Geometric Algebra Techniques in Mathematics and Physics

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UPC

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> Calculer sur les concepts de la géométrie selon les règles d'une algèbre a été depuis longtemps le but des recherches de nombreux mathématiciens, comme Leibnitz qui en rêva ou comme Carnot qui s'y essaya.
>
> CASANOVA-1976

- B. Olinde Rodrigues (1795–1851). Rotation group (1840).
- W. R. Hamilton (1805-1865). Quaternions (1843).
- H. Grassmann (1809-1877). *Ausdehnungslehre* (1844, 1862).
- B. Riemann (1826-1866). Riemann sphere (related to spinors).
- W. K. Clifford (1845-1879). Geometric product (1878).
- R. Lipschitz (1832-1903). Lipschitz groups (1880).
- G. Peano (1858-1932). *Saggio di calcolo geometrico* (1896).
- K. Vahlen (1869-1945). Geometric product formula (1897).
- J. W. Gibbs (1839-1903). *Vector analysys* (1901).

- Study (1862-1930) & E. Cartan (1869-1951). *Nombres complexes* (1904).
- H. Weyl (1885-1955). Group representations (1926).
- W. Pauli (1900-1958). Pauli matrices (1927).
- P. Dirac (1902-1984). Dirac equation (1928).
- R. Brauer (1901-1977) & H. Weyl. *Spinors in n dimensions* (1935).
- E. Cartan. *Leçons sur la théorie des spineurs* (1937).
- M. Riesz (1886-1969). Dirac's equation in GR (1953).
- M. Riesz. *Clifford numbers and spinors* (1958).
- C. Chevalley (1909-1984). *The algebraic theory of spinors* (1954).
- E. Artin (1898-1962). *Geometric algebra* (1957).

Olinde Rodrigues

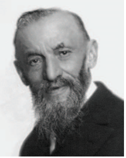
Hamilton

Grassmann

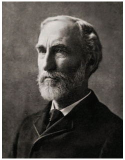
Riemann

Clifford

Lipschitz

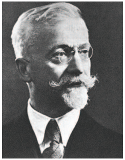


Peano



Gibbs

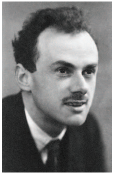
Study



E. Cartan

Weyl

Pauli



Dirac

Brauer

Riesz

Chevalley

E. Artin

- **Preliminary comments**. Notations and conventions.
- **GA ingredients**. Exterior product. Geometric product. Involutions. Inner product.
- **Examples**. The algebras $\mathcal{G}_2$ and $\bar{\mathcal{G}}_2$. Matrix representations of $\mathcal{G}_2$ and $\bar{\mathcal{G}}_2$. Quaternions.
- **Linking $\mathcal{G}_n$ to geometry**. A quote of Feynman. Euclidean geometry revisited. Rotors. The Lipschitz groups Γ_n and $\widetilde{\Gamma}_n$. The groups Pin_n and Spin_n. Oriented area in E_2. Euler's spinor formula. Composition of rotors (after Hestenes). Hodge duality. $\mathcal{G}_3$ and the Pauli algebra. Rotations of the rotors $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$. The cross product. Rotations about any axis. Vector algebra form of Euler's rotor. Olinde Rodrigues' formulas.
- **Appendix**. Why not *trinions*? Beyond quaternions.
- **References**

K denotes a field of characteristic $\neq 2$. Its elements are called *scalars*. The basic choices are $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$.

Let $n \geqslant 1$ be an integer, $\boldsymbol{e} = e_1, \ldots, e_n$ a sequence of n distinct *symbols*. For each sequence $K = k_1, \ldots, k_r \in \{1, \ldots, n\}$ $(0 \leqslant r \leqslant n)$, let e_K denote the *word* $e_{k_1} \cdots e_{k_r}$.

Now consider the vector space $\Lambda(\boldsymbol{e})$ freely spanned by the e_I with I strictly increasing (in which case we say that I is a *multiindex*).

- $\dim \Lambda(\boldsymbol{e}) = 2^n$;
- $\Lambda(\boldsymbol{e}) = \oplus_{r=0}^{r=n} \Lambda^r(\boldsymbol{e})$,

where $\Lambda^r(\boldsymbol{e})$, called the space of *r-vectors*, is the subspace of $\Lambda(\boldsymbol{e})$ spanned by the e_I with I of length r $(|I| = r)$.

Finally let $\widehat{e}_K \in \Lambda(\boldsymbol{e})$ be the vector 0 if K has repeated indexes and $(-1)^K e_{\omega(K)}$ otherwise, where $\omega(K)$ is the result of reordering K in increasing order and $(-1)^K$ is the sign of the permutation K of $\omega(I)$.

The *exterior product* $\wedge : \Lambda(\boldsymbol{e}) \times \Lambda(\boldsymbol{e}) \to \Lambda(\boldsymbol{e})$ is defined as the unique bilinear map such that $e_I \wedge e_J = \widehat{e}_{I,J}$, where I, J denotes the *concatenation* of I and J.

Note that $e_I \wedge e_J = 0$ if and only if $I \cap J \neq \emptyset$.

It turns out that the exterior product is associative, with unit e (e_I for I the empty sequence!) and *skew-commutative*, that is $e_j \wedge e_i = -e_i \wedge e_j$. Or, more generally, if $x \in \Lambda^r(\boldsymbol{e})$ and $y \in \Lambda^s(\boldsymbol{e})$, then $y \wedge x = (-1)^{rs} x \wedge y$.

Examples

- $\widehat{e}_{3,1,2} = e_{1,2,3}$ and $\widehat{e}_{3,2,1} = -e_{1,2,3}$.
- $e_{1,3} \wedge e_{2,3,5} = 0$ (3 is a repeated index).
- $e_{2,3,5} \wedge e_{1,4} = e_{1,2,3,4,5}$, for $(-1)^{2,3,5,1,4} = +1$.
- $e_{2,5} \wedge e_{1,4} = -e_{1,2,4,5}$, for $(-1)^{2,5,1,4} = -1$.
- $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}$.

Let $\boldsymbol{q} = q_1, \ldots, q_n \in \mathbf{K}$ and set, for each multiindex $I = i_1 < \cdots < i_r$, $q_I = q_{i_1} \cdots q_{i_r}$ (the product of $q_{i_1}, \ldots, q_{i_r}$ in $\mathbf{K}$). Note that in particular $q = q_\emptyset = 1$.

The (relative to $\boldsymbol{q}$) *geometric product* $\Lambda(\boldsymbol{e}) \times \Lambda(\boldsymbol{e}) \to \Lambda(\boldsymbol{e})$, which will be denoted by juxtaposition of its factors, is the only bilinear map such that

$$e_I e_J = (-1)^{s(I,J)} q_{I \cap J} e_{I+J},$$

where $I + J$ denotes the *odd-sum* of I and J, namely $(I \cup J) - (I \cap J)$ rearranged in increasing order and $s(I, J)$ is the number of *inversions* in the sequence I, J.[1]

Examples

- If $i \in \{1, \ldots, n\}$, $e_i^2 = e_i e_i = (-1)^{s(i,i)} q_i e = q_i$.
- If $i, j \in \{1, \ldots, n\}$ and $i < j$, $e_i e_j = (-1)^{s(i,j)} q e_{i,j} = e_{i,j}$, but $e_j e_i = (-1)^{s(j,i)} q e_{i,j} = -e_{i,j}$.
- $e_{1,3} e_{2,3,5} = (-1)^{s(1,3,2,3,5)} q_3 e_{1,2,5} = -q_3 e_{1,2,5}$.

[1] For two summands, $I + J$ coincides with *symmetric difference* $I \Delta J$.

The map $\mathbf{K} \to \Lambda^0(\boldsymbol{e})$, $\lambda \mapsto \lambda e$, is an isomorphism. Henceforth we will identify $\Lambda^0(\boldsymbol{e})$ and $\mathbf{K}$. In particular, we will write $1 = 1_{\mathbf{K}}$ instead of e.

Theorem

The geometric product is associative with multiplicative unit $e = 1$.

Proof. For a direct proof, we refer to Artin-1957. We will follow a more conceptual approach in the coming lectures. □

We will denote $\Lambda_{\boldsymbol{q}}(\boldsymbol{e})$ the exterior algebra $\Lambda(\boldsymbol{e})$ endowed with the geometric product relative to $\boldsymbol{q}$.

Clifford considered the case $\bar{\mathcal{G}}_n = \mathcal{G}_{\bar{n}} = \Lambda_{-\mathbf{1}_n}(\boldsymbol{e})$ in Clifford-1878 (so that $e_i^2 = -1$) and the case $\mathcal{G}_n = \Lambda_{\mathbf{1}_n}(\boldsymbol{e})$ in Clifford-1882 (so that $e_i^2 = 1$).

Examples. (1) $\mathcal{G}_1 \simeq \mathbf{R} \oplus \mathbf{R} = 2\mathbf{R}$, $a + be_1 \mapsto (a+b, a-b)$.

(2) $\bar{\mathcal{G}}_1 \simeq \mathbf{C}$, $a + be_1 \mapsto a + bi$.

Parity involution

The linear automorphism α of $\Lambda(\boldsymbol{e})$ defined by

$$\alpha(e_I) = (-1)^r e_I \quad (r = |I|)$$

is an *involutive* automorphism for the exterior and the geometric product.[2] Instead of $\alpha(x)$, we usually write x^α.

As a consequence, $\Lambda_{\boldsymbol{q}}(\boldsymbol{e}) = \Lambda^+_{\boldsymbol{q}}(\boldsymbol{e}) \oplus \Lambda^-_{\boldsymbol{q}}(\boldsymbol{e})$, where $\Lambda^+_{\boldsymbol{q}}(\boldsymbol{e}) = \{x \in \Lambda_{\boldsymbol{q}}(\boldsymbol{e}) \,|\, x^\alpha = x\}$ and $\Lambda^-_{\boldsymbol{q}}(\boldsymbol{e}) = \{x \in \Lambda_{\boldsymbol{q}}(\boldsymbol{e}) \,|\, x^\alpha = -x\}$.

Moreover, $\Lambda^+_{\boldsymbol{q}}(\boldsymbol{e}) = \oplus_{j=0}^{n//2} \Lambda^{2j}(\boldsymbol{e})$ is a subalgebra (the *even subalgebra)* of both the exterior product and the geometric product.[3]

Note that $\dim \Lambda^+_{\boldsymbol{q}}(\boldsymbol{e}) = \dim \Lambda^-_{\boldsymbol{q}}(\boldsymbol{e}) = 2^{n-1}$.

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[2] It suffices to prove it for products of the form $e_I \wedge e_J$ and $e_I e_J$, in which cases it follows straightforwardly from the defining formulas.

[3] $n//2 = \lfloor n/2 \rfloor$ is the integer quotient of n by 2.

Reversion involution

Given a multiindex $I = i_1, \dots, i_r$, we let $\widetilde{I}$ denote I in reversed order, that is, $\widetilde{I} = i_r, \dots, i_1$. Since restoring the original order amounts to $\binom{r}{2}$ transpositions, and since $\binom{r}{2} \equiv r//2 \mod 2$, we see that

$$\hat{e}_{\widetilde{I}} = (-1)^{r//2}\hat{e}_I.$$

The linear automorphism τ of $\Lambda(\boldsymbol{e})$ defined by

$$\tau(e_I) = (-1)^{r//2} e_I$$

is an *involutive antiautomorphism* (the *reversion involution*) for the exterior and the geometric product. The scheme of the proof is similar to the one used for the parity involution. Instead of $\tau(x)$, we usually write x^τ or $\widetilde{x}$.[4]

[4] In symbols, $(xy)^\tau = y^\tau x^\tau$.

Clifford involution

The *Clifford involution* is the antiautomorphism κ (of the exterior and geometric products) defined as $\kappa(x) = \tau(\alpha(x)) = \alpha(\tau x)$. It is also denoted x^κ or $\bar{x}$.

Since $r//2 + r \equiv \binom{r}{2} + r = \binom{r+1}{2} \equiv (r+1)//2 \mod 2$, for $x \in \Lambda^r_{\boldsymbol{q}}(\boldsymbol{e})$ we have

$$x^\kappa = (-1)^{(r+1)//2} x.$$

Note that the signs of α, τ, κ for $r = 4j, 4j+1, 4j+2, 4j+3 \equiv 0, 1, 2, 3 \mod 4$ are

	0	1	2	3
α	$+$	$-$	$+$	$-$
τ	$+$	$+$	$-$	$-$
κ	$+$	$-$	$-$	$+$

Let $E_n = \Lambda^1(\boldsymbol{e}) = \langle e_1, \dots, e_n \rangle$ (its elements will be called *vectors*). If $v = v^1 e_1 + \cdots + v^n e_n$,

$$Q(v) = q_1(v^1)^2 + \cdots + q_n(v^n)^2$$

is a quadratic form of E_n. Its associated scalar product $v \cdot v'$ is given by[5]

$$v \cdot v' = q_1 v^1 v'^1 + \cdots + q_n v^n v'^n.$$

The *inner product* (or *contraction*), $x \cdot x'$, is a bilinear map

$$\Lambda_q(\boldsymbol{e}) \times \Lambda_q(\boldsymbol{e}) \to \Lambda_q(\boldsymbol{e})$$

that generalizes the scalar product. For its definition we only need to consider the case when x and x' are simple multivectors, say

$$x = v_1 \wedge \cdots \wedge v_r, \quad y = v'_1 \wedge \cdots \wedge v'_s.$$

[5] $2v \cdot v' = Q(v + v') - Q(v) - Q(v')$.

The rules for computing $(v_1 \wedge \cdots \wedge v_r) \cdot (v'_1 \wedge \cdots \wedge v'_s)$ are as follows.

- If $r = 0$ (so x is a scalar, say λ), $\lambda \cdot y = \lambda y$. There is a similar rule for the case $s = 0$.
- If $r = 1 \leq s$ (so x is a vector, say $v = v_1$), $x \cdot y = \sum_{k=1}^{k=s} (-1)^{k-1} (v \cdot v'_k) v'_1 \wedge \cdots \wedge v'_{k-1} \wedge v'_{k+1} \wedge \cdots \wedge v'_s$. This means that $v\cdot$ acts as a (left) skew-derivation of the exterior product. When $s = 1$, the inner product is just the scalar product of vectors.
- If $s = 1 \leq r$ (so y is a vector, say $v' = v'_1$), $x \cdot v' = \sum_{k=1}^{k=r} (-1)^{r-k} (v_k \cdot v') v_1 \wedge \cdots \wedge v_{k-1} \wedge v_{k+1} \wedge \cdots \wedge v_r$. This means that $\cdot v'$ acts as a right skew-derivation of the exterior product. This agrees with $(-1)^{r-1} v' \cdot x$.
- If $2 \leqslant s \leqslant r$, $x \cdot y = (x \cdot v'_1) \cdot (v'_2 \wedge \cdots \wedge v'_s)$.

In the following lectures we will give precise formulas to evaluate the inner product. In particular we will see that for $r = s$, it does not matter whether we evaluate from the left or from the right, and that the result is the scalar $\det G(x, y)$, where $G(x, y)$ is the matrix whose entries are $v_i \cdot v'_j$. This means that $x \cdot y = y \cdot x$ when x and y are homogeneous of the same degree. For $r \neq s$, however, the commutativity does not hold in general, for if $s \leqslant r$, then $x \cdot y = (-1)^{rs+s} y \cdot x$.

Remark. We have denoted the inner product (following Hestenes-1966 and Casanova-1976) using only the dot ($\cdot$). But there are authors that use the symbols $x \rfloor y$ (when $r \leqslant s$) and $x \lfloor y$ (when $r \geqslant s$).

Some relations among the exterior, geometric and inner products

If $v \in E_n$ and x is any homogeneous multivector of degre r,

- $vx = v \cdot x + v \wedge x$ and $xv = x \cdot v + x \wedge v$.
- $2v \wedge x = vx + (-1)^r xv$.
- $2v \cdot x = vx - (-1)^r xv$.

In particular we see that two vectors commute (anticommute) if and only if they are parallel (perpendicular).

The aim of GA is the study of the structure $\Lambda_{\boldsymbol{q}}(\boldsymbol{e})$ (and others that are more general and to which we will devote the coming lectures) with the three products (exterior, geometric and interior) and to develop methods for its application to a variety of fields and problems.

In what follows of this lecture, we will consider in detail several examples for low n, including $\mathcal{G}_n$ and $\bar{\mathcal{G}}_n$ for $n = 2, 3$.

The linear basis of $\mathcal{G}_2$ (and of $\bar{\mathcal{G}}_2$) is $1, e_1, e_2, e_{12} \doteq \boldsymbol{i}$. The tables for the geometric product, however, are quite different:

$\mathcal{G}_2$	e_1	e_2	$\boldsymbol{i}$
e_1	1	$\boldsymbol{i}$	e_2
e_2	$-\boldsymbol{i}$	1	$-e_1$
$\boldsymbol{i}$	$-e_2$	e_1	-1

$\bar{\mathcal{G}}$	e_1	e_2	$\boldsymbol{i}$
e_1	-1	$\boldsymbol{i}$	$-e_2$
e_2	$-\boldsymbol{i}$	-1	e_1
$\boldsymbol{i}$	e_2	$-e_1$	-1

In both cases the even subalgebra $\langle 1, \boldsymbol{i}\rangle$ is isomorphic to $\mathbf{C}$, $a + b\boldsymbol{i} \mapsto a + bi$, and $\boldsymbol{i}$ anticommutes with the *vectors*, that is, the elements of $E_2 = \Lambda^1(\boldsymbol{e}) = \langle e_1, e_2\rangle$, but the action of $\boldsymbol{i}$ on E_2 is different: in $\mathcal{G}_2$, $\boldsymbol{i}\{e_1, e_2\} = \{-e_2, e_1\}$, while in $\bar{\mathcal{G}}_2$ we have $\boldsymbol{i}\{e_1, e_2\} = \{e_2, -e_1\}$. Multiplying by $\boldsymbol{i}$ on the left yields a rotation of amplitude $\pi/2$ with different orientations: clockwise in the case of $\mathcal{G}_2$ and counterclockwise in the case of $\bar{\mathcal{G}}$. On the other hand, if $x = x_0 + x_1 e_1 + x_2 e_2 + x_3 \boldsymbol{i}$, then $x^\alpha = x_0 - x_1 e_1 - x_2 e_2 + x_3 \boldsymbol{i}$, $x^\tau = x_0 + x_1 e_1 + x_2 e_2 - x_3 \boldsymbol{i}$ and $x^\kappa = x_0 - x_1 e_1 - x_2 e_2 - x_3 \boldsymbol{i}$.

G_2 can be represented by the matrices

$$e_1, e_2 \mapsto E_1 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, E_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

In detail:

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 \boldsymbol{i} \mapsto X = \begin{pmatrix} x_0 + x_1 & x_2 + x_3 \\ x_2 - x_3 & x_0 - x_1 \end{pmatrix},$$

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto x = \tfrac{1}{2}[(a+d) + (a-d)e_1 + (b+c)e_2 + (b-c)\boldsymbol{i}]$$

Thus $\mathcal{G}_2 \simeq \mathbf{R}(2)$, the algebra of 2×2 real matrices. In terms of X, the involutions act as follows

$$X^\alpha = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = det(X)(X^T)^{-1}, \quad X^\tau = X^T, \quad X^\kappa = det(X)X^{-1}.$$

From the multiplication table it is clear that $\bar{\mathcal{G}}_2 \simeq \mathbf{H}$, the field of quaternions, via (for example) $e_1 \doteq \boldsymbol{j}$, $e_2 \doteq \boldsymbol{k}$.

Now $\mathbf{H}$ is algebra-isomorphic with the (real) subalgebra $\mathcal{H}$ of $\mathbf{C}(2)$ of matrices

$$X = \begin{pmatrix} a+di & c+bi \\ -c+bi & a-di \end{pmatrix} = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}, \quad a,b,c,d \in \mathbf{R}.$$

The *Pauli-like matrices* $E_0 = I_2$,

$$E_1 = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma_1, \quad E_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} = i\sigma_2, \quad E_3 = \begin{pmatrix} i & \\ & -i \end{pmatrix} = i\sigma_3$$

form a linear basis of $\mathcal{H}$ and the linear map determined by $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \mapsto E_0, E_1, E_2, E_3$ gives in fact an insomorphism of algebras $\mathbf{H} \simeq \mathcal{H}$ because the E_k satisfy the relations $E_k^2 = -I_2$, $E_1E_2 = -E_2E_1 = E_3$ (and cyclic permutations) in exact correspondence with Hamilton's relations $\boldsymbol{i}^2 = -1$, $\boldsymbol{i}\boldsymbol{j} = -\boldsymbol{j}\,\boldsymbol{i} = \boldsymbol{k}$ (and cyclic permutations).

It is easy to check that if $X \in \mathcal{H}$ corresponds to $x = a + d\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} \in \mathbf{H}$, then its Clifford conjugate $\bar{x} = a - d\boldsymbol{i} - b\boldsymbol{j} - c\boldsymbol{k}$ corresponds to $X^\dagger$ (the transpose of the complex conjugate or *Hermitian adjoint* of X).

The square norm of x, $Q(x) = x\bar{x} = a^2 + b^2 + c^2 + d^2$, corresponds to $XX^\dagger = \det(X)\sigma_0$, so that $Q(x) = \det(X)$.

Note that for $x \neq 0$, $\bar{x}/Q(x)$ is the inverse of x.

Remark. The real matrices in $\mathcal{H}$ are precisely those of the space $\langle E_0, E_2\rangle$. This space is a subalgebra of $\mathbf{R}(2)$ which is isomorphic to $\mathbf{C}$, $1, i \mapsto E_0, E_2$. But notice that $\langle E_0, E_2\rangle$ is different from the image of the even subalgebra $\bar{\mathcal{G}}_2^+ \simeq \mathbf{C}$ of $\bar{\mathcal{G}}_2$ under the isomorphism $\bar{G}_2 \simeq \mathbf{H}$. Later in this lecture we are going to study a more satisfying realization of $\mathbf{H}$ using $\mathcal{G}_3$.

"The most remarkable formula in mathematics is:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is our jewel. We may *relate the geometry to the algebra* by representing complex numbers in a plane

$$x + iy = re^{i\theta}$$

This is the *unification of algebra and geometry*."

R. Feynman, *Lecture Notes in Physics*, Volume I, Part 1.

Comment. Emphasis not in the original. Note that Euler's formula works with no change by taking the 'imaginary unit' to be the geometric $\boldsymbol{i}$.

