}_2 = \langle 1, e_1, e_2, e_{12} = \boldsymbol{i} \rangle$, $\boldsymbol{i}^2 = -1$ (Gauss algebra).

$\mathcal{P} = \mathcal{G}_3 = \langle 1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123} = \boldsymbol{i} \rangle$, $\boldsymbol{i}^2 = -1$ (Pauli).
$e_{23} = \boldsymbol{i} e_1 = e_1 \boldsymbol{i}$, $e_{31} = \boldsymbol{i} e_2 = e_2 \boldsymbol{i}$, $e_{12} = \boldsymbol{i} e_3 = e_3 \boldsymbol{i}$.
General element: $(\alpha + \beta \boldsymbol{i}) + (v + w \boldsymbol{i})$ $(\alpha, \beta \in \mathbf{R},\ v, w \in E_2)$.
$G_2^+ = \{\alpha + w\boldsymbol{i}\} = \mathbf{H}$ (quaternion field).

$E_{1,3} = \langle e_0, e_1, e_2, e_3 \rangle$. In $\mathcal{D} = \mathcal{G}_{1,3}$ (Dirac algebra), set:
$\boldsymbol{i} = e_{0123}$, $\boldsymbol{\sigma}_k = e_k e_0$. Then
$\mathcal{D} = \langle 1, e_1, e_2, e_3, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \mathbf{i}\boldsymbol{\sigma}_1, \mathbf{i}\boldsymbol{\sigma}_2, \mathbf{i}\boldsymbol{\sigma}_3, e_1\mathbf{i}, e_2\mathbf{i}, e_3\boldsymbol{i}, \boldsymbol{i} \rangle$
A general element has the form $(\alpha + \beta\mathbf{i}) + (v + w\mathbf{i}) + (\boldsymbol{E} + \mathbf{i}\boldsymbol{B})$,
$(\alpha, \beta \in \mathbf{R},\ v, w \in E_3,\ \boldsymbol{E}, \boldsymbol{B} \in \mathcal{E} = \langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3 \rangle)$.
$\mathcal{D}^+ = \langle 1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3, \boldsymbol{i}\boldsymbol{\sigma}_1, \boldsymbol{i}\boldsymbol{\sigma}_2, \boldsymbol{i}\boldsymbol{\sigma}_3, \boldsymbol{i} \rangle \simeq \mathcal{P}(\mathcal{E})$.
Its elements have the form $(\alpha + \beta\boldsymbol{i}) + (\boldsymbol{E} + \boldsymbol{i}\boldsymbol{B})$.
Note: $\boldsymbol{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$.

EM with GA

With the notations introduced in the section **EM with differential forms** (slide 27), Maxwell's equations are equivalent to $dF = 0$ and $\delta F = J$, which in turn can be written, since J is a 1-form, as a single equation: $(d+\delta)F = J$.

The main point here is that the *GA operator* $\partial = dt\partial_t + dx\partial_x + dy\partial_y + dz\partial_z$ allows to encode Maxwell's equations in the GA equation $\partial F = J$. In fact, $\partial \wedge F = dF$, $\partial \cdot F = \delta F$ and hence $\partial F = \partial \cdot F + \partial \wedge F = \delta F + dF = (\delta + d)F$.

It is important to note, in view of a general formalism, that we can combine $\boldsymbol{E}$ and $\boldsymbol{B}$ into the bivector $F = \boldsymbol{E} + \boldsymbol{iB} \in \mathcal{D}^2$ (electromagnetic bivector, or *Faraday bivector*), ρ and $\boldsymbol{J}$ into the vector $-\rho e_0 + J_x e_1 + J_y e_2 + J_y e_3$, and express Maxwell's equations as $\partial F = J$, where $\partial = e^0\partial_0 + e^1\partial_1 + e^2\partial_2 + e^3\partial_3$. Here $e^0 = e_0$ and $e^k = -e_k$ $(k = 1, 2, 3)$, so that $e^i \cdot e_j = \delta^i_j$.

For details, see [14, §1 and §3].

