

On Leibniz' *characteristica universalis*
–considered as a thinking mood–

S. Xambó

UPC

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Abstract

Leibniz' idea of a *characteristica universalis* (CU) will be presented in a historical perspective and illustrated in the realm of geometric algebra and geometric calculus with a sample of applications to geometry and physics. The main thread will be that Leibniz idea can be regarded as a productive *thinking mood*. The talk will end with some reflections aimed at highlighting the wide scope and generality of Leibniz' idea.

Background themes

- **Intrinsic thinking.**
- Drive for **unification** (inter-connections between ideas, systems of ideas, models in physics, ...).
- Asking questions, producing conjectures, phrasing arguments, ... (*ars inveniendi*).
- The development and accomplishments of **Geometric Algebra** (a case-study).
- Collective effort and much to be done.

1970's

- Lectures of Juan B. Sancho (master Sancho) on 'geometric algebra' (E. Artin, *Geometric algebra*, 1957).
- From synthetic (axiomatic) projective geometry to its algebraic realization (1974).
- Relativity groups ($\gamma = (1 - v^2/V^2)^{-1/2}$).
- Struggle to overcome customary spacetime presentations.
- *Álgebra lineal geometrías lineales* (1977, 1978).
Volume of parallelepipeds:

$$V(x_1, \dots, x_r) = \sqrt{\det(X^T \cdot X)}, \quad X = x_1, \dots, x_r.$$

Cross product $x = x_1 \times \cdots \times x_{n-1}$ in E_n :

- Perpendicular to its factors;
- x_1, \dots, x_{n-1}, x positively oriented;
- $|x| = V(x_1 \times \cdots \times x_{n-1})$.

Angle α between $U = \langle u_1, \dots, u_k \rangle$ and $V = \langle v_1, \dots, v_k \rangle$:

$$\cos(\alpha) = |\det(U^T \cdot V)| / \sqrt{\det(U^T \cdot U) \det(V^T \cdot V)}.$$

1980's and 1990's

- Enumerative geometry
(Shubert calculus. Halphen's Refinement).
- Connections with string theory.
- UCM/FM: Algebra, Algebraic Geometry, Arithmetic Algebraic Geometry
- UPC/FME: Geometry, Mathematical models in Physics, Error-correcting codes.
- Project Omega.

2000's

- **Wiris**. The CC package. *Block Error-Correcting Codes: A Computational Primer*.
- Projects WebALT (<http://www.webalt.net/>) and JEM (Joining Educational Mathematics).
- Project VISIO.

2010's

- WCCI 2010. Plenary lectures volume (see <http://www-ma2.upc.es/sxd/ICAIB/wcci-2010.html>).
- Project mOltO (<http://www.molto-project.eu/>): “to develop a set of tools for translating texts between multiple languages in real time with high quality.” **MGL** (w J. Saludes).
- Symposium Juan B. Sancho Guimerá, *in memoriam* (2014). *Escondidas sendas de la geometría proyectiva a la física cuántica*.
- MFO, *Realities of Imaginary* (Leibniz Gemeinschaft).
- GAT, Mexico lectures (March 2015).
- AGACSE 2015 (<http://www-ma2.upc.edu/agacse2015/>). Dedicated to David Hestenes. Second edition of his *Space-time algebra*. Review in the NL of the EMS (December 2015).
- GAT, Valladolid lectures, first part (November 2015).

- Leibniz' Congress (January 2016)
- GAT, Valladolid lectures, second part (May 2016).
- Editor of two special issues of AACA stemming from AGACSE 2015 (to appear in 2016).
- Direction of the Summer Research School “Lluís Santaló” on Geometric Algebra and its Applications to Science and Engineering (IUP, Santander, August 2016). Teachers: David Hestenes, Anthony Lasenby, Joan Lasenby, Leo Dorst.
- *Multilingual tools for mathematics* (with J. Saludes), to appear in “The language of mathematics – computational, linguistic and logical aspects” (special issue of SDV).

