**2019 Interdisciplinary Colloquium
in Topology and its Applications**

A LIGHT DREAM

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Dedicated to Eusebio Corbacho and Elena Martín Peinador on the occasion of their jubilee

ABSTRACT. The study of light has a long and distinguished history that has occupied the best minds in each epoch. One of the major boosts occurred when MAXWELL subsumed all previous wave-theory views in his unification of optics and electromagnetism, and an even more radical impulse with the advent of relativity (EINSTEIN) and quantum theory (PLANCK), which uncovered the wave-particle duality of light and later of all physical particles (DE BROGLIE). The goal of this note is to report on recent work in the area of geometric algebra as a base for an inquiry aimed to understand in that formalism recent knotted vortex electromagnetic solutions and to seek analogous solutions for the DIRAC equation following seminal work of HESTENES.

1. Research after retirement?

The common wisdom is that retirement, for a researcher, even when imposed unconditionally, can be celebrated as the archway to a season in which to rejoice by concentrating the available energy on those matters that one really likes, and which often were neglected at the bottom of some drawer by the wandering course of life.[1]

This popular view could easily lead to a long analysis about how different people organize themselves to create a jubilation atmosphere around themselves. The known examples, many still active in their eighties and nineties, would be wonderful witnesses of the main assertion.[2]

They could also distill practical advice for 'beginners' about what to foster and what to avoid; about how to cope with new realities such as a weaker interaction with proximate colleagues and a vanishing contact with student classes; about how to best face the new financial circumstances and its impact on the life of a researcher; or, in general, about how to graft meaning to this passage of life.

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[1] Expressed as an echo of GOETHE's verse "Des Lebens labyrintisch irren Lauf" in *Faust*'s dedication (Zueignung).

[2] An extraordinary example is Robert J. FINKELSTEIN, the doctoral advisor of David HESTENES. He was born 26 March 1916 and the date of his last post in arxiv, *Are Dyons the Preons of the Knot Model?*, is 6 September 2018! *My Century of Physics*, written on the occasion of his 100th aniversary, goes over all his significant experiences threaded in nearly nine decades of productive research, of which more than three in compulsory retirement. This autobiography, and many other biographical details, can be downloaded from his Wikipedia profile.

Not being aware of the existence of any systematic analysis of that kind, nor of initiatives aimed at appropriate coordinating efforts, the most expedient way I find on this occasion is to provide a few snapshots of my own strifes in the first three years of retirement, 2015-2018, and of my dreams for the period 2019-2021. For definiteness, I will advance a few broad brush strokes about what I will just call my Light dream.

At the start, it has to be recognized that a Light dream has many facets that contribute to make it an attractive subject, particularly for a mature age. Since light is so deeply connected with vision and awareness, it has deep cultural connotations in all civilizations; its most engaging history spans millennia and in each epoch it has attracted the brightest minds; its symbiosis of mathematics and experimentation remains, to a good extent, much closer to human grasp than many of the grander undertakings in experimental physics; it continues to be the focus of present day fundamental theoretical and experimental science; and on the philosophical side, it provides excellent illustrations about the nature of knowledge and how it is produced and refined.

2. And in the daylight?

Primarily, the study of light belongs to physics.[3] Modern textbooks, like for instance [18], distinguish four nested domains. The most encompassing, and also the most recent, is *quantum optics* (or photonics). For classical (i.e., non-quantum) optics, the most complete picture is provided by Maxwell's *electromagnetic theory*, as *wave optics* can be considered as a scalar approximation of Maxwell's optics and *geometrical optics*, or *ray optics*, as the limit of wave optics when the wavelengths are negligible with respect to the dimensions of the objects in the experimental setting.[4]

Of course, the historical development unfolded in the opposite direction. Ray optics was studied already in ancient times and as a science it evolved as an application of Euclid's geometry. Wave optics did not appear until wave phenomena first observed in water waves or in acoustics, such as diffraction and interference, were seen to occur also for light. But the greatest jump in the understanding of light was Maxwell's discovery that its nature is electromagnetic. This theory provided a deeper understanding of many aspects of light, as for example the nature of color and of the polarization states, and a wealth of new phenomena, as for example the energy density of electromagnetic fields. The final stage began with the realization of the quantum behavior of light in the structure of the black body radiation (Planck), the quantum explanations of the photoelectric effect (Einstein) and of the atomic spectra (Bohr), and the realization that matter could transform into electromagnetic radiation and vice-versa (Dirac), an idea

[3] The *Penguin Dictionary of Physics* defines **optics** as "the branch of physics concerned with the study of light, its production, propagation, measurement, and properties".

