Universidad de Almería
3rd International WS on GA

GA perspectives On Morley's theorem

S. Xambó

UPC: FME & IMTech

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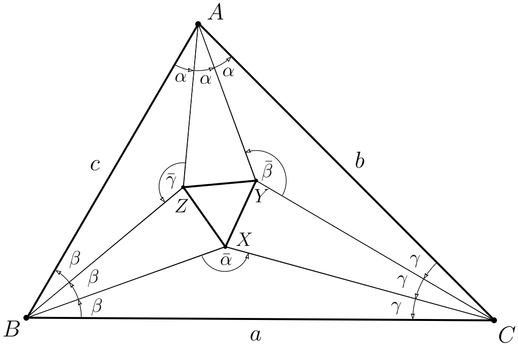


Figure 1: Let ABC be a triangle in the Euclidean plane. Define the angles α, β, γ by trisecting the triangle angles $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and produce the triangle XYZ (*Morley's triangle* of ABC). *Theorem*: XYZ *is equilateral*.

Remarks .$\bar{\alpha} = \pi - (\beta + \gamma) = \pi - (\frac{\pi}{3} - \alpha) = \frac{2\pi}{3} + \alpha$. Similarly, $\bar{\beta} = \frac{2\pi}{3} + \beta$, $\bar{\gamma} = \frac{2\pi}{3} + \gamma$.

$\diamond$**1** (The geometric product)**.** Let $(\mathcal{G}, \wedge)$ be the Grassmann algebra of $\mathcal{E}$. Then there is a ***unique*** *bilinear* ***associative*** *unital product* $xx' \in \mathcal{G}$ $(x, x' \in \mathcal{G})$, such that

$$vv' = v \cdot v' + v \wedge v' \tag{1}$$

for all $v, v' \in \mathcal{E}$.

- This result is general, [2], in the sense that the approach works if $\mathcal{E}_2$ is replaced by a quadratic space $\mathcal{E}_{r,s}$ of any signature (r, s).
- Eq. (1) implies that $v^2 = v \cdot v$, and $v'v = -vv'$ if (and only if) $v \cdot v' = 0$ ($\Leftrightarrow v \perp v'$). In particular, *any non-zero vector v is invertible*: $v^{-1} = \frac{v}{v \cdot v} = \frac{v}{|v|^2}$. This feature is paramount in geometric algebra, as computations in it have the flavor of computations in $\mathbb{R}$, except that the geometric product is not commutative.

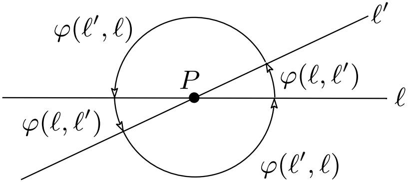
- If u, u' is a positive orthonormal basis of $\mathcal{E}$, the bivector $\boldsymbol{i} = uu'$ (unit area) only depends on the orientation of $\mathcal{E}$ and $\boldsymbol{i}^2 = -1$.
- The even algebra $\mathcal{G}^+ = \mathcal{G}^0 \oplus \mathcal{G}^2 = \mathbb{R} \oplus \mathbb{R}\boldsymbol{i} = \mathbb{C} \simeq \mathbb{R} + \mathbb{R}i = \mathbf{C}$.
- $\mathcal{E} \to \mathcal{E}$, $v \mapsto v\boldsymbol{i}$, is a positive $\frac{\pi}{2}$-rotation, while $v \mapsto \boldsymbol{i}v$ is a negative $\frac{\pi}{2}$-rotation.[1]
- $\mathcal{E} \to \mathcal{E}$, $v \mapsto ve^{\boldsymbol{i}\varphi} = e^{-\boldsymbol{i}\varphi}v$ is a positive rotation of amplitude φ.[2]
- If $z = re^{\boldsymbol{i}\varphi} \in \mathbb{C}$, the map $\mathcal{E} \to \mathcal{E}$, $v \mapsto vz = \bar{z}v$ is a *rotodilation*.
- If u is a non-zero vector, the map $\mathcal{E} \to \mathcal{E}$, $v \mapsto -uvu^{-1}$ is the *reflection* about the line $u^\perp$.

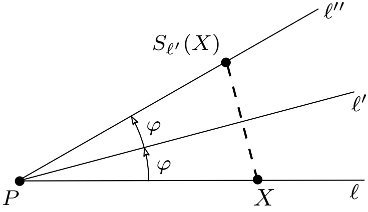
Notation: $u_\varphi = e^{\boldsymbol{i}\varphi}$.

[1] $u \mapsto u\boldsymbol{i} == uuu' = u'$, $u' = u'\boldsymbol{i} = u'uu' = -uu'u' = -u$.

[2] $ue^{\boldsymbol{i}\varphi} = u\cos\varphi + u\boldsymbol{i}\sin\varphi = u\cos\varphi + u'\sin\varphi$; $u'e^{\boldsymbol{i}\varphi} = -u\sin\varphi + u'\cos\varphi$.

- Symmetry S_ℓ about the line $\ell = P + \langle u \rangle$:
  $S_\ell(X) = P - u(X - P)u^{-1}$.
- Rotation $R_{P,\varphi}$ about P of amplitude φ:
  $R_{P,\varphi}(X) = P + (X - P)e^{\boldsymbol{i}\varphi}$.
- If ℓ and ℓ' are non-parallel lines, $S_{\ell'} S_\ell = R_{P,2\varphi}$, where $P = \ell \cap \ell'$ and $\varphi = \varphi(\ell, \ell')$.