The Dirac equation after Hestenes

The original Dirac equations were written in terms of 4×4 complex matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ that provided a matrix representation of $\mathcal{D}$ determined by $e_i \mapsto \gamma_i$. The space on which these matrices act, $\mathbf{C}^4$, was the space of *Dirac spinors*; the *wave function* was map $\psi : E_{1,3} \to \mathbf{C}^4$; and the Dirac equation was derived as a "relativistic Schrödinger equation for the electron wave function" (Klein-Gordon equation).

It turs out, however, that GA shows that *the complex matrices are superfluous, as the only crucial fact required is that they satisfy Clifford's relations*. And after that, the analysis reveals that the role of $\mathbf{C}^4$ must be played by the space $\mathcal{D}^+$ (which has complex dimension 4) and hence that the wave function is to be thought as a *spinor field*, the name for a function $\psi : E_{1,3} \to \mathcal{D}^+$.

As is customary, instead of e_0, e_1, e_2, e_3 used so far, we will use $\gamma_0, \gamma_1, \gamma_2, \gamma_3$.

The final conclusion is that the *Dirac equation* is morphed into the following equation for the spinor field ψ:

$$\partial\psi i\hbar - \frac{e}{c}A\psi = m_e c\psi\gamma_0,$$

where c is the speed of light, e is the electron charge and m_e its mass. In this equation i is not $\sqrt{-1}$, but the bivector $i = \gamma_{21}$, and A is the *electromagnetic potential*, a vector field such that $\partial \wedge A = F$ and $\partial \cdot A = 0$.

As expressed by D. Hestenes, this equation "reveals geometric structure in the Dirac theory that is so deeply hidden [even inaccessible] in the matrix version that it remains unrecognized by QED experts to this day".

See [14, §3.3], [20], [21, §6.2 and §6.3], which include a comprehensive survey of applications.

Remark

$$i = \gamma_2\gamma_1 = \boldsymbol{i}\gamma_3\gamma_0 = \boldsymbol{i}\boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2.$$

This 2-area element is a geometric imaginary unit that replaces the (ungeometric) imaginary unit $\sqrt{-1}$ in the original Dirac equation.

The first important advantage of the GA formulation of the Dirac equation is that $\psi(x)$ admits a decomposition of the form

$$\psi = \rho^{1/2} e^{\boldsymbol{i}\beta/2} R,$$

where $\rho = \rho(x)$ is a positive real number, $\beta = \beta(x) \in [0, 2\pi)$ and $R = R(x)$ is a *rotor* (that is, $R\widetilde{R} = 1$). Note that this expression has eight degrees of freedom: $1 + 1 + 6$.

Define $e_\mu = e_\mu(x) = R\gamma_\mu\widetilde{R}$ (*comoving frame*). Since R is a rotor, this s an orthonormal frame field in $E_{1,3}$ with the same orientation and temporal orientation as the reference frame γ_μ.

Note that $\psi\gamma_\mu\widetilde{\psi} = \rho e_\mu$, because $\boldsymbol{i}$ anticommutes with vectors and $\widetilde{\boldsymbol{i}} = \boldsymbol{i}$:

$$\psi\gamma_\mu\widetilde{\psi} = \rho e^{\boldsymbol{i}\beta/2}R\gamma_\mu\widetilde{R}e^{\boldsymbol{i}\beta/2} = \rho e^{\boldsymbol{i}\beta/2}e^{-\boldsymbol{i}\beta/2}R\gamma_\mu\widetilde{R} = \rho e_\mu.$$

In particular, $\psi\gamma_0\widetilde{\psi} = \rho v$, where $v = e_0$, is the *Dirac current*.

The vector

$$s = \tfrac{\hbar}{2}R\gamma_3\widetilde{R} = \tfrac{\hbar}{2}e_3 \qquad (1)$$

is the *spin vector*.