[4] *loc. cit.:* "The ray treatment of light is called *geometrical optics* as distinct from *physical optics*, which attempts to explain the objective phenomena of light.

that was fully developed in quantum electro-dynamics (QED for short, which is the most precise theory in all the history of science).

In the main, my Light dream envisions a detailed exploration of the physics of light by paying a close attention to the mathematics involved. The main guiding principle is that mathematics is an irreplaceable asset when the goal is to understand physics, light physics in the present case. Ideally, this stance should help mathematics students to get a stronger appreciation of physics and students of physics to get a firmer grip of the mathematics involved.

3. What mathematics?

From the point of view of mathematics, in principle Euclidean geometry and calculus suffice for most of Maxwell's theory, but its real glory is not revealed until it is phrased as a field theory on Minkowski's spacetime or in more general spacetimes. Here is a nutshell summary. Let t, x, y, z denote inertial coordinates (with units so that $c = 1$). Let $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$ be the electric and magnetic fields. Let ρ the charge density and $\boldsymbol{J} = (J_x, J_y, J_z)$ the current density vector. Combine $\boldsymbol{E}$ and $\boldsymbol{B}$ into the *electromagnetic field* $F = \widehat{\boldsymbol{E}}dt + \widetilde{\boldsymbol{B}}$, where $\widehat{\boldsymbol{E}} = E_x dx + E_y dy + E_z dz$ and $\widetilde{\boldsymbol{B}} = B_x dydz + B_y dzdx + B_z dxdy$, and ρ and $\boldsymbol{J}$ into the *electromagnetic current* $J = \rho dt + \widehat{\boldsymbol{J}}$. Then Maxwell's equations are equivalent to the equations (see [24] for details and further references)

$$dF = 0, \quad \delta F = J,$$

where d is the exterior differential and δ the codifferential (adjoint to d).[5] In the exterior algebra of differential forms, $\Lambda = \Lambda(dt, dx, dy, dz)$, these two equations can be written as a single equation, namely $(d + \delta)F = J$.

As it turns out, Euclidean geometry, or orthogonal geometry of any non-degenerate signature (Lorentz' or Mikowski's geometries in particular), are best phrased in the language of geometric algebra (GA), cf. [23] and [13, §1 and §3]. For example, the natural reincarnation of the operator $d + \delta$ in GA is the Dirac operator ∂ and Maxwell's equations are equivalent to $\partial F = J$, where now J is vector and F a bivector. See [13, §3.3], and note how to write (§3.4) Hestenes' version of the Dirac equation with the same operator. Moreover, the quantum mechanical formalisms can be cogently and gainfully phrased in the language of GA (see [8]). These statements amount to a long story in themselves and they play as harbingers of powerful research ideas and methods (see, for instance, [22] for historical background and the second edition of [6] for the views of a long standing leader in the field; see also [23, §6.2 and §6.3], which include a comprehensive survey of applications).

The author's conclusion is that the best mathematical choice available to pursue the Light dream is the GA formalism. The advantages are not readily seen in most

[5] To note that the equations $d\psi = \delta\psi = 0$ make sense on any Riemannian manifold for any differential form ψ, and that they characterize the harmonic forms on that manifold (Hodge). See [19].

ray optics, particularly in those areas in which the required geometry is elementary plane geometry, but its power is already clearly manifest in the treatment of electromagnetic radiation and it appears to provide essential theoretical insights in the case of photonics. Moreover, the formalism is also excellent for the treatment of gravitational waves as advanced in [12]. For the topological aspects, see §5.