Bertrand Russell: *A History of Western Philosophy*. Allen & Unwin, 1946.

Leibniz was a **firm believer in the importance of logic**, not only in its own sphere, but as a basis of metaphysics. He did work on mathematical logic which would have been enormously important if he had published it; he would, in that case, have been the founder of mathematical logic, which would have become known a century and a half sooner than it did in fact. He abstained from publishing, because he kept on finding evidence that Aristotle's doctrine of the syllogism was wrong on some points; **respect for Aristotle** made it impossible for him to believe this, so he mistakenly supposed that the errors must be his own.

Nevertheless he cherished throughout his life the hope of discovering a kind of generalized mathematics, which he called *characteristica universalis*, by means of which thinking could be replaced by calculation. 'If we had it,' he says, 'we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis.' 'If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other (with a friend as witness, if they liked): Let us calculate (*Calcuemus*).

Gaston Casanova: *L'algèbre vectorielle. Que sais-je? 1657*, Presses universitaires de France, 1976.

Calculer sur les concepts de la géométrie selon les règles d'une algèbre a été depuis longtemps le but des recherches de nombreux mathématiciens, comme Leibnitz qui en rêva..." (Introduction, first sentence).

Michael J. Crowe: *A history of vector analysis*. Dover, 1994 (slightly corrected reprint of the Dover edition of 1985, republication of the original 1967 Notre Dame Press). Paper 2002 with the same title (lecture about the book)

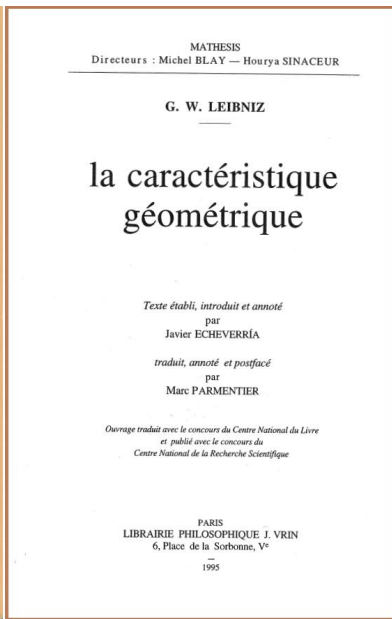
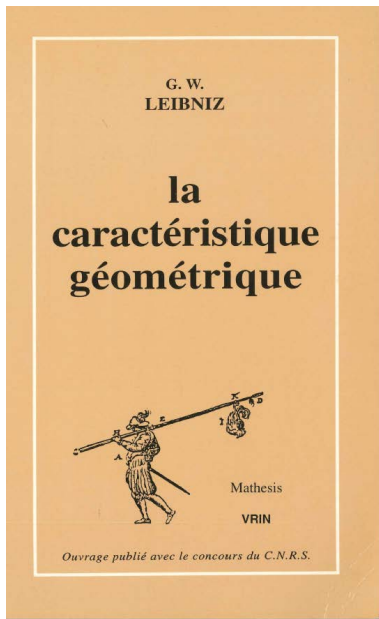
Leibniz made many contributions to mathematics; among the less well known is his concept of a **geometry of situation**. In this regard Leibniz discussed the possibility of creating a system which would serve as a direct method of space analysis. Although the details of this idea were never fully worked out by Leibniz, he advanced far enough to be ranked as a conceptual **forerunner of the first vectorial analysts**. Moreover his essay, which was first published in 1833, played a part in the later history of vectorial analysis.

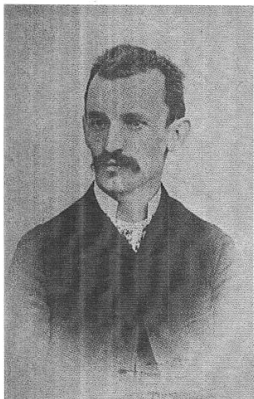
Leibniz' main ideas were contained in a letter dated September 8, 1679, and written to Christian Huygens. In this letter Leibniz wrote:

I am still not satisfied with algebra... I believe that, so far as geometry is concerned, we need still **another analysis** which is distinctly geometrical or linear and **which will express situation directly as algebra expresses magnitude directly**. And I believe that I have found the way and that we can represent figures and even machines and movements by characters, as algebra represents numbers or magnitudes.

