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Spinning spinors with GA for one century and beyond

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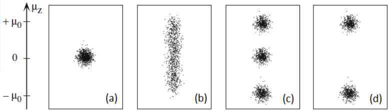
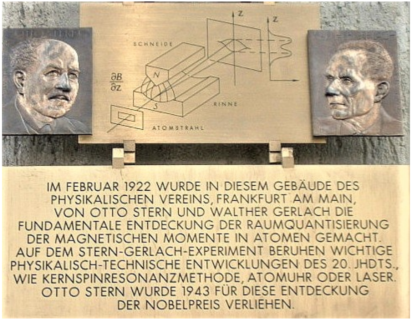
https://web.mat.upc.edu/sebastia.xambo/99/xambo-spinning.pdf

Commemoration

(Adapted from Fig. 8.2 in [1])



- One hundred years of the crucial experiments by Stern and Gerlach:
  Experimental evidence of the directional quantization in a magnetic field [2]



See the animation in the Wikipedia 'Stern–Gerlach_experiment' article explaining the contrast between the behavior of a classical magnet and that of a quantum spin.

Epistemological analysis: *Wrong theory—Right experiment: The significance of the Stern-Gerlach experiments* [3]

■ Five hundred years of the first 'spin' around the Earth (Fernão de Magalhães/Fernando de Magallanes/Ferdinand Magellan–Juan Sebastián Elcano)

"The most promising words ever written on the maps of human knowledge are *terra incognita*–unknown territory" (*The Discoverers* [4], page XVI).

Geometric algebra

Notations and background facts

N

- $\mathcal{G} = \mathcal{G}_\eta(V)$ denotes the *geometric algebra* of a vector space V endowed with a quadratic form η. Let $n = \dim(V)$.
- $\mathcal{G}_{r,s,t}$ if V is a *real* vector space and η has signature (r, s, t) $(r + s + t = n)$.
- $\mathcal{G}_{r,s}$ if η is non-degenerate $(t = 0)$, so $r + s = n$; $\mathcal{G}_n$ if $s = 0$.

Remark: We let $\eta(v, v') = \frac{1}{2}\big(\eta(v + v') - \eta(v) - \eta(v')\big)$, the *polarized form* of η, so $\eta(v, v) = \eta(v)$.

$\mathcal{G}$ comes equipped with several essential features:

Grade decomposition: $\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \mathcal{G}^2 \oplus \cdots \oplus \mathcal{G}^{n-1} \oplus \mathcal{G}^n$: any $x \in \mathcal{G}$ has a unique expression $x = x_0 + x_1 + x_2 + \cdots + x_{n-1} + x_n$ with $x_k \in \mathcal{G}^k$ (x_k is often denoted $\langle x \rangle_k$, and $\langle x \rangle_0 = \langle x \rangle$). The elements of the linear space $\mathcal{G}^k$ are said to have *grade* k.

$\dim(\mathcal{G}^k) = \binom{n}{k}$, so $\dim(\mathcal{G}) = 2^n$.

$\mathcal{G}^0$ is the *scalar field* ($\mathbf{R}$, for instance), and $\mathcal{G}^1 = V$ (*vectors*).

In general the elements of $\mathcal{G}^k$ are called *k-vectors* (*bivectors*, *trivectors*,... for $k = 2, 3, \ldots$), and the elements of $\mathcal{G}$, *multivectors*.

Since $\dim(\mathcal{G}^n) = 1$, the elements of $\mathcal{G}^n$ are called *pseudoscalars*.

Outer or $\wedge$ product. $x \wedge x' \in \mathcal{G}$, $x, x' \in \mathcal{G}$. It is *unital*, *bilinear*, and *associative*. Moreover, $x \wedge x' \in \mathcal{G}^{k+k'}$ when $x \in \mathcal{G}^k, x' \in \mathcal{G}^{k'}$.

In particular $x = v_1 \wedge \cdots \wedge v_k \in \mathcal{G}^k$ if $v_1, \ldots, v_k \in V$.

Fundamental property. $x = v_1 \wedge \cdots \wedge v_k \neq 0$ if and only if $v_1, \ldots, v_k$ are linearly independent, and in this case x is said to be a *k-blade*.

The relation $v \wedge v = 0$ for any $v \in V$ implies that $v' \wedge v = -v \wedge v'$ ($v, v' \in V$) and, in general, $x' \wedge x = (-1)^{kk'} x \wedge x'$ if $x \in \mathcal{G}^k, x' \in \mathcal{G}^{k'}$.