In the case of $\mathcal{G}_n$, the quadratic form Q on E_n is positive definite (*Euclidean space*). Setting, as usual, $|v| = +\sqrt{Q(v)}$, then for non-zero vectors $v, v' \in E_n$ there is a unique $\theta = \theta(v, v') \in [0, \pi]$ such that

$$v \cdot v' = |v||v'| \cos\theta.$$

Projection. Let u be a non-zero vector. Then for any vector u, the orthogonal projection of v on $\langle u \rangle$, $\pi_u(v)$, is given by the formula

$$\pi_u(v) = (v \cdot u)u^{-1}.$$

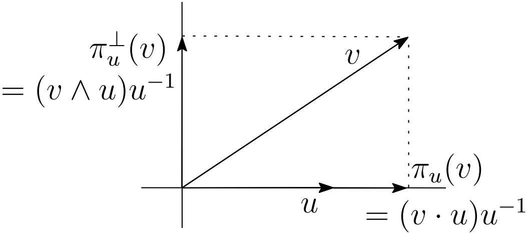
Proof. The right hand side is linear in v and its value is clearly 0 for $v \in u^{\perp}$. On the other hand, its value for $v = u$ is
$(u \cdot u)u^{-1} = u^2 u^{-1} = u$. □

Rejection. The difference $\pi_u^\perp(v) = v - \pi_u(v)$ is orthogonal to u (for $u^{-1}\cdot u = (u/u^2)\cdot u = 1$ and hence $(v - (v\cdot u)u^{-1})\cdot u = 0$) and sometimes it is called the *rejection* of v from $\langle u\rangle$.

Now we have:

$$\pi_u^\perp(v) = (v\wedge u)u^{-1}.$$

Proof. We know that $v\cdot u = vu - v\wedge u$. Hence
$\pi_u^\perp(v) = v - (v\cdot u)u^{-1}) = v - (vu - v\wedge u)u^{-1} = (v\wedge u)u^{-1}$. □



Axial symmetries and reflections

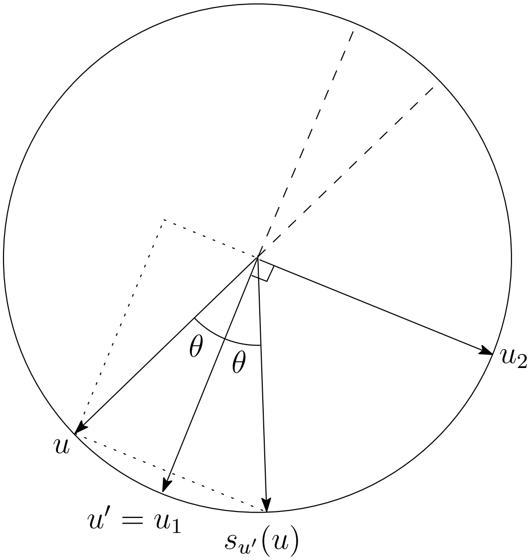
Proposition. If u is a non-zero vector, then the map $s_u : E_n \to E_n$, $v \mapsto uvu^{-1}$ is the *axial symmetry* with respect to the line (*axis*) $\langle u \rangle$.

Proof. Since $uu^{-1} = 1$, $s_u(u) = u$. If $v \in u^\perp$, then u and v anticommute and hence $s_u(v) = uvu^{-1} = -vuu^{-1} = -v$. Thus s_u is the linear map that leaves u fix and is $-\mathrm{Id}$ on $u^\perp$. □

Corollary. If u is a non-zero vector, then the map $m_u : E_n \to E_n$, $v \mapsto -uvu^{-1}$ ($m_u = -s_u$) is the *reflection* across the hyperplane $u^\perp$.

Proof. Indeed, m_u is the identity on $u^\perp$ and maps u to $-u$. □

Proposition. Let u and u' be non-zero vectors, $u \nparallel u'$, and set $\theta = \theta(u, u')$. Then the map $\rho_{u,u'} = s_{u'} s_u = m_{u'} m_u$ is the rotation of amplitude 2θ on the (oriented) plane $U = \langle u, u' \rangle$.



$$u = u_1 \cos\theta - u_2 \sin\theta \qquad s_{u'}(u) = u_1 \cos\theta + u_2 \sin\theta$$

Proof. Without loss of generality, we may assume that u and u' are unit vectors, so they are their own inverses. Since $m_u = -s_u$ and $m_{u'} = -s_{u'}$, the relation $s_{u'}s_u = m_{u'}m_u$ is clear.

Let $L = U^{\perp} = u^{\perp} \cap u'^{\perp}$. It is clear that $\rho_{u,u'}$ is the identity on L and that it leaves U invariant. Therefore it suffices to show that the restriction of $\rho = \rho_{u,u'}$ to the plane U is a rotation of amplitude 2θ. But the restriction of s_u and $s_{u'}$ to U are the *reflections* across $\langle u\rangle$ and $\langle u'\rangle$, respectively, and the composition of two reflections is a rotation, so that it is enough to calculate $\theta' = \theta(u, \rho(u)) = \theta(u, s_{u'}(u))$. To that end, let $u_1 = u'$ and $u_2 \in u'^{\perp} \cap U$ be unitary with u_1, u_2 defining the same orientation of U as u, u'. Then $u = \cos(\theta)u_1 - \sin(\theta)u_2$, $s_{u'}(u) = \cos(\theta)u_1 + \sin(\theta)u_2$, so $\cos(\theta') = u \cdot s_{u'}(u) = \cos^2(\theta) - \sin^2(\theta) = cos(2\theta)$ and $\sin(\theta') = 2\sin(\theta)\cos(\theta) = \sin(2\theta)$. □

Let u, u' be non-zero vectors and set $R = u'u \in \mathcal{G}_n^+$. We say that R is the *rotor* defined by u and u' on account of the following fact:

$$\rho_{u,u'}(v) = (s_{u'}s_u)(v) = u'(uvu^{-1})u'^{-1} = RvR^{-1}.$$

Since $\rho_{u,u'}$ only depends on R, we will write ρ_R to denote it. But it is important to remember that the amplitude θ_R of the rotation ρ_R is $2\theta(u, u')$.[6] If we only know R, θ_R can be obtained as the angle $\theta(v, \rho_R(v))$, where v is any vector in the plane of the rotation.

Composition. If R and S are rotors, $(\rho_R \circ \rho_S)(v) = (RS)v(RS)^{-1}$.

[6] For an interesting analysis about the historical difficulty of uncovering the significance of this factor of 2, see Altman-1989.

The non-zero vectors generate a subgroup Γ_n of the multiplicative group $\mathcal{G}_n^\times$ of invertible elements of $\mathcal{G}_n$ (with respect to the geometric product). The subgroup Γ_n^+ of even elements of Γ_n is the subgroup generated by the rotors and we have a homomorphism

$$\rho : \Gamma_n^+ \to \mathrm{SO}_n, \quad \psi \mapsto \rho_\psi,$$

where $\rho_\psi(v) = \psi v \psi^{-1}$.

The homomorphism ρ extends to a homomorphism $\widetilde{\rho} : \Gamma_n \to \mathrm{O}_n$, $\phi \mapsto \widetilde{\rho}_\phi$, where $\widetilde{\rho}_\phi(v) = (-1)^{|\phi|}\phi v \phi^{-1}$.

For a non-zero vector u, $\widetilde{\rho}_u = m_u$. Since the reflections across hyperplanes generate O_n, $\widetilde{\rho}$ is surjective. The homomorphism ρ is also surjective, because rotations are products of an even number of reflections.

The group Pin_n is defined as the subgroup of Γ_n generated by the unit vectors and $\mathrm{Spin}_n = \mathrm{Pin}_n \cap \Gamma_n^+$. Since $m_{\lambda u} = m_u$ for any non-zero vector u and scalar λ, any reflection can be written as m_u with u a unit vector. It follows that the homomorphisms

$$\widetilde{\rho} : \mathrm{Pin}_n \to \mathrm{O}_n \text{ and } \rho : \mathrm{Spin}_n \to \mathrm{SO}_n$$

are surjective.

These homomorphisms will be defined and studied under more general assumptions in the coming lectures. In particular we will see that their kernel is $\{\pm I_n\}$.

The area $A(v,v')$ of the parallelogram $[v,v']$ defined by $v,v' \in E_2$ is given by the formula $|v||v'|\sin\theta$, $\theta = \theta(v,v')$.

On the other hand, $v \wedge v' = D(v,v')\boldsymbol{i}$, where $\boldsymbol{i} = e_1 e_2$ and $D(v,v') = v_1 v_2' - v_2 v_1'$ is bilinear and skew-symmetric.

Now we have

$$v^2 {v'}^2 = v v' v' v = (v \cdot v')^2 - (v \wedge v')^2 = (v \cdot v')^2 + D(v,v')^2$$

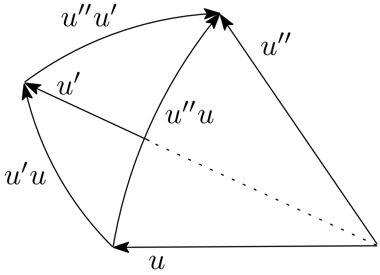
and hence

$$D(v,v')^2 = v^2 {v'}^2 - (v \cdot v')^2 = |v|^2 |v'|^2 (1 - \cos^2\theta) = |v|^2 |v'|^2 \sin^2\theta.$$

We conclude that $D(v,v') = \pm A(v,v')$, with the sign depending on the orientation of v, v' (relative to e_1, e_2).

Let $u, u' \in E_n$ be linearly independent and $U = \langle u, u' \rangle$. Let R be the rotor $u'u$, so that $\rho_R(v) = RvR^{-1}$ for $v \in E_n$. If $\theta = \theta(u, u') \in (0, \pi)$, then ρ_R is the rotation in U (extended by the identity on $U^\perp$) of amplitude $\alpha = 2\theta$ in the sense determined by the orientation of U given by u, u'. Pick an orthonormal basis u_1, u_2 of U with the same orientation as u, u' and let $\boldsymbol{i}_U = u_1 u_2 = u_1 \wedge u_2$ (the unit area in U), which satisfies $\boldsymbol{i}_U^2 = -1$. Then $u \wedge u' = |u||u'| \sin(\theta) \boldsymbol{i}_U$ and $u' \wedge u = -|u||u'| \sin(\theta) \boldsymbol{i}_U$. Consequently, $R = u'u = |u||u'|(\cos\theta - \sin\theta\, \boldsymbol{i}_U) = |u||u'|e^{-\theta \boldsymbol{i}_U} = |u||u'|e^{-\frac{1}{2}\alpha \boldsymbol{i}_U}$ and $R^{-1} = u^{-1}{u'}^{-1} = |u|^{-2}|u'|^{-2}uu' = |u|^{-1}|u'|^{-1}e^{\frac{1}{2}\alpha \boldsymbol{i}_U}$. Finally,

$$\rho_R(v) = e^{-\frac{1}{2}\alpha \boldsymbol{i}_U} v e^{\frac{1}{2}\alpha \boldsymbol{i}_U}.$$



If $R = u'u$ and $R' = u''u'$ are rotors, where u, u', u'' are unit vectors (there is no loss of generality with this assumption), then $R'R = u''u'u'u = u''u \doteq R''$. This is illustrated in the figure. In particular we have that $\rho_{R'} \circ \rho_R = \rho_{R''}$. It has to be remembered, however, that the rotation amplitudes of R, R' and R'' are $2\theta(u, u')$, $2\theta(u', u'')$ and $2\theta(u, u'')$.

The algebra $\mathcal{G} = \mathcal{G}_3$ has dimension 8. The spaces of scalars ($\mathcal{G}^0 = \mathbf{R}$) and *pseudoscalars* ($\mathcal{G}^3$) have dimension 1 and are generated by 1 and $\boldsymbol{i} = e_1 e_2 e_3$. The space of vectors ($E_3 = \mathcal{G}^1$) and of bivectors ($\mathcal{G}^2$) have dimension 3 and are generated by e_1, e_2, e_3 and $e_2 e_3, e_3 e_1, e_1 e_2$, respectively.

These generators can be written in a more compact form using the relations

$$e_2 e_3 = \boldsymbol{i} e_1 = e_1 \boldsymbol{i},\ \ e_3 e_1 = \boldsymbol{i} e_2 = e_2 \boldsymbol{i},\ \ e_1 e_2 = \boldsymbol{i} e_3 = e_3 \boldsymbol{i}$$

which show that $\boldsymbol{i}$ lies in the center of $\mathcal{G}$ and that the map $G^1 \to \mathcal{G}^2$, $v \mapsto \boldsymbol{i} v \doteq v^*$, is an isomorphism, with inverse the map $w \mapsto -\boldsymbol{i} w$. These isomorphisms, which are isometries, are a special case of *Hodge duality*.

The algebra $\mathcal{G}_3$ admits the (complex) matrix representation $1, e_1, e_2, e_3 \mapsto I_2, \sigma_1, \sigma_2, \sigma_3$, where the σ_k are the *Pauli matrices*:

$$\sigma_1 = \sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\ \sigma_2 = \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix},\ \sigma_3 = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

Therefore $\mathcal{G}_3 \simeq \mathbf{C}(2)$. Note that $\boldsymbol{i} = e_1 e_2 e_3 \mapsto \sigma_1\sigma_2\sigma_3 = iI_3$.

Proof. It is immediate to check that $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}$. □

The Pauli represention of $\mathcal{G}_3$, or any other matrix representation for that matter, is not needed to understand $\mathcal{G}_3$ and its applications. The advantages of working directly with $\mathcal{G}_3$, which can be regarded as the 'true' Pauli algebra, have been noticed already and will be further highlighted in the considerations that follow. The story will repeat later on when we study the Dirac algebra.

Multiplication table of $\mathcal{G} = \mathcal{G}_3$

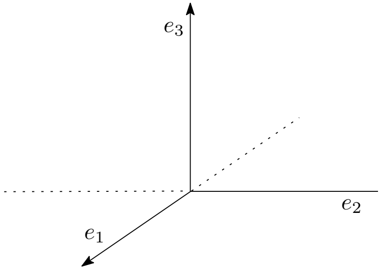
$\mathcal{G}_3$	e_1	e_2	e_3	$\boldsymbol{i}e_1$	$\boldsymbol{i}e_2$	$\boldsymbol{i}e_3$	$\boldsymbol{i}$
e_1	1	$\boldsymbol{i}e_3$	$-\boldsymbol{i}e_2$	$\boldsymbol{i}$	$-e_3$	e_2	$\boldsymbol{i}e_1$
e_2	$-\boldsymbol{i}e_3$	1	$\boldsymbol{i}e_1$	e_3	$\boldsymbol{i}$	$-e_1$	$\boldsymbol{i}e_2$
e_3	$\boldsymbol{i}e_2$	$-\boldsymbol{i}e_1$	1	$-e_2$	e_1	$\boldsymbol{i}$	$\boldsymbol{i}e_3$
$\boldsymbol{i}e_1$	$\boldsymbol{i}$	$-e_3$	e_2	-1	$-\boldsymbol{i}e_3$	$\boldsymbol{i}e_2$	$-e_1$
$\boldsymbol{i}e_2$	e_3	$\boldsymbol{i}$	$-e_1$	$\boldsymbol{i}e_3$	-1	$-\boldsymbol{i}e_1$	$-e_2$
$\boldsymbol{i}e_3$	$-e_2$	e_1	$\boldsymbol{i}$	$-\boldsymbol{i}e_2$	$\boldsymbol{i}e_1$	-1	$-e_3$
$\boldsymbol{i}$	$\boldsymbol{i}e_1$	$\boldsymbol{i}e_2$	$\boldsymbol{i}e_3$	$-e_1$	$-e_2$	$-e_3$	-1

We see that $\langle 1, \boldsymbol{i} \rangle \simeq \mathbf{C}$ is the center of $\mathcal{G}$.

We also see that the even subalgebra $\mathcal{G}^+ = \langle 1, \boldsymbol{i}e_1, \boldsymbol{i}e_2, \boldsymbol{i}e_3 \rangle$ is isomorphic to the quaternion field $\mathbf{H} = \langle 1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K} \rangle$, with $\boldsymbol{i}e_1 = e_2e_3, \boldsymbol{i}e_2 = e_3e_1, \boldsymbol{i}e_3 = e_1e_2 \mapsto \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}$.

See the slide 43 for further features about $\mathbf{H}$ deduced from this representation.

Since $\langle e_2, e_3\rangle^\perp = \langle e_1\rangle$ and $\theta(e_2, e_3) = \pi/2$, the rotation produced by the rotor ***I*** has axis $\langle e_1\rangle$ and amplitude $2\theta = \pi$. In other words, it is the axial symmetry with respect to the axis $\langle e_1\rangle$. In a similar way we find that ***J*** and ***K*** yield the axial symmetries with respect to the axes $\langle e_2\rangle$ and $\langle e_3\rangle$, respectively.



If v and v' are vectors, let $v \times v'$ be the vector such that

$$\boldsymbol{i}(v \times v') = v \wedge v' \text{ or } v \wedge v' = -\boldsymbol{i}(v \wedge v').$$

In particular we have, if j, k, l is a cyclic permutation of $1, 2, 3$,

$$e_j \times e_k = -\boldsymbol{i}(e_j \wedge e_k) = -\boldsymbol{i}(\boldsymbol{i} e_l) = e_l.$$

Lemma

$$v \times v' = -\boldsymbol{i} v \cdot v'.$$

Proof. Since both sides are linear in v, it is enough to check the formula for $v = e_j$. In this case, the left hand side is $e_j \times v' = v'_k e_l - v'_l e_k$, while the right hand side is $-\boldsymbol{i} e_j \cdot v' = -e_k e_l \cdot v' = (v' \cdot e_k)e_l - (v' \cdot e_l)e_k = v'_k e_l - v'_l e_k$. □

Remark. Since $\boldsymbol{i}$ reverses sign when we reverse the orientation of the basis, we see that $v \times v'$ also reverses sign when we reverse the orientation. This is usually described by saying that the cross product is an *axial vector* to distinguish it from the *polar* vectors (the vectors in E_3) whose nature is independent of the space orientation.

Mixed product

$$(v \times v') \cdot v'' = -i(v \wedge v' \wedge v'') = \det(v, v', v'').$$

Proof. Since $v \times v'$ is a vector,

$$2(v \times v') \cdot v'' = (v \times v')v'' + v''(v \times v').$$

Using $v \times v' = -\boldsymbol{i}(v \wedge v')$, and that $\boldsymbol{i}$ is a central element, we get

$$2(v \times v') \cdot v'' = -\boldsymbol{i}(v \wedge v')v'' - \boldsymbol{i}v''(v \wedge v') = -2\boldsymbol{i}v \wedge v' \wedge v''.$$

To finish, use $v \wedge v' \wedge v'' = \boldsymbol{i}\det(v, v', v'')$. □

Double cross product

$$(v \times v') \times v'' = (v \cdot v'')v' - (v' \cdot v'')v.$$

Proof. Indeed, $(v \times v') \times v'' = -i(v \times v') \cdot v'' = -(v \wedge v') \cdot v''$ and $(v \wedge v') \cdot v'' = (v' \cdot v'')v - (v \cdot v'')v'$. □

Geometrically, the cross-product of two linearly independent vectors is determined by the following properties:

1) $v \times v'$ is orthogonal to v and to v'.
2) Its length is equal to $A(v, v')$.
3) $v, v', v \times v'$ is positively oriented.

Proof. The mixed product formula gives 1). As for 2), we have $|v \times v'|^2 = |v \wedge v'|^2 = (v \wedge v') \cdot (v \wedge v') = A(v, v')^2$. Finally,

$$\det(v, v', v \times v') = (v \times v') \cdot (v \times v') > 0.$$ □

Let u be a unit vector and α a real number. Let $\rho_{u,\alpha}$ be the rotation about u of amplitude α. Then the following variation of Euler's spinor formula holds:

$$\rho_{u,\alpha}(v) = e^{-\frac{1}{2}\boldsymbol{i}u\alpha} v e^{\frac{1}{2}\boldsymbol{i}u\alpha}.$$

Proof. Let u_1, u_2 be perpendicular unit vectors in $U = u^{\perp}$ such that $u_1 \times u_2 = u$. If we let $\boldsymbol{i}_U = u_1 \wedge u_2$, then we know that

$$\rho_{u,\alpha}(v) = e^{-\frac{1}{2}\boldsymbol{i}_U\alpha} v e^{\frac{1}{2}\boldsymbol{i}_U\alpha}.$$

Now note that $\boldsymbol{i}_P = u_1 \wedge u_2 = \boldsymbol{i}(u_1 \times u_2) = \boldsymbol{i}u$. □

Example. The rotor for $\rho_{e_1,\pi}$ is $e^{-\frac{1}{2}\boldsymbol{i}e_1\pi} = e^{-\boldsymbol{I}\frac{\pi}{2}} = \boldsymbol{I}$. Similarly, $\boldsymbol{J}$ and $\boldsymbol{K}$ are the rotors for $\rho_{e_2,\pi}$ and $\rho_{e_3,\pi}$, respectively, in accord with the slide 36.

Lemma (Vector algebra form of Euler's rotor)

$$\rho_{u,\alpha}(v) = (1-\cos\alpha)(v\cdot u)u + v\cos\alpha + (u\times v)\sin\alpha.$$

Proof. Since this expression is linear in v, and its value for $v=u$ is u, it is enough to consider the case in which v is orthogonal to u. In that case, v anticommutes with u and

$$e^{-\frac{1}{2}\boldsymbol{i}u\alpha}ve^{\frac{1}{2}\boldsymbol{i}u\alpha} = ve^{\boldsymbol{i}u\alpha} = v\cos\alpha + v\boldsymbol{i}u\sin\alpha = v\cos\alpha + (u\times v)\sin\alpha$$

as $v\boldsymbol{i}u = \boldsymbol{i}vu = -\boldsymbol{i}(u\wedge v) = u\times v$.