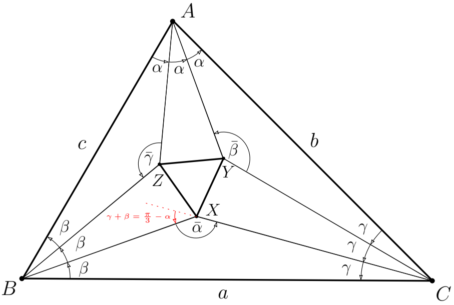
$\ell'' = S_{\ell'}(\ell) = (S_{\ell'} S_\ell)(\ell), \; S_{\ell'} S_\ell = R_{P,2\varphi}$

$\diamond$**2** (Fixed point of the composition of two rotations)**.**
The composition $R = R_{Q,\psi}R_{P,\varphi}$ has a unique fixed point, $F = F(R)$, if $u_\varphi u_\psi \neq 1$,

$$F = P + (Q - P)\frac{1 - u_\psi}{1 - u_\varphi u_\psi}, \tag{2}$$

and in this case $R = R_{F,\varphi+\psi}$.

Otherwise, $R_{Q,\psi}R_{P,\varphi} = T_v$, where $v = (Q - P)(1 - u_\psi)$.



◇**3.** Let $R_A = R_{A,2\alpha}, R_B = R_{B,2\beta}, R_C = R_{C,2\gamma}$. Then:

(1) $R_A = S_{AZ}S_{AB} = S_{AC}S_{AY}$
$R_B = S_{BX}S_{BC} = S_{BA}S_{BZ}$
$R_C = S_{CY}S_{CA} = S_{CB}S_{CX}$

(2) $R_BR_C = S_{BX}S_{CX} = R_{X,2(\frac{\pi}{3}-\alpha)}$: $X = F(R_BR_C)$
$R_CR_A = S_{CY}S_{AY} = R_{X,2(\frac{\pi}{3}-\beta)}$: $Y = F(R_CR_A)$
$R_AR_B = S_{AZ}S_{BZ} = R_{Z,2(\frac{\pi}{3}-\gamma)}$: $Z = F(R_AR_B)$

◇**4.** Let $u_A = e^{2\boldsymbol{i}\alpha}$, $u_B = e^{2\boldsymbol{i}\beta}$, $u_C = e^{2\boldsymbol{i}\gamma}$. Then:

$$\begin{aligned}
X &= C + (B-C)\tfrac{1-u_B}{1-u_B u_C} \\
&= C + (B-C)\tfrac{\sin\beta}{\sin(\beta+\gamma)}e^{-\boldsymbol{i}\gamma} = B + (C-B)\tfrac{\sin\gamma}{\sin(\beta+\gamma)}e^{\boldsymbol{i}\beta}, \\
Y &= A + (C-A)\tfrac{1-u_C}{1-u_C u_A} \\
&= A + (C-A)\tfrac{\sin\gamma}{\sin(\gamma+\alpha)}e^{-\boldsymbol{i}\alpha} = C + (A-C)\tfrac{\sin\alpha}{\sin(\gamma+\alpha)}e^{\boldsymbol{i}\gamma} \\
Z &= B + (A-B)\tfrac{1-u_A}{1-u_A u_B} \\
&= B + (A-B)\tfrac{\sin\alpha}{\sin(\alpha+\beta)}e^{-\boldsymbol{i}\beta} = A + (B-A)\tfrac{\sin\beta}{\sin(\alpha+\beta)}e^{\boldsymbol{i}\alpha}.
\end{aligned}$$

The first expressions are a direct consequence of equation (2). For the second expressions, say of X, use that

$$\begin{aligned}
1-u_B &= 1-e^{2\boldsymbol{i}\beta} = 1-\cos 2\beta - \boldsymbol{i}\sin 2\beta = 2\sin^2\beta - 2\boldsymbol{i}\sin\beta\cos\beta \\
&= -2\boldsymbol{i}\sin\beta(\cos\beta + \boldsymbol{i}\sin\beta) = -2\boldsymbol{i}\sin\beta\ e^{\boldsymbol{i}\beta},
\end{aligned}$$

hence $\frac{1-u_B}{1-u_B u_C} = \frac{\sin\beta\ e^{\boldsymbol{i}\beta}}{\sin(\beta+\gamma)e^{\boldsymbol{i}(\beta+\gamma)}} = \frac{\sin\beta}{\sin(\beta+\gamma)}e^{-\boldsymbol{i}\gamma}$.

Finally, the third expressions, say of X, is derived from the second as follows:

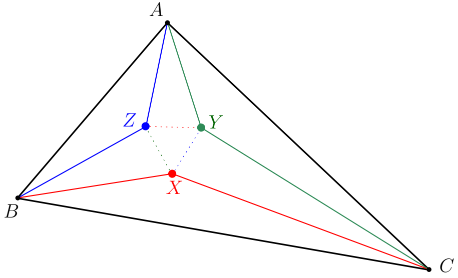
$$C+(B-C)\tfrac{\sin\beta}{\sin(\beta+\gamma)}e^{-\boldsymbol{i}\gamma} = B+(C-B)-(C-B)\tfrac{\sin\beta}{\sin(\beta+\gamma)}e^{-\boldsymbol{i}\gamma}$$

$$= B+(C-B)(1-\tfrac{\sin\beta}{\sin(\beta+\gamma)}e^{-\boldsymbol{i}\gamma}) \text{ and}$$

$$\sin(\beta+\gamma)-\sin\beta\ e^{-\boldsymbol{i}\gamma} = \sin\beta\cos\gamma+\cos\beta\sin\gamma-\sin\beta\cos\gamma+\boldsymbol{i}\sin\beta$$

$$= \sin\gamma(\cos\beta+\boldsymbol{i}\sin\beta) = \sin\gamma\ e^{\boldsymbol{i}\beta}. \quad \square$$

