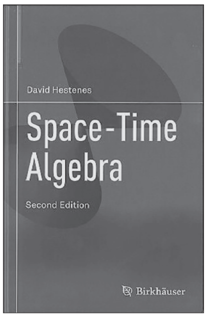
Book Reviews



David Hestenes

Space-Time Algebra (second edition)

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Reviewer: Sebastià Xambó-Descamps

The 6th Conference on Applied Geometric Algebras in Computer Science and Engineering[1] (AGACSE) was dedicated to David Hestenes (Arizona State University) in recognition of his masterly and sustained leadership for half a century, particularly at the interface of mathematics and physics. The dedication was celebrated with the launch[2] of a second edition of his *Space-Time Algebra* (Gordon and Breach, 1966). David was present during the whole week and the standing ovation after his keynote lecture,[3] culminating in his recitation of the stirring call to action from Tennyson's Ulysses,[4] was a very moving moment for all participants. The David Hestenes Prize was established for the best work submitted by a young researcher and was awarded to *Lei Huang* (Academy of Science, Beijing, China) for "Elements of line geometry with geometric algebra".[5] His work shows how to bring the power of geometric algebra to bear on 3D projective geometry, thus linking new mathematical theory with very practical applications in computer science. *Pierre-Philippe Dechant* (University of York, UK) and *Silvia Franchini* (University of Palermo, Italy) were finalists with the works "The E_8 geometry from a Clifford perspective" and "A family of embedded coprocessors with native geometric algebra support",[6] respectively. The conference was preceded, for the first time, by a 2-day Summer School to better prepare the less experienced and it was attended by two thirds of the conference participants. The next AGACSE will be in Campinas, Brazil, in 2018.

David Hestenes (July 2015, during his keynote lecture).

From left to right: David Hestenes, Lei Huang, Silvia Franchini, Pierre-Philippe Dechant, Sebastià Xambó-Descamps, Eduardo Bayro-Corrochano.

Space-Time Algebra (STA) is actually a reprint of the first edition, but with two precious new items: a foreword by Anthony Lasenby[7] and a preface by the author "after fifty years". It was a landmark in 1966 and it is as fresh today as it was then in its "attempt to simplify and clarify the [mathematical] language we use to express ideas about space and time", a language that "introduces novelty of expression and interpretation into every topic" (the quotations are from the preface to the first edition). This language is usually called *geometric algebra* (GA), a term introduced by W.K. Clifford in his successful synthesis of ideas from H. Grassmann and W.R. Hamilton. In Part I of STA, GA is advanced and honed into a resourceful mathematical system capable of expressing geometric and physical concepts in an intrinsic, efficient and unified way. Two special cases are worked through in detail: the geometric algebras of 3D Euclidean space (Pauli algebra) and 4D Minkowski space[8] (Dirac algebra). These geometric algebras are then used, in a real tour-de-force, to elicit the deep geometric structure of relativistic physics. This takes the remaining four parts of the book: Electrodynamics, Dirac fields (including spinors and the Dirac equation), Lorentz transformations and Geometric calculus (including novel principles of global and local relativity, gauge transformations and spinor derivatives). There are also four short appendixes,

[1] AGACSE 2015, 27–31 July, Barcelona, Spain: http://www-ma2.upc.edu/agacse2015/
[2] Suggested as an 80th birthday gift by Leo Dorst, cooperatively backed by Eduardo Bayro-Corochano, Joan Lasenby, Eckhard Hitzer and the author of this review, and enthusiastically embraced by Springer, each participant received a copy by courtesy of the Catalan Mathematical Society and the Royal Spanish Mathematical Society.
[3] *Fifty Years with Geometric Algebra: a retrospective.*
[4] "Made weak by time and fate, but strong in will / To strive, to seek, to find, and not to yield" (last two verses).
[5] Joint work with Hongbo Li, Lei Dong, and Changpeng Shao.
[6] Co-authored by Antonio Gentile, Filippo Sorbello, Giorgio Vassallo and Salvatore Vitabile.
[7] Professor of Astrophysics and Cosmology at the Cavendish Laboratory, Cambridge University. Co-author of the superb treatise [1].
[8] A real vector space with a metric of signature (+, –, –, –).

A to D, which amount to a supplement of the GA part. We will come back to them below.