In matrix form, say $x' = xM$,

$$M = \begin{pmatrix} a^2+(1-a^2)\cos\alpha & ab\delta + c\sin\alpha & ac\delta - b\sin\alpha \\ ba\delta - c\sin\alpha & b^2+(1-b^2)\cos\alpha & bc\delta + a\sin\alpha \\ ca\delta + b\sin\alpha & cb\delta - a\sin\alpha & c^2+(1-c^2)\cos\alpha \end{pmatrix}$$

where $u\equiv(a,b,c)$ and $\delta = 1-\cos\alpha$. □

To determine the composition $\rho_{u',\alpha'} \circ \rho_{u,\alpha}$, it is enough to compute its rotor, say $e^{\frac{1}{2}\boldsymbol{i}u''\alpha''}$, as the product of the corresponding rotors:

$$e^{\frac{1}{2}\boldsymbol{i}u''\alpha''} = e^{\frac{1}{2}\boldsymbol{i}u'\alpha'} e^{\frac{1}{2}\boldsymbol{i}u\alpha}.$$

This relation can be written in the form

$\cos\frac{\alpha''}{2} + \boldsymbol{i}u''\sin\frac{\alpha''}{2} = (\cos\frac{\alpha'}{2} + \boldsymbol{i}u'\sin\frac{\alpha'}{2})(\cos\frac{\alpha}{2} + \boldsymbol{i}u\sin\frac{\alpha}{2})$ which itself is equivalent to the equations

$$\cos\tfrac{\alpha''}{2} = \cos\tfrac{\alpha'}{2}\cos\tfrac{\alpha}{2} - (u \cdot u')\sin\tfrac{\alpha'}{2}\sin\tfrac{\alpha}{2}$$

$$u''\sin\tfrac{\alpha''}{2} = u\sin\tfrac{\alpha}{2}\cos\tfrac{\alpha'}{2} + u'\cos\tfrac{\alpha}{2}\sin\tfrac{\alpha'}{2} + (u \times u')\sin\tfrac{\alpha}{2}\sin\tfrac{\alpha'}{2}$$

There are two solutions to the first equation ($\pm\alpha''$), and hence two solutions $\pm u''$ to the second equation, but since $\rho_{-u,-\alpha} = \rho_{u,\alpha}$, they determine the same rotation.

Let us return to the realization of the quaternion field $\mathbf{H}$ as the even algebra $\mathcal{G}_3^+$ (slide 35). A quaternion x can be written in the form $x = s + \boldsymbol{i}ut$, where u is a unit vector and $s, t \in \mathbf{R}$. Then $\bar{x} = s - \boldsymbol{i}ut$, because $\boldsymbol{i}u$ is a bivector. Thus $Q(x) = x\bar{x} = s^2 + t^2$ and the inverse x^{-1} of a non-zero quaternion x is $x^{-1} = \bar{x}/Q(x)$. Note that $|x| = \sqrt{s^2 + t^2}$ is the norm on $\mathbf{H}$ associated to the (Euclidean) symmetric bilinear form $\langle x|y\rangle = \frac{1}{2}(x\bar{y} + y\bar{x})$.

Given two quaternions $x = s + \boldsymbol{i}ut$ and $x' = s' + \boldsymbol{i}u't'$, we have $Q(xx') = xx'\bar{x}'\bar{x} = Q(x)Q(x')$ and as consequence $|xx'| = |x||x'|$.

The explicit form of xx' is given by the expression

$$xx' = ss' + \boldsymbol{i}u't's + \boldsymbol{i}uts' - uu'tt'.$$

It follows that $xx' - x'x = (u'u - uu')tt' = 2(u \wedge u')tt'$. Thus x and x' commute if and only if one of them is scalar or else $u' = \pm u$.

Why not *trinions*

If there were 3D 'numbers' $a + b\boldsymbol{i} + c\boldsymbol{j}$ analogous to complex numbers $a + b\boldsymbol{i}$ in 2D, then in particular we would have $\boldsymbol{i}\boldsymbol{j} = a + b\boldsymbol{i} + c\boldsymbol{j}$ for some $a, b, c \in \mathbf{R}$. Multiplying by $\boldsymbol{i}$, we obtain

$$-\boldsymbol{j} = a\boldsymbol{i} - b + c(a + b\boldsymbol{i} + c\boldsymbol{j}),$$

which is equivalent to

$$-b + ca + (a + cb)\boldsymbol{i} + (1 + c^2)\boldsymbol{j} = 0,$$

and this contradicts the assumed (linear) independence of $1, \boldsymbol{i}, \boldsymbol{j}$.

Beyond quaternions?

A key result is *Hurwitz's theorem* (1898): there are exactly four normed real division algebras: **R**,**C**, **H** and **O** (Cayley's octonion algebra, wich is non-associative). See Baez-2002.

Before each lecture, I will try to upload the pdf slides to SLP-GAT. In particular, you will find there details for references, an in particular for those mentioned in the course slides: References for SLP-GAT.

Artin 1957

Casanova 1976

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Geometric Algebra Techniques in Mathematics and Physics

S. Xambó

UPC

SLP · 9-13 March · 2015

- **Introduction**. Objectives. Notations and conventions.
- **Exterior algebra**. Exterior powers. Exterior product. ΛE. Universal property of ΛE. On the *Ausdehnungslehere*. The skew-derivation $\widehat{\xi}$. Dimension of the exterior algebra. Creation (or inflation) operators. Functoriality.
- **Duality**. Duality pairing. Duality formula. $\Lambda^r(E^*) \simeq (\Lambda^r E)^*$.
- **Clifford algebra**. The bilinear form q. Annihilating (or contracting) operators. Key lemma. The algebra $C_q(E)$. The canonical linear map $C_q(E) \to \Lambda E$. Grade involution and $C_q^+ E$. The algebras C_n and $\bar{C}_n$. The algebra $C_{1,1}$.
- **Lipschitz groups**. Clifford automorphisms. The group Γ_q. Adjoint representation of Γ_q. Twisted Clifford operators. The group $\widetilde{\Gamma}_q$. Adjoint representation of $\widetilde{\Gamma}_q$. $\mathrm{Pin}_q(E)$ and $\mathrm{Spin}_q(E)$.
- **References**

> Until recently I was unacquainted with the Ausdehnungslehre [...]. I may, perhaps, therefore be permitted to express my profound admiration of that extraordinary work and my conviction that its principles will exercise a vast influence upon the future of science.
>
> CLIFFORD-1878

The ground ingredient in our presentation is the exterior algebra ΛE of a vector space E of finite dimension n over a commutative field K. When E is endowed with a quadratic form, ΛE can be enriched with an associative product (the *geometric product* or *Clifford product*) for which all non-isotropic vectors are invertible. As illustrated in the introductory lecture, we can regard ΛE as a stage for defining geometric objects and relations among them, and the geometric product, together with the inner product that will be introduced in general in next lecture, as an efficient toolbox for defining transformations of the geometric objects with no need of coordinates or matrices.

E and F denote vector spaces.

- $L(E,F)$: the vector space of linear maps from E to F.
- $E^* = L(E,K)$: *dual space* of E.
- $\mathrm{End}_K(E) = L(E,E)$: the vector space of *endomorphisms* of E. It is an associative algebra with the product given by the composition of endomorphisms: $(f,g) \mapsto f \circ g$. The algebra with the *opposite product* ($(f,g) \mapsto g \circ f$) will be denoted $\mathrm{End}_K^{\mathrm{op}}(E)$.
- $E^r = E \times \overset{r)}{\cdots} \times E$.
- $L_r(E;F)$: the vector space of multilinear maps $E^r \to F$. Note that $L_1(E;F) = L(E,F)$. By convention, $L_0(E;F) = F$.
- $A_r(E;F)$: the vector space of skew-symmetric (also called *alternating*) multilinear maps $E^r \to F$. Note that $A_1(E;F) = L(E,F)$. By convention, $A_0(E;F) = F$).

In order to check that a multilinear map $f \in L_r(E;F)$ is skew-symmetric, it is enough to show that $f(e_1,\dots,e_r)$ changes sign when any two of the (arbitrary) vectors $e_1,\dots,e_r$ are interchanged.

- We will also use the *tensor powers* T^rE ($r \geqslant 0$) of E (we will refer to the elements of T^rE as *tensors* of order r). By convention, $T^0E = K$ and for $r > 0$ there is, for any vectors $e_1,\dots,e_r \in E$, a well defined element $e_1 \otimes \cdots \otimes e_r \in T^rE$, called the *tensor product* of $e_1,\dots,e_r$, such that:

1) the map $E^r \to T^rE$, $(e_1,\dots,e_r) \mapsto e_1 \otimes \cdots \otimes e_r$, is multilinear;
2) for any multilinear map $f : E^r \to F$, F a vector space, there exists a unique *linear* map $f^t : T^rE \to F$ such that

$$f^t(e_1 \otimes \cdots \otimes e_r) = f(e_1,\dots,e_r).$$

Furthermore, there exists a unique bilinear map (*tensor product*)

$$T^rE \times T^sE \to T^{r+s}E, \quad (x, y) \mapsto x \otimes y,$$

such that

$$(e_1 \otimes \cdots \otimes e_r) \otimes (e'_1 \otimes \cdots \otimes e'_s) = e_1 \otimes \cdots \otimes e_r \otimes e'_1 \otimes \cdots \otimes e'_s.$$

The *tensor algebra* is the vector space

$$TE = \bigoplus_{r \geqslant 0} T^rE = K \oplus E \oplus T^2E \oplus \cdots$$

endowed with the tensor product. It is an associative graded algebra.

For any integer $r > 0$, the r-th *exterior power* of E, denoted $\Lambda^r E$, is characterized as follows:

- $\Lambda^0 E = K$ and $\Lambda^1 E = E$.
- If $r > 1$, for any elements $e_1, \ldots, e_r \in E$ there is a well defined element

$$e_1 \wedge \cdots \wedge e_r \in \Lambda^r E,$$

called the *exterior product* of $e_1, \ldots, e_r$, and the map

$$(e_1, \ldots, e_r) \mapsto e_1 \wedge \cdots \wedge e_r$$

is multilinear and skew-symmetric. □

Universal property of $\Lambda^r E$

If $f : E^r \to F$ is a skew-symmetric multilinear map with values in a vector space F, then there is a unique *linear* map $\widehat{f} : \Lambda^r E \to F$ such that $\widehat{f}(e_1 \wedge \cdots \wedge e_r) = f(e_1, \ldots, e_r)$. In other words, $f \mapsto \widehat{f}$ provides a natural isomorphism $A_r(E; F) \simeq L(\Lambda^r E, F)$. Note that this isomorphism also holds for $k = 0, 1$.

- $\Lambda^r E = 0$ for any $r > n$.

Exterior product

Given integers $r, s \geqslant 0$, there is a unique bilinear map

$$\Lambda^r E \times \Lambda^s E \to \Lambda^{r+s} E, \quad (x, y) \mapsto x \wedge y,$$

such that $(e_1 \wedge \cdots \wedge e_r) \wedge (e'_1 \wedge \cdots \wedge e'_s) \mapsto e_1 \wedge \cdots \wedge e_r \wedge e'_1 \wedge \cdots \wedge e'_s$. This map is called *exterior* (or *outer*) *product*.

$$\Lambda E = \bigoplus_{r=0}^{n} \Lambda^r E = K \oplus E \oplus \Lambda^2 E \oplus \cdots \oplus \Lambda^n E.$$

with the exterior product is a graded associative algebra (the *exterior algebra* of E). This implies that $e_1 \wedge \cdots \wedge e_r = e_1 \wedge \cdots \wedge e_r$ and hence there is no need to distinguish the operator symbols $\wedge$ and $\wedge$.

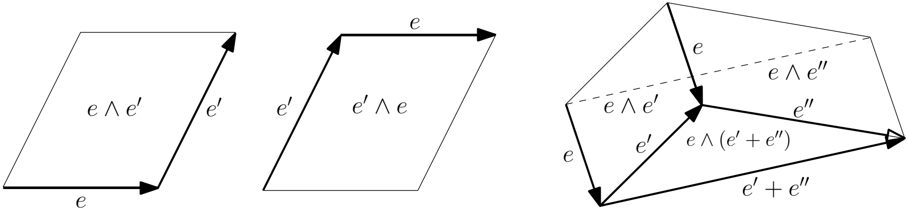
If $x \in \Lambda E$, there is a unique decomposition $x = x_0 + x_1 + \cdots + x_n$ with $x_r \in \Lambda^r E$. The term x_r is called the *grade r* component of x (there are authors that denote it by $\langle x \rangle_r$). The grade 0 and 1 components are also referred to as *scalar* and *vector* components. Since $\dim_K \Lambda^n E = 1$, x_n is also called the *pseudoscalar* component of x.

The exterior product is *grade-commutative* (or *skew-commutative*, or *supercommutative*), which means that if $x \in \Lambda^r E$ and $y \in \Lambda^s E$, then

$$y \wedge x = (-1)^{rs} x \wedge y. \quad (1)$$

In particular, $e' \wedge e = -e \wedge e'$, for all $e, e' \in E$.

If $i : E \to A$ is a linear map with values in an algebra A and i satisfies $i(e)^2 = 0$ for any $e \in E$, then there exists a unique homomorphism of algebras $j : \Lambda E \to A$ such that $j(e) = i(e)$ for all $e \in E$.



An *r-blade* is an r-vector of the form $e_1 \wedge e_2 \wedge \cdots \wedge e_r$, $e_1, \ldots, e_r \in E$. Any r-vector is the sum of a finite number of r-blades, but in general there are r-vectors that are not r-blades.

The exterior algebra is also known as *Grassmann algebra*, inasmuch as it is the algebraic structure that Hermann G. Grassmann (1809-1877) discovered in pursuing his *Ausdehnungslehre*, or extension theory (cf. Grassmann-2000).

Just as the vectors $e \in E = \Lambda^1 E$ represent oriented extensions of dimension 1, the elements of $\Lambda^r E$, which are called *r-vectors*, represent oriented extensions of dimension r.

For example, if $e, e' \in E$, then the 2-vector (or *bivector*) $e \wedge e'$ represents the 2-dimensional extension associated to the oriented parallelogram defined by e and e'. The oriented condition of the notion of extension is echoed in this case by the rule $e' \wedge e = -e \wedge e'$ $(e, e' \in E)$.

Let $\xi \in E^* = L(E, K)$ (*dual space* of E). Then there exists a unique *skew-derivation* $\widehat{\xi}$ of ΛE such that $\widehat{\xi}(e) = \xi(e)$ for all $e \in E$. This skew-derivation satisfies

$$\widehat{\xi}(e_1 \wedge \cdots \wedge e_r) = \sum_{k=1}^{r} (-1)^{k-1} \xi(e_k)\, e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_r \quad (*)$$

for any $e_1, \ldots, e_r \in E$. The map $\widehat{\xi}$ is graded of degree -1 and $\widehat{\xi}^{\,2} = 0$.

Given a vector sequence $e = e_1, \ldots, e_m \in E$, and a sequence of indexes $I = i_1, \ldots, i_r \in \{1, \ldots, m\}$, we write

$$\widehat{e_I} = e_{i_1} \wedge \cdots \wedge e_{i_r} \in \Lambda^r E.$$

We say that I is *multiindex* if $i_1 < \cdots < i_r$.

If $\boldsymbol{e} = e_1, \dots, e_n \in E$ is a basis of E, then the blades $\widehat{e}_I$, when I runs over the set of multiindices (of length r), form a basis of ΛE ($\Lambda^r E$). In particular, $\dim_K \Lambda^r E = \binom{n}{r}$ and $\dim_K \Lambda E = 2^n$.

In terms of the L1, the map $\Lambda(\boldsymbol{e}) \to \Lambda(E)$ is an isomorphism. On the left, the e_j are regarded as symbols but on the right they are vectors in E.

If $e \in E$, we define the linear map $\mu_e : \Lambda E \to \Lambda E$ by the formula

$$\mu_e(x) = e \wedge x.$$

The linear map μ_e is graded of degree 1 and $\mu_e^2 = 0$.

Since the exterior product is multilinear, the map

$$E \to \mathrm{End}_K(\Lambda E), \quad e \mapsto \mu_e,$$

is linear and extends in a unique way to an algebra homomorphism $\Lambda E \to \mathrm{End}_K(E)$, say $x \mapsto \mu_x$, and it can be easily checked that $\mu_x(y) = x \wedge y$ for all $x, y \in \Lambda E$.

If $f : E \to F$ is a linear map, for any $r \geqslant 0$ there is unique linear map $\Lambda^r f : \Lambda^r E \to \Lambda^r F$ such that

$$(\Lambda^r f)(e_1 \wedge \cdots \wedge e_r) = f(e_1) \wedge \cdots \wedge f(e_r).$$

Gluing the $\Lambda^r f$ for the different r we get a linear map $\Lambda f : \Lambda E \to \Lambda F$, and this map is a homomorphism of algebras.

Main/grade/parity involution

The linear automorphism $E \to E$, $e \mapsto -e$, yields a linear automorphism $\alpha : \Lambda^r E \to \Lambda^r E$ such that

$$(e_1 \wedge \cdots \wedge e_r) \mapsto (-e_1) \wedge \cdots \wedge (-e_r) = (-1^r) e_1 \wedge \cdots \wedge e_r.$$

The corresponding automorphism of ΛE, which will also be denoted by α, acts on even grades as the identity and on odd grades as minus the identity: if $x = \sum_{r \geqslant 0} x_r \in \Lambda E$ is the grade decomposition of x, then $\alpha(x) = \sum_{r \geqslant 0} (-1)^r x_r$.

Even subalgebra and superalgebra structure

$\Lambda^{+} E = \sum_{j\geqslant 0} \Lambda^{2j} E$ is a (graded) subalgebra of ΛE. In fact,

$$\Lambda^{+} E = \{x \in \Lambda E \,|\, x^{\alpha} = x\}.$$

The graded subspace

$$\Lambda^{-} E = \{x \in \Lambda E \,|\, \alpha(x) = -x\} = \sum_{j\geqslant 0} \Lambda^{2j+1} E$$

is called the *odd subspace* of ΛE (it is not a subalgebra), and

$$\Lambda E = \Lambda^{+} E \oplus \Lambda^{-} E.$$

This decomposition is a $\mathbf{Z}_2$-grading, in the sense that

$$\Lambda^{+}\Lambda^{+} \subseteq \Lambda^{+}, \quad \Lambda^{+}\Lambda^{-}, \Lambda^{-}\Lambda^{+} \subseteq \Lambda^{-}, \text{ and } \Lambda^{-}\Lambda^{-} \subseteq \Lambda^{+}.$$

With respect to this grading, the exterior product is *supercommutative*: $x \wedge y = (-1)^{|x||y|} y \wedge x$, where $|x| = 0$ if $x \in \Lambda^{+} E$ and $|x| = 1$ if $x \in \Lambda^{-} E$.

Reversion involution

Since the map $E^r \mapsto \Lambda^r$ such that $(e_1, \ldots, e_r) \mapsto e_r \wedge \cdots \wedge e_1$ is multilinear and skew-symmetric, there is a unique linear map

$$\tau : \Lambda^r E \to \Lambda^r E, \quad \text{such that} \quad e_1 \wedge \cdots \wedge e_r \mapsto e_r \wedge \cdots \wedge e_1.$$

Gluing these maps for $r \geqslant 0$ we get a linear automorphism $\tau : \Lambda E \to \Lambda E$ which is an involutive *anti-automorphism* of the exterior product: $\tau(x \wedge y) = \tau(y) \wedge \tau(x)$ (it is called the *main anti-automorphism* or the *reversion involution* of ΛE). Since

$$e_r \wedge \cdots \wedge e_1 = (-1)^{\binom{r}{2}} e_1 \wedge \cdots \wedge e_r,$$

the restriction of τ to $\Lambda^r E$ amounts to multiplication by $(-1)^{\binom{r}{2}}$.

Clifford involution

The anti-automorphism $x \mapsto \kappa(x) = \bar{x}$, where $\bar{x} = \widetilde{x}^{\alpha}$ is called the *Clifford involution* or *Clifford conjugation*. It coincides with $\widetilde{x^{\alpha}}$ and on grade r elements it reduces to multiplication by $(-1)^{\binom{r+1}{2}}$.

Remark. The integers $\binom{r}{2} = r(r-1)/2$ and $r//2$ have the same parity, where $r//2$ is the integer quotient of r by 2, or the integer part of $r/2$. Thus we have that $\widetilde{x} = (-1)^{r//2}x$ and $\bar{x} = (-1)^{(r+1)//2}x$ for all $x \in \Lambda^r E$.

The patterns of these signs for the consecutive integers $4k, 4k+1, 4k+2, 4k+3$ are $++--$ and $+--+$, respectively.

Consider the linear map $E^* \to \mathrm{End}_K^{\mathrm{op}}(\Lambda E)$ such that $\xi \mapsto \widehat{\xi}$.

Since $\widehat{\xi}^2 = 0$, the universal property of the exterior algebra tells us that there exists a unique homomorphism of algebras $\Lambda E^* \to \mathrm{End}_K^{\mathrm{op}}(\Lambda E)$, $\phi \mapsto \widehat{\phi}$, which agrees with $\xi \mapsto \widehat{\xi}$ for $\xi \in E^*$.

If $x \in \Lambda E$ and $\phi \in \Lambda E^*$, instead of $\widehat{\phi}(x)$ we will simply write $\phi(x)$. Thus we have a bilinear *duality pairing* $\Lambda E^* \times \Lambda E \to \Lambda E$ or, more specifically

$$\Lambda^r E^* \times \Lambda^s E \to \Lambda^{s-r} E.$$

Since this map vanishes for $r > s$, in the remaining of this section we will assume that $r \leqslant s$.

Next step will be to find a practical formula for evaluating $\phi(x)$.

Remark. Instead of $\phi(x)$, some authors write $\phi \lrcorner x$ (*left contraction*).

- If r and s are positive integers, $r \leqslant s$, $\mathscr{I}_{r,s}$ will denote the set of multiindices $I \subseteq \{1, \ldots, s\}$ of length r.
- Given $I \in \mathscr{I}_{r,s}$, we set $I' = \{1, \ldots, s\} - I$. In the special case in which I has a single element k, instead of $\{k\}'$ we will write k', so that $k' = \{1, \ldots, s\} - \{k\}$.
- The number of inversions in the sequence (I, I') will be denoted $t(I)$, so that $(-1)^{t(I)}$ is the sign of the permutation (I, I') of $\{1, \ldots, s\}$.
- If M is an $r \times s$ matrix and $I \in \mathscr{I}_{r,s}$, M_I will denote the $r \times r$ submatrix of M formed with the columns whose indexes are the elements of I.

Let $\xi_1, \ldots, \xi_r \in E^*$ and $e_1, \ldots, e_s \in E$, $r \leqslant s$. Then we have:

$$(\xi_1 \wedge \cdots \wedge \xi_r)(e_1 \wedge \cdots \wedge e_s) = \sum_{I \in \mathscr{I}_{r,s}} (-1)^{t(I)} \det(M_I)\, \widehat{e}_{I'},$$

where M is the $r \times s$ matrix whose entries are the scalars $\xi_i(e_j)$, for $i = 1, \ldots, r$, $j = 1, \ldots, s$.