- Knowing X, Y, Z, we can draw the trisectors:



- Let us set (cf. Figure 1):
$$\boldsymbol{a} = C - B, \quad \boldsymbol{b} = A - C, \quad \boldsymbol{c} = B - A,$$
so that $\boldsymbol{a} = a\boldsymbol{a}^\star$, $\boldsymbol{b} = b\boldsymbol{b}^\star$, $\boldsymbol{c} = c\boldsymbol{c}^\star$ ($\boldsymbol{a}^\star, \boldsymbol{b}^\star, \boldsymbol{c}^\star$ unit vectors).

Then we have the following relations:
$$\boldsymbol{b}^\star = -\boldsymbol{a}^\star e^{-\boldsymbol{i}3\gamma}, \quad \boldsymbol{c}^\star = -\boldsymbol{b}^\star e^{-\boldsymbol{i}3\alpha}, \quad \boldsymbol{a}^\star = -\boldsymbol{c}^\star e^{-\boldsymbol{i}3\beta}. \tag{3}$$
Notice, for example, that $\boldsymbol{b}^\star = \boldsymbol{a}^\star e^{\boldsymbol{i}(\pi-3\gamma)} = -\boldsymbol{a}^\star e^{-\boldsymbol{i}3\gamma}$.

θ'	θ''	θ^*	θ^{**}
$\frac{\pi}{3}+\theta$	$\frac{2\pi}{3}+\theta$	$\frac{\pi}{3}-\theta$	$\frac{2\pi}{3}-\theta$

$$\boxed{\sin(3\theta) = 4\sin\theta\sin\theta^*\sin\theta^{**}}$$

- Since clearly $\theta' + \theta^{**} = \theta'' + \theta^* = \pi$, then $\sin\theta^{**} = \sin\theta'$, $\cos\theta^{**} = -\cos\theta'$, $\sin\theta^* = \sin\theta''$, $\cos\theta^* = -\cos\theta''$. Note that according to the convention declared in Figure 1 the value θ'' agrees with $\bar{\theta}$. We also have that $\sin(\beta+\gamma) = \sin(\frac{\pi}{3}-\alpha) = \sin\alpha^*$ and similarly $\sin(\gamma+\alpha) = \sin\beta^*$, $\sin(\alpha+\beta) = \sin\gamma^*$.

With the notations and conventions declared in the previous slide, the third expression of Z and the second of Y in $\diamond\mathbf{4}$ take the form

$$Z = A + \boldsymbol{c}^\star c \frac{\sin\beta}{\sin\gamma^*} e^{\boldsymbol{i}\alpha}, \quad Y = A - \boldsymbol{b}^\star b \frac{\sin\gamma}{\sin\beta^*} e^{-\boldsymbol{i}\alpha}.$$

Since $\boldsymbol{c}^\star = -\boldsymbol{b}^\star e^{-\boldsymbol{i}3\alpha}$, by (3), we readily get

$$Z - Y = \boldsymbol{b}^\star \left(\frac{b\sin\gamma}{\sin\beta^*} - \frac{c\sin\beta}{\sin\gamma^*} e^{-\boldsymbol{i}\alpha} \right) e^{-\boldsymbol{i}\alpha}.$$

Now use the sine theorem to replace $b = \delta\sin 3\beta$ and $c = \delta\sin 3\gamma$, and then the boxed *trig identity* to obtain

$$Z - Y = 4\delta\sin\beta\sin\gamma\, \boldsymbol{b}^\star (\sin\beta^{**} - \sin\gamma^{**} e^{-\boldsymbol{i}\alpha}) e^{-\boldsymbol{i}\alpha}.$$

Since $\beta^{**} = \gamma' + \alpha$, $\sin\beta^{**} = \sin\gamma'\cos\alpha + \cos\gamma'\sin\alpha$. On the other hand, $\sin\gamma^{**} e^{-\boldsymbol{i}\alpha} = \sin\gamma'\cos\alpha - \boldsymbol{i}\sin\gamma'\sin\alpha$. Therefore,

$$\sin\beta^{**} - \sin\gamma^{**} e^{-\boldsymbol{i}\alpha} = \cos\gamma'\sin\alpha + \boldsymbol{i}\sin\gamma'\sin\alpha = \sin\alpha\ e^{\boldsymbol{i}\gamma'}.$$

Since $e^{\boldsymbol{i}\gamma'} e^{-\boldsymbol{i}\alpha} = e^{\boldsymbol{i}(\gamma'-\alpha)} = e^{\boldsymbol{i}(\beta+2\gamma)}$, we get:

$$Z - Y = L\,\boldsymbol{b}^{\star} e^{\boldsymbol{i}(\beta+2\gamma)}, \tag{4}$$

where $L = 4\delta \sin\alpha \sin\beta \sin\gamma$.