Basis. Let $e_1, \ldots, e_n$ be a basis of V. Let $\mathcal{I}_k = \{I = i_1, \ldots, i_k\}$ be the set of *k-multiindices* ($1 \leqslant i_1 < \cdots < i_k \leqslant n$), and for each I, set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$. Then $\{e_I \mid I \in \mathcal{I}_k\}$ is a basis of $\mathcal{G}^k$. N

Inner or $\cdot$ product. $x \cdot x' \in \mathcal{G}$ for all $x, x' \in \mathcal{G}$. It is *bilinear*, but it is *neither unital nor associative*. And it is not *commutative*.[1]

It is determined by the following rules:

- $\lambda \cdot x = x \cdot \lambda = 0$ for any $x \in \mathcal{G}$ and any scalar λ.
- $v \cdot v' = \eta(v, v')$, $v \cdot (v_1 \wedge v_2) = \eta(v, v_1)v_2 - \eta(v, v_2)v_1$, ...
  $v \cdot (v_1 \wedge \cdots \wedge v_k) = \sum_{j=i}^{k} (-1)^{j-1} \eta(v, v_j) v_1 \wedge \cdots \wedge v_{j-1} \wedge v_{j+1} \wedge \cdots \wedge v_k$
- If $x \in \mathcal{G}^k$, $x' \in \mathcal{G}^j$, and $k \geqslant j \geqslant 1$, then
  $$x \cdot x' = (-1)^{(k-j)j} x' \cdot x \text{ (\textit{commutation rule})}.$$
  Example: $(v_1 \wedge \cdots \wedge v_k) \cdot v = (-1)^{k-1} v \cdot (v_1 \wedge \cdots \wedge v_k)$.
- If $1 < k \leqslant j$, $x' \in \mathcal{G}^j$, $(v_1 \wedge v_2 \wedge \cdots v_k) \cdot x' = (v_1 \wedge \cdots v_{k-1}) \cdot (v_k \cdot x')$

[1] It extends the inner product of vectors, but it is $\mathcal{G}$-valued, not scalar valued, and it is *not commutative*, so some people are bewildered by this usage of the 'dot'.

Relation to η. The form η extends naturally to $\mathcal{G}$. This extension is managed by the following rules: $\mathcal{G}^k \perp \mathcal{G}^j$ for $j \neq k$ and, as will be seen on page 13, $\eta(x,x') = (-1)^{k/\!\!/2}(x \cdot x')$ if $x, x' \in \mathcal{G}^k$. Here $k/\!\!/2 = \lfloor k/2 \rfloor$. Its usefulness is that it has the same parity as $\binom{k}{2}$.

Notation. If $\boldsymbol{e} = e_1, \ldots, e_n$ is a basis of V, and I is a multiindex, $\eta(e_I)$ will be abriged to η_I. In particular, $\eta_j = \eta(e_j)$, and if $\boldsymbol{e}$ is *orthogonal*, then $\eta_I = \eta_{i_1} \cdots \eta_{i_k}$.

In terms of $\boldsymbol{e}$, the inner product is determined by bilinearity, the commutation rule, and the following fact: If I, I' are multiindices and $|I| \leqslant |I'|$, then

$$e_I \cdot e_{I'} = \begin{cases} 0 & \text{when } I \not\subset I' \\ \eta_I \sigma(I,I') e_{I'-I} & \text{otherwise} \end{cases}$$

Geometric product. $xx' \in \mathcal{G}$ for all $x, x' \in \mathcal{G}$. It is *unital*, *bilinear* and *associative*.

It is determined by bilinearity and the following laws:

- (*Contraction*) $v^2 = \eta(v)$.
- (*Transference*) $v(v_1 \wedge \cdots \wedge v_k) = v \wedge v_1 \wedge \cdots \wedge v_k$ if $\eta(v, v_j) = 0$ $(j = 1, \ldots, k)$.

The contraction rule is equivalent to *Clifford's relations*:

- $vv' + v'v = 2\eta(v, v') = 2v \cdot v'$ for all $v, v' \in V$.

In particular it follows that $vv' = -v'v$ if and only if v and v' are orthogonal.

- If $v_1, \ldots, v_k \in V$ are pairwise orthogonal, then

$$v_1 \cdots v_k = v_1 \wedge \cdots \wedge v_k.$$

In particular, if $e_1, \ldots, e_n$ is an *orthogonal* basis of V, then $e_I = e_{i_1} \cdots e_{i_k}$ for any multiindex $I = i_1, \ldots, i_k$ (not necessarily in increasing order).

Recursive formulas. For any $v \in V$ and any $x \in \mathcal{G}$:

- $vx = v \cdot x + v \wedge x$ and $xv = x \cdot v + x \wedge v$.

Artin's formula. Let $e_1, \ldots, e_n$ be an *orthogonal* basis of V and I, I' multiindices. Then

- $e_I e_{I'} = \eta_{I \cap I'} \sigma(I, I') e_{I \Delta I'}$, where $I \Delta I'$ is the *symmetric difference* of I and I', sorted in increasing order.

Remark. Since any blade can be written as the wedge of pairwise orthogonal vectors, the previous slide implies that any blade can be expressed as the geometric product of pairwise ortogonal vectors.

Parity involution. It is the *automorphism* of $\mathcal{G}$, $x \mapsto \hat{x}$, determined by the condition $\hat{v} = -v$ for $v \in V$. For $x \in \mathcal{G}^k$, $\hat{x} = (-1)^k x$.

- For all $x, x' \in \mathcal{G}$, $\widehat{x \wedge x'} = \hat{x} \wedge \hat{x}'$, $\widehat{x \cdot x'} = \hat{x} \cdot \hat{x}'$, $\widehat{xx'} = \hat{x}\hat{x}'$.