I believe that by this method one could treat mechanics almost like geometry...

I have no hope that we can get very far in physics until we have found such a **method**...





Giuseppe Peano

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- René Descartes (1596-1650)
- Isaac Newton (1642-1727)
- Gottfried W. Leibniz (1646-1716)
- Leonhard Euler (1707-1783)
- Joseph-Louis Lagrange (1736-1813))

- August F. Möbius (1790-1868)
- B. Olinde Rodrigues (1795–1851). Rotation group (1840).
- Giusto Bellavitis (1803-1880). Method of *equipollences*.
- W. R. Hamilton (1805-1865). **Quaternions** (1843).
- H. Grassmann (1809-1877). **Ausdehnungslehre** (1844, 1862).
- B. Riemann (1826-1866). Riemann sphere (related to spinors).
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Olinde Rodrigues



Hamilton



Grassmann



Riemann



Clifford



Lipschitz



Peano



Whitehead



Gibbs



Study



E. Cartan



Weyl



Pauli



Dirac



Brauer



Riesz



Chevalley



E. Artin

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E. Witt



D. Hestenes



J. Helmstetter



P. Lounesto



A. Lasenby



J. Lasenby



L. Dorst



C. Doran



David Hestenes (2015)

Notations: E_2 , an oriented Euclidean plane; e_1, e_2 an arbitrary positive orthonormal basis ($e_k^2 = e_k \cdot e_k = 1$, $e_1 \cdot e_2 = 0$).

The exterior (or Grassmann) algebra of E_2 , ΛE_2 , is the sum of

$$\Lambda^0 E_2 = \mathbf{R} \text{ (scalars),}$$

$$\Lambda^1 E_2 = E_2 = \langle e_1, e_2 \rangle = \{ \lambda_1 e_1 + \lambda_2 e_2 \} \text{ (vectors), and}$$

$$\Lambda^2 E_2 = \mathbf{R} e_{12} \text{ (bivectors), also pseudoscalars; } e_{12} = e_1 \wedge e_2).$$

The bivector $i = e_{12}$ represents the **unit area**.

The GA of E_2 , denoted $\mathcal{G}(E_2)$, is the algebra ΛE_2 endowed with the unique associative product (*geometric product*) that satisfies

$$(P) \quad xx' = x \cdot x' + x \wedge x' \quad (x, x' \in E_2).$$

Remark. If $x \perp x' \Leftrightarrow x \cdot x' = 0$, then $xx' = x \wedge x' = -x' \wedge x = -x'x$. Thus *orthogonal vectors anticommute*.

If $x \parallel x' \Leftrightarrow x \wedge x' = 0$, then $xx' = x \cdot x' = x' \cdot x = x'x$. Thus *parallel vectors commute*.

In particular, $x^2 = x \cdot x = |x|^2$ for any $x \in E_2$.

Any non-zero $x \in E_2$ is *invertible* and $x^{-1} = x/|x|^2$.

Example. $i = e_{12} = e_1 \wedge e_2 = e_1 e_2$. Hence

$$i^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1.$$

The subspace $\mathcal{G}^+ = \Lambda^0 + \Lambda^2 = \mathbf{R} + \mathbf{R}i$ is a subalgebra of \mathcal{G} (the *even* subalgebra). Since $i^2 = -1$, its elements are entitled to be called *complex numbers* and $\mathbf{C} = \mathcal{G}^+$ the *field* of complex numbers.