The expressions for $X - Z$ and $Y - X$ are obtained from (4) by cyclic permutations:

$$X - Z = L\,\boldsymbol{c}^{\star} e^{\boldsymbol{i}(\gamma+2\alpha)},$$

$$Y - X = L\,\boldsymbol{a}^{\star} e^{\boldsymbol{i}(\alpha+2\beta)}.$$

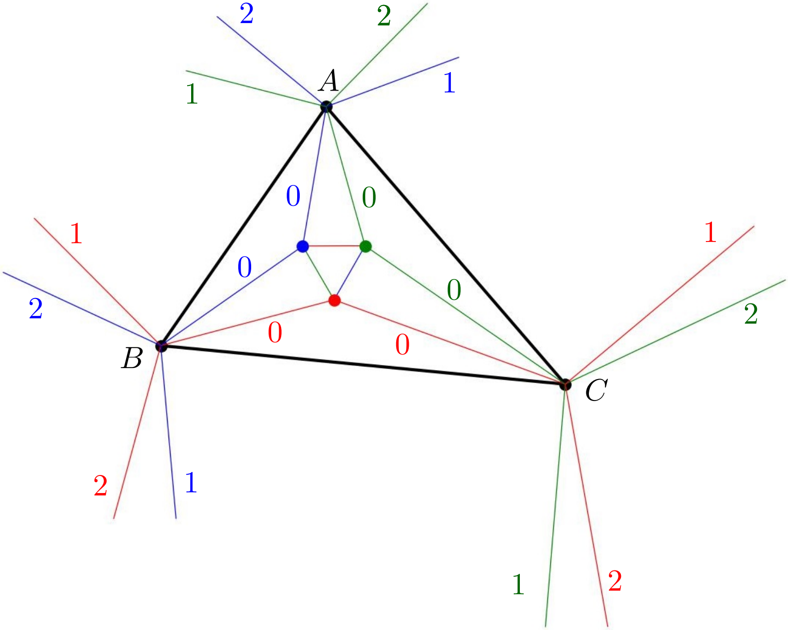
In particular, $XY = YZ = ZX = L$ and XYZ is equilateral. $\square$

Using the relations (3), the directions of the sides of XYZ relative to $\boldsymbol{a}^{\star}$, say, are as follows:

$$Z - Y = L\boldsymbol{a}^{\star} e^{\boldsymbol{i}(\pi+\beta-\gamma)}$$

$$X - Z = L\boldsymbol{a}^{\star} e^{-\boldsymbol{i}(\alpha+2\gamma)}$$

$$Y - X = L\boldsymbol{a}^{\star} e^{\boldsymbol{i}(\alpha+2\beta)}.$$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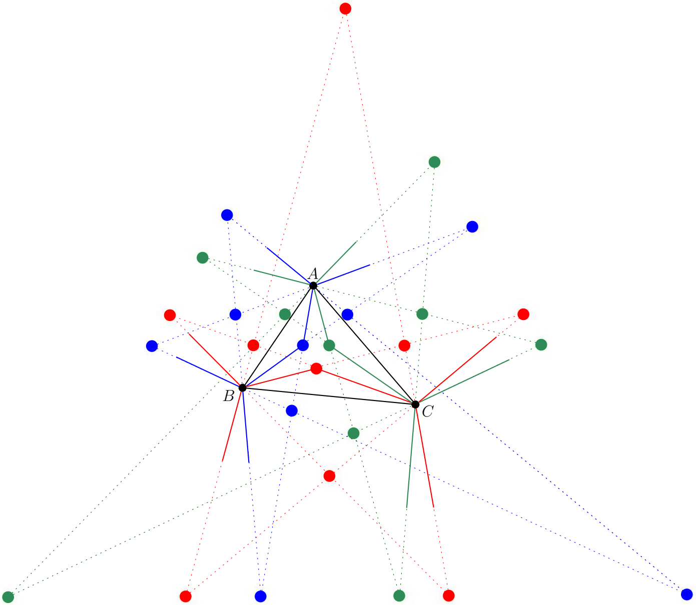
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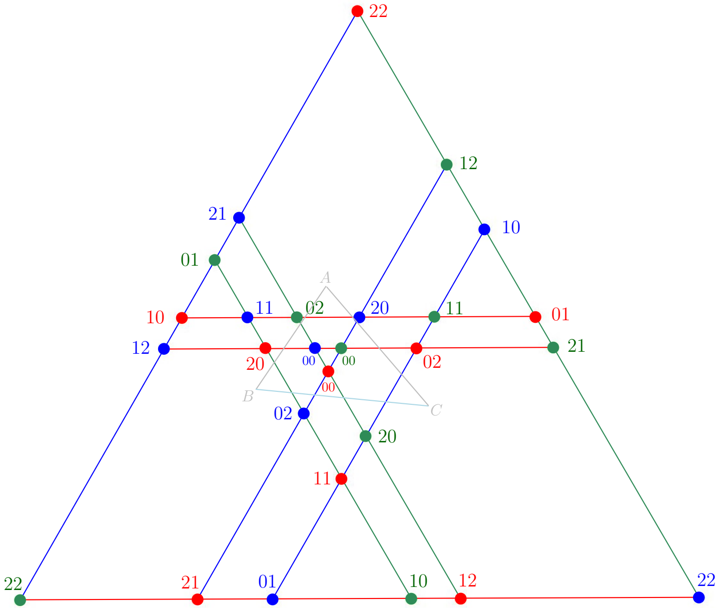
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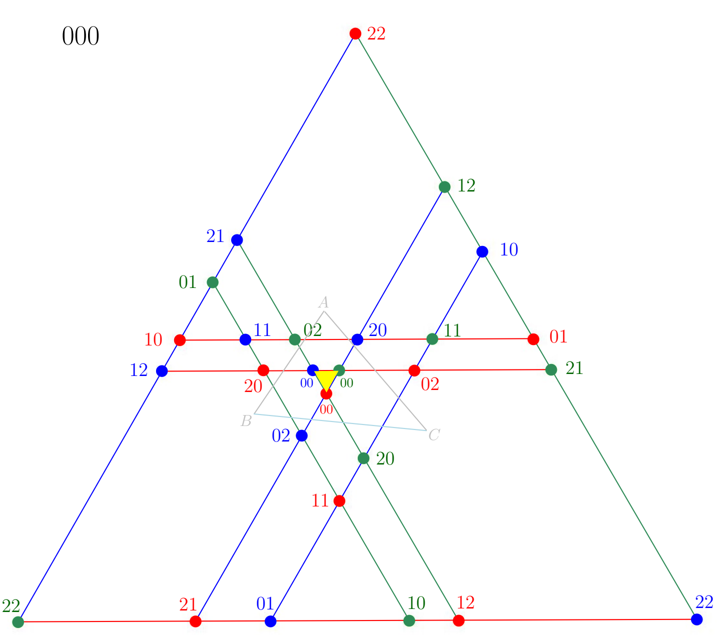
A Morley gallery