Even subalgebra

- $\mathcal{G}^+ = \{x \in \mathcal{G} \,|\, \hat{x} = x\}$, $\mathcal{G}^- = \{x \in \mathcal{G} \,|\, \hat{x} = -x\}$, $\mathcal{G} = \mathcal{G}^+ \oplus \mathcal{G}^-$. N

Reverse involution. It is the *antiautomorphism* of $\mathcal{G}$, $x \mapsto \tilde{x}$, determined by the condition $\tilde{v} = v$ for $v \in V$. For $x \in \mathcal{G}^k$, $\tilde{x} = (-1)^{k/\!\!/2} x$.

- For all $x, x' \in \mathcal{G}$, $\widetilde{x \wedge x'} = \tilde{x}' \wedge \tilde{x}$, $\widetilde{x \cdot x'} = \tilde{x}' \cdot \tilde{x}$, $\widetilde{xx'} = \tilde{x}'\tilde{x}$.

The parity and reverse involutions commute, as they act by a sign in each grade. The composition of the two is called the *Clifford involution* and is denoted by $\bar{x}$. Thus we have $\bar{x} = \tilde{\hat{x}} = \hat{\tilde{x}}$. If $x \in \mathcal{G}^k$, $\bar{x} = (-1)^k(-1)^{k(k-1)/2} x = (-1)^{k(k+1)/2} x = (-1)^{(k+1)/\!\!/2} x$.

- $2v \wedge x = vx + \hat{x}v, \quad 2v \cdot x = vx - \hat{x}v$ (*Riesz formulas*).
- $v \cdot (xx') = (v \cdot x)x' + \hat{x}(v \cdot x')$
- If $x \in \mathcal{G}^j$ and $x' \in \mathcal{G}^k$, then $(xx')_i$ is 0 unless i is in the range $|j-k|, \ldots, j+k$ and has the same parity as $j+k$.
- $(xx')_{|j-k|} = x \cdot x'$ if $j, k > 0$ and $(xx')_{j+k} = x \wedge x'$.

η *in terms of the geometric product*. For all $x, x' \in \mathcal{G}$, $\eta(x, x') = (\tilde{x}x')_0 = (x\tilde{x}')_0$, and $\eta(x) = (\tilde{x}x)_0 = (x\tilde{x})_0$.

If $x, x' \in \mathcal{G}^k$, $\eta(x, x') = (-1)^{k/\!\!/2}(xx')_0$ and $\eta(x) = (-1)^{k/\!\!/2}(x^2)_0$.

If x is a k-blade, then $\tilde{x}x$ is already a scalar (express x as the geometric product of k vectors that are pairwise orthogonal) and $\eta(x) = \tilde{x}x = (-1)^{k/\!\!/2}x^2$. In particular we see that x is invertible if and only if $x^2 \neq 0$, or if and only if $\eta(x) \neq 0$, and in this case we have $x^{-1} = x/x^2 = \tilde{x}/\eta(x)$.

Pseudoscalars

Let $\boldsymbol{e} = e_1, \ldots, e_n$ be an orthonormal basis of $V = V_{r,s}$ and define

$$\mathbf{i}_{\boldsymbol{e}} = e_1 \wedge \cdots \wedge e_n \in \mathcal{G}^n.$$

We will say that $\mathbf{i}_{\boldsymbol{e}}$ is the *pseudoescalar* associated to $\boldsymbol{e}$. Note that the metric formula gives us that

$$\eta(\mathbf{i}_{\boldsymbol{e}}) = \eta(e_1) \cdots \eta(e_n) = (-1)^s.$$

If $\boldsymbol{e}' = e'_1, \ldots, e'_n$ is another orthonormal basis of V, then

$$\mathbf{i}_{\boldsymbol{e}'} = \delta \mathbf{i}_{\boldsymbol{e}},$$

where $\delta = \det_{\boldsymbol{e}}(\boldsymbol{e}')$ is the determinant of the matrix of the vectors $\boldsymbol{e}'$ with respect to the basis $\boldsymbol{e}$. Now the equalities

$$\eta(\mathbf{i}_{\boldsymbol{e}}) = \eta(\mathbf{i}_{\boldsymbol{e}'}) = \eta(\delta \mathbf{i}_{\boldsymbol{e}}) = \delta^2 q(\mathbf{i}_{\boldsymbol{e}})$$

allow us to conclude that $\delta = \pm 1$. This means that, up to sign, there is a unique pseudoscalar. The distinction of one of them amounts to a choice of an *orientation* for V.

Let $\mathbf{i} \in \mathcal{G}^n$ be a pseudoescalar and $\mathcal{G}^\times$ the group of invertible multivectors with respect to the geometric product. Then

1. $\mathbf{i} \in \mathcal{G}^\times$, $\quad \mathbf{i}^{-1} = (-1)^s \tilde{\mathbf{i}} = (-1)^s (-1)^{n/\!\!/2} \mathbf{i}$, $\quad \mathbf{i}^2 = (-1)^{n/\!\!/2}(-1)^s$.

2. *Hodge duality*. For any $x \in \mathcal{G}^k$, $\mathbf{i}x, x\mathbf{i} \in \mathcal{G}^{n-k}$ (by Artin's formula) and the maps $x \mapsto \mathbf{i}x$ and $x \mapsto x\mathbf{i}$ give linear isomorphisms $\mathcal{G}^k \to \mathcal{G}^{n-k}$. The inverse maps are given by $x \mapsto \mathbf{i}^{-1}x$ and $x \mapsto x\mathbf{i}^{-1}$, respectively.