For $r = 1$, this formula agrees with the antiderivation $\widetilde{\xi_1}$.

When $s = r$, the result is the scalar $\det(M)$.

Example. $(\xi_1 \wedge \xi_2)(e_1 \wedge e_2 \wedge e_3) =$

$$\begin{vmatrix} \xi_1(e_2) & \xi_1(e_3) \\ \xi_2(e_2) & \xi_2(e_3) \end{vmatrix} e_1 - \begin{vmatrix} \xi_1(e_1) & \xi_1(e_3) \\ \xi_2(e_1) & \xi_2(e_3) \end{vmatrix} e_2 + \begin{vmatrix} \xi_1(e_1) & \xi_1(e_2) \\ \xi_2(e_1) & \xi_2(e_2) \end{vmatrix} e_3$$

The map $\Lambda^r E^* \to (\Lambda^r E)^*$, $\phi \mapsto \phi(\ )$, is a linear isomorphism.

Fix a basis $e_1, \dots, e_n$ of E and let $e^1, \dots, e^n \in E^*$ be the dual basis (that is $e^i(e_j) = \delta^i_j$). The r-blades $\widehat{e}_I$, when $I = (i_1 < \cdots < i_r)$ runs over the multiindices of order r, form a basis of $\Lambda^r E$. Similarly, the dual r-blades $\widehat{e}^J$, when $J = (j_1 < \cdots < j_r)$ runs over the multiindices of order r, form a basis of $\Lambda^r E^*$.

Moreover, by the duality formula

$$\widehat{e}^J(\widehat{e}_I) = \delta^I_J$$

the image of the basis $\{\widehat{e}^J\}$ of $\Lambda^r E^*$ in $(\Lambda^r E)^*$ is the dual basis of the basis $\{\widehat{e}_I\}$ of $\Lambda^r E$.

From now on we will assume that we have a fixed linear map $q : E \to E^*$ or, equivalently, a *bilinear map* on E, the two views being related by the equation

$$q(e, e') = q(e)(e'), \quad e, e' \in E.$$

Remark. This relation establishes a canonical linear map

$$L(E, E^*) \to L_2(E; K),$$

which is an isomorphism (the inverse map is determined by $q(e) = q(e, \cdot)$).

Eventually we will require that q is symmetric, and sometimes also non-degenerate, but we can go a long way without these assumptions.

The map $\delta_e : \Lambda E \to \Lambda E$ is defined as the skew-derivation associated to $\widetilde{e} = q(e) \in E^*$:

$$\delta_e(x) = \widetilde{e}(x).$$

Such maps δ_e, which are graded of degree -1, are called *annihilating operators*.

The map

$$E \to \mathrm{End}_K(\Lambda E), \quad e \mapsto \delta_e,$$

is linear and satisfies $\delta_e^2 = 0$ for all $e \in E$.

In next statement we will use δ_e and the creation operators μ_e, which also satisfy $\mu_e^2 = 0$.

For any $e \in E$,

$$(\mu_e + \delta_e)^2 = q(e,e)\mathrm{Id}_{\wedge E}.$$

Proof. Expanding the square, we get:

$$(\mu_e + \delta_e)^2 = \mu_e^2 + \delta_e^2 + \mu_e\delta_e + \delta_e\mu_e = \mu_e\delta_e + \delta_e\mu_e.$$

Now

$$(\delta_e\mu_e)(x) = \widetilde{e}(e \wedge x) = \widetilde{e}(e)x - e \wedge \widetilde{e}(x) = q(e,e)x - \mu_e(\delta_e(x)),$$

so that

$$(\mu_e\delta_e + \delta_e\mu_e)(x) = q(e,e)x = q(e,e)\mathrm{Id}_{\wedge E}(x) \quad \text{for all } x.$$

This, together with the previous relation, completes the proof. □

Consider now the linear map

$$\lambda : E \to \mathrm{End}_K(\Lambda E), \quad e \mapsto \lambda_e = \mu_e + \delta_e.$$

This map can be extended in a unique way, by the universal property of the tensor algebra, to a homomorphism of algebras $\lambda : TE \to \mathrm{End}_K(\Lambda E)$, a map that satisfies (and is determined by) the relation

$$\lambda(e_1 \otimes \cdots \otimes e_r) = \lambda_{e_1} \circ \cdots \circ \lambda_{e_r}, \quad e_1, \ldots, e_r \in E.$$

Since the elements of the form $t_e = e \otimes e - q(e,e)1_K$ belong to the kernel of λ, λ induces a unique algebra homomorphism

$$\bar{\lambda} : C_q E \to \mathrm{End}_K(\Lambda E), \quad \bar{\lambda}(\bar{t}) = \lambda(t)$$

where $C_q E$ denotes the quotient of TE by the bilateral ideal I_q generated by the tensors t_e, $e \in E$, and $\bar{t} \in C_q E$ denotes the image of $t \in TE$ under the quotient map $TE \to C_q E$.

The algebra C_qE, which in the literature is often denoted $C\ell(E,q)$ or by other similar symbols, is called the *Clifford algebra* of q (or of (E,q)) and its product (the *Clifford product*) will be denoted by juxtaposition of its factors.

Note that if $e, e' \in E$, then we have the *Clifford relations*

$$\bar{e}^2 = q(e,e), \qquad \bar{e}\bar{e}' + \bar{e}'\bar{e} = q(e,e') + q(e',e).$$

The first equality is a direct consequence of the fact that $\bar{t}_e = 0$. For the second, expand $(\bar{e}+\bar{e}')^2$ and $q(e+e', e+e')$ and use the first relation for e, e' and $e+e'$.

Now $\bar{\lambda}$ induces a linear map

$$\wedge : C_q E \to \Lambda E, \quad \bar{t} \mapsto \bar{\lambda}(\bar{t})1_K = \lambda(t)1_K.$$

Lemma. Let $\bar{E} \subseteq C_q(E)$ be the image of $E = T^1 E$ under the quotient map $\pi : TE \to C_q E$. Then the quotient map induces an isomorphism $E \simeq \bar{E}$ and $\wedge$ induces an isomorphism $\bar{E} \simeq \Lambda^1 E = E$.

Proof. Indeed, for any $e \in E$ we have

$$\wedge(\bar{e}) = \bar{\lambda}(\bar{e})1_K = \lambda_e 1_K = \mu_e 1_K + \delta_e 1_K = e.$$

This shows that the composition

$$T^1 E = E \xrightarrow{\pi} \bar{E} \xrightarrow{\wedge} E = \Lambda^1 E$$

is the identity and from this the two claims follow immediately. □

Identifying E, $T^1 E$, $\bar{E}$ and $\Lambda^1 E$, the *Clifford relations* take the form

$$e^2 = q(e,e), \qquad ee' + e'e = q(e,e') + q(e',e).$$

If $e_1, \ldots, e_n$ is a basis of E, we set $e_I = e_{i_1} \cdots e_{i_r} \in C_q E$. Remember that we have also set $\widehat{e_I} = e_{i_1} \wedge \cdots \wedge e_{i_r} \in \Lambda^r E$.

Theorem. The linear map $\wedge : C_q E \to \Lambda E$ is an isomorphism.

Proof. The main lemmas in the proof are (1) that the set $\mathcal{B} = \{e_I\}$, where I runs over the set of multiindices taken from $\{1, \ldots, n\}$, generates $C_q E$ as a vector space, so that $\dim_K C_q(E) \leq 2^n$, and (2) that $\wedge$ is surjective, so that $\dim_K C_q(E) \geq 2^n$.

Proof of (2). The surjectivity can be established by induction by showing that the image of $\wedge$ contains $\Lambda^r E$ if it contains $\Lambda^j E$ for $j < r$. Since this is clearly true for $r = 0$ and $r = 1$, we can assume that $r > 1$.

Let $e_1, \ldots, e_r \in E$ and consider $\wedge(e_1 \cdots e_r) = \lambda_{e_1} \circ \cdots \circ \lambda_{e_r} 1_K$, where, by definition, $\lambda_{e_j} = \mu_{e_j} + \delta_{e_j}$. It follows that the term of highest grade in the expansion of $\wedge(e_1 \cdots e_r)$ is $\mu_{e_1} \circ \cdots \circ \mu_{e_r} 1_K = e_1 \wedge \cdots \wedge e_r$.

By the induction hypothesis, there exists $x \in C_qE$ such that $\wedge(e_1 \cdots e_r) = e_1 \wedge \cdots \wedge e_r + \wedge(x)$, where all the grades involved in $\wedge(x)$ are lower than r. This shows that $e_1 \wedge \cdots \wedge e_r$ belongs to the image of $\wedge$ and $\Lambda^r E$ is therefore contained in the image of $\wedge$.

Proof of (1). Since the Clifford product is multilinear, it is clear that the products of the form $e_{j_1} \cdots e_{j_r}$ ($r \geqslant 0$, $j_1, \ldots, j_r \in \{1, \ldots, n\}$) generate C_qE as a vector space. So it will be enough to show that such products are linear combinations of elements e_I in $\mathcal{B}$ with $|I| \leqslant r$. Given that this claim is tautological for $r = 0$ and $r = 1$, we move on to the case $r > 1$ and proceed by induction on r. The induction hypothesis allows us to assume that $e_{j_2} \cdots e_{j_r}$ is a linear combination of elements e_I from $\mathcal{B}$ with $|I| \leqslant r - 1$ and therefore it will be enough to show that a product of the form $e_j e_I$, I a multiindex of order $s \leqslant r - 1$, is a linear combination of elements in $\mathcal{B}$.

In fact we can show a more precise claim: if $I = \{i_1, \ldots, i_s\}$, $e_j e_I$ is a linear combination of $e_{I_1}, \ldots, e_{I_s}$ and, if $j \notin I$, of $e_{\bar{I}}$, where $I_k = I - \{i_k\}$ and $\bar{I} = I \cup j$ (arranged in increasing order). Let us argue by cases. If $j < i_1$, $e_j e_I = e_{\bar{I}}$. If $j = i_1$, then $e_j e_{i_1} = e_j^2 = q(e_j, e_j)$ and $e_j e_I$ is a scalar multiple of e_{I_1}. If $j > i_1$, then $e_j e_{i_1} = -e_{i_1} e_j + \rho$, $\rho = q(e_j, e_{i_1}) + q(e_{i_1}, e_j) \in K$ (Equation (**??**)) and $e_j e_I = -e_{i_1} e_j e_{i_2} \cdots e_{i_s} + \rho e_{I_1}$. By induction, $e_j e_{i_2} \cdots e_{i_s} = e_j e_{I'}$ ($I' = I_1$) is a linear combination of the $e_{I'_k}$, $k = 2, \ldots, s$, and, if $j \notin I'$, of $e_{I' \cup \{j\}}$. Finally note that $e_{i_1} e_{I'_k} = e_{I_k}$ and $e_{i_1} e_{I' \cup \{j\}} = e_{I \cup j} = e_{\bar{I}}$. □

Remark. The linear isomorphism $\wedge : C_q E \to \Lambda E$ is an *algebra* isomorphism if and only if $q = 0$.

The grade involution of ΛE is also an involution of $C_q(E)$, for in the tensor algebra $\alpha(e \otimes e - q(e,e)) = e \otimes e - q(e,e)$. Therefore,

$$C_q^+(E) = \{x \in C_q(E) \mid x^\alpha = x\}$$

is a subalgebra of $C_q(E)$ (the *even subalgebra*). Under the linear isomorphism $C_q E \simeq \Lambda E$,

$$C_q^+ E \simeq \Lambda^+ E.$$

The *odd subspace* $C_q^- E$ is defined as $\{x \in C_q(E) \mid x^\alpha = -x\}$ and $C_q^- E \simeq \Lambda^- E$.

Let q be the standard bilinear form of $\mathbf{R}^n$,

$$q(x,y) = x^T y = x_1 y_1 + \cdots + x_n y_n,$$

and set $C_n = C_q(\mathbf{R}^n)$ and $\bar{C}_n = C_{-q}(\mathbf{R}^n)$.

If $\boldsymbol{e} = e_1, \ldots, e_n$ is the standard basis of $\mathbf{R}^n$, then both in C_n and in $\bar{C}_n$ we have $e_i e_j = -e_j e_i$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, but $e_i^2 = 1$ in C_n and $e_i^2 = -1$ in $\bar{C}_n$. So C_n and $\bar{C}_n$ are isomorphic to the algebras $C_{\mathbf{1}_n}(\boldsymbol{e})$ and $C_{-\mathbf{1}_n}(\boldsymbol{e})$ introduced and in L1.

In essence, these are the algebras introduced by Clifford (1882 and 1878). Note that $e_I = \widehat{e_I}$ for any multiindex I. This is established in a more general setting in a later slide, but the main idea of the proof can be seen in the following computation, where $j \neq k$:

$$e_j e_k = (\mu_{e_j} + \delta_{e_j})(\mu_{e_k} + \delta_{e_k})1_K = (\mu_{e_j} + \delta_{e_j})e_k = e_j \wedge e_k + \widetilde{e}_j(e_k) = e_j \wedge e_k$$

as $\widetilde{e}_j(e_k) = q(e_j, e_k) = 0$.

In addition to the algebras C_1, $\bar{C}_1$, C_2 and $\bar{C}_2$ studied in L1, it will also play an inportant role the Clifford algebra $C_{1,1}$ of $(\mathbf{R}^2, q)$, where $q(x,y) = x_1y_1 - x_2y_2$.

In this case the generators $1, e_1, e_2, e_1e_2$ satisfy

$$e_1^2 = 1,\ e_2^2 = -1,\ e_2e_1 = -e_1e_2,\ (e_1e_2)^2 = 1.$$

It follows that $C_{1,1} \simeq \mathbf{R}(2)$,

$$a + be_1 + ce_2 + de_1e_2 \mapsto \begin{pmatrix} a+b & c+d \\ -c+d & a-b \end{pmatrix}.$$

The even subalgegra $C_{1,1}^+$ is isomorphic to C_1.

> The mainspring of scientific thought is not an external goal toward which one must strive, but the pleasure of thinking.
>
> A. EINSTEIN, ~ 1918

We will write $C_q^\times(E)$ to denote the group of invertible elements of $C_q(E)$.

A vector $e \in E$ is *isotropic* if $q(e,e) = 0$. Otherwise it is said to be *non-isotropic* (or also *anisotropic*).

Inverse of a non-isotropic vector. If a vector $e \in E$ is non-isotropic, then $e \in C_q^\times(E)$ and $e^{-1} = q(e,e)^{-1}e$.

We will write $E^\times$ to denote the set of non-isotropic vectors, so $E^\times = E \cap C_q^\times(E)$.

For any $u \in C_q^\times(E)$, the map

$$\rho_u : C_q(E) \to C_q(E), \quad x \mapsto uxu^{-1}$$

is an algebra automorphism. Moreover, the map

$$C_q^\times(E) \to \mathrm{Aut}(C_q(E)), \quad u \mapsto \rho_u$$

is a group homomorphism.

Proof. The computation

$$\rho_u(xy) = uxyu^{-1} = uxu^{-1}uyu^{-1} = \rho_u(x)\rho_u(y)$$

proves the first part. For the second part, if $u, v \in C_q^\times(E)$ and $x \in C_q(E)$,

$$\rho_{uv}(x) = uvx(uv)^{-1} = uvxv^{-1}u^{-1} = u(vxv^{-1})u^{-1} = \rho_u(\rho_v(x)),$$

which shows that $\rho_{uv} = \rho_u \circ \rho_v$. □

The *Lipschitz group* of q, denoted $\Gamma_q = \Gamma_q(E)$, is the subgroup of $C_q^\times(E)$ generated by $E^\times$. Thus the elements u of Γ_q have the form $u = e_1 \cdots e_r$, with $e_1, \ldots, e_r \in E^\times$. It is also clear that

$$u^{-1} = e_r^{-1} \cdots e_1^{-1}.$$

Lemma. If $u \in \Gamma_q(E)$, then $\rho_u(E) = E$.

Proof. By definition of Γ_q, and the fact that ρ is a homomorphism, it is enough to show the relation for $u \in E^\times$. But in this case we have, for any $e \in E$,

$$ueu^{-1} = \rho u^{-1} - e = q(u,u)^{-1}\rho u - e \in E,$$

where $\rho = q(e,u) + q(u,e)$. □

If $u \in \Gamma_q$, we have a linear automorphism

$$\rho_u : E \to E.$$ [1]

So we have a group homomorphism

$$\rho : \Gamma_q \to \mathrm{GL}(E),\ u \mapsto \rho_u.$$

In other words, ρ is a *representation* of Γ_q by linear automorphisms of E.

We will say that ρ is the *adjoint* (or *principal*, or *vector*) *representation* of Γ_q.

[1] There is no harm in using the same symbol as for the corresponding automorphism of $C_q(E)$.

For any $u \in C_q^\times(E)$, the map

$$\widetilde{\rho}_u : C_q(E) \to C_q(E), \quad x \mapsto u^\alpha x u^{-1}$$

is a *linear* automorphism of $C_q(E)$.

We will say that it is the *twisted Clifford operator* associated to u.

Note that if $u \in \Gamma_q$, then $\widetilde{\rho}_u = (-1)^{|u|}\rho_u$.

Finally, the map

$$C_q^\times(E) \to \mathrm{GL}(C_q(E)), \quad u \mapsto \widetilde{\rho}_u$$

is a group homomorphism.

Proof. The first part is obvious and the proof of the last part is a short computation similar to the proof of the last assertion on the slide 36. □

We define the *twisted Lipschitz group* of q, denoted $\widetilde{\Gamma}_q = \widetilde{\Gamma}_q(E)$, as the group formed with the *even* and *odd* elements $u \in C_q^{\times}(E)$ such that

$$\widetilde{\rho}_u(e) = u^{\alpha} e u^{-1} \in E \text{ for all } e \in E.$$

Note that with this definition the condition $u^{\alpha} e u^{-1} \in E$ is equivalent to $ueu^{-1} \in E$, for $u^{\alpha} = \pm u$.

Again, a vector $u \in E^{\times}$ belongs to $\widetilde{\Gamma}_q$, for $\widetilde{\rho}_u = -\rho_u$. Since $E^{\times}$ generates $\Gamma_q(E)$, in particular we have the inclusion $\Gamma_q(E) \subseteq \widetilde{\Gamma}_q(E)$. In fact $\Gamma_q(E)$ is a normal subgroup of $\widetilde{\Gamma}_q(E)$.

If $u \in \widetilde{\Gamma}_q$, we have a linear automorphism[2]

$$\widetilde{\rho}_u : E \to E$$

So we have a group homomorphism

$$\widetilde{\rho} : \widetilde{\Gamma}_q \to \mathrm{GL}(E),\ u \mapsto \widetilde{\rho}_u.$$

In other words, $\widetilde{\rho}$ is a *representation* of $\widetilde{\Gamma}_q$ by linear automorphisms of E.

We will say that it is the *twisted adjoint representation* of $\widetilde{\Gamma}_q$.

[2] There is no harm in using the same symbol as for the corresponding operator of $C_q(E)$.

The group $\mathrm{Pin}_q(E)$ is defined as the subgroup of $C_q^\times(E)$ generated by the vectors $u \in E$ such that $q(u,u) = \pm 1$. Clearly, $\mathrm{Pin}_q(E) \subseteq \Gamma_q(E)$.

We also write $\mathrm{Spin}_q(E) = \mathrm{Pin}_q(E) \cap C_q^+(E)$.

The restriction of the adjoint representation $\rho : \Gamma_q(E) \to \mathrm{GL}(E)$ to $\mathrm{Pin}_q(E)$ and $\mathrm{Spin}_q(E)$ gives linear representations

$$\rho : \mathrm{Pin}_q(E) \to \mathrm{GL}(E) \text{ and } \rho : \mathrm{Spin}_q(E) \to \mathrm{GL}(E)$$

(there is no harm in using the same symbol ρ in all cases).

Artin 1957

Chevalley 1954

Hestenes 1986

Hitzer 2011

Figueroa O'Farrill 2010

Lawson-Michelsohn 1989

Riesz 1958

Geometric Algebra Techniques in Mathematics and Physics

S. Xambó

UPC

SLP · 9-13 March · 2015

Introduction. Nature and aims of GA.

Functoriality of $\Lambda_q(E)$. The map Λf, $f \in \mathrm{End}(E)$. Orthogonal maps. Functoriality of the geometric product. Involutions.

Inner product. Inner product of blades. Multivector metric.

Further relations and examples. Riesz formulas. δ_e is a skee-derivation of the geometric product. Grade decomposition of a product.

Orthogonal systems. Definitions and conventions. Key propositions. Orthogonal bases of $\Lambda_q(E)$. Reduction formula. Artin/Vahlen product formula. Alternative definition of the multivector and norm. Antisymmetrization of the geometric product. Non-degenerate metrics.

Pseudoscalars and Hodge duality. $\boldsymbol{i_e}$. Playing with a pseudo-scalar.

Adjoint representations. $\rho : \Gamma_q \to \mathrm{O}_q$ and $\widetilde{\rho} : \widetilde{\Gamma}_q \to \mathrm{O}_q$. Axial and reflection symmetries. Generating rotations. Fundamental exact sequences. Pin and Spin exact sequences.

By the *geometric algebra* of q we will understand the structure $\Lambda_q E$ obtained by endowing the exterior algebra ΛE with the *Clifford product* (or *geometric product*, or simply *product*) through the canonical linear isomorphism

$$\wedge : C_q E \to \Lambda E$$

and with the *interior product* $x \cdot y$ defined later in this lecture.

Note that from the definition of the linear map $\wedge : C_q E \to \Lambda E$, we get the following *key formula* for the computation of a geometric product of the form ex, for $e \in E$ and $x \in \Lambda E$:

$$ex = \lambda_e x = e \wedge x + \widetilde{e}(x).$$

In general terms, the study of geometric algebra (GA for short) consists in spelling out the interrelations between these three products (exterior, geometric and interior) and also the procedures for its application to specific situations.

Suppose that E is a vector space equipped with bilinear symmetric form q and that $f : E \to E$ is a linear map.

Let $\Lambda f : \Lambda E \to \Lambda E$ be the algebra homomorphism induced by f.

Lemma

$$\Lambda f \circ \mu_e = \mu_{f(e)} \circ \Lambda f, \text{ for all } e \in E.$$

Proof. For any $x \in \Lambda E$,

$$(\Lambda f \circ \mu_e)(x) = \Lambda f(e \wedge x) = f(e) \wedge \Lambda f(x) = (\mu_{f(e)} \circ \Lambda f)(x).$$ □

We say that f is *q-orthogonal* if $q(f(e), f(e')) = q(e, e')$ for all $e, e' \in E$. If in addition $f \in \mathrm{GL}(E)$ (in other words, f is a linear automorphism of E), we say that f is a *q-isometry*, or simply an *isometry* if q can be understood from the context.