Triangle ijk**:** $ij\,jk\,ki$**,** $i, j, k \in \{0, 1, 2\}$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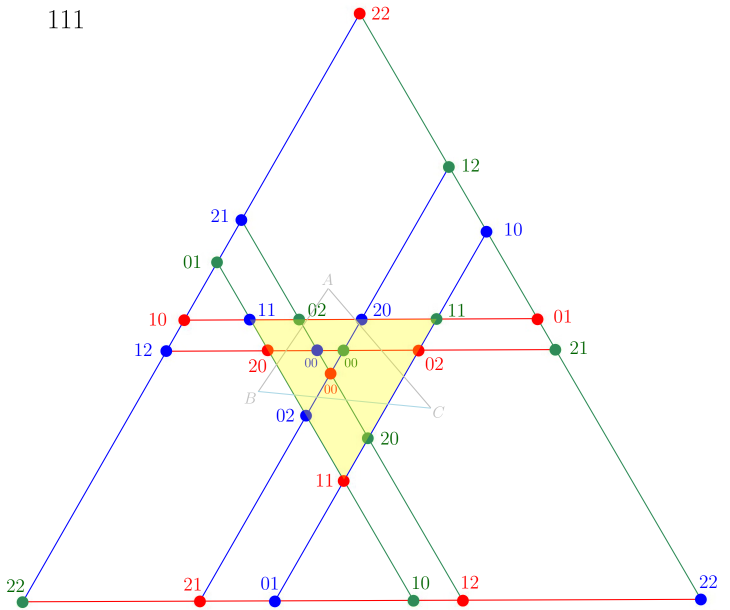
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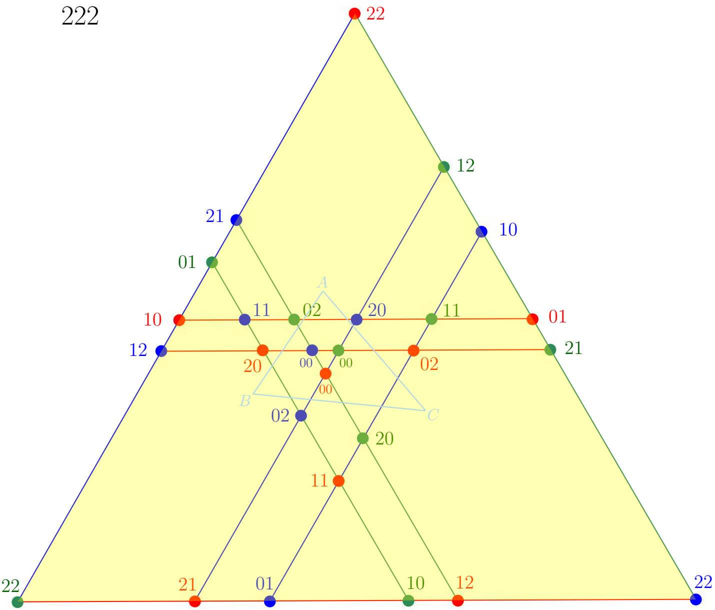
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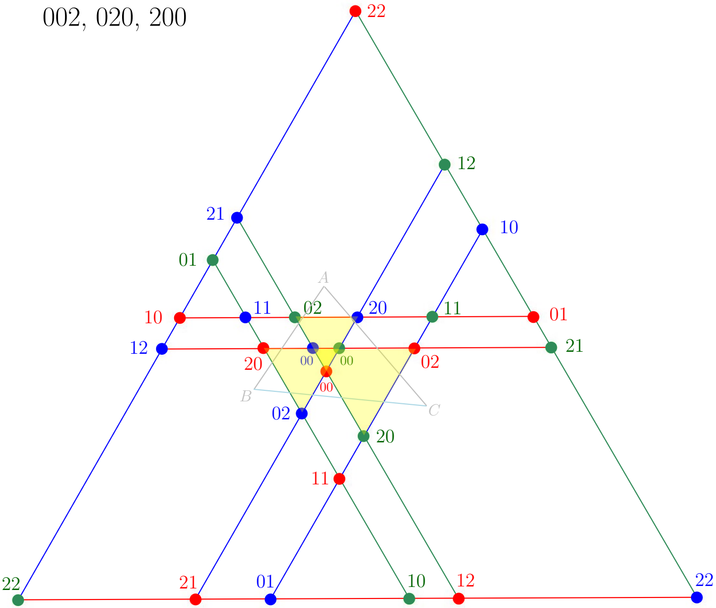
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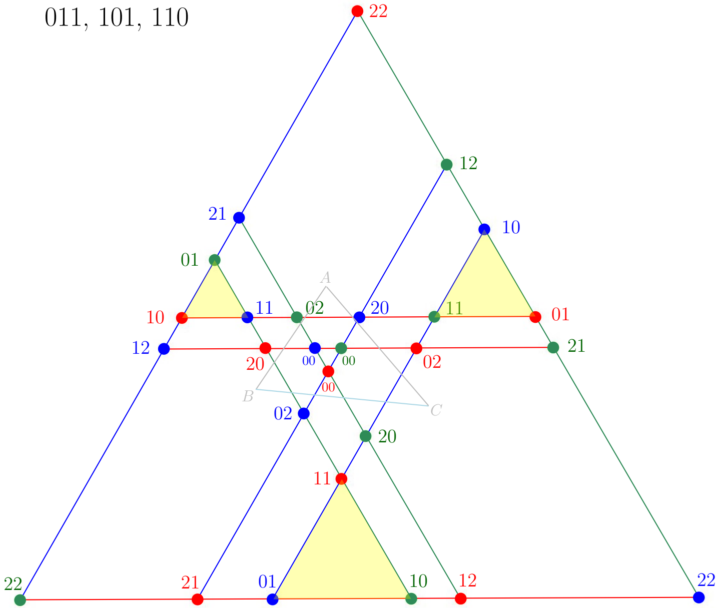
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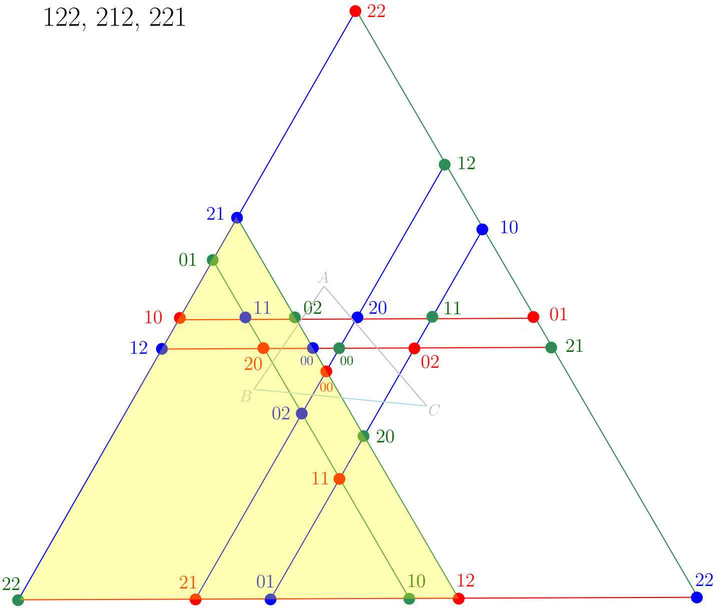
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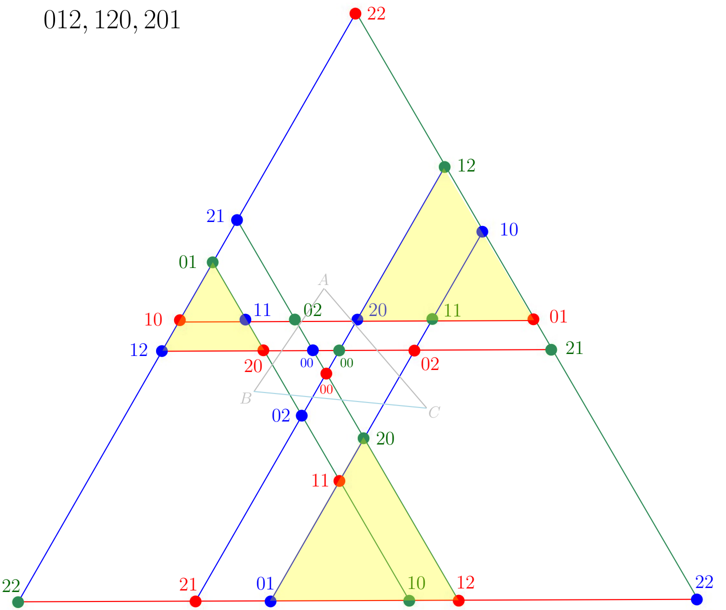
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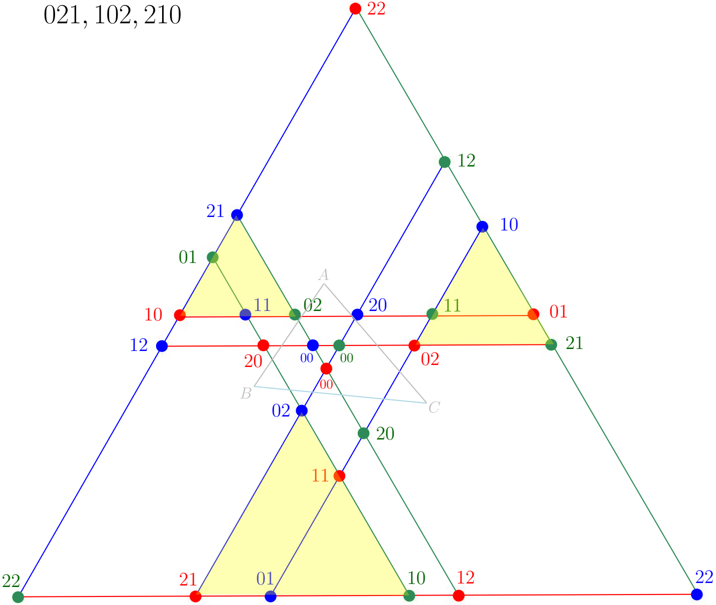
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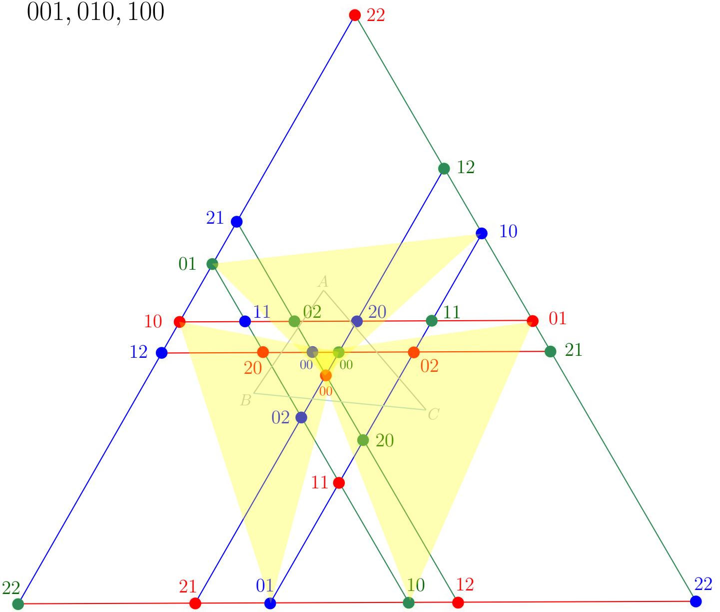
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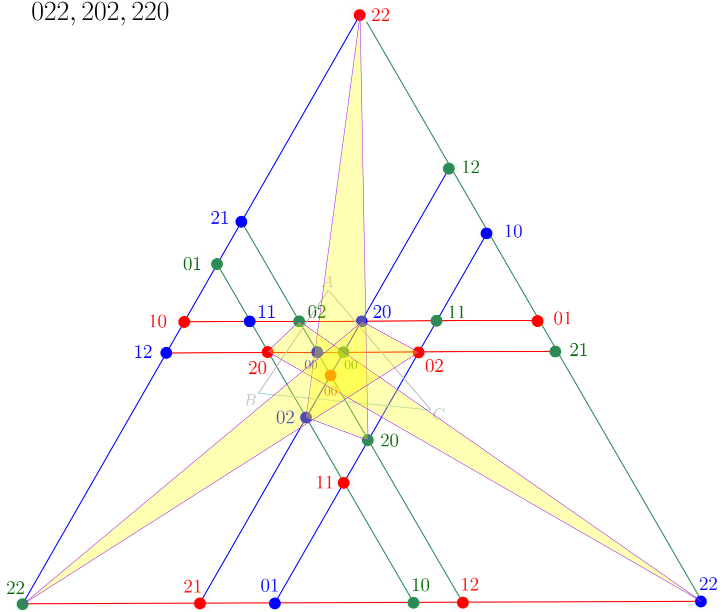
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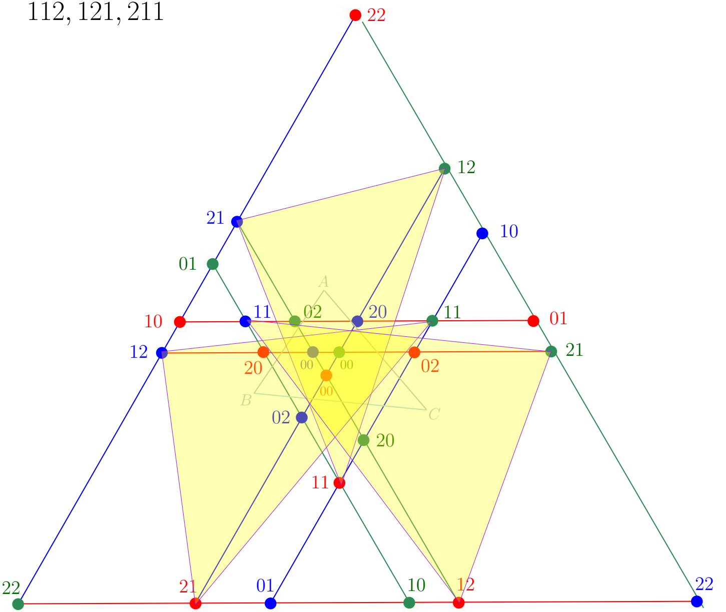
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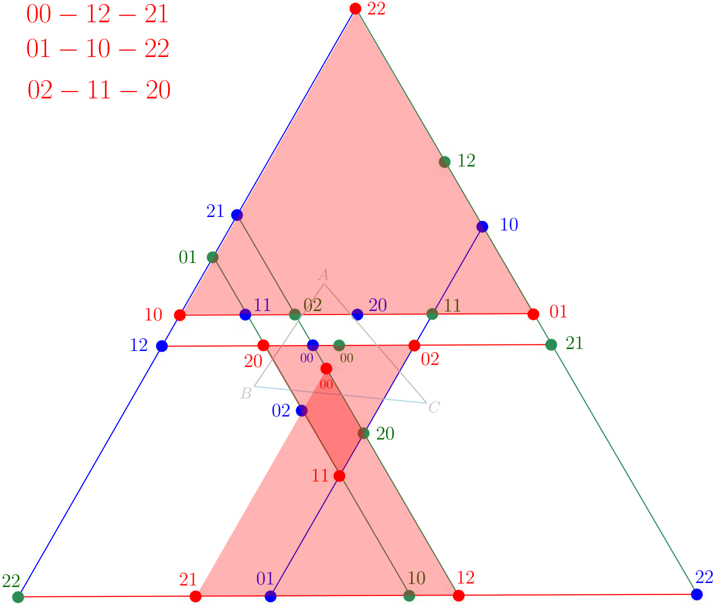