The rotor R transforms the unit i to $\iota = Ri\widetilde{R}$, which is the (comoving) space-like plane quantity $e_2 e_1$ and $S = \frac{\hbar}{2}\iota$ can be called the *spin bivector*. The relation to the spin vector is as follows:

$$S = \boldsymbol{i} s v.$$

Proof $\boldsymbol{i} s v = \frac{\hbar}{2}\boldsymbol{i} R\gamma_3\widetilde{R}R\gamma_0\widetilde{R} = \frac{\hbar}{2}R\boldsymbol{i}\gamma_3\gamma_0\widetilde{R} = \frac{\hbar}{2}Ri\widetilde{R} = \frac{\hbar}{2}\iota = S.$ □

With $R = e^{i(k\cdot x)}$, we have a 'monochromatic spinor' (yes, $\boldsymbol{i}$ and i)

$$\psi = \rho^{1/2} e^{\boldsymbol{i}\beta/2} e^{i(k\cdot x)}.$$

A straightforward computation shows that the condition for this wave to satisfy the real Dirac equation is that

$$\hbar k = m_e c v e^{-\boldsymbol{i}\beta}$$

This implies that $\cos(\beta) = \pm 1$. As for monochromatic electromagnetic waves, the condition for constant phase in the moving frame is $v \cdot x = c\tau$, and so

$$\hbar k \cdot x = \pm m_e c(v \cdot x) = \pm m_e c^2 \tau$$

which yields the de Broglie frequency $m_e c^2/\hbar$ of the electron.

A closer analysis shows that the vector e_1 turns in the plane ι with frequency $2m_e c^2/\hbar$, which is the *zitterbewegung* frequency of Schrödinger, with period 4.0466×10^{-21}s.

Backpack for a voyage

Concerning topology, it appears when looking at different aspects of theories.

- The fundamental group of relevant groups plays an important role (see Xambo-2018-spins [21], §6.1.7, summarized before).

- Another source of topological structure can emerge when looking at the nature and properties of the solutions to given equations. In the case of electromagnetism, the most significant references here are DeKlerk-2016 [4] and DeKlerk-et-3-2017 [5]. See also Arrayas-Bouwmeester-Trueba-2017 [1] and the many references cited therin. Other works: Lomonaco-1996 [15], Kauffman-2001 [10], Kedia-PeraltaSalas-2018 [11]. For the case of gravitational waves, see Lasenby-2019 [13].

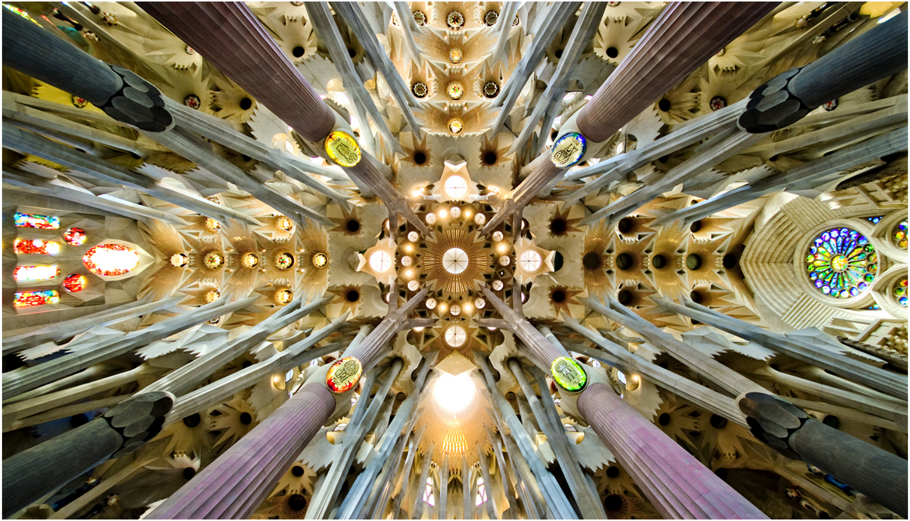
- Topology is important when there is interest in general space-times, Rodrigues-DeOliveira-2016 [18], or Burns-2019 [3] for the case of general backgrounds for EM.