Lemma. If f is q-orthogonal, then

$$\Lambda f \circ \delta_e = \delta_{f(e)} \circ \Lambda f, \text{ for all } e \in E.$$

Proof. Both expressions $\Lambda f \circ \delta_e$ and $\delta_{f(e)} \circ \Lambda f$ are skew-derivations of ΛE. To show that they are equal, it is enough to see that they agree on E. But this is a direct consequence of the definitions: on one hand

$$(\Lambda f \circ \delta_e)(e') = q(e, e'),$$

and on the other

$$(\delta_{f(e)} \circ \Lambda f)(e') = \delta_{f(e)}(f(e')) = q(f(e), f(e')) = q(e, e'). \qquad \square$$

Theorem 1. If f is q-orthogonal, then Λf is also a homomorphism of the geometric product:

$$\Lambda f(xy) = \Lambda f(x)\,\Lambda f(y) \text{ for all } x, y \in \Lambda E.$$

Proof. It is enough to show that if $e_1, \ldots, e_r \in E$, then

$$(\Lambda f)(e_1 \cdots e_r) = f(e_1) \cdots f(e_r).$$

Since this is obviously true for $r = 1$, we can assume that $r > 1$ and proceed by induction. Let $e = e_1$ and $x = e_2 \cdots e_r$. Then

$$e_1 \cdots e_r = ex = \lambda_e(x),\ \lambda_e = \mu_e + \delta_e, \text{ and so}$$

$$\Lambda f(e_1 \cdots e_r) = (\Lambda f \circ \lambda_e)(x) = \lambda_{f(e)}(\Lambda f(x)) = f(e_1)\Lambda f(x),$$

The end of the proof is now immediate because

$$\Lambda f(x) = f(e_2) \cdots f(e_r)$$

by the induction hypothesis. □

The grade involution α of ΛE is also an involution of the geometric product.

Proof. This has already been proved in L2. For a variation, note that $e \mapsto -e$ is a q-isometry and so the statement is a direct corollary of the Theorem above. □

The reversion anti-automorphism τ of ΛE is also an anti-automorphism of the geometric product:

$$\tau(xy) = \tau(y)\tau(x), \quad \text{or } \widetilde{xy} = \widetilde{y}\,\widetilde{x}.$$

Proof. The reversion anti-automorphism of the tensor algebra leaves invariant the generators $t_e = e \otimes e - q(e,e)$ of the ideal I_q such that $C_q(E) = T(E)/I_q$, so that the reversion anti-automorphism of $T(E)$ descends to $C_q(E)$. □

Corollary. The Clifford anti-automorphism of ΛE is also an anti-automorphism of the geometric product. □

The fact that the parity involution is also an automorphism of the geometric product implies that the even subalgebra $\Lambda^+(E)$ of the exterior algebra is also a subalgebra of $\Lambda_q(E)$. Thus we will denote it also by $\Lambda_q^+(E)$ and say that it is the *even geometric algebra* of q.

In the context of geometric algebra, the involution τ is also called, for reasons that have to do with matrix representations (cf. L1), *hermitian conjugation*, and sometimes it is denoted $x^\dagger$.

To specify the *inner product* $x \cdot y$ (also called *interior product* of two multivectors $x, y \in \Lambda E$, it is enough to take care of the case $x \in \Lambda^r E$ and $y \in \Lambda^s E$, for then the general case is determined by bilinearity.

The inner product is called *contraction* by some authors, and in fact, as indicated in L1, they distinguish two flavors: *left contraction* and *right contraction*, often denoted $x \lrcorner y$ and $x \llcorner y$ (cf. Riesz-1993, Lounesto-1993, Lounesto-1997). Following Hestenes-1966, however, we will not need to distinguish between the two, and hence we will use a single symbol $x \cdot y$. The point is that this expression will be evaluated differently according to whether $r \leqslant s$ or $r \geqslant s$.

When $r = s$, we will show that both ways give the same answer, and that they yield the same value as the natural extension of the scalar product q to $\Lambda_q(E)$. Thus in this case the inner product is symmetric, a property that in general is not satisfied when $r \neq s$.

First assume $r \leq s$. The inner product

$$\Lambda^r E \times \Lambda^s E \to \Lambda^{s-r} E$$

is defined as the composition

$$\Lambda^r E \times \Lambda^s E \to \Lambda^r E^* \times \Lambda^s E \to \Lambda^{s-r} E,$$

where the map on the left is $\Lambda^r q \times \mathrm{Id}$ and the map on the right is the duality pairing studied in L2.

Example. If $e \in E$ and $x \in \Lambda E$ (x not necessarily homogeneous),

$$e \cdot x = q(e)(x) = \widetilde{e}(x).$$

This, together with the key formula (L2) yield the equation

$$ex = e \wedge x + e \cdot x.$$

Consistency of notation: the inner product of two vectors is the same as their dot product, so we are not using a single symbol for two different meanings.

Theorem 2. Let $e_1, \ldots, e_r, e'_1, \ldots, e'_s \in E$. Then

(1) $(e_1 \wedge \cdots \wedge e_r) \cdot (e'_1 \wedge \cdots \wedge e'_s) = (e_1 \wedge \cdots \wedge e_{r-1}) \cdot (e_r \cdot (e'_1 \wedge \cdots \wedge e'_s))$

(2) If G is the $r \times s$ matrix whose entries are the scalars $e_i \cdot e'_j$, for $i = 1, \ldots, r$, $j = 1, \ldots, s$, then

$$(e_1 \wedge \cdots \wedge e_r) \cdot (e'_1 \wedge \cdots \wedge e'_s) = \sum_{I \in \mathscr{I}_{r,s}} (-1)^{t(I)} \det(G_I)\, \widehat{e'}_{I'},$$

where $t(I)$ is the number of inversions in the sequence (I, I').

(3) In the special case $s = r$, the result is the scalar $\det(G)$.

Proof. Part (1) is a direct consequence of the definitions; (2) and (3) are a reformulation of the duality formula (L2). □

We would like to have a rule analogous to rule (1) in the Theorem 2, say

$$(e_1 \wedge \cdots \wedge e_r) \cdot (e'_1 \wedge \cdots \wedge e'_s) = ((e_1 \wedge \cdots \wedge e_r) \cdot e'_1) \cdot (e'_2 \wedge \cdots \wedge e'_s).$$

Moreover, this rule should be consistent with Theorem 2 (3), in the sense that it should produce, when applied recursively, the same value. The operation $\cdot e'_1$ should act as a skew-derivation, as $e_r \cdot$ does in a, but it has to be a *right* skew-derivation (otherwise, as we will see, the required consistency when $r = s$ would not hold):

$$\begin{aligned}(e_1 \wedge \cdots \wedge e_r) \cdot e'_1 &= \textstyle\sum_{k=r}^{k=1} (-1)^{r-k} (e_k \cdot e'_1) \widehat{e}_{k'} \\ &= (-1)^{r-1} \textstyle\sum_{k=1}^{k=r} (-1)^{k-1} (e_k \cdot e'_1) \widehat{e}_{k'} \\ &= (-1)^{r-1} e'_1 \cdot (e_1 \wedge \cdots \wedge e_r).\end{aligned}$$

This rule yields, when we move all the e''s to the left, the value

$$(-1)^{rs-\binom{s+1}{2}}(e'_s \wedge \cdots \wedge e'_1).(e_1 \wedge \cdots \wedge e_r),$$

as the accumulated number of sign changes is

$$(r-1)+\cdots+(r-s) = rs - \binom{s+1}{2}.$$

Reordering the e''s leads to the expression

$$(e_1 \wedge \cdots \wedge e_r)\cdot(e'_1 \wedge \cdots \wedge e'_s) = (-1)^{s(r+1)}(e'_1 \wedge \cdots \wedge e'_s).(e_1 \wedge \cdots \wedge e_r),$$

for that reordering produces $(-1)^{\binom{s}{2}}$ additional sign changes, which means that the sign in front becomes $(-1)^{rs-s} = (-1)^{rs+s}$. Now we can apply Theorem 2 (2) to conclude:

$$(e_1 \wedge \cdots \wedge e_r)\cdot(e'_1 \wedge \cdots \wedge e'_s) = (-1)^{rs+s}\sum_{J\in\mathscr{I}_{s,r}}(-1)^{t(J)}\det(G_J^T)\,\widehat{e}_{J'},$$

where G^T is the matrix formed with the scalars $e'_j \cdot e_i$, which is the transpose of the matrix $G = (e_i \cdot e'_j)$ introduced before.

Remark. In the case $r = s$, the sign in front is $(-1)^{r(r+1)} = +1$, while the sum reduces to the scalar $\det(G^T) = \det(G)$, so that the formula yields the same value as Theorem 2 (3). □

As a consequence of the preceding discussions, we have a precise statement about the behavior of the inner product when we exchange its factors.

Swaping the inner product factors

If $x \in \Lambda^r E$ and $y \in \Lambda^r E$, then

$$x \cdot y = \sigma(r,s)\, y \cdot x,$$

where $\sigma(r,s) = (-1)^{rs+r}$ if $r \leqslant s$ and $\sigma(r,s) = (-1)^{rs+s}$ if $r \geqslant s$.

Notations: $f \in \mathrm{End}_K(E)$ is q-orthogonal and x, y are elements of ΛE.

Theorem 3. $\Lambda f(x \cdot y) = (\Lambda f(x)) \cdot (\Lambda f(y))$. So $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$.

Proof. By bilinearity we may assume that $x \in \Lambda^r E$ and $y \in \Lambda^s E$ are blades, say

$$x = e_1 \wedge \cdots \wedge e_r, \quad y = e'_1 \wedge \cdots \wedge e'_s.$$

Moreover, by the swapping rule we only need to consider the case in which $r \leqslant s$, and rule (1) in Theorem 2 tells us that

$$x \cdot y = (\delta_{e_1} \circ \cdots \circ \delta_{e_r})(y).$$

Now apply Λf to this expression, and use the functoriality of δ, to get

$$\begin{aligned} \Lambda f(x \cdot y) &= (\delta_{f(e_1)} \circ \cdots \circ \delta_{f(e_r)})(f(e'_1) \wedge \cdots \wedge f(e'_s)) \\ &= (f(e_1) \wedge \cdots \wedge f(e_r)) \cdot (f(e'_1) \wedge \cdots \wedge f(e'_s)) \\ &= (\Lambda f(x)) \cdot (\Lambda f(y)). \end{aligned}$$

□

$\tau(x \cdot y) = \tau(y) \cdot \tau(x)$.

Proof. By bilinearity, we can assume that x and y are as in the proof of the previous statement. Again we can assume that $r \leqslant s$. We will proceed by induction on r. So the first step is to show that

$$\tau(e_1 \cdot (e'_1 \wedge \cdots \wedge e'_s)) = \tau(e'_1 \wedge \cdots \wedge e'_s) \cdot e_1.$$

The left hand side is equal to

$$\begin{aligned}\tau & \left(\textstyle\sum_{k=1}^{s}(-1)^{k-1}(e_1 \cdot e'_k)\widehat{e'}_{k'}\right) \\ & = \textstyle\sum_{k=1}^{s}(-1)^{k-1}(e_1 \cdot e'_k)\tau(\widehat{e'}_{k'}) \\ & = \textstyle\sum_{k=1}^{s}(-1)^{k-1}(e_1 \cdot e'_k)e'_s \wedge \cdots \wedge e'_{k+1} \wedge e'_{k-1} \wedge \cdots \wedge e'_1.\end{aligned}$$

Similarly, the right hand side is equal to

$$(e'_s \wedge \cdots \wedge e'_1) \cdot e_1 = \textstyle\sum_{k=1}^{s}(-1)^{k-1}(e'_k \cdot e_1)e'_s \wedge \cdots \wedge e'_{k+1} \wedge e'_{k-1} \wedge \cdots \wedge e'_1$$

and we see that it coincides with the left hand side.

The case $r > 1$ is now readily settled using rule (1) in Theorem 2 and induction on r. Indeed, if we put $x' = e_1 \wedge \cdots \wedge e_{r-1}$, $e = e_r$, then

$$\begin{aligned}
\tau(x \cdot y) &= \tau((x' \wedge e) \cdot y) \\
&= \tau(x' \cdot (e \cdot y)) \\
&= \tau(e \cdot y) \cdot \tau(x') && \text{(induction hypothesis)} \\
&= (\tau(y) \cdot e) \cdot \tau(x') && \text{(case } r = 1) \\
&= \tau(y) \cdot (e \wedge \tau(x')) \\
&= \tau(y) \cdot \tau(x' \wedge e) \\
&= \tau(y) \cdot \tau(x).
\end{aligned}$$

Next relation is an immediate consequence of the preceding two statements:

$$\kappa(x \cdot y) = \kappa(y) \cdot \kappa(x). \qquad \square$$

If $e \in E$, we know $(*)$ $ex = e \cdot x + e \wedge x$. Now we will see that $xe = x \cdot e + x \wedge e$.

Proof. A simple coomputation using the properties of the reversion involution τ and the formula $(*)$:

$$\begin{aligned} xe &= \tau\left(e\tau(x)\right) = \tau(e \cdot \tau(x) + e \wedge \tau(x)) \\ &= \tau(\tau(x \cdot e) + \tau(x \wedge e)) = x \cdot e + x \wedge e. \end{aligned}$$

The functoriality statements show that for any q-orthogonal map $f : E \to E$, Λf is is an algebra endomorphism of $\Lambda_q(E)$. This endomorphism is an automorphism if f is a q-isometry. This proves that we have a group homomorphism (clearly injective)

$$\mathrm{O}_q(E) \to \mathrm{Aut}(\Lambda_q(E)),$$

where $\mathrm{O}_q(E)$ denotes the group of q-isometries of E.

Definition. The vector metric q (the symmetric map $q : E \to E^*$ that we fixed at the beginning of this lecture) extends to a metric on ΛE in a natural way:

$$\Lambda q : \Lambda E \to \Lambda E^* \simeq (\Lambda E)^*.$$

Instead of $\Lambda q(x, y) = (\Lambda q)(x)(y)$, we will write $\langle x|y\rangle$. This pairing is bilinear and symmetric and $\langle e|e'\rangle = e \cdot e'$ when $e, e' \in E$.

The fact that the isomorphism $\Lambda E^* \simeq (\Lambda E)^*$ is the direct sum of isomorphisms $\Lambda^r E^* \simeq (\Lambda^r E)^*$ implies that $\Lambda^r E$ and $\Lambda^s E$ are Λq-orthogonal when $r \neq s$. Consequently, for the determination of the metric $\langle x|y\rangle$ we can assume that x and y belong to the same exterior power, say $x, y \in \Lambda^r E$. Owing to the bilinearity, we can further assume that x and y are blades.

The metric formula. If $x = e_1 \wedge \cdots \wedge e_r$ and $x' = e'_1 \wedge \cdots \wedge e'_r$ ($e_1, \ldots, e_r, e'_1, \ldots, e'_r \in E$), then

$$\langle x|x'\rangle = \det(G) = x \cdot x',$$

where G is the $r \times r$ matrix such that $G_{ij} = e_i \cdot e'_j$.

Proof. Indeed, since

$$\langle x|x'\rangle = (\Lambda^r q(x))(x') \text{ and } \Lambda^r q(x) = \widetilde{e}_1 \wedge \cdots \wedge \widetilde{e}_r,$$

by the duality formula we get that $\langle x|x'\rangle = \det(G)$, where $G_{ij} = \widetilde{e}_i(e'_j) = e_i \cdot e'_j$. But this determinant agrees with the inner product $x \cdot x'$ by the formula giving the inner product of blades. □

The metric norm. The *metric norm* $Q(x)$ of a multivector x is defined by $Q(x) = \langle x|x\rangle$. If $x = \sum_{r=0}^n x_r$ is the grade decomposition of x, then $Q(x) = \sum_{r=0}^n Q(x_r)$. In the case when x is an r-blade, say $x = e_1 \wedge \cdots \wedge e_r$, then $Q(x) = \det(G)$, where $G_{ij} = e_i \cdot e_j$.

Lemma

$$ex + (-1)^r xe = 2e \wedge x \text{ and } ex - (-1)^r xe = 2e \cdot x.$$

Proof. We know that

$$ex = e \cdot x + e \wedge x \text{ and } xe = x \cdot e + x \wedge e.$$

Now $x \cdot e = (-1)^{r+1} e \cdot x$ and $x \wedge e = (-1)^r e \wedge x$ and hence $xe = (-1)^r(-e \cdot x + e \wedge x)$, or

$$(-1)^r xe = -e \cdot x + e \wedge x.$$

To obtain the two equalities in the statement it suffices to add and subtract the expressions for ex and $(-1)^r xe$. □

These formulas are the basis to express the exterior and interior products in terms of the geometric product. For the case of two vectors $e, e' \in E$ they give the relations

$$2e \wedge e' = ee' - e'e, \quad 2e \cdot e' = ee' + e'e.$$

Lemma

$$e \cdot (xy) = (e \cdot x)y + x^{\alpha}(e \cdot y).$$

Proof. $e\cdot = \delta_e : E \to K$ extends to a unique skew derivation of the tensor algebra $T(E)$, which means that

$$e \cdot (x \otimes y) = (e \cdot x) \otimes y + x^{\alpha} \otimes (e \cdot y).$$

So it suffices to observe that this extension vanishes on the expressions $t_x = x \otimes x - q(x,x)$, $x \in E$. But this is clear because $e\cdot$ kills scalars and $e \cdot (x \otimes x) = (e \cdot x)x - x(e \cdot x) = 0$.

Note that the extension of $e\cdot$ to a skew-derivation of $T(E)$ follows from the (necessary) relation

$$e\cdot(e_1 \otimes \cdots \otimes e_r) = \sum_{k=1}^{r}(-1)^{k-1}(e\cdot e_k)e_1 \otimes \cdots \otimes e_{k-1} \otimes e_{k+1} \otimes \cdots \otimes e_r,$$

which in turn is well defined because the right-hand side is multilinear in $e_1, \ldots, e_r$. □

Theorem 3. Let $x \in \Lambda^r E$ and $y \in \Lambda^s E$ and set $z = xy$. Then the indices $k \in \{0, 1, \ldots, n\}$ such that $z_k \neq 0$ have the form $k = |r - s| + 2i$, $i \geqslant 0$ and $k \leqslant r + s$.

Furthermore, $z_{|r-s|} = x \cdot y$ and $z_{r+s} = x \wedge y$.

Proof. It is enough to prove the statement when x and y are blades and $r \leqslant s$. So let $e_1, \ldots, e_r, e'_1, \ldots, e'_s \in E$, and set $x = e_1 \wedge \cdots \wedge e_r$ and $y = e'_1 \wedge \cdots \wedge e'_s$. Then we can express the product $z = xy$ in the following form:

$$z = \lambda_x(y) = (\lambda_{e_1} \circ \cdots \circ \lambda_{e_r})(y) = (\mu_{e_1} + \delta_{e_1}) \cdots (\mu_{e_r} + \delta_{e_r})(y).$$

If in the expansion of this expression we choose i μ's $(0 \leqslant i \leqslant r)$ and $r - i$ δ's, we get a term of grade $s + i - (r - i) = s - r + 2i$. The highest possible grade is when $i = r$, and in this case it is plain that the term z_{r+s} is $x \wedge y$. On the other hand the minimum grade is attained when $i = 0$, so $k = s - r$, and in this case it is also clear that $z_{s-r} = x \cdot y$. □

Remark. In the case that $e \in E$ and $x \in \Lambda^r(E)$, the relations $(ex)_{r-1} = e \cdot x$ and $(ex)_{r+1} = e \wedge x$ are a direct consequence of $ex = e \cdot x + e \wedge x$. Similarly, $xe = x \cdot e + x \wedge e$ imply that $(xe)_{r-1} = x \cdot e$ and $(xe)_{r+1} = x \wedge e$.

Note also that the case $r \geq s$ in the previous theorem can be deduced by induction on s. For $s = 0$, it is tautological and the case $s = 1$ has been established in the previous paragraph. So assume that $s > 1$. Then, with the same notations as in the proof above, and with $y' = e'_2 \wedge \cdots \wedge e'_s$ (so that $y = e'_1 \wedge y'$) we have

$$\begin{aligned} x \cdot y &= (x \cdot e'_1) \cdot y' = (xe'_1)_{r-1} \cdot y' \\ &= (xe'_1 y')_{(r-1)-(s-1)} = (xe'_1 y')_{r-s} \\ &= (xy)_{r-s} + (x(e'_1 \cdot y'))_{r-s} = (xy)_{r-s} \end{aligned}$$

because the minimum grade of $x(e'_1 \cdot y')$ is $r - (s-2) = r - s + 2$.

Two vectors $e, e' \in E$ are *q-orthogonal* if and only if $q(e, e') = 0$. Note that this implies that two orthogonal vectors anti-commute: $ee' + e'e = q(e, e') = 0$. Note also that two *parallel* vectors commute (if $e' = \alpha e$, then both ee' and $e'e$ are equal to αe^2).

If $e_1, \ldots, e_n \in E$ and $q(e_i, e_j) = 0$ for $i \neq j$, we say that the sequence $\{e_1, \ldots, e_r\}$ is an *orthogonal system*.

It is an easy exercise to prove that any metric q admits an orthogonal basis if $2 \neq 0$ in K, and that if $2 = 0$ in K then there are metrics for which there are no orthogonal basis.[1] Henceforth we will avoid such metrics, which means that in characteristic 2 no metrics will be considered that do not admit an orthogonal basis.

[1] For the metric $q(x, y) = x_1 y_2 + x_2 y_1$ in $E = \mathbf{Z}_2^2$, no pair of distinct non-zero vectors is orthogonal. On the other hand, $(1, 0), (0, 1) \in E$ is an orthogonal system for the metric $x_1 y_1 + x_2 y_2$. Note that both metrics are non-degenerate.