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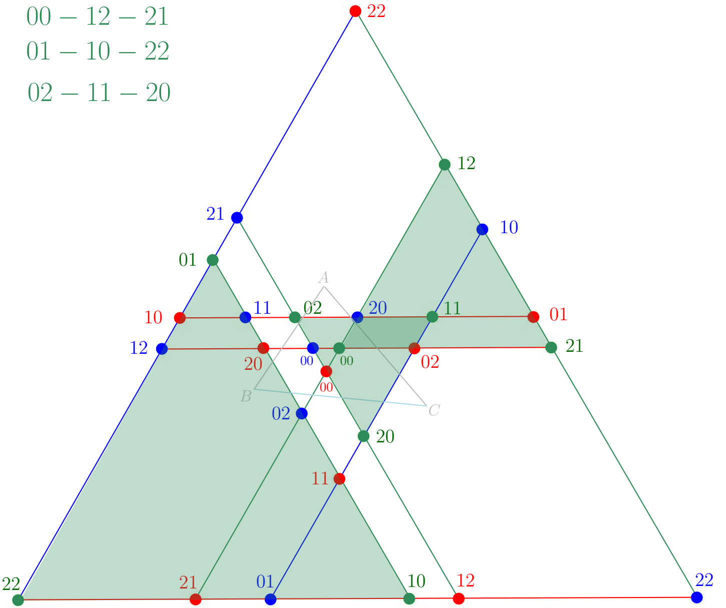