Although the presentation of GA in STA, and in later works of Hestenes and many others, is framed in a set of quite natural algebraic axioms, it turns out that the approach may come across as unusual for some tastes, which perhaps explains why the book is not as well known among theoretical physicists as it surely deserves. For example, in an otherwise meritorious paper,[9] E.T Jaynes declares (an admittedly extreme view that may be saying more about himself than about STA):

> *It is now about 25 years since I started trying to read David Hestenes' work on space-time algebra. All this time, I have been convinced that there is something true, fundamental, and extremely important for physics in it. But I am still bewildered as to what it is, because he writes in a language that I find indecipherable; his message just does not come through to me. Let me explain my difficulty, not just to display my own ignorance, but to warn those who work on space-time algebra: nearly all physicists have the same hang-up, and you are never going to get an appreciative hearing from physicists until you learn how to explain what you are doing in plain language that makes physical sense to us.*

Fortunately, STA was 'discovered' in the late 1980s by people like Stephen Gull, Anthony Lasenby and others, in Cambridge and elsewhere (see [2], the references therein, and [3]), an eventuality which led to a flourishing of new ideas, results and applications in many fields (see, for example, [1, 4, 5, 6]).

GA, as espoused in STA, seems not to be very well known in mathematical circles either, this time because it may perhaps be perceived as a closed, short-range structure, or even because its presentation may be found not to follow the formal strictures of the trade. As avowed by the vast existing literature, the first perception is untenable, even if one takes into account only its service to mathematics, or even only to geometry. Concerning formalities, there is no doubt that a mathematically minded approach may extract a meaningful and satisfying picture of GA, as this does not (logically) depend on the physics. Assuming basic knowledge of the Grassmann (or exterior) algebra, here is a possible sketch of such a picture. The geometric algebra $\Lambda_g E$ of a (real) vector space E of finite dimension n, equipped with a symmetric bilinear form g (the *metric*), is the exterior algebra

$$\Lambda E = \Lambda^0 E \oplus \Lambda^1 E \oplus \Lambda^2 E \oplus \cdots \oplus \Lambda^n E \ (\Lambda^0 E = \mathbb{R},\ \Lambda^1 E = E)^{10}$$

enriched with the *inner product* $x \cdot y$ and the *geometric product* xy, which in turn can be explained as follows. To define the inner product $x \cdot y$, we may assume that

$$x = e_1 \wedge \cdots \wedge e_r \in \Lambda^r E,\ y = e'_1 \wedge \cdots \wedge e'_s \in \Lambda^s E$$
$$(e_1, \ldots, e_r, e'_1, \ldots, e'_s \in E,\ r,s \geq 1).$$

If $r = 1$ (say $x = e \in E$) then $e \cdot y$ is defined as the left contraction of e and y, namely

$$e \cdot y = \Sigma_{k=0}^{k=s} (-1)^k g(e, e'_k)\, e'_1 \wedge \cdots \wedge e'_{k-1} \wedge e'_{k+1} \wedge \cdots \wedge e'_s.$$

For example,

$$e \cdot e' = g(e, e') \text{ and } e \cdot (e'_1 \wedge e'_2) = g\,(e, e'_1)\, e'_2 - g\,(e, e'_2)\, e'_1.$$

If $1 < r \leq s$ then $x \cdot y$ can be defined recursively by the relation

$$x \cdot y = (e_2 \wedge \cdots \wedge e_r) \cdot (e_1 \cdot y).$$

In the case $r \geq s$, analogous formulas using the right contraction $x \cdot e$ lead to the rule

$$x \cdot y = (-1)^{rs+s}\, y \cdot x).$$

We see that $x \cdot y \in \Lambda^{|r-s|} E$ for $x \in \Lambda^r E$, $y \in \Lambda^s E$. If $r = s$, then $x \cdot y = g(x,y)$, where we use the same symbol g for the natural extension of the metric to ΛE, so that, in particular, $x \cdot y = y \cdot x$, as required by the formula above. But note that if $r \neq s$ then $g(x,y) = 0$, whereas $x \cdot y$ may be non-zero and may be the opposite of $y \cdot x$ (precisely when s is odd and r even).