Proposition . If $e_1, \ldots, e_r \in E$ is an orthogonal system, then

$$e_1 e_2 \cdots e_r = e_1 \wedge e_2 \wedge \cdots \wedge e_r.$$

Proof. By induction with respect to r. Since for a single vector $e \in E$ the claim is obvious, we can assume that $r > 1$ and that the relation holds for orthogonal systems of $r - 1$ vectors. Then we have

$$\begin{aligned} e_1 e_2 \cdots e_r &= e_1(e_2 \wedge \cdots \wedge e_r) \quad \text{(by induction)} \\ &= e_1 \wedge e_2 \wedge \cdots \wedge e_r + \delta_{e_1}(e_2 \wedge \cdots \wedge e_r) \\ &= e_1 \wedge e_2 \wedge \cdots \wedge e_r, \end{aligned}$$

for

$$\delta_{e_1}(e_2 \wedge \cdots \wedge e_r) = \sum_{k=2}^{r} (-1)^{k-1} \widetilde{e}_1(e_k) e_2 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1} \wedge \cdots \wedge e_r$$

and $\widetilde{e}_1(e_k) = e_1 \cdot e_k = 0$ for $k = 2, \ldots, r$. □

Lemma. If $e_1, \ldots, e_n$ is an orthogonal basis of E, then the blades $e_I = \widehat{e}_I$, where I runs over the set of multiindices $I \subseteq \{1, \ldots, n\}$, form an orthogonal basis of $\Lambda_q(E)$. Moreover, $Q(e_I) = q_I$, where $q_I = q(e_{i_1}, e_{i_1}) \cdots q(e_{i_r}, e_{i_r})$.

Proof. The equality $e_I = \widehat{e}_I$ is justified by the previous Proposition. Then the metric formula implies that $\langle \widehat{e}_I | \widehat{e}_J \rangle$ is 0 if $I \neq J$ and that $\langle \widehat{e}_I | \widehat{e}_I \rangle = q_I$. □

Now we will use this lemma to see that the computations of the Clifford product take the simplest form when we know a q-orthogonal basis $e_1, \ldots, e_n \in E$.

- N: The set of indices $\{1, \ldots, n\}$.
- K: a sequence of indices $k_1, \ldots, k_r \in N$.
- $e_K = e_{k_1} \cdots e_{k_r}$.
- $l_j = l_j(K)$, for $j \in N$: the number of times that j appears in K.
- $\widehat{K}$: the multiindex such that $j \in \widehat{K}$ if and only if l_j is odd.
- $t(K)$: the number of pairs $i, j \in N$ such that $i < j$ and $k_i > k_j$.
- $\bar{q}_K$: $\prod_{j=1}^{n} q(e_j, e_j)^{l_j//2}$, where $l_j//2$ denotes the integer quotient of l_j by 2 (it is $l_j/2$ if l_j is even and $(l_j - 1)/2$ if l_j is odd).

Lemma (Reduction formula)

$$e_K = (-1)^{t(K)} \bar{q}_K \, e_{\widehat{K}}.$$

Proof. Since two distinct contiguous factors in e_K anticommute, it follows that

$$e_K = (-1)^{t(K)} \prod_{j=1}^{n} e_j^{l_j}.$$

If $l_j = 2l'_j + r_j$, $r_j \in \{0,1\}$, it is clear that

$$e_j^{l_j} = q_j^{l'_j} e_j^{r_j}.$$

Consequently,

$$e_K = (-1)^{t(K)} \prod_{j=1}^{n} q_j^{l'_j} \prod_{j=1}^{n} e_j^{r_j},$$

and this coincides with the expression in the statement as a direct consequence of the definitions. □

Corollary 1. If I and J are multiindices, then

$$e_I e_J = (-1)^{t(I,J)} q_{I\cap J}\, e_{I\triangle J},$$

where $\triangle$ denotes the *symmetric difference* operation.

Note that this formula was taken in L1 as the basis for the *ad hoc* definition of the Clifford product in $\Lambda(\boldsymbol{e})$.

Corollary 2

$$e_J e_I = (-1)^c (-1)^{rs} e_I e_J,$$

where $r = |I|, s = |J|, c = |I \cap J|$.

Proof. There are rs pairs (i_k, j_l) $(k = 1, \ldots, r,\ j = 1, \ldots, s)$. The number of pairs with $i_k > i_l$ is $t(I, J)$, the number of pairs with $i_k < i_l$ is $t(J, I)$, and there are c pairs such that $i_k = j_l$ (coincidences). Thus $rs = t(I, J) + t(J, I) + c$ and $t(J, I) \equiv rs + c + t(I, J) \mod 2$. Now the claim is immediate, for $J \cap I = I \cap J$ and $J \triangle I = I \triangle J$. □

Remark. What we have called Artin's formula was already discovered by Vahlen-1897 (for the case where $e_i^2 = -1$, $i = 1, \ldots, n$, and in another guise; cf. Lounesto-1993 for interesting historical remarks). Instead of e_I, Vahlen wrote (using here a slightly different notation) $e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n}$, where $i_1, i_2, \ldots, i_n \in \{0, 1\}$. Then the product formula in question for two such 'monomials' can be expressed as follows:

$$e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n} = (-1)^{\sum_{k \geqslant l} i_k j_l} e_1^{i_1+j_1} e_2^{i_2+j_2} \cdots e_n^{i_n+j_n}.$$

where the exponent sums are modulo 2. If we allow the more general relation $e_i^2 = q_i \in K$ $(i = 1, \ldots, n)$, then the formula takes the form

$$e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n} = (-1)^{\sum_{k > l} i_k j_l} e_1^{i_1+j_1} e_1^{i_2+j_2} \cdots e_n^{i_n+j_n},$$

now with the exponents added as integers.

Proof

The key to show this is that

$$e_1^{i_1} e_2^{i_2} \cdots e_n^{i_n} e_1^{j_1} e_2^{j_2} \cdots e_n^{j_n} = (-1)^{\sum_{l<n} i_n j_l} e_1^{i_1} e_2^{i_2} \cdots e_{n-1}^{i_{n-1}} e_1^{j_1} e_2^{j_2} \cdots e_{n-1}^{j_{n-1}} e_n^{i_n+j_n},$$

which itself is easily checked. Note that if $i_n = 0$ the relation is obvious and that if $i_n = 1$, then moving the e_n of the left factor just after the e_{n-1} of the right factor introduces $\sum_{l<n} j_l$ sign changes. □

Proposition . $\langle x|y\rangle = (x^\tau y)_0$. In particular, $Q(x) = (x^\tau x)_0$.

Proof. Theorem 3 implies that $(x^\tau y)_0 = 0$ if $x \in \Lambda_q^r(E)$, $y \in \Lambda_q^s(E)$ and $r \neq s$. Since the expression $(x^\tau y)_0$ is bilinear, to show the claimed equality it is enough to check it when x and y are any pair of elements taken from a basis of $\Lambda_q^r(E)$.

If we choose an orthogonal basis $e_1, \ldots, e_n$ of E, then we can use the basis $\{e_I = \widehat{e_I}\}$, where I runs over $\mathscr{I}_{r,n}$. By the Lemma on slide 27, we know that this basis is orthogonal and that $Q(e_I) = q_I$. On the other hand, it is clear that $(e_I^\tau e_I)_0 = e_I^\tau e_I = q_I = Q(e_I)$, while for $J \neq I$ we get $(e_I^\tau e_J)_0 = \pm(e_I e_J)_0 = 0$ by Artin's formula. □

Remark. We have $(x^\tau y)_0 = (xy^\tau)_0$, because

$$(x^\tau y)_0 = \langle x|y\rangle = \langle y|x\rangle = (y^\tau x)_0 = ((x^\tau y)^\tau)_0 = (x^\tau y)_0.$$

In the last step we use that τ preserves the grading.

Proposition . If $e_1, \ldots, e_r \in E$, then

$$\sum_I (-1)^I e_{i_1} \cdots e_{i_r} = r!\, e_1 \wedge \cdots \wedge e_r,$$

where $I = i_1, \cdots, i_r$ runs over all permutations of $1, \ldots, r$.

Proof. The left hand side is a multilinear skew-symmetric function of $(e_1, \ldots, e_r)$. By the universal property of the exterior product, there exists a unique linear map $a : \Lambda^r E \to \Lambda_q E$ such that

$$a(e_1 \wedge \cdots \wedge e_r) = \sum_I (-1)^I e_{i_1} \cdots e_{i_r}.$$

If now $e_1, \ldots, e_r$ is an orthogonal system, then all terms in the sum are equal to $e_1 \wedge \cdots \wedge e_r$, so that in this case

$$a(e_1 \wedge \cdots \wedge e_r) = r!\, e_1 \wedge \cdots \wedge e_r.$$

In particular we have that if $e_1, \ldots, e_n$ is an orthogonal basis of E, then $a(\widehat{e}_I) = r!\, \widehat{e}_I$ for any multiindex I of rank r. Since these $\widehat{e}_I$ form a basis of $\Lambda^r E$, we actually have that $a(x) = r!\, x$ for all $x \in \Lambda^r E$. □

Definition

The metric $q : E \to E^*$ is said to be *non-degenerate* if it is an isomorphism, which is equivalent to say that $\ker(q) = \{0\}$.

In terms of the dot product, q is non-degenerate if and only if, given any $e \in E$, $e \cdot e' = 0$ for all e' implies $e = 0$.

In terms of the matrix G of q with respect to a basis $e_1, \ldots, e_n$ (so $G_{ij} = e_i \cdot e_j$), q is non-degenerate if and only if $\det(G) \neq 0$. Actually G is the matrix of q with reference to the basis $e_1, \ldots, e_n$ of E and the dual basis $e^1, \ldots, e^n$ of E^*.

If q is non-degenerate, then the induced multivector metric is also non-degenerate. In fact, $\Lambda^r q : \Lambda^r E \to \Lambda^r E^*$ is an isomorphism for all r.

Definition

Given a basis $\boldsymbol{e} = e_1, \ldots, e_n$ of E, let

$$\boldsymbol{i}_{\boldsymbol{e}} = e_1 \wedge \cdots \wedge e_n \in \Lambda^n E.$$

We will say that it is the *pseudoscalar* associated to $\boldsymbol{e}$. Note that by the metric formula we have:

$$Q(\boldsymbol{i}_{\boldsymbol{e}}) = \det(G),\ \ G_{ij} = e_i \cdot e_j.$$

If $\boldsymbol{e}' = e_1', \ldots, e_n'$ is another basis of E, then

$$\boldsymbol{i}_{\boldsymbol{e}'} = \det_{\boldsymbol{e}}(\boldsymbol{e}')\boldsymbol{i}_{\boldsymbol{e}},$$

where $\det_{\boldsymbol{e}}(\boldsymbol{e}')$ is the determinant of the matrix of the vectors $\boldsymbol{e}'$ with respect to the basis $\boldsymbol{e}$.

Remark

Without further structure on E, on the field K, or on both, we do not have any clue for distinguishing one pseudoescalar from another.

For example, is it possible to select a pseudoscalar of norm ± 1?

In general it is not possible, for if we pick any pseudoscalar $\boldsymbol{i}$, then any other pseudoscalar has the form $\boldsymbol{i}' = \lambda \boldsymbol{i}$, $\lambda \in K$, $\lambda \neq 0$, and for $\boldsymbol{i}'$ to have norm ± 1 we would have to solve for λ the equation $\lambda^2 Q(\boldsymbol{i}) = \pm 1$. But this equation does not have a solution unless $\pm Q(\boldsymbol{i})^{-1}$ is a square in K, a condition that is not always satisfied.

But there are some general properties concerning the behavior of pseudoscalars that can be formulated for any of them and which will be very handy in the following.

Assume that the metric q is non-degenerate and let $\boldsymbol{i} \in \Lambda_q^n(E)$ be any non-zero pseudoscalar. Then we have:

Theorem 4

1) $\boldsymbol{i} \in C_q^\times(E)$, $\boldsymbol{i}^{-1} = Q(\boldsymbol{i})^{-1}\boldsymbol{i}^\tau = (-1)^{n//2}Q(\boldsymbol{i})^{-1}\boldsymbol{i}$ and $\boldsymbol{i}^2 = (-1)^{n//2}Q(\boldsymbol{i})$.
2) For any $x \in \Lambda_q^r(E)$, we have $\boldsymbol{i}x, x\boldsymbol{i} \in \Lambda_q^{n-r}(E)$ and the maps $x \mapsto \boldsymbol{i}x$ and $x \mapsto x\boldsymbol{i}$ are linear isomorphisms $\Lambda_q^r(E) \to \Lambda_q^{n-r}(E)$. The inverse maps are $x \mapsto \boldsymbol{i}^{-1}x$ and $x \mapsto x\boldsymbol{i}^{-1}$, respectively.
3) $\boldsymbol{i} \in \Gamma_q$. Therefore the map $E \to E$ such that $e \mapsto \boldsymbol{i}e\boldsymbol{i}^{-1}$ is a q-isometry.
4) If n is odd, $\boldsymbol{i}$ commutes with all elements of $\Lambda_q(E)$. This is also expressed by saying that $\boldsymbol{i}$ belongs to the *center* of $\Lambda_q(E)$.
5) If n is even, $\boldsymbol{i}$ commutes with even multivectors and anticommutes with odd multivectors.

6) If $Q(\boldsymbol{i}) = 1$, then the maps defined in (2) are isometries (*Hodge duality*).

Proof. (1) Since $Q(\boldsymbol{i}) = \boldsymbol{i}^\tau \boldsymbol{i}$ and $Q(\boldsymbol{i}) \neq 0$, we see that $\boldsymbol{i} \in C_q^\times(E)$ and that $\boldsymbol{i}^{-1}$ is given by the formula (1).

(2) Choose an orthogonal basis $\boldsymbol{e} = e_1, \ldots, e_n$ of E. Then

$$\boldsymbol{i} = \lambda \boldsymbol{i}_{\boldsymbol{e}} = \lambda e_1 \cdots e_n = \lambda e_N,$$

for some $\lambda \in K$ $(N = \{1, \ldots, n\})$. Now for any multiindex I of order r, Artin's formula shows that $e_I \boldsymbol{i}, \boldsymbol{i} e_I \in \Lambda^{n-r}(E)$.

(3) Obvious.

(4) and (5) are a direct consequence of the Corollary 2 on slide 30:

$$e_j \boldsymbol{i} = e_j e_N = (-1)^{n+1} e_N e_j = (-1)^{n+1} \boldsymbol{i} e_j,$$

so $\boldsymbol{i}$ commutes (anticommutes) with all vectors for odd n (for n even).

(6) Let us compute $\langle x\boldsymbol{i}|y\boldsymbol{i}\rangle$, for $x, y \in \Lambda^r E$, using the alternative definition of the norm:

$$\begin{aligned}\langle x\boldsymbol{i}|y\boldsymbol{i}\rangle &= ((x\boldsymbol{i})^\tau y\boldsymbol{i})_0 = (\boldsymbol{i}^\tau x^\tau y\boldsymbol{i})_0 \\ &= \boldsymbol{i}^\tau (x^\tau y)_0 \boldsymbol{i} = \boldsymbol{i}^\tau \langle x|y\rangle \boldsymbol{i} = \langle x|y\rangle \boldsymbol{i}^\tau \boldsymbol{i} \\ &= \langle x|y\rangle Q(\boldsymbol{i}) = \langle x|y\rangle.\end{aligned}$$

In the third step we have used that $z \mapsto \boldsymbol{i}^\tau z\boldsymbol{i}$ preserves grades, a fact that follows from (2). That $\langle \boldsymbol{i}x|\boldsymbol{i}y\rangle = \langle x|y\rangle$ is even simpler, because

$$(\boldsymbol{i}x)^\tau \boldsymbol{i}y = x^\tau \boldsymbol{i}^\tau \boldsymbol{i}y = Q(\boldsymbol{i})x^\tau y.$$

This completes the proof. □

Remark. If $Q(\boldsymbol{i}) = -1$, then the maps $\Lambda^r E \to \Lambda^{n-r} E$ such that $x \mapsto x\boldsymbol{i}$ and $x \mapsto \boldsymbol{i}x$ are *antiisometries*. Indeed, the proof above shows that $\langle x\boldsymbol{i}|y\boldsymbol{i}\rangle = -\langle x|y\rangle$.

Toward the end of L2, we described the adjoint and twisted adjoint representations $\rho : \Gamma_q \to \mathrm{GL}(E)$ and $\widetilde{\rho} : \widetilde{\Gamma}_q \to \mathrm{GL}(E)$. The aim of this section is to establish further properties of these representations.

We will assume that q is non-degenerate and that $2 \neq 0$ in K.

Writing $\mathrm{O}_q = \mathrm{O}_q(E)$ to denote the group of q-isometries of E, which is called the *orthogonal group* of q, then we can start with a simple observation:

Lemma. The linear automorphisms $\widetilde{\rho}_u \in \mathrm{GL}(E)$, for $u \in \widetilde{\Gamma}_q$, are q-isometries, which means that the representation $\widetilde{\rho}$ is actually a group homomorphism

$$\widetilde{\rho} : \widetilde{\Gamma}_q \to \mathrm{O}_q.$$

Since $\widetilde{\rho}_u = (-1)^{|u|}\rho_u$, for $u \in \Gamma_q$, we also have a homomorphism

$$\rho : \Gamma_q \to \mathrm{O}_q.$$

Proof

Let us compute $Q(\widetilde{\rho}_u(e))$ for $u \in \widetilde{\Gamma}_q$ and $e \in E$:

$$\begin{aligned} Q(\widetilde{\rho}_u(e)) &= (\widetilde{\rho}_u(e))^2 = (u^\alpha e u^{-1})^2 \\ &= (ueu^{-1})^2 = ue^2u^{-1} \\ &= e^2 = Q(e), \end{aligned}$$

where in the third step we have used that $u^\alpha = (-1)^{|u|}u$. The proof follows because Q determines, when $2 \neq 0$ in K, the bilinear form q by the polarization formula $2q(e, e') = Q(e+e') - Q(e) - Q(e')$. □

Given $u \in E^{+}$, we have a decomposition $E = \langle u \rangle \perp u^{\perp}$, where $u^{\perp}$ is the hyperplane $\{e \in E \mid u \cdot e = 0\}$. The *axial symmetry* with respect to u is the linear map s_u such that

$$s_u(u) = u \text{ and } s_u(e) = -e \text{ if } e \in u^{\perp}.$$

The linear map $m_u = -s_u$ satisfies

$$m_u(u) = -u \text{ and } m_u(e) = e \text{ for } e \in u^{\perp},$$

and it is called the *reflection* in the direction u or across the the hyperplane $u^{\perp}$.

Warning. The more familiar term *reflection across* the hyperplane $u^{\perp}$ is acceptable because we assume that the metric is non-degenerate, for in that case $u^{\perp}$ determines the line $\langle u \rangle$ as $u^{\perp\perp}$. For degenerate metrics this need not be true. Indeed, simple examples show that we may have $\dim u^{\perp\perp} > 1$ and hence $u^{\perp}$ does not determine uniquely the line $\langle u \rangle$.

Lemma

If $u \in E^{+}$, then $s_u = \rho_u$ and $m_u = \widetilde{\rho}_u$.

Proof. We have $\rho_u(u) = uuu^{-1} = u$ and for $e \in u^{\perp}$,

$$\rho_u(e) = ueu^{-1} = -euu^{-1} = -e,$$

where we have used that two orthogonal vectors anticommute. This proves the first part.

On the other hand $\widetilde{\rho}_u = -\rho_u$ (since u is odd) and hence

$$m_u(e) = -s_u(e) = -\rho_u(e) = \widetilde{\rho}_u(e).$$

This completes the proof. □

If we set $\mathrm{SO}_q = \mathrm{O}_q^+$ to denote the subgroup of O_q formed with the isometries that have determinant $+1$ (we will say that it is the *rotation group*, or the *special orthogonal group* of q), then we have:

Lemma

If $u, v \in E^+$, then

$$s_u \circ s_v = m_u \circ m_v \in \mathrm{SO}_q.$$

Proof. The relation $s_u \circ s_v = m_u \circ m_v$ is a direct consequence of the definitions. On the other hand, it is clear that $\det(m_u) = -1$ and therefore $\det(m_u \circ m_v) = 1$. □

The action of $m_u \circ m_v = s_u \circ s_v$ on a vector e is given by

$$e \mapsto uvev^{-1}u^{-1} = ReR^{-1},$$

where $R = uv$. Since this map is a rotation, expressions of the form $R = uv$, where $u, v \in E^\times$, are called *rotors*.

Remark

If n is odd, then we have

$$\det(\rho_u) = \det(s_u) = (-1)^{n-1} = 1.$$

Thus in this case the image of ρ is contained in SO_q and there is no hope to obtain in this way the elements of O_q that are not in SO_q. Overcoming this defect is the job of the twisted Lipschitz group, as established in next result.

Theorem 5. The image of $\widetilde{\rho}:\widetilde{\Gamma}_q \to \mathrm{GL}(E)$ is $\mathrm{O}_q(E)$ and its kernel is $K^\times$. So we have an exact sequence

$$1 \to K^\times \hookrightarrow \widetilde{\Gamma}_q \xrightarrow{\widetilde{\rho}} \mathrm{O}_q \to 1.$$

Proof. We know that the image of $\widetilde{\rho}$ is contained in O_q. Now $\widetilde{\rho}_u = m_u$ for any $u \in E^\times$, where m_u is the reflection in the direction u. Therefore, if $u_1, \ldots, u_k \in E^\times$, and we set $u = u_1 \cdots u_k \in \Gamma_q$, then

$$\widetilde{\rho}_u = \widetilde{\rho}_{u_1} \circ \cdots \circ \widetilde{\rho}_{u_k} = m_{u_1} \circ \cdots \circ m_{u_k}.$$

This shows that the image of $\widetilde{\rho}$ contains all the isometries that can be expressed as the product of reflections in the direction of non-isotropic vectors. But by the Cartan-Dieudonné theorem, any q-isometry can be expressed in this way (even with $k \leqslant n$) and consequently the image of $\widetilde{\rho}$ contains O_q. Thus the image of $\widetilde{\rho}$ is O_q.

It remains to prove that $\ker(\widetilde{\rho}) = K^+$. If $\lambda \in K^+$, then $\widetilde{\rho}_\lambda = \mathrm{Id}$, as $\widetilde{\rho}_\lambda(e) = \lambda^\alpha e \lambda^{-1} = e$ for all $e \in E$. So $K^+ \subseteq \ker(\widetilde{\rho})$.