3. For any vector v, $v\mathbf{i} = (-1)^{n-1}\mathbf{i}v$. So $\mathbf{i}$ commutes with all the elements of $\mathcal{G}$ if n is odd (this is expressed by saying that $\mathbf{i}$ is a *central* element of $\mathcal{G}$). If n is even, $\mathbf{i}$ commmutes (anticommutes) with even (odd) multivectors.

4. If $\eta(\mathbf{i}) = 1$ ($\eta(\mathbf{i}) = -1$), the Hodge duality maps are *isometries* (*antiisometries*).

The formula $\mathbf{i}^2 = (-1)^{n/\!\!/2}(-1)^s$ implies, setting $\nu = s - r \pmod 8$,

$$\mathbf{i}^2 = \begin{cases} 1 & \text{if } \nu = 0,3,4,7 \quad (0,3 \mod 4), \\ -1 & \text{if } \nu = 1,2,5,6 \quad (1,2 \mod 4). \end{cases} \tag{1}$$

In fact from $\nu = s - r \pmod 8$ we have $s = r + \nu \pmod 8$, $n = r + s = 2r + \nu \pmod 8$, and the sign of $\mathbf{i}^2$ coincides with the parity of

$$n/\!\!/2 + s = r + \nu/\!\!/2 + r + \nu \mod 8,$$

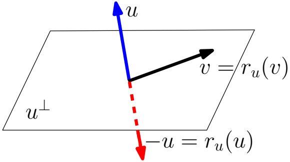
which is the parity of $(3\nu)/\!\!/2$, or more simply of $(\nu + 1)/\!\!/2$.

Note. Since $n \equiv \nu \pmod 2$, $\mathbf{i}$ is central if and only if ν is odd.

Spinors and their kinds

Basic observation. If $u \in V = V_{r,s}$ is a non-null vector (i.e. $u^2 \neq 0$), the expression $uvu^{-1} \in V$ for any $v \in V$, and the *linear* map $r_u : v \mapsto -uvu^{-1} = \hat{u}vu^{-1}$ is the *reflection* about the hyperplane $u^{\perp} = \{v \in V \mid u \cdot v = 0\}$.

Proof. If $v \in u^{\perp}$, $r_u(v) = \hat{u}vu^{-1} = -\hat{u}u^{-1}v = v$. On the other hand, $r_u(u) = \hat{u}uu^{-1} = -u$. □



- $\{r_u \mid u \in V, u^2 = \pm 1\}$ is the set of all reflections in hyperplanes, with the redundancy $r_{-u} = r_u$ only.

The groups $\mathrm{Pin}_{r,s}$ *and* $\mathrm{Spin}_{r,s}$. They are the subgroups of $\mathcal{G}^{\times}_{r,s}$ whose elements are products of unit vectors (*pinors*) and of an even number of unit vectors (*spinors*), respectively ($\mathrm{Spin}_{r,s} = \mathrm{Pin}^{+}_{r,s}$).

Basic fact. If $\mathfrak{p} \in \mathrm{Pin}_{r,s}$, $r_{\mathfrak{p}}(v) = \hat{\mathfrak{p}} v \mathfrak{p}^{-1} \in V$ for any $v \in V$, and the linear map $v \mapsto r_{\mathfrak{p}}(v)$ belongs to $\mathrm{O}_{r,s}$ (the group of *orthogonal transformations*, also called *isometries*, of V).

Indeed, if $\mathfrak{p} = u_1 \cdots u_m$, $u_1, \ldots, u_m$ unit vectors, then $r_{\mathfrak{p}}$ is the composition of the reflections $r_{u_1}, \ldots, r_{u_m}$, and hence it belongs to $\mathrm{O}_{r,s}$. Moreover, if $\mathfrak{p}$ is a spinor, then $r_{\mathfrak{p}} \in \mathrm{SO}_{r,s}$.

By the Cartan-Dieudonné theorem, the maps $\mathrm{Pin}_{r,s} \to \mathrm{O}_{r,s}$ and $\mathrm{Spin}_{r,s} \to \mathrm{SO}_{r,s}$ given by $\mathfrak{p} \mapsto r_{\mathfrak{p}}$ are onto, and it turns out that $r_{\mathfrak{p}} = Id$ if and only if $\mathfrak{p} = \pm 1$.

Rotors. $\mathrm{Spin}^0 = \mathrm{Spin}^0_{r,s}$ is the group of *rotors*, i.e. spinors $\mathfrak{p}$ such that $\mathfrak{p}\tilde{\mathfrak{p}} = 1$. If $\mathrm{SO}^0_{r,s}$ is the connected component of $\mathrm{Id} \in \mathrm{SO}_{r,s}$, then $\mathrm{Spin}^0_{r,s} \twoheadrightarrow \mathrm{SO}^0_{r,s}$.

Note. If u and u' are unit vectors of opposite signs, the spinor uu' is not a rotor: $(uu')(uu')^{\sim} = uu'u'u = u^2 u'^2 = -1$.