The geometric product xy may be characterised as the only bilinear *associative* product such that

$$ex = e \cdot x + e \wedge x,$$

for any $e \in E$ and any $x \in \Lambda E$.[11] Note that $e^2 = e \cdot e = g\,(e,e)$, so that e is invertible if it is non-isotropic ($g\,(e,e) \neq 0$), $e^{-1} = g(e,e)^{-1} e$, which means that, in GA, *division by non-isotropic vectors is a legal operation*. This fact, together with the associativity of the geometric product, explains why operating with (multi)vectors is so natural and agile. Note also that $e e' = e \wedge e' = -e' e$ if (and only if) e and e' are orthogonal ($g\,(e,e') = 0$). With this approach, all the GA formulas in STA, and others obtained afterwards, can be established. Here are some examples. For $e \in E$ and $x \in \Lambda E$, the relation $xe = x \cdot e + x \wedge e$ also holds. If $x \in \Lambda^r E$, $y \in \Lambda^s E$ and $z = xy$ then $z_k \neq 0$ implies that $k = |r-s| + 2j$, $j = 0, \ldots, \min(r,s)$. In particular, the minimum and maximum possible degrees are $|r-s|$ and $r+s$. In fact, it happens that $(xy)_{|r-s|} = \tilde{x} \cdot y$ if $r \leq s$, $x \cdot \tilde{y}$ if $r \geq s$, and $(xy)_{r+s} = x \wedge y$, where $\tilde{x}$ is the result of reversing the order of the factors of x. Thus, we see that the geometric product determines the inner and outer products.[12] Another very useful fact is that if the vectors $e_1, \ldots, e_r$ are pairwise orthogonal then

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9 E.T. Jaynes: Scattering of light by free electrons as a test of quantum theory. In *The electron: New theory and experiment* (D. Hestenes and A. Weingarthofer, eds.), 1–20. Kluwer, 1991.

10 Its product $x \wedge y$ is the *exterior* or *outer* product. Its elements, which are called *multivectors*, have the form $x = x_0 + x_1 + \cdots + x_n$, with $x_r \in \Lambda^r E$ (the r-vector part of x).

11 This fundamental formula is what most upset Jaynes, who vehemently objected to its non-homogeneous character: $e \cdot x \in \Lambda^{r-1} E$ and $e \wedge x \in \Lambda^{r+1} E$ when $x \in \Lambda^r E$.

12 Notice, however, that for these relations to make sense, we need to know the grading, which is not naturally defined by means of the geometric product. Here the grading is taken to be a lower level structure, as the definition of the graded algebra ΛE only depends on the vector space structure of E. Actually this fact is one of the great ideas bequeathed by Hermann Grassmann.

$$e_1 \cdots e_r = e_1 \wedge \cdots \wedge e_r.$$

Let us turn back to the foreword and the new preface of STA. In the foreword, A. Lasenby states that:

> *This small book started a profound revolution in the development of mathematical physics, one which has reached many working physicists already, and which* stands poised to bring about far-reaching change in the future.

The roots of this potential are clearly delineated in Hestenes' preface, "with the confidence that comes from decades of hindsight", by asserting four "**Claims** for STA as formulated in this book" (boldface emphases in the original):

(1) *STA enables a unified,* ***coordinate-free*** *formulation for all of relativistic physics, including the Dirac equation, Maxwell's equation and general relativity.*
(2) *Pauli and Dirac matrices are represented in STA as* ***basis vectors*** *in space and spacetime respectively, with no necessary connection to spin.*
(3) *STA reveals that the* ***unit imaginary*** *in quantum mechanics has its origin in spacetime geometry.*
(4) *STA reduces the mathematical divide between classical, quantum and relativistic physics, especially in the use of* ***rotors*** *for rotational dynamics and gauge transformations.*

Before briefly commenting on these claims, here is a simple geometric example that neatly illustrates the core aspects of (3) and (4). It is about the representation in GA of rotations in the ordinary Euclidean space as explained in STA, Appendix C. Let E_3 be the Euclidean 3-space and $e_1, e_2, e_3 \in E_3$ an orthonormal basis. Denote $e_j e_k = e_j \wedge e_k$ by e_{jk}, with a similar meaning for e_{jkl}. Then $i = e_{123}$ commutes with vectors and hence with any element of ΛE_3, and $i^2 = -1$.[13] Since $ie_1 = e_{23}$, $ie_2 = e_{31}$ and $ie_3 = e_{12}$, we have a linear isomorphism $E_3 = \Lambda^1 E_3 \to \Lambda^2 E_3$, $u \mapsto b = iu$. The inverse isomorphism $\Lambda^2 E_3 \to E_3$ is given by $b \mapsto u = -ib$.[14] Now, let $u \in E_3$ be a unit vector and $\alpha \in \mathbb{R}$ (an angle). Then the linear map $\rho_{u,\alpha}: E_3 \to E_3$ defined by