4. How I got here?

At this point it may be permissible to say a few words about how I got here, particularly because this will delineate what is the departing state in pursuing the Light dream. From the teaching experience, the most valuable is having taught many courses of *Geometry* (cf. [20]) and of *Mathematical Models of Physics* (cf. [21]). In Geometry, for example, the treatment of the focal properties of conics, particularly of the parabola, and of the mirror symmetries as a foundation of the Euclidean group, have deep reminiscences of the mathematical and optical proficiency achieved by the ancient Greeks (Euclid, Archimedes, Apollonius, et cetera).

In the case of MMF, I have already mentioned how to write Maxwell's equations as a single equation in the exterior algebra of forms or in the geometric algebra of Minkowski's space-time. Another key chapter was devoted to Hamiltonian mechanics, with emphasis on the Hamilton's principle of stationary action and its relation to Fermat's principle of stationary optical path.

Beyond teaching, the most salient experiences in between (2009-2018) have been the increased awareness of the significance of GA and my involvement and engagements in trying to master it. There is no need to repeat here the account in [23, §6.0], except to stress once more the acknowledgment of Hestenes leading teachings for over half a century.

Some of the lessons learned from the activities in that period inspired the books [13] (particularly chapters 1-3) and [23]. The writing of those works went in parallel with the organization of the Conference AGACSE 2018, which was again a privileged vantage point for learning new results and techniques (see, for instance, [25] and [26]) and which continues to this day with the editorial work of the corresponding Proceedings. Twenty-four papers were received and among them a few that are wonderful stimuli for the Light project.

5. Resources and conclusion

The main reference works singled out to pursue the Light dream are [7] and [17]. They are very different, but for the present concerns their contrast is regarded as an asset. The Hestenes work [7] is a twenty-page preprint, is staged on the Minkowski space $\mathcal{M}_{1,3}$, the main formalism is the geometric algebra $\mathcal{G}_{1,3}$ of that space, and its main goal is to set up an ambitions Maxwell-Dirac theory that fuses the GA versions of Maxwell's and Dirac's equations. It has "implications for the electron's anomalous magnetic moment, structure of the photon, and extension to a 'standard Model' wherein all elementary particles are composed of electrons". It includes comments by the author that further enhance its inspirational value, as for example in these assessments: "Considering the enormous scope of Maxwell-Dirac theory, our treatment has many loose ends and is best regarded as *a guide for further research*, though the model of electron as a point particle with an inherent periodicity is an essential feature in all variants of the theory" and "With electrons modeled as vacuum singularities, *it is natural to consider the topology of more complex vacuum singularities to model the whole zoo of elementary particles, including photons*" (emphasis added). Its briefness, however, is a mirage, because it presupposes having mastered several of the author's own papers and also several insightful works contributed by other authors.

On the other hand, [17] is a very thorough treatise whose main language is differential geometry. The GA formalism is encoded in the language of spinor bundles. By "many faces of Maxwell, Dirac and Einstein's equations", the authors mean "the many different ways in which those equations can be presented using different mathematical theories". To a large extent, it is more concerned in, e.g., "providing mathematical unity to the theory of Maxwell and Dirac fields" than to delve into the question of whether they are different sides of the same coin. For a summary of the contents, see [26].

Concerning topology, it appears when looking at different aspects of theories. If there is a relevant group, usually its fundamental group plays an important role (for orthogonal and spinorial groups, see the synopsis results collected in [23, §6.1.7]). Another source of topological structure can emerge when looking at the nature and properties of the solutions to given equations. In the case of electromagnetism, the most representative references for us are [1] and [3, 4], together with many references cited therin. See also [14], [9], [10]. Finally, let us just mention that topology can be important when there is interest in general space-times, [17], or in quantum gravity or in cosmology, [12].

As a conclusion, here is a rough description of the main aspects on which to focus the searchlight. (1) To phrase in GA the vortex solutions for the electromagnetic field, particularly [1]. (2) To carry out a close mathematical analysis of the unified MAXWELL-DIRAC theory presented in [7]. For this, [17] and references in experimental physics such as [5], [11], [16], [2] and [15] are expected to provide convenient touchstones. (3) To phrase in GA the knotted vortex electromagnetic solutions discovered in [4]. (4) To seek a generalization of (3) for the theory in (2).

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