Representations. A *representation* of $\mathcal{G} = G_{r,s}$ in a vector space S assigns to any $x \in \mathcal{G}$ a linear map $X : S \to S$ in such a way that the passage $x \mapsto X$ preserves sums and products: $x + x'$ and xx' are mapped to $X + X'$ and XX', respectively. Any such representation produces representations of $\mathrm{Pin}_{r,s}$ and $\mathrm{Spin}_{r,s}$ in an obvious way. If -1 is represented as the identity of S, we actually have representations of $\mathrm{O}_{r,s}$ and $\mathrm{SO}_{r,s}$, respectively. Otherwise the representation is said to be *spinorial* and in this case the elements of S are also called *spinors*.

The *vector representation* $\mathfrak{s} \mapsto r_{\mathfrak{s}}$ of $\mathrm{Spin}_{r,s}$ is clearly not spinorial (to wit, *vectors are not spinors*).

A particular case of spinorial representation is when S is a linear subspace of $\mathcal{G}$ and $X\mathfrak{z} = x\mathfrak{z}$. For this to make sense, it is required that $x\mathfrak{z} \in S$ for all $x \in \mathcal{G}$ and all $\mathfrak{z} \in S$, i.e., that S is a (left) *ideal* of $\mathcal{G}$. Among these, the most relevant are when S is *minimal*, which means that S does not contain any ideals other than $\{0\}$ and S. N

Classifications

Classes of $\mathcal{G}_{r,s}$ and $\mathcal{G}^+_{r,s}$
The horologion
Physics spinors

ν	0	1	2	3	4	5	6	7
F_ν	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$2\mathbf{H}$	$\mathbf{H}$	$\mathbf{C}$	$\mathbf{R}$	$2\mathbf{R}$
d_ν	1	2	4	8	4	2	1	2
k_ν	0	1	2	3	2	1	0	1
$\bar{\mathcal{G}}_\nu$	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$2\mathbf{H}$	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$2\mathbf{R}(8)$
$\mathcal{G}_\nu$	$\mathbf{R}$	$2\mathbf{R}$	$\mathbf{R}(2)$	$\mathbf{C}(2)$	$\mathbf{H}(2)$	$2\mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$

Table 1: For each integer ν mod 8 there is a basic algebra form that we denote by F_ν (second row). The third and fourth row contain the dimensions $d_\nu = \dim F_\nu = 2^{k_\nu}$ and the exponents k_ν. For any signature (r, s), $\mathcal{G}_{r,s} \simeq F_\nu(m)$, where $\nu = s - r$ mod 8 and where m is determined from the relation $2^n = d_\nu m^2$, or $m = 2^{(n-k_\nu)/2}$. The fifth row applies the prescription to $\bar{\mathcal{G}}_\nu = \mathcal{G}_{0,\nu}$. For example, $\bar{\mathcal{G}}_3 \simeq 2\mathbf{H}(m)$, with $2^3 = 8m^2$, hence $m = 1$ and $\bar{\mathcal{G}}_3 \simeq 2\mathbf{H}$. Similarly, $\bar{\mathcal{G}}_5 \simeq \mathbf{C}(m)$, where $2^5 = 2 \times m^2$, and so $\bar{\mathcal{G}}_5 \simeq \mathbf{C}(4)$. The sixth row describes $\mathcal{G}_\nu = \bar{\mathcal{G}}(8-\nu)$. Adapted from [5, Table 6.1].

```
F =['R','C','H', '2H', 'H', 'C','R','2R']
K =[0,1,2,3,2,1,0,1]

def Grs(r,s):
    nu = (s-r) % 8
    k = K[nu]
    m = 2**((r+s-k)//2)
    if m==1: A = F[nu]
    else: A = F[nu]+'('+str(m)+')'
    return A

def Gn(n):
    L = []
    if n==0: return [F[0]]
    for r in range(n+1):
    s = n-r
    L += [Grs(r,s)]              # [(G(0,n),...,G(n,0))]
return L
```

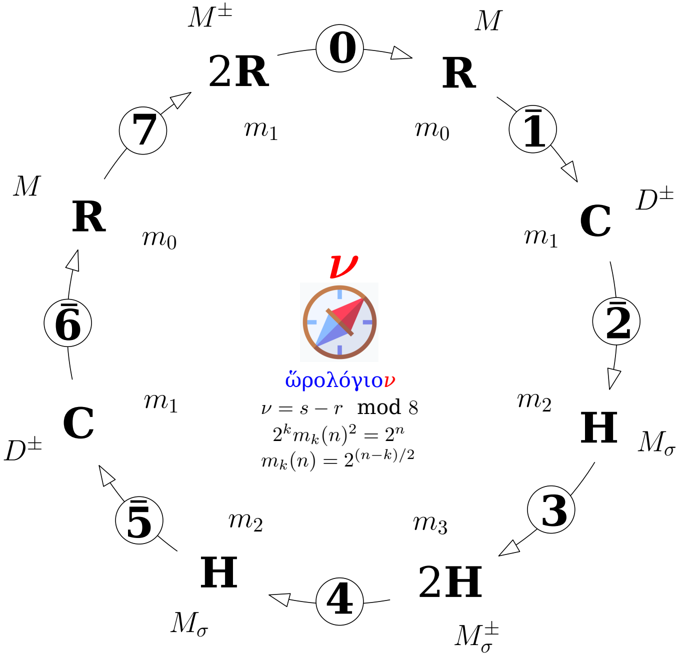
```
for nu in range(0,16): print(Grs(0,nu))

R,     C,     H,     2H,     H(2),  C(4),  R(8),   2R(8)

R(16), C(16), H(16), 2H(16), H(32), C(64), R(128), 2R(128)

for n in range(0,8): print(Gn(n))

R
C,     2R
H,     R(2),  R(2)
2H,    C(2),  2R(2), C(2)
H(2),  H(2),  R(4),  R(4),  H(2)
C(4),  2H(2), C(4),  2R(4), C(4),  2H(2)
R(8),  H(4),  H(4),  R(8),  R(8),  H(4), H(4)
2R(8), C(8),  2H(4), C(8),  2R(8), C(8), 2H(4), C(8)
```