- (1) To phrase in GA the vortex solutions for the electromagnetic field: rfArrayas-Bouwmeester-Trueba-2017, and relevant references therein.

- (2) To carry out a close mathematical analysis of Hestenes' unified MAXWELL-DIRAC theory presented in Hestenes-2019 [9]. For this, Rodrigues-DeOliveira-2016 [18] and references in experimental physics such as Glauber-2007 [7], Kleckner-Irvine-2013 [12], Riek-et-8-2015 [17], BeneaChelmus-Settembrini-Scalari-Faist-2019 [2] and Moskalenko-Ralph-2019 [16] are expected to provide convenient touchstones.

- (3) To phrase in GA the **knotted** vortex electromagnetic solutions discovered in [5].

- (4) To seek a generalization of (3) for the theory in (2).



Antoni Gaudí's Light dream?

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Dedicated to Eusebio Corbacho and Elena Martín Peinador on the occasion of their jubilee

P

Recuerdo de la Jornada de Sage y Python en Jarandilla de la Vera. Depués vinieron, en noviembre de 2015, las tesis de Emilio Estévez Martínez y María del Carmen Somoza López, en las compartimos unos días magníficos, en lo científico y en lo cultural: Eusebio, Mª del Carmen, Emilio, María Jesús, Elena Martín, Paco Botana,...

P

Jornada de homenaje a José María Montesinos Amilibia, 9 de septiembre de 2015. Organizada por Elena Martín Peinador, José Manuel Sanjurjo y Marco Castrillón.

Main motivation

Profound insights are slow in coming. What few we have took over three thousand years to glean, even though the pace is ever quickening. It is marvelous indeed to watch the answer subtly change while the question immutably remains—*What is light*? Hecht-2017 [8] Optics, page 17.

Hestenes-2019 [9], *Maxwell-Dirac theory*: "The spectacular success of quantum electrodynamics (QED) gives physicists great confidence in Maxwell's equation on the one hand and Dirac's equation on the other, yet something is missing in relations between them. With his usual penetrating insight, Einstein focused on the crux of the problem: *is a delusion to think of electrons and the fields as two physically different, independent entities. Since neither can exist without the other, there is only one reality to be described, which happens to have two different aspects; and the theory ought to recognize this from the start instead of doing things twice*".

Dirac-1931 [6], *Quantised Singularities in the Electromagnetic Field*: "The steady progress of physics requires for its theoretical formulation a mathematics that gets continually more advanced [...] a mathematics that continually shifts its foundations and gets more abstract[...] The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics".

This is not unlike Hilbert's program for axiomatizing physics.S

P

Remark on Rømer's reasoning NEWTON's theory predicted the time at which any one of the Jupiter's satellites would be eclipsed by the planet. The main observation of RØMER, who concentrated on Io, was that the *observed* time of the eclipse varied along the year with respect to the calculated time. Let t_P and t_Q be the observed times when the Earth is between the Sun and Jupyter (position P) and when the Sun is between the Earth and Jupyter, position Q (roughly half a year later).

Then $\Delta t = t_Q - t_P$ is the time spent by light in going from P to Q, or L/c, where L is the diameter of the Earth's orbit. Since Δt and L are known, this was enough to get c.

Rømer's reasoning was a little more involved, as the observation of Jupyter at Q is not possible, so he had to measure the eclipse times at other points of the Earth orbit and carry out a suitable interpolation. P

In 1818 Fresnel entered a competition sponsored by the French Academy. His paper on the theory of diffraction ultimately won first prize and the title *Mémoire Couronné*, but not until it had provided the basis for a rather interesting story. The judging committee consisted of Pierre Laplace, Jean B. Biot, Siméon D. Poisson, Dominique F. Arago, and Joseph L. Gay-Lussac—a formidable group indeed. *Poisson, who was an ardent critic of the wave description of light, deduced a remarkable and seemingly untenable conclusion from Fresnel's theory. He showed that a bright spot would be visible at the center of the shadow of a circular opaque obstacle, a result that he felt proved the absurdity of Fresnel's treatment*. P