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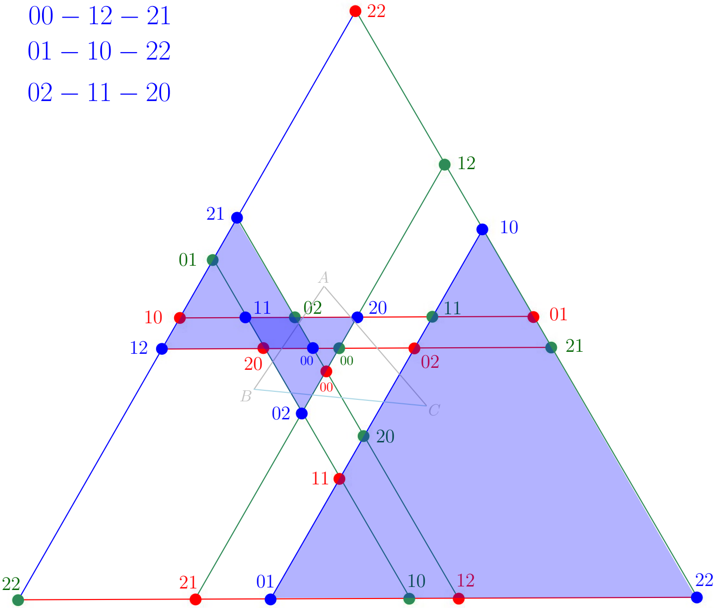