$M^{\pm}$
2R
0
R
M
$\bar{1}$
m_1
m_0
7
M
R
m_0
C
$D^{\pm}$
m_1
ν
$\bar{2}$
$\bar{6}$
ὥρολόγιον
$\nu = s - r \mod 8$
$2^k m_k(n)^2 = 2^n$
$m_k(n) = 2^{(n-k)/2}$
m_1
C
$D^{\pm}$
m_2
H
M_σ
3
$\bar{5}$
m_2
m_3
H
4
2H
M_σ
$M_\sigma^{\pm}$

The labels in the outer ring of the *horologion* correspond to the five different basic forms as explained in next table. In the last three columns (ν odd), $\mathbf{i}$ is a central element and $\mathbf{i}^2 = -1$ for the $\mathbf{C}$ form and $\mathbf{i}^2 = 1$ for the $2\mathbf{R}$ and $2\mathbf{H}$ forms.

ν	$0, \bar{6}$	$\bar{2}, 4$	$\bar{1}, \bar{5}$	7	3
F_ν	$\mathbf{R}$	$\mathbf{H}$	$\mathbf{C}$	$2\mathbf{R}$	$2\mathbf{H}$
m	m_0	m_2	m_1	m_1	m_3
S	$\mathbf{R}^{m_0}$	$\mathbf{H}^{m_2}$	$\mathbf{C}^{m_1}, \bar{\mathbf{C}}^{m_1}$	$\pm\mathbf{R}^{m_1}$	$\pm\mathbf{H}^{m_3}$
Label	M	M_σ	$D^\pm$	$M^\pm$	$M_\sigma^\pm$

Table 2: The integer $m_k = 2^{(n-k)/2}$ is the *order of the matrices in the algebra class* and hence also the *dimension of the ground space* S on which the matrices act. The labels M, M_σ and D stand for *Majorana*, *symplectic Majorana*, and *Dirac* (or *Weyl-Dirac*), respectively. The action of $\mathbf{i}$ on $\pm\mathbf{R}^{m_1}$, and on $\pm\mathbf{H}^{m_3}$ is by ± 1. The action of $\mathbf{i}$ on $\mathbf{C}^{m_1}$ and $\bar{\mathbf{C}}^{m_1}$ is by $\pm i$.

One century of spinors

[6] (uhlenbeck-goudsmit-1926): Spinning electrons and the structure of spectra

[7] (pauli-1927): Zur Quantenmechanik des magnetischen Elektrons

[8] (dirac-1928): The quantum theory of the electron, I, II

[9] (weyl-1928): Gruppentheorie und Quantenmechanik

[10] (vanderwaerden-1929): Spinoranalyse

[11] (mercier-1935): Expression des Equations de l'Electromagnetisme au Moyen des Nombres de Clifford

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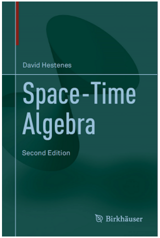

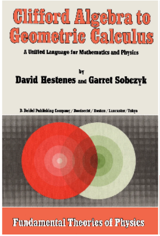

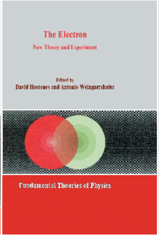

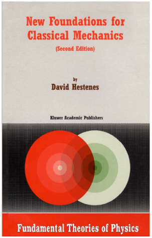

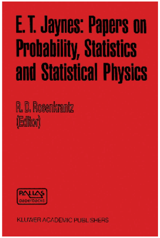


Clifford Algebras and their Applications in Mathematical Physics
Edited by A. Micali, R. Boudet and J. Helmstetter
Kluwer Academic Publishers
Fundamental Theories of Physics
Clifford Algebras and their Applications in Mathematical Physics
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Fundamental Theories of Physics
Geometric Computing for Perception Action Systems
Concepts, Algorithms, and Scientific Applications
Eduardo Bayro Corrochano
Geometric Algebra with Applications in Science and Engineering
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Garret Sobczyk
Editors
G. Sommer (Ed.)
Geometric Computing with Clifford Algebras
Theoretical Foundations and Applications in Computer Vision and Robotics
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Clifford Algebras and Spinors
Pertti Lounesto
Cambridge University Press
Applications of Geometric Algebra in Computer Science and Engineering
Leo Dorst
Chris Doran
Joan Lasenby
Editors
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Geometric Algebra for Physicists
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Eduardo Bayro Corrochano
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Robot Modelling and Control
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Sebastià Xambó-Descamps Ed.
Systems, Patterns and Data Engineering with Geometric Calculi
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[24] (nofech-2020): Biquaternionic Dirac Equation Predicts Zero Mass for Majorana Fermions