To show the converse inclusion, suppose that $u \in \widetilde{\Gamma}_q$ is an element of $\ker(\widetilde{\rho})$. Then $\widetilde{\rho}_u = \mathrm{Id}$, which means that $(-1)^{|u|} u e u^{-1} = e$, or $(-1)^{|u|} u e = e u$, for all $e \in E$. In particular we will have, if we pick an orthogonal basis $\boldsymbol{e} = e_1, \ldots, e_n$ of E, $(-1)^{|u|} u e_j = e_j u$ for $j = 1, \ldots, n$. Using the basis $\{e_I\}$ of $\Lambda_q(E)$ asociated to $\boldsymbol{e}$, it is clear that we can write, for any given j, $u = u' + e_j u''$, with u' and u'' not involving e_j, and hence we have that the condition $(-1)^{|u|} u e_j = e_j u$ takes the form $(-1)^{|u|} u' e_j + (-1)^{|u|} e_j u'' e_j = e_j u' + e_j^2 u''$. But $(-1)^{|u|} u' e_j = e_j u'$, for $|u'| = |u|$, and so we get $(-1)^{|u|} e_j u'' e_j = e_j^2 u''$. Since the parity of u'' is opposite to the parity of u, this boils down to the relation $-e_j^2 u'' = e_j^2 u''$. Thus we conclude that $u'' = 0$ and so u does not involve e_j. Since j was arbitrary, it follows that u must be a scalar. □

Corollary. The group $\widetilde{\Gamma}_q$ is the subgroup of $\Lambda_q^\times(E)$ generated by $K^\times$ and $E^\times$. This can also be expressed by the formula $\widetilde{\Gamma}_q = K^\times \Gamma_q$.

Proof. In the proof of the previous theorem we have seen that $\widetilde{\rho} : \Gamma_q \to \mathrm{O}_q$ is surjective. This implies that any element of $\widetilde{\Gamma}_q$ has the form λu, with $u \in \Gamma_q$ and $\lambda \in \ker(\widetilde{\rho}) = K^\times$. □

If we set Γ_q^+ to denote the even part of Γ_q, then its image under $\widetilde{\rho}$ is, again by the Cartan-Dieudonné theorem, the subgroup $\mathrm{O}_q^+ = \mathrm{SO}_q$ of O_q consisting of the q-isometries that have determinant $+1$. It follows that $\widetilde{\rho}^{-1}(\mathrm{SO}_q) = K^\times \Gamma_q^+$, which is, by the Corollary above, the even subgroup $\widetilde{\Gamma}_q^+$ of $\widetilde{\Gamma}_q$. To summarize:

Corollary. The sequence

$$1 \to K^\times \hookrightarrow \widetilde{\Gamma}_q^+ \xrightarrow{\widetilde{\rho}} \mathrm{SO}_q \to 1$$

is exact and $\widetilde{\Gamma}_q^+ = K^\times \Gamma_q^+$. □

Corollary (The quotient $\widetilde{\Gamma}_q/\Gamma_q$). If we set $K_0^\times = K^\times \cap \Gamma_q = K^\times \cap \Gamma_q^+$, then there is a canonical isomorphism $K^\times/K_0^\times \simeq \widetilde{\Gamma}_q/\Gamma_q$. Furthermore, if ${K^\times}^2 = \{\lambda^2 \mid \lambda \in K^\times\}$ is the subgroup of squares of $K^\times$, then ${K^\times}^2 \subseteq K_0^\times$ and consequently $K^\times/K_0^\times$ is a quotient of $K^\times/{K^\times}^2$.

Proof. The map $K^\times \to \widetilde{\Gamma}_q/\Gamma_q$ (the inclusion $K^\times \hookrightarrow \widetilde{\Gamma}_q = K^\times\Gamma_q$ followed by the quotient map $\widetilde{\Gamma}_q \to \widetilde{\Gamma}_q/\Gamma_q$) is surjective and its kernel is $\Gamma_q \cap K^\times = K_0^\times$. We therefore have a canonical isomorphism $K^\times/K_0^\times \simeq \widetilde{\Gamma}_q/\Gamma_q$.

For the second statement, first note that if $u \in E^{\times}$, then

$$q(u,u) = u^2 \in \Gamma_q \cap K^{\times} = K_0^{\times}.$$

In particular, for any $\lambda \in K^{\times}$,

$$\lambda^2 = q(\lambda u, \lambda u)/q(u,u) \in K_0^{\times},$$

and this completes the proof. □

A field K is said to be a *spin* field (Lawson-Michelsohn-1989) if $2 \neq 0$ in K and for any $a \in K$ at least one of the equations $\lambda^2 = \pm a$ has a solution $\lambda \in K$. Any algebraically closed field, and in particular $\mathbf{C}$, is spin. The real field $\mathbf{R}$ is spin, as for any $a \in \mathbf{R}$ either $a > 0$ or $-a \geqslant 0$. Another example are the fields $\mathbf{Z}_p$, where p is a prime number of the form $4k+3$ ($p = 3, 7, 11, 19, \ldots$).

Theorem 6. Assume that K is a spin field. Let $U = \{\pm 1\}$ if $\sqrt{-1} \notin K$ and $U = \{\pm 1, \pm\sqrt{-1}\}$ if $\sqrt{-1} \in K$. Then the sequences

$$1 \to U \to \mathrm{Pin}_q(E) \xrightarrow{\widetilde{\rho}} \mathrm{O}_q(E) \to 1$$

$$1 \to U \to \mathrm{Spin}_q(E) \xrightarrow{\widetilde{\rho}} \mathrm{SO}_q(E) \to 1$$

are exact.

Proof

If $\lambda = u_1 \cdots u_k \in \mathrm{Pin}_q$ (so $u_j^2 = \pm 1$) is in the kernel of $\widetilde{\rho}$, then we must have $\lambda \in K$. But $\lambda^2 = \pm u_1^2 \cdots u_k^2 = \pm 1$. This shows that in both sequences the kernel is U.

To finish the proof it is enough to see that any reflection in the direction of a vector $u \in E^\times$ can be realized as the reflection $m_{\hat{u}}$ for a vector $\hat{u}$ such that $\hat{u}^2 = \pm 1$.

To see this, note that $m_u = m_{\lambda u}$ for any non-zero scalar λ, and for $(\lambda u)^2 = \lambda^2 u^2$ to be ± 1 it is necessary and sufficient that $\lambda^2 = \pm (u^2)^{-1}$. But this relation has at least one solution $\lambda \in K$ if K is spin. □

Artin 1957

Chevalley 1954

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Geometric Algebra Techniques in Mathematics and Physics

S. Xambó

UPC

SLP · 9-13 March · 2015

Introduction. Definitions, notations and conventions.

Lorentzian GA. $\mathcal{G} = \mathcal{G}_{1,3} = \Lambda_\eta E$. Involutions and $\mathcal{G}^+$. Complex structure. Interpretation of the even subalgebra. Polar form of a bivector. Generating Lorentz isometries.

Space-time kinematics. Space-time paths. Proper time. Relative vectors. Relative velocity.

Lorentz transformations. Lorentz boosts. Lorentz group.

Appendix A. The Dirac representation of $\mathcal{G}$.

Appendix B. $\mathrm{SU}_2 \to \mathrm{SO}_3$. Action of $\mathbf{H}^\times$ on $V = \langle \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K} \rangle$. The homomorphism $\mathrm{SU}_2 \to \mathrm{SO}_3$

References.

Historically, the spacetime algebra was the first modern implementation of geometric algebra. This is because it provides a *sythetic* framework for studying spacetime physics.

DORAN-LASENBY-2003, CH. 5.

A *Lorentzian spacetime* is a real quadratic space (E, η) of signature $(1,3)$. We express this by writing $E_{1,3}$ instead of E. We also write $x \cdot y$ to mean $\eta(x, y)$. The elements $x \in E$ are called *events*. We will use the customary terms *time-like*, *space-like* and *light-light* to refer to vectors such that $\eta(x, x)$ is positive, negative or null, respectively.

An (inertial) *frame* of $E_{1,3}$ is an orthonormal basis $\gamma = \gamma_0, \gamma_1, \gamma_2, \gamma_3$:

$$\gamma_0 \cdot \gamma_0 = 1,\ \gamma_0 \cdot \gamma_j = 0,\ \gamma_j \cdot \gamma_k = \delta_{j,k}\ (j, k \in \{1, 2, 3\}.$$

Or, in the familiar relativistic notations,

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}\ (\mu, \nu \in \{0, 1, 2, 3\}).$$

The components of an event x in the frame γ are denoted x^μ, and so $x = x^\mu\gamma_\mu$. Instead of x^0 we often write ct, so that $x = ct\gamma_0 + x^j\gamma_j$.

The *reciprocal frame* of γ is the frame $\gamma^0, \gamma^1, \gamma^2, \gamma^3$, where $\gamma^0 = \gamma_0$ and $\gamma^j = -\gamma_j$. The components of an event x in the reciprocal frame are denoted x_μ, so that $x = x_\mu\gamma^\mu$. Clearly, $x_0 = x^0$ and $x_k = -x^k$.

Remark

In Dirac's theory, the symbols γ_μ are certain 4×4 matrices (the *Dirac matrices*), but here they are just vectors. The Dirac matrices produce a concrete representation of the spacetime algebra (the geometric algebra of $E_{1,3}$), so that we can say that the spacetime algebra encodes Dirac's algebra without matrices (see Appendix, slide 31). The beauty and usefulness of this approach will be apparent along the way, much in the same way as it happened with the treatment of quaternions by geometric algebra in previous lectures.

We will write $\mathcal{G} = \mathcal{G}_{1,3}$ to denote the geometric algebra $\Lambda_\eta E$. In terms of the frame γ, the basic computational rule is

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}.$$

Let $\mathbf{i} = \boldsymbol{i}_\gamma$ be the pseudo-scalar unit associated to the frame γ:

$$\mathbf{i} = \gamma_0\gamma_1\gamma_2\gamma_3.$$

By L3.30, *Corollary* 2, $\mathbf{i}$ anticommutes with vectors and trivectors and commutes with scalars, bivectors and pseudo-scalars. In a compact form, $x\mathbf{i} = (-1)^r\mathbf{i}x$ for $x \in \mathcal{G}^r$. Moreover, by the general results presented in L3 (Theorem 4 and Remark after its proof), we have:

Proposition

1) $\mathbf{i}^2 = Q(\mathbf{i}) = \det(\mathrm{diag}(+,-,-,-)) = -1.$
2) The Hodge duality map $\mathcal{G}^r \to \mathcal{G}^{4-r}$, $x \mapsto x^* = x\mathbf{i} = (-1)^r\mathbf{i}x$ is an antiisometry for $r = 0, 1, 2, 3$. This implies that the signatures of $\mathcal{G}^2$, $\mathcal{G}^3$ and $\mathcal{G}^4$ are $(3,3)$, $(3,1)$ and $(0,1) = \bar{1}$. □

Let $\sigma_1 = \gamma_1\gamma_0 = \gamma_{10}$, $\sigma_2 = \gamma_2\gamma_0 = \gamma_{20}$, $\sigma_3 = \gamma_3\gamma_0 = \gamma_{30}$. Then a short computation shows that $\sigma_j^* = \sigma_j\mathbf{i} = -\gamma_k\gamma_l = -\gamma_{kl}$ (j, k, l a cyclic permutation of $(1, 2, 3)$).[1] Explicitely,

$$\sigma_1^* = -\gamma_{23},\ \sigma_2^* = -\gamma_{31},\ \sigma_3^* = -\gamma_{12}.$$

The σ_j and σ_j^* have signatures -1 and $+1$, respectively, and together form a basis of $\mathcal{G}^2$.

The $\gamma_\mu^* = \gamma_\mu\mathbf{i}$ form a basis of $\mathcal{G}^3$ and they have signatures -1 for $\mu = 0$ and $+1$ otherwise. Note that

$$\gamma_0^* = \gamma_{123},\ \gamma_1^* = \gamma_{023},\ \gamma_2^* = \gamma_{031},\ \gamma_3^* = \gamma_{012}.$$

With these notations, we finally have

$$\mathbf{i} = \gamma_{0123}.$$

[1] This is often condensed as $\sigma_j^* = -\epsilon_{jkl}\gamma_k\gamma_l$, where ϵ_{jkl} denotes the sign of the permutation jkl of 123 (Levi-Civita symbol).

These facts are summarized in the following table:

G	γ_0	γ_1	γ_2	γ_3	σ_1	σ_2	σ_3	σ_1^*	σ_2^*	σ_3^*	γ_0^*	γ_1^*	γ_2^*	γ_3^*	$\mathbf{i}$
γ_0	1	$\bar\sigma_1$	$\bar\sigma_2$	$\bar\sigma_3$	$\bar\gamma_1$	$\bar\gamma_2$	$\bar\gamma_3$	$\bar\gamma_1^*$	$\bar\gamma_2^*$	$\bar\gamma_3^*$	$\mathbf{i}$	$\bar\sigma_1^*$	$\bar\sigma_2^*$	$\bar\sigma_3^*$	γ_0^*
γ_1	σ_1	-1	$\bar\sigma_3^*$	σ_2^*	$\bar\gamma_0$	γ_3^*	$\bar\gamma_2^*$	$\bar\gamma_0^*$	$\bar\gamma_3$	γ_2	σ_1^*	-$\mathbf{i}$	σ_3	$\bar\sigma_2$	γ_1^*
γ_2	σ_2	σ_3^*	-1	$\bar\sigma_1^*$	$\bar\gamma_3^*$	$\bar\gamma_0$	γ_1^*	γ_3	$\bar\gamma_0^*$	$\bar\gamma_1$	σ_2^*	$\bar\sigma_3$	-$\mathbf{i}$	σ_1	γ_2^*
γ_3	σ_3	$\bar\sigma_2^*$	σ_1^*	-1	γ_2^*	$\bar\gamma_1^*$	$\bar\gamma_0$	$\bar\gamma_2$	γ_1	$\bar\gamma_0^*$	σ_3^*	σ_2	$\bar\sigma_1$	-$\mathbf{i}$	γ_3^*
σ_1	γ_1	γ_0	$\bar\gamma_3^*$	γ_2^*	1	σ_3^*	$\bar\sigma_2^*$	$\mathbf{i}$	$\bar\sigma_3$	σ_2	γ_1^*	γ_0^*	γ_3	$\bar\gamma_2$	σ_1^*
σ_2	γ_2	γ_3^*	γ_0	$\bar\gamma_1^*$	$\bar\sigma_3^*$	1	σ_1^*	σ_3	$\mathbf{i}$	$\bar\sigma_1$	γ_2^*	$\bar\gamma_3$	γ_0^*	γ_1	σ_2^*
σ_3	γ_3	$\bar\gamma_2^*$	γ_1^*	γ_0	σ_2^*	$\bar\sigma_1^*$	1	$\bar\sigma_2$	σ_1	$\mathbf{i}$	γ_3^*	γ_2	$\bar\gamma_1$	γ_0^*	σ_3^*
σ_1^*	$\bar\gamma_1^*$	$\bar\gamma_0^*$	$\bar\gamma_3$	γ_2	$\mathbf{i}$	$\bar\sigma_3$	σ_2	-1	$\bar\sigma_3^*$	σ_2^*	γ_1	γ_0	$\bar\gamma_3^*$	γ_2^*	$\bar\sigma_1$
σ_2^*	$\bar\gamma_2^*$	γ_3	$\bar\gamma_0^*$	$\bar\gamma_1$	σ_3	$\mathbf{i}$	$\bar\sigma_1$	σ_3^*	-1	$\bar\sigma_1^*$	γ_2	γ_3^*	γ_0	$\bar\gamma_1^*$	$\bar\sigma_2$
σ_3^*	$\bar\gamma_3^*$	$\bar\gamma_2$	γ_1	$\bar\gamma_0^*$	$\bar\sigma_2$	σ_1	$\mathbf{i}$	$\bar\sigma_2^*$	σ_1^*	-1	γ_3	$\bar\gamma_2^*$	γ_1^*	γ_0	$\bar\sigma_3$
γ_0^*	-$\mathbf{i}$	σ_1^*	σ_2^*	σ_3^*	$\bar\gamma_1^*$	$\bar\gamma_2^*$	$\bar\gamma_3^*$	γ_1	γ_2	γ_3	1	$\bar\sigma_1$	$\bar\sigma_2$	$\bar\sigma_3$	$\bar\gamma_0$
γ_1^*	$\bar\sigma_1^*$	$\mathbf{i}$	$\bar\sigma_3$	σ_2	$\bar\gamma_0^*$	$\bar\gamma_3$	γ_2	γ_0	$\bar\gamma_3^*$	γ_2^*	σ_1	-1	$\bar\sigma_3^*$	σ_2^*	$\bar\gamma_1$
γ_2^*	$\bar\sigma_2^*$	σ_3	$\mathbf{i}$	$\bar\sigma_1$	γ_3	$\bar\gamma_0^*$	$\bar\gamma_1$	γ_3^*	γ_0	$\bar\gamma_1^*$	σ_2	σ_3^*	-1	$\bar\sigma_1^*$	$\bar\gamma_2$
γ_3^*	$\bar\sigma_3^*$	$\bar\sigma_2$	σ_1	$\mathbf{i}$	$\bar\gamma_2$	γ_1	$\bar\gamma_0^*$	$\bar\gamma_2^*$	γ_1^*	γ_0	σ_3	$\bar\sigma_2^*$	σ_1^*	-1	$\bar\gamma_3$
$\mathbf{i}$	$\bar\gamma_0^*$	$\bar\gamma_1^*$	$\bar\gamma_2^*$	$\bar\gamma_3^*$	σ_1^*	σ_2^*	σ_3^*	$\bar\sigma_1$	$\bar\sigma_2$	$\bar\sigma_3$	γ_0	γ_1	γ_2	γ_3	-1

This is the multiplication table of $\mathcal{G}$ in terms of the described basis. The bar over symbols indicates minus sign, not Clifford conjugation.

A product $\gamma_I \gamma_J$ is determined by the rule L3.30, *Corollary* 1. In this case it yields $\pm\gamma_K$, $K = I \bigtriangleup J$ (symmetric difference) and $\pm 1 = (-1)^\nu$, where ν is the sum of the number of elements in $\{1,2,3\} \cap I \cap J$ and the number of inversions in the sequence I, J.

By L3.30, *Corollary* 2, the table is symmetric up to sign, because $\gamma_J \gamma_I = (-1)^c (-1)^{rs} \gamma_I \gamma_J$, where $r = |I|$, $s = |J|$ and $c = |I \cap J|$. The result can be summarized as follows: $\gamma_J \gamma_I = -\gamma_I \gamma_J$ if one of the following two cases occurs:

- $c = 1, 3$ and r or s is even
- $c = 0, 2$ and both r and s are odd.

Otherwise $\gamma_J \gamma_I = \gamma_I \gamma_J$.

Example. $\gamma_1 \sigma_2^* = \gamma_1 \gamma_{13} = -\gamma_3$.

Example. Consider $\sigma_2\gamma_3^* = \gamma_{20}\gamma_{012} = \gamma_2\gamma_{12} = \gamma_1$. We can also argue that the result must be $\pm\gamma_1$. Since there are three inversions (20 twice and 21), and 2 is the only -1 index in common, we get $\sigma_2\gamma_3^* = \gamma_1$. Since the product shares two indices (0 and 2), and only one factor is odd, we conclude that $\gamma_3^*\sigma_2 = \gamma_1$ as well.

Example. We have defined $\gamma_I^* = \gamma_I\mathbf{i}$, and have observed that $\mathbf{i}\gamma_I = \pm\gamma_I^*$, the sign being $+1$ when $|I|$ is even and -1 when it is odd. This simplifies the computation of products in which one of the factors is a Hodge dual. Here are a couple of illustrations:

$$\sigma_3^*\gamma_3 = \sigma_3\mathbf{i}\gamma_3 = -\sigma_3\gamma_3\mathbf{i} = \gamma_0\gamma_3\gamma_3\mathbf{i} = -\gamma_0\mathbf{i} = -\gamma_0^*.$$

$$\gamma_1^*\sigma_2^* = \gamma_1\mathbf{i}\sigma_2\mathbf{i} = -\gamma_1\sigma_2 = \gamma_1\gamma_0\gamma_2 = -\gamma_0\gamma_1\gamma_2 = -\gamma_3^*,$$

and similarly $\sigma_2^*\gamma_1^* = \gamma_3^*$.

If $x = x_0 + x_1 + x_2 + x_3 + x_4 \in \mathcal{G}$, the three involutions α, τ, κ of $\mathcal{G}$ act as follows:

$$x^\alpha = x_0 - x_1 + x_2 - x_3 + x_4,$$

$$x^\tau = x_0 + x_1 - x_2 - x_3 + x_4,$$

$$x^\kappa = x_0 - x_1 - x_2 + x_3 + x_4.$$

The elements of the even subalgebra $\mathcal{G}^+$ have the form $x = x_0 + x_2 + x_4$ and in this case $x^\tau = x^\kappa = x_0 - x_2 + x_4$. The elements of the odd subspace $\mathcal{G}^-$ have the form $x = x_1 + x_3$ and in this case $x^\alpha = -x$ and $x^\tau = x_1 - x_3$.

Lemma. 1) The multivector x has the form $x_0 + x_4$ if and only if

$$x = x^\alpha \text{ and } x = x^\tau.$$

2) The multivector $x \in \mathcal{G}^1$ (or $x = x_1$) if and only if $x^\alpha = -x$ and $x^\tau = x$. □

The subspace $\langle 1, \mathbf{i}\rangle = \mathcal{G}^0 + \mathcal{G}^4$ is a subalgebra isomorphic to $\mathbf{C}$. We will say that this is the algebra of *complex scalars*. Henceforth, $\mathbf{C}$ will denote this algebra. By the Lemma, $\mathbf{C} = \{x \in \mathcal{G} \,|\, x = x^\alpha = x^\tau\}$. A typical complex scalar will be denoted $\alpha + \beta\mathbf{i}$, $\alpha, \beta \in \mathbf{R}$.

The space $\mathcal{G}^- = \mathcal{G}^1 + \mathcal{G}^3 = \mathcal{G}^1 + \mathcal{G}^1\mathbf{i}$ is closed under multiplication by $\mathbf{i}$, and hence by complex scalars, and will be called the space of *complex vectors*. A basis of this $\mathbf{C}$-space is $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. A typical complex vector will be denoted $a + b\mathbf{i}$, $a, b \in \mathcal{G}^1$. Note that $\gamma_0\mathcal{G}^- = \mathcal{G}^-\gamma_0 = \mathcal{G}^+$.

The space $\mathcal{G}^2$ of bivectors is closed under multiplication by $\mathbf{i}$ and hence it is a $\mathbf{C}$-space. As a basis of this $\mathbf{C}$-space we may take $\sigma_1, \sigma_2, \sigma_3$. A typical bivector will be denoted $\boldsymbol{x} + \boldsymbol{y}\mathbf{i}$, $\boldsymbol{x}, \boldsymbol{y} \in \langle\sigma_1, \sigma_2, \sigma_3\rangle$.