In the traditional view, 2D ‘vectors’ are taken to be the same thing as ‘complex numbers’ ($a + bi$). In GA, vectors ($E_2 = \mathcal{G}^-$) and complex numbers ($\mathbf{C} = \mathcal{G}^+$) are different structures, but E_2 can still be seen as a \mathbf{C} -vector space (of dimension 1) because $iE_2 = E_2$. Note that $ie_1 = -e_2$, $ie_2 = e_1$.

If $\theta \in \mathbf{R}$, θi is an intrinsic geometric notion of **the** angle of amplitude θ in E_2 (note that as i does not depend on the choice of e_1, e_2).

Indeed, we can form the unit complex number

$$e^{\theta i} = \cos(\theta) + i \sin(\theta)$$

and with the usual computation we see that

$$x \mapsto xe^{\theta i}$$

is the rotation of $x \in E_2$ by the angle θ (or rather θi):

$$e_1 e^{\theta i} = e_1 \cos(\theta) + e_2 \sin(\theta)$$

$$e_2 e^{\theta i} = -e_1 \sin(\theta) + e_2 \cos(\theta)$$

“The most remarkable formula in mathematics is:

$$e^{i\theta} = \cos \theta + i \sin \theta$$

This is our jewel. We may *relate the geometry to the algebra* by representing complex numbers in a plane

$$x + iy = re^{i\theta}$$

This is the *unification of algebra and geometry.*”

R. Feynman, *Lecture Notes in Physics*, Volume I, Part 1.

Comment. Emphasis not in the original. Note that Euler’s formula works with no change by taking the ‘imaginary unit’ to be the geometric *i*.

Notations: E_3 , an oriented Euclidean space; e_1, e_2, e_3 an arbitrary positive orthonormal basis.

We set $e_j \wedge e_k = e_{jk}$ and $e_1 \wedge e_2 \wedge e_3 = e_{123}$. The trivector $\mathbf{i} = e_{123}$ represents the unit volume.

The exterior (or Grassmann) algebra of E_3 , ΛE_3 , is the sum of

$$\Lambda^0 E_3 = \mathbf{R} \text{ (*scalars*);}$$

$$\Lambda^1 E_3 = E_3 = \langle e_1, e_2, e_3 \rangle \text{ (*vectors*);}$$

$$\Lambda^2 E_3 = \langle e_{23}, e_{31}, e_{12} \rangle \text{ (*bivectors*); and}$$

$$\Lambda^3 E_3 = \mathbf{Ri} \text{ (*trivectors* or *pseudoscalars*).$$

The GA of E_3 , denoted $\mathcal{G}(E_3)$, is the algebra ΛE_3 endowed with the unique associative product (*geometric product*) that satisfies the rule (P). So orthogonal vectors anticommute and parallel vectors commute.

The computation of products is straightforward since it happens that

$$e_j e_k = e_{jk} \text{ and } e_j e_k e_l = e_{jkl} \text{ if } j, k, l \text{ are distinct.}$$

Example. $\mathbf{i}^2 = -1$, for $\mathbf{i}^2 = e_1 e_2 e_3 e_1 e_2 e_3 = e_2 e_3 e_2 e_3 = -1$.

Example (Hodge duality). The map $E_3 \rightarrow \Lambda^2 E_3$, $x \mapsto x^* = x\mathbf{i}$ is a linear isomorphism. The inverse isomorphism is given by $y \mapsto y^* = -y\mathbf{i}$.