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[28] (kalkan-li-schroecker-siegele-2022): The Study Variety of Conformal Kinematics

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[31] (wieser-song-2022): Formalizing Geometric Algebra in Lean

Outlook

New ideas
Questions and projects

- Advance in the application of geometric algebra and geometric calculus to deep learning models (see the collection [32] and the references therein)
- Revisit infinity: Spinors in Hilbert space [33, 34, 35, 36, 37] and Infinite Clifford algebras [38].
- Explore discrete mathematics applications of geometric algebra over the integers, or over the rational numbers, or over finite fields.
- Explore physics applications of geometric algebra over a *p*-adic field (see the survey [39]).
- [40] (A Light Dream). "... geometric algebra as a base for an inquiry aimed to understand in that formalism recent knotted vortex electromagnetic solutions and to seek analogous solutions for the Dirac equation following seminal work of Hestenes."

Thank you!

Thanks to the organizers and program chairs!

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“In February 1922, in these buildings of the Frankfurt Physics association, Otto Stern and Walther Gerlach made the fundamental discovery of the space quantization of atomic magnetic moments. Important physico-technical developments in the 20th century are based on the Stern-Gerlach experiment” (translation).

Otto Stern (1888-1969) emigrated to the USA in 1933 and was awarded the 1943 Nobel Prize in Physics (after 82 nominations since 1925, second only to the 84 of Arnold Sommerfeld, who did not receive it) “for his contribution to the development of the molecular ray method and his discovery of the magnetic moment of the proton” (not for the Stern-Gerlach experiment).

“The classical theory predicts that the atomic magnets assume all possible directions with respect to the direction of the magnetic field. On the other hand, the quantum theory predicts that we shall find only two directions parallel and antiparallel to the field (new theory, the old one gave also the direction perpendicular to the field)” (from the Nobel lecture). P

My sketchy presentation of GA, besides explaining notations and conventions, is a wish to harmonize three aspects that seem relevant for the ICACGA purposes:

(1) The applicability side, as unfolded in the last hundred years; for the most part, it has been based on real geometric algebras.

(2) The computational side, by highlighting the structural and algorithmic relationships between notions.

(3) The mathematical side, by considering the mathematical structures involved and how they fit together.

Although real geometric algebras will keep being central in most applications, it seems sensible to allow, for other possible inquiries, scalar fields other than **R**, as for instance *finite fields* (for applications connected with discrete mathematics), or the *rational field* **Q**, for infinite-precision arithmetic and symbolic-numeric processing. And, why not?, a *p-adic field* for more esoteric applications in mathematical physics. See the survey [39] (dragovich-et4-2017) (*p-Adic mathematical physics, the first 30 years*).

The geometric algebra formalism and its geometric semantics form a perfect match in familiar contexts. After all, this is the main reason why geometric algebra is so valuable. In less familiar grounds, the validity of the algebraic formalism is the only sure asset, and it is this asset that confers 'geometric intuitions' to the studied objects and their relations. P

In terms of this basis, the exterior product is determined by bilinearity and the expressions

$$e_I \wedge e_{I'} = \begin{cases} 0 & \text{if } I \cap I' \neq \emptyset \\ \sigma(I, I') e_{I+I'} & \text{otherwise} \end{cases}$$

Here $I + I'$ denotes the sequence I, I' sorted in increasing order and $\sigma(I, I') = (-1)^{t(I,I')}$, where $t(I, I')$ is the number of transpositions required for this sorting.

P

The grade decomposition of any element of $\mathcal{G}^+$ only has even grade terms.

$\mathcal{G}^+$ is a subalgebra of $\mathcal{G}$ in the sense of the geometric product, but also in the sense of the wedge product and of the inner product.

The elements of $\mathcal{G}^-$ are those whose grade decomposition only contains odd grade terms. It is only a linear subspace, but $\mathcal{G}^+\mathcal{G}^- = \mathcal{G}^-\mathcal{G}^+ = \mathcal{G}^-$ and $\mathcal{G}^-\mathcal{G}^- \subseteq \mathcal{G}^+$.

P

Thus *spinor* may mean an element $\mathfrak{s} \in \mathrm{Spin}_{r,s}$, in which case it acts on V (and on $\mathcal{G}$) by the *spinorial* (or *sandwich*) formula $\mathfrak{s}x\mathfrak{s}^{-1}$, or an element $\mathfrak{z}$ of a a spinorial space S, in which case it is acted upon by elements $x \in \mathcal{G}$ (instead of $X(\mathfrak{z})$ we may simply write $x\mathfrak{z}$). If S is a minimal left of $\mathcal{G}$, the expression $x\mathfrak{z}$ is the geometric product. [43] (hestenes-2015-sta) (§12 and §19).