- By intersecting angle trisectors of the same color we get 9 Morley points of that color, 27 in all.
- In Morley's constellation we find 9 Morley lines: three lines parallel to YZ (depicted in red), three parallel to ZX (in green) and three parallel to XY (in blue). Each of these lines contains six Morely points, two of each color. The relation $3 \times (3 \times 6) = 2 \times 27$ just says that each Morley point belongs to precisely two Morley lines.
- There are 27 tricolor Morley triangles, 18 equilateral and 9 non-equilateral, together with 9 equilaterals monochrome Morley triangles, three of each color. The sides of the 27 equilateral triangles are parallel to the sides of XYZ. Of the 18 Morley tricolor triangles, 9 are positively oriented and 9 negatively oriented.
- The 9 Morley triangles $ij - jk - ki$ that are not equilateral are those for which $i + j + k \equiv 1 \mod 3$.

Courtesy of [3], Remark 1, the red point $ij = \ell_B^i \cap \ell_C^j$ of the MC ($i, j \in \{0, 1, 2\}$) is the fixed point of the composition

$$R_{B,u_B\omega^i} R_{C,u_C\omega^{-j}}, \quad \omega = e^{2\pi \boldsymbol{i}/3}$$

(this is a generalization of $\diamond\mathbf{3}(2)$, as there the case 00 was established). Analogous formulas are valid for the green and blue points of the MC.

Comment. In the said Remark 1 it is stated that "one obtains in this way the 18 nondegenerate (*sic*) equilateral triangles of variants of Morley's theorem", with no insight into the fact that the sides of all 18 triangles are parallel to the sides of the Morley triangle XYZ, and no explanation of what 'nondegenerate' might mean (as we saw above, they are in fact non-equilateral tricolor Morley triangles), and no reference to the remaining nine equilateral triangles (in our terms, the monochrome Morley triangles).

References I

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[2] S. Xambó-Descamps, "Geometry and Physics with Geometric Algebra," *Geometric Mechanics*, vol. 2(3), no. 3, pp. 337–383, 2025.

[3] A. Connes, "A new proof of Morley's theorem," *Publications Mathématiques de l'IHÉS*, vol. S88, pp. 43–46, 1998.

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