$$\rho_{u,\alpha}(x) = e^{-1/2\alpha iu}\, x\, e^{1/2\alpha iu}$$

is the rotation about the vector u of angle α. Indeed, by the usual expansion of the exponential we get

$$e^{\pm\alpha iu/2} = \cos(\alpha/2) \pm iu \sin(\alpha/2),$$

and the formula follows on noting that u is fixed, as it commutes with either exponential, and that if x is orthogonal to u then $e^{-\alpha iu/2} x\, e^{\alpha iu/2} = xe^{\alpha iu}$ (as $ux = -ux$), which is the result of rotating x in the plane $u^\perp$ by an angle α in the sense of the orientation given by u. To see this, let u_1, u_2 be an orthonormal basis of $u^\perp$ such that $-i(u_1 \wedge u_2) = u$, i.e. $iu = u_1 u_2$ (Hodge duality). Then $(u_1 u_2)^2 = -1$, $e^{\alpha iu} = \cos(\alpha) + u_1 u_2 \sin(\alpha)$, and the claim follows by a simple calculation of $u_1 e^{\alpha iu}$ and $u_2 e^{\alpha iu}$. Note that the GA formula for $\rho_{u,\alpha}(x)$ greatly facilitates the computation of the composition $\rho_{u',\alpha'} \rho_{u,\alpha}$ of two rotations, for it is reduced to the (brief) GA computation of $e^{\alpha iu/2} e^{\alpha' iu'/2}$. This yields, as shown in Appendix C, the remarkable formulas for the angle and axis of the composite rotation.[15]

As the example above shows, complex numbers appear in GA not as formal entities but with a surprising geometric meaning. The significance of point (3) is that this also happens in physics, where the i appearing in, say, the Schrödinger and Dirac equations is revealed to be subtly and significantly related to GA entities. The GA form of the E_3 rotations also illustrates interesting aspects of (4). Expressions such as

$$R = e^{-\alpha iu/2} = \cos(\alpha/2) - iu \sin(\alpha/2) \in \Lambda^0 E + \Lambda^2 E$$

(this is the *even* subalgebra of ΛE) are called *rotors* and the rotation $\rho_{u,\alpha}$ is given by $x \mapsto RxR^{-1}$.

As proved in STA, Part IV, Lorentz transformations (rotations of Minkowski's space) may also be described by rotors $R = e^{-b/2}$, where b is a bivector. With respect to an inertial frame, the rotor can be resolved as a product of one spatial rotor, which gives a rotation in the Euclidean 3-space associated to that frame, and a time-like rotor, which gives a Lorentz boost in that frame. The main resource here is the marvellous way in which the Pauli algebra of that Euclidean space is embedded in the Dirac algebra.

As for claim (2), note that in all these interpretations and calculations, the customary matrix representation of the Pauli and Dirac algebras plays no role, and work with coordinate systems and coordinates is unnecessary. The notion of spin, and its role in particle physics, is also greatly clarified and improved.

Paraphrasing a quote from [2] devoted to physicists, let me finish by expressing the hope that also mathematicians not yet knowing STA will find a number of surprises, and even that they will be surprised that there are so many surprises!

The reviewer thanks Leo Dorst for the improvements made possible by his comments, suggestions and corrections after reading a first draft.

References

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[2] S. Gull, A. Lasenby and C. Doran: Imaginary numbers are not real – the geometric algebra of spacetime. *Foundations of Physics* 23/9 (1993), 1175–1201. "[…] a tutorial introduction to the ideas of geometric algebra, concentrating on its physical applications".
[3] P. Lounesto: *Clifford algebras and spinors*. LMS LNS 239. Cambridge University Press, 1997.

[13] With the familiar interpretation of the exterior algebra, the element i is the unit volume, so that this relation infuses geometric meaning to the 'imaginary unit'.

[14] These are examples of the ease with which Hodge duality is managed with GA.

[15] These formulas were first obtained by Olinde Rodrigues by means of extensive analytical calculations: *J. Math. Pures Appl.* 5 (1840), 380–440.

[4] L. Dorst, D. Fontijne and S. Mann: *Geometric algebra for computer science: An object-oriented approach to geometry.* Elsevier/Morgan Kaufman, 2007. Revised edition 2009.
[5] C. Perwass: *Geometric algebra with applications in engineering.* Springer, 2009.
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