"The universe seems to be made out of vectors and spinors rather than scalars" [44, Preface] (*Quantum Field Theory, A Tourist Guide for Mathematicians*)

Quotation from [18] (rashevskii-1957), *The theory of spinors*:

"The theory of spinors, especially in several dimensions, is only weakly represented in our mathematical literature. The present article aims to remedy this deficiency to a certain extent. From our point of view, the theory of spinors is in the first instance the theory of a linear representation of a Clifford algebra, and only incidentally the theory of a linear representation of a rotation group." P

Ahead of the arrow of any *hour* ν, we have the basic form F_ν of $\mathcal{G}_{r,s}$, where $\nu = s - r \mod 8$. The form $F_{\nu-1}$ of $\mathcal{G}^+_{r,s}$ can be read at the tail of the ν-arrow. To specify the order m of the matrices it is convenient to use the notation $m_k = 2^{(n-k)/2}$, $k = k_\nu$ (here $k \in \{0, 1, 2, 3\}$). For example, $\mathcal{G}_{3,1} \simeq F_6(m) = \mathbf{R}(m_0)$, where $m_0 = 2^{4/2} = 4$, which says that $\mathcal{G}_{3,1}$ is isomorphic, as an algebra, to the matrix algebra $\mathbf{R}(4)$. On the other hand, $\mathcal{G}^+_{3,1}$ is isomorphic to $\mathbf{C}(m_1) = \mathbf{C}(2)$, because $\nu = 6$, $F_5 = \mathbf{C}$, and $m_1 = 2^{(3-1)/2} = 2$. The values $\nu = 1, 2, 5, 6$ (or $\nu = 1, 2 \mod 4$) have been marked with an overbar to indicate that $\mathbf{i}^2 = -1$. For the other values, $\mathbf{i}^2 = 1$. The labels M and M_σ stand for *real* and *symplectic* (or *quaternionic*) *Majorana*, respectively, and D for *Dirac*. There is one label for each of the five different basic forms as shown in the table on next slide.

P

No history here of the XIX century pioneers: Grassmann, Hamilton, Clifford, Lifschitz, Peano (see for instance in *A history of vector analysis*, [45]).

[6] (uhlenbeck-goudsmit-1926): Spinning electrons and the structure of spectra (introduce *electron spin* as an intrinsic angular momentum).

[7] (pauli-1927): Zur Quantenmechanik des magnetischen Elektrons (via the *Pauli matrices* $\sigma_1, \sigma_2, \sigma_3$, he discovers $\mathcal{G}_3$ in the form of $\mathbf{C}(2)$.

[8] (dirac-1928): The quantum theory of the electron, I, II (via the *Dirac matrices* $\gamma_0, \gamma_1, \gamma_2, \gamma_3$, he discovers $\mathcal{G}_{1,3}$ in the form of a subalgebra of $\mathbf{C}(4)$.

[9] (weyl-1928): Gruppentheorie und Quantenmechanik

[10] (vanderwaerden-1929): Spinoranalyse

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"E. Cartan developed a general method of constructing irreducible representations of O_n (or any other semi-simple group) by considering the infinitesimal operations, and he found[2] as the building stones of the whole edifice the tensor representations Γ_f together with *one further double-valued representation,* $\Delta : o \to S(o)$ *of degree* $n/\!\!/2$. The quantities of kind Δ are called *spinors*. In the four-dimensional world this kind of quantities has come to its due honors by Dirac's theory of the spinning electron. Cartan, according to his standpoint, states the transformation law $S(o)$ of spinors only for the infinitesimal rotations o. Here we shall give a simple finite description of the representation Δ and shall derive from it by the simplest algebraic means the main properties of the spinors. [...] One of the chief results will be that Dirac's equations of the motion of an electron and the expression for the electric current are ulniquely determilned even in the case of arbitrary dimensionality."

[13] (cartan-1937): Leçons sur la théorie des spineurs (2 volumes)

[14] (wigner-1939): On Unitary Representations of the Inhomogeneous Lorentz Group P

[2] *Bulletin Société Mathématique de France*, vol. 41 (1913), p. 53.

From the Foreword by Anthony Lasenby to the 2015 edition: "This small book started a profound revolution in the development of mathematical physics, one which has reached many working physicists already, and which stands poised to bring about far-reaching change in the future". "For someone such as myself, however, coming from a background of cosmology and astrophysics, this realization, which I gathered from this short book and David's subsequent papers, was a revelation, and showed me that one could cut through pages of very unintuitive spin calculations in Dirac theory, which only experts in particle theory would be comfortable with, and replace them with a few lines of intuitively appealing calculations with rotors, of a type that for example an engineer, also using Geometric Algebra, would be able to understand and relate to immediately in the context of rigid body rotations."

From the author's Preface after fifty years: "I am pleased to report that STA is as relevant today as it was when first published. I regard nothing in the book as out of date or in need of revision. Indeed, it may still be the best quick introduction to the subject. It retains that first blush of compact explanation from someone who does not know too much."

From the author's Introduction: "Pauli and, subsequently, Dirac introduced matrix representations of Clifford algebra for physical but not for geometrical reasons".

"The mathematical language used in any field of physics is crucial in shaping one's thoughts and understanding of the physics" (Terje Vold, 1990, review of *New Foundations of Classical Mechanics*).

P

[46] (micali-boudet-helmstetter-1992)
[47] (brackx-delanghe-serras-1993)
[48] (bayro-2001),
[49] (bayro-sobczyk-2001)
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[58] (vaz-rocha-2016)
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