We thus see that a typical multivector has the form

$$(\alpha + \beta\mathbf{i}) + (a + b\mathbf{i}) + (\boldsymbol{x} + \boldsymbol{y}\mathbf{i}).$$

Let $\mathcal{E}$ be the space $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$, so that $\mathcal{G}^2 = \mathcal{E} + \mathcal{E}\mathbf{i}$. With the multivector metric, $\mathcal{E}$ is Euclidean and $\sigma_1, \sigma_2, \sigma_3$ is an orthonormal basis. We will say that it is the *relative* (Euclidean) space. If necessary, we will denote it by $\mathcal{E}(\gamma_0)$ to underline that it is a frame-dependent space. The geometric algebra of $\mathcal{E}$ will be denoted $\mathcal{P}$ (the Pauli algebra).

Proposition

(1) The even algebra $\mathcal{G}^+$ is isomorphic to $\mathcal{P}$ and the pseudoscalar of $\mathcal{P}$ coincides with $\mathbf{i}$.

(2) The linear grading of $\mathcal{P}$ is given by

$$\mathcal{P}^0 = \mathbf{R},\ \mathcal{P}^1 = \mathcal{E},\ \mathcal{P}^2 = \mathcal{E}\mathbf{i},\ \mathcal{P}^3 = \mathbf{R}\mathbf{i}.$$

Proof. The $\sigma_1, \sigma_2, \sigma_3$ generate $\mathcal{G}^+$ as an $\mathbf{R}$-algebra, for $\sigma_j\sigma_k = -\sigma_l^*$ (j, k, l a cyclic permutation of $1, 2, 3$) and $\boldsymbol{i}_\sigma = \sigma_1\sigma_2\sigma_3 = \mathbf{i}$. Now (1) follows from this and the relations $\sigma_j^2 = 1$ and $\sigma_j\sigma_k + \sigma_k\sigma_j = 0$ (if $k \neq j$) and (2) is straightforward. □

Given a bivector $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\mathbf{i}$, we have

$$\boldsymbol{z}^2 = |\boldsymbol{x}|^2 - |\boldsymbol{y}|^2 + 2(\boldsymbol{x} \cdot \boldsymbol{y})\mathbf{i} \in \mathbf{C}.$$

In particular we see that $\boldsymbol{z}^2 \in \mathbf{R}$ if and only if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$, in which case we say that $\boldsymbol{z}$ is *plain* (or *simple*). We also say that $\boldsymbol{z}$ is *positive*, *null* or *negative* according to whether $|\boldsymbol{x}|^2 > |\boldsymbol{y}|^2$, $|\boldsymbol{x}|^2 = |\boldsymbol{y}|^2$, or $|\boldsymbol{x}|^2 < |\boldsymbol{y}|^2$. If $\boldsymbol{x} \cdot \boldsymbol{y} \neq 0$ (hence also $\boldsymbol{z}^2 \neq 0$), we say that $\boldsymbol{z}$ is *slanted* (or *composite*).

Examples. (1) A non-zero $\boldsymbol{x} \in \mathcal{E}$ is plain and positive ($\boldsymbol{x}\boldsymbol{x} = |\boldsymbol{x}|^2 > 0$) and $\boldsymbol{x}\mathbf{i}$ is plain and negative ($(\boldsymbol{x}\mathbf{i})^2 = -\boldsymbol{x}^2 = -|x|^2 < 0$).

(2) If $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{E}$ are two unit orthogonal vectors, like σ_1 and σ_2, then $\boldsymbol{u} + \boldsymbol{v}\mathbf{i}$ is null.

Remark. The Lorentzian norm of $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}\mathbf{i}$ is $|y|^2 - |\boldsymbol{x}|^2$, because $Q(z) = \langle z|z\rangle = (zz^\tau)_0 = -(z^2)_0 = |y|^2 - |\boldsymbol{x}|^2$.

Lemma. If $z^2 \neq 0$, then there exists a unique plain positive bivector z' and a unique $\alpha \in (-\pi/2, \pi/2]$ such that $z = z' e^{\mathbf{i}\alpha}$ (*polar decomposition of* z).

Proof. First let us show existence. If z is plain, then z^2 is real and non-zero. If $z^2 > 0$, it suffices to take $z' = z$ and $\alpha = 0$, and if $z^2 < 0$, then we can take $z' = -z\mathbf{i}$ and $\alpha = \frac{\pi}{2}$, for $z'^2 = -z^2 > 0$ and $z = z'\mathbf{i} = z' e^{\mathbf{i}\frac{\pi}{2}}$.

So we may assume that z is slanted. Then we can write, $z^2 = \rho^2 e^{2\mathbf{i}\alpha}$, with $\rho \in \mathbf{R}$, $\rho > 0$, and $\alpha \in (0, \pi)$. So $z^2 = (\rho e^{\mathbf{i}\alpha})^2$. Now define z' as follows: if $\alpha \leq \pi/2$, set $z' = z e^{-\mathbf{i}\alpha}$, in which case $z'^2 = (z e^{-\mathbf{i}\alpha})^2 = z^2 e^{-2\mathbf{i}\alpha} = \rho^2 > 0$ and $z = z' e^{\mathbf{i}\alpha}$; and if $\pi/2 < \alpha < \pi$, define $z' = z e^{\mathbf{i}(\pi - \alpha)}$, in which case $z'^2 = z^2 e^{2\pi\mathbf{i}} e^{-2\mathbf{i}\alpha} = z^2 e^{-2\mathbf{i}\alpha} = \rho^2 > 0$ and $z = z' e^{\mathbf{i}(\alpha - \pi)}$ (note that $-\pi/2 < \alpha - \pi < 0$).

As for uniqueness, suppose that we have $z'e^{\mathbf{i}\alpha} = z''e^{\mathbf{i}\beta}$, with z' and z'' plain and positive and $\alpha, \beta \in (-\pi/2, \pi/2]$. Without loss of generality we may assume that $\alpha \leq \beta$, which implies that $z' = z''e^{\mathbf{i}(\beta-\alpha)}$ with $-\pi < \beta - \alpha < \pi$. Taking squares and using that z'^2 and z''^2 are real and positive, we conclude that $e^{\mathbf{i}(2\beta-2\alpha)}$ is real and positive. In the range of $2\beta - 2\alpha$, namely $(-2\pi, 2\pi)$, the only possibilities for $e^{\mathbf{i}(2\beta-2\alpha)}$ to be real are $2\beta - 2\alpha = 0$ or $2\beta - 2\alpha = \pm\pi$, and of these, only the first (equivalent to $\beta = \alpha$) yields a positive value. □

Corollary

With the same notations and assumptions as in the Lemma, if z is slanted, then $z = z_1 + \mathbf{i}z_2$ with z_1 and z_2 plain and positive.

Proof. Indeed, we have $z = z'e^{\mathbf{i}\alpha} = z'\cos\alpha + \mathbf{i}z'\sin\alpha$, and both $z_1 = z'\cos\alpha$ and $z_2 = z'\sin\alpha$ are plain and positive. □

Remark. This Corollary explains why instead of *slanted* some authors use the term *composite*, and consequently the term *simple* for plain (the non-slanted).

Definition. Given the *polar decomposition* $\boldsymbol{z} = \boldsymbol{z}' e^{\mathbf{i}\alpha}$ of a non-null bivector $\boldsymbol{z}$, we have $\boldsymbol{z}'^2 > 0$ and $\boldsymbol{z}^2 = \boldsymbol{z}'^2 e^{2\alpha\mathbf{i}}$. We define the *magnitude* of $\boldsymbol{z}$, $|\boldsymbol{z}|$, as $\sqrt{\boldsymbol{z}'^2}$, so that $|\boldsymbol{z}| > 0$, $\boldsymbol{z}^2 = |\boldsymbol{z}|^2 e^{2\alpha\mathbf{i}}$, and $|\boldsymbol{z}| = |\boldsymbol{z}'|$. The angle $\alpha = \alpha(\boldsymbol{z})$ will be called the *slant angle* of $\boldsymbol{z}$.

Remark. In terms of the angle $\theta = \theta(\boldsymbol{x}, \boldsymbol{y})$, the magnitude of $\boldsymbol{z}$ is given by the formula

$$|\boldsymbol{z}|^2 = (|x|^2 - |y|^2)^2 + 4|x|^2|y|^2 \cos^2\theta.$$

Proof. From the definitions we have that $|\boldsymbol{z}|^2$ is equal to

$$|\boldsymbol{z}^2| = |\boldsymbol{x}^2 - \boldsymbol{y}^2 + 2(\boldsymbol{x}\cdot\boldsymbol{y})\mathbf{i}|^2,$$

which is equal to $(\boldsymbol{x}^2 - \boldsymbol{y}^2)^2 + 4(\boldsymbol{x}\cdot\boldsymbol{y})^2$, and $\boldsymbol{x}\cdot\boldsymbol{y} = |x||y|\cos\theta$. □

Remark

Let $z \in \mathcal{G}^2$ be a bivector. If $z^2 \neq 0$, then z is invertible (so $z \in \mathcal{G}^2 \cap \mathcal{G}^\times$) and

$$z^{-1} = z/z^2 = -\tilde{z}/z^2.$$

Indeed, we have seen that $z^2 \in \mathbf{C}$ (so z^2 commutes with with all even elements, and in particular with all bivectors) and hence $z(z/z^2) = 1$. Since $\tilde{z} = -z$, we can also write $z^{-1} = -\tilde{z}/z^2$.

Lemma

Let $z \in \mathcal{G}^2$ and assume $z^2 \neq 0$.

Let $L_z : \mathcal{G} \to \mathcal{G}$ be the automorphism of $\mathcal{G}$ defined by

$$L_z x = zxz^{-1}.$$

Then $L_z\mathcal{G}^1 = \mathcal{G}^1$.

Proof

Let $x \in \mathcal{G}^1$ and put $y = L_z(x) = zxz^{-1}$. To check that $y \in \mathcal{G}^1$, it suffices to see that $y^\alpha = -y$ and $\widetilde{y} = y$ (by the Lemma on slide 11):

$$y^\alpha = z^\alpha x^\alpha (z^\alpha)^{-1} = -zxz^{-1} = -y,$$

$$y^\tau = (z^\tau)^{-1} x^\tau z^\tau = z^{-1}xz = zxz/z^2 = y. \quad \square$$

Lemma. With the same assumptions as in the previous Lemma, the induced **R**-linear map $L_z : \mathcal{G}^1 \to \mathcal{G}^1$ is a proper Lorentz isometry ($L_z \in \mathrm{O}_\eta^+$).

Proof. The computation

$$y^2 = zxz^{-1}zxz^{-1} = zx^2z^{-1} = x^2$$

shows that L_z preserves que Lorentz quadratic form and therefore it is a Lorentz isometry.

On the other hand, using that $\mathbf{i}$ commutes with bivectors,

$$L_z(\mathbf{i}) = z\mathbf{i}z^{-1} = \mathbf{i}.$$

But we also have

$$L_z(\mathbf{i}) = \det(L_z)$$

and hence $\det(L_z) = 1$. □

Let $x = x(s)$ be a *parametrized curve*, or *path*, in $E = E_{1,3}$.

Lemma. The sign of dx/ds^2 is invariant under strictly monotonous reparametrizations $s = s(\tau)$.

Proof. Since $dx/d\tau = (dx/ds)(ds/d\tau)$, and $ds/d\tau$ is a non-zero scalar, $(dx/d\tau)^2 = (ds/d\tau)^2(dx/ds)^2$ shows that the signs of $(dx/d\tau)^2$ and $(dx/ds)^2$ are the same. □

If we regard (as we will) two paths differing in a strictly monontonous reparameterization as the same (geometric) *curve* (or *trajectory*), the Lemma says that there is a well defined *sign* associated to any curve.

A path $x = x(s)$ is said to be *timelike* (*lightlike* or *null*, *spacelike*) if $(dx/ds)^2 > 0$ ($(dx/ds)^2 = 0$, $(dx/ds)^2 < 0$).

Timelike paths

If $x = x(s)$ is a timelike curve, the quantity

$$\tau(s) = \int_0^s \left(\frac{dx}{ds}(\xi) \cdot \frac{dx}{ds}(\xi)\right)^{1/2} d\xi$$

does not depend on the parametrization of the curve and will be called *proper time* on the curve.

Since $\tau(s)$ is a strictly increasing function of s, it has an inverse, $s = s(\tau)$. Then we can consider the parametrization $x(\tau) = x(s(\tau))$ by proper time. We will denote $dx/d\tau$ by $\dot{x}$ and we will say that it is the (unit) *tangent vector* of the path.

Lemma. The unit tangent vector satisfies $\dot{x}^2 = 1$.

Proof. Let $\alpha(\xi) = \left(\frac{dx}{ds}(\xi) \cdot \frac{dx}{ds}(\xi)\right)^{1/2}$, so that $d\tau/ds = \alpha(s)$ and $ds/d\tau = 1/\alpha(s(\tau))$. Then

$$\dot{x}^2 = \left(\frac{dx}{d\tau}\right)^2 = \left(\frac{ds}{d\tau}\right)^2 \left(\frac{dx}{ds}(s(\tau))\right)^2 = \alpha(s(\tau))^{-2}\alpha(s(\tau))^2 = 1. \quad \square$$

Remark. The path $x(\tau) = \tau\gamma_0$ represents the space-time trajectory of a particle at rest at the γ_0 frame. Since $\dot{x} = \gamma_0$ and $\gamma_0^2 = 1$, τ is the proper time of that particle. More generally, the Lemma indicates that $\dot{x}$ is to be regarded as the *instantaneous rest frame* of the path, and that the proper time is the time measured along the path by the instanteneous rest frame.

Lightlike paths

For a timelike path, there is no preferred parameter, proper time is 0.

Spacelike paths

There is a preferred parameter s such that $(dx/ds)^2 = -1$. This parameter measures *proper distance*.

The bivector $\boldsymbol{x} = x \wedge \gamma_0 \in \mathcal{E}$ will be called the *relative vector* (with respect to the frame γ) of the event x. This satisfies that

$$\boldsymbol{x}\gamma_0 = x^k\gamma_k,$$

for $\boldsymbol{x}\gamma_0 = (x \wedge \gamma_0)\gamma_0 = (x\gamma_0 - (x \cdot \gamma_0))\gamma_0 = x - x^0\gamma_0 = x^k\gamma_k$.

We have

$$x\gamma_0 = x \cdot \gamma_0 + x \wedge \gamma_0 = t + \boldsymbol{x},$$

where we write $t = x^0$. So

$$\begin{aligned} x^2 &= x\gamma_0\gamma_0 x = (x \cdot \gamma_0 + x \wedge \gamma_0)(x \cdot \gamma_0 + \gamma_0 \wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2. \end{aligned}$$

Let $v = v(\tau)$ be the *proper velocity* of a particle $x = x(\tau)$, so that $v = dx/d\tau)$. Set $u = \gamma_0$. Then

$$vu = v\gamma_0 = \tfrac{d}{d\tau}(xv) = \tfrac{d}{d\tau}(t + \boldsymbol{x}),$$

and consequently

$$\tfrac{dt}{d\tau} = v \cdot u, \quad \tfrac{d\boldsymbol{x}}{d\tau} = v \wedge u.$$

Let $\boldsymbol{v}$ be the *relative velocity*, so $\boldsymbol{v} = d\boldsymbol{x}/dt$. Then we have:

$$\boldsymbol{v} = d\boldsymbol{x}/dt = (d\boldsymbol{x}/d\tau)(d\tau/dt) = \tfrac{v\wedge u}{v\cdot u}.$$

Since $Q(v \wedge u) = 1 - (v \cdot u)^2$, it follows that

$$Q(\boldsymbol{v}) = 1 - (v \cdot u)^{-2} < 1.$$

This gives $v \cdot u = 1/\sqrt{1 - Q(\boldsymbol{v})}$ (the Lorentz factor γ of $\boldsymbol{v}$).

Note also that $v = vuu = (v \cdot u + v \wedge u)u = \gamma(1 + \boldsymbol{v})u$.

The Lorentz bost

$$t' = \gamma(t - \beta x),\ x' = \gamma(x - \beta t)$$

is equivalent to the frame transformation

$$\gamma_0' = \gamma(\gamma_0 + \beta\gamma_1),\ \gamma_1' = \gamma(\gamma_1 + \beta\gamma_0).$$

Note that

$$(\gamma_0')^2 = \gamma^2(1-\beta^2) = 1,\ (\gamma_1')^2 = \gamma^2(\beta^2 - 1) = -1,\ \gamma_0' \cdot \gamma_1' = 0,$$

which show that the transformation $\gamma_\mu \to \gamma_\mu'$ is a Lorentz isometry.

Introduce the angle α so that $\tanh(\alpha) = \beta$. Then

$$\gamma = (1 - \tanh^2(\alpha))^{-/2} = \cosh(\alpha),\ \text{and}$$
$$\begin{aligned}\gamma_0' &= \cosh(\alpha)\gamma_0 + \sinh(\alpha)\gamma_1\\ &= (\cosh(\alpha) + \sinh(\alpha)\gamma_1\gamma_0)\gamma_0 = e^{\alpha\sigma_1}\gamma_0.\end{aligned}$$

Similarly, $\gamma_1' = \cosh(\alpha)\gamma_1 + \sinh(\alpha)\gamma_0 = e^{\alpha\gamma_1\gamma_0}\gamma_1 = e^{\alpha\sigma_1}\gamma_1$.

Now we can see that the Lorentz boost can be expressed as follows:

$$\gamma'_\mu = e^{\frac{1}{2}\alpha\sigma_1}\gamma_\mu e^{-\frac{1}{2}\alpha\sigma_1}.$$

Indeed, σ_1 commutes with γ_2 and γ_3, and they are fixed by the right hand side expression, in agreement with the Lorentz boost. On the other hand, σ_1 anticommutes with γ_0 and γ_1, and so for $\mu = 0, 1$ that expression is equal to $e^{\alpha\sigma_1}\gamma_\mu$, also in agreement with the Lorentz boost.

Note that this is a special case of the Lemma proved on slide 15.

We can represent a spacetime point x as the Hermitian matrix

$H(x) = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0+x^3 \end{pmatrix}$. In this representation, the Lorentz quadratic form is the determinant: $\det(h(x)) = Q(x)$.

Given $A \in \mathrm{SL}_2(\mathbf{C})$, then $AH(x)A^\dagger$ is again a hermitian matrix, say $H(L_A(x))$, and

$Q(L_A(x)) = \det(AH(x)A^\dagger) = \det(H(x)) = Q(x)$. It follows that L_A is a Lorentz isometry. Moreover, the map $\mathrm{SL}_2(\mathbf{C}) \to O_{1,3}$ is a group homomorphism. The image of this homomorphism turns out to be the connected component of the identity of $O_{1,3}$, and its kernel is $\{\pm 1\}$. From this it follows that $\mathrm{SL}_2(\mathbf{C}) \simeq \mathrm{Spin}_{1,3}$.

This construction is analogous to the identification of SU_2 as Spin_3 (cf. Appendix B).

Proposition. Define

$$\Gamma_0 = \begin{pmatrix} \sigma_0 & \\ & -\sigma_0 \end{pmatrix} \text{ and } \Gamma_k = \begin{pmatrix} & -\sigma_k \\ \sigma_k & \end{pmatrix},\ k = 1, 2, 3.$$

Then there exists an algebra isomorphism $\mathcal{G} \to \mathbf{R}(4)$ such that $\gamma_\mu \mapsto \Gamma_\mu$.

Proof. The Γ_μ satisfy the Clifford relations $\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\eta_{\mu\nu}I_4$. This follows from the Clifford relations $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}$ satisfied by the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ and straightfoward matrix computations. So there is an algebra homomorphism $\mathcal{G} \to \mathbf{R}(4)$ such that $\gamma_\mu \mapsto \Gamma_\mu$. Finally, this homomorphism is an isomorphism: the images Γ_I of the γ_I (I running over the multiindeces of $0, 1, 2, 3$) turn out to be lineraly independent. □

Definition. Given $x \in \mathbf{H}^\times$, let $\rho_x : \mathbf{H} \to \mathbf{H}$ denote the the automorphism of $\mathbf{H}$ defined by $\rho_x(y) = xyx^{-1}$.

Theorem1. The map ρ_x satisfies that $\rho_x(V) = V$ and the map $\rho_x : V \to V$ belongs to $\mathrm{SO}(V) \simeq \mathrm{SO}_3$. Furthermore, the sequence

$$1 \to \mathbf{R}^\times \to \mathbf{H}^\times \xrightarrow{\rho} \mathrm{SO}_3 \to 1$$

is exact.

Proof. To show that $v' = \rho_x(v) \in V$ when $v \in V$, it is enough to show that v'^2 is real and non-positive:

$$v'^2 = (xvx^{-1}xvx^{-1}) = xv^2x^{-1} = v^2,$$

which is real and non-positive. Now

$$Q(v') = Q(xvx^{-1}) = Q(x)Q(v)Q(x)^{-1} = Q(v)$$

says $\rho_x \in \mathrm{O}(V)$, so $\det(\rho_x) = \pm 1$. Since $\mathbf{H}^\times$ is connected and $x \mapsto \det(\rho_x)$ is continuous, $\rho_x \in \mathrm{SO}(V)$.

Since the elements $x \in \mathbf{H}^{\times}$ such that $\rho_x = \mathrm{Id}$ satisfy $xvx^{-1} = v$ for all $v \in V$, we see that $\ker(\rho)$ is the center of $\mathbf{H}^{\times}$ and so $\ker(\rho) = \mathbf{R}^{\times}$. Finally ρ is surjective because for a vector $v \in V$ we have $\rho_v = m_v$ (the reflection in the direction v with mirror $v^{\perp}$) and these reflections generate SO_3 by the Cartan-Dieudonné theorem. □

If we restrict ρ to $\mathbf{H}_1 = \{x \in \mathbf{H}^{\times} \mid Q(x) = 1\}$, then $\rho : \mathbf{H}_1 \to \mathrm{SO}_3$ is still surjective (by the same argument), but its kernel is reduced to $\mathbf{R}^{\times} \cap \mathbf{H}_1 = \{\pm 1\}$. Since in addition $\mathbf{H}_1 = \mathrm{SU}_2$, we have:

Corollary. We have an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{SU}_2 \xrightarrow{\rho} \mathrm{SO}_3 \to 1.$$

Riesz 1958

Hestenes-1966

Casanova 1976

Doran-Lasenby-2003

Garling 2011

Dressel-Bliokh-Nori-2014

Geometric Algebra Techniques in Mathematics and Physics

S. Xambó

UPC

SLP · 9-13