For instance,

$$e_2^* = e_2 \mathbf{i} = e_2 e_1 e_2 e_3 = -e_1 e_3 = e_{31},$$

$$e_{31}^* = -e_{31} \mathbf{i} = -e_2 \mathbf{i}^2 = e_2.$$

Theorem. Let $u \in E_3$ be a unit vector and $\theta \in \mathbf{R}$. Then we have:

- If i_u be the unit pseudoscalar of the plane u^\perp , then $i_u = \mathbf{i}u = u\mathbf{i} = u^*$.
- If we let $R = R_{u,\theta} = e^{\frac{1}{2}\theta i_u}$ (called *rotor* of the angle $\frac{1}{2}\theta i_u$), then $R_{u,\theta} = \cos(\frac{1}{2}\theta) + \mathbf{i}u \sin(\frac{1}{2}\theta)$ and $R^{-1} = \cos(\frac{1}{2}\theta) - \mathbf{i}u \sin(\frac{1}{2}\theta)$.
- If $x \in E_3$, $R^{-1}xR \in E_3$ and the linear map $\rho_{u,\theta} : x \mapsto R^{-1}xR$ is the rotation about u of amplitude θ .

Let u_1, u_2 be a positive orthonormal basis of u^\perp , which means that u_1, u_2, u is a positive basis of E_3 . Then $i_u = u_1 u_2 = u_1 u_2 u^2 = \mathbf{i}u$. The other claims in the first and second points are straightforward.

For the proof of the third point, note that x commutes with R (and R^{-1}) if x is parallel to u , so that for such x we have $\rho_{u,\theta}(x) = x$. On the other hand, if $x \in u^\perp$, then $R^{-1}x = xR$ (use the expressions in the second point) and therefore $\rho_{u,\theta}(x) = xR^2 = xe^{\theta i_u} = xe^{\theta i_u}$, which is the result of rotating x in u^\perp by an amplitude θ .

To determine the composition $\rho_{u',\theta'} \circ \rho_{u,\theta}$, it is enough to compute its rotor, say $e^{\frac{1}{2}\mathbf{i}u''\theta''}$, as the product of the corresponding rotors:

$$e^{\frac{1}{2}\mathbf{i}u''\theta''} = e^{\frac{1}{2}\mathbf{i}u'\theta'} e^{\frac{1}{2}\mathbf{i}u\theta}.$$

This relation can be written in the form

$\cos \frac{\theta''}{2} + \mathbf{i}u'' \sin \frac{\theta''}{2} = (\cos \frac{\theta'}{2} + \mathbf{i}u' \sin \frac{\theta'}{2})(\cos \frac{\theta}{2} + \mathbf{i}u \sin \frac{\theta}{2})$ which itself is equivalent to the equations

$$\cos \frac{\theta''}{2} = \cos \frac{\theta'}{2} \cos \frac{\theta}{2} - (u \cdot u') \sin \frac{\theta'}{2} \sin \frac{\theta}{2}$$

$$u'' \sin \frac{\theta''}{2} = u \sin \frac{\theta}{2} \cos \frac{\theta'}{2} + u' \cos \frac{\theta}{2} \sin \frac{\theta'}{2} + (u \times u') \sin \frac{\theta}{2} \sin \frac{\theta'}{2},$$

where $u \times u' = -\mathbf{i}(u \wedge u')$ (*cross product*).

There are two solutions to the first equation ($\pm\theta''$), and hence two solutions $\pm u''$ to the second equation, but since $\rho_{-u,-\theta} = \rho_{u,\theta}$, they determine the same rotation.

Let $i_1 = -e_1 i = -e_2 e_3$, $i_2 = -e_2 i = -e_3 e_1$, $i_3 = -e_3 i = -e_1 e_2$.
 Then $i_k^2 = -1$ ($k = 1, 2, 3$) and $i_1 i_2 = -i_2 i_1 = i_3$, $i_2 i_3 = -i_3 i_2 = i_1$,
 $i_3 i_1 = -i_1 i_3 = i_2$.

Example. $i_3 i_1 = e_3 i e_1 i = -e_3 e_1 = i_2$.

Theorem. The even subalgebra $\mathcal{G}^+ = \langle 1, i_1, i_2, i_3 \rangle$ is the field of (geometric) quaternions.

References: <http://www-ma2.upc.edu/sxd/--GAT-IMUVA/>

As long as algebra and geometry were separated, their progress was slow and their uses limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together toward perfection.

Joseph-Louis Lagrange (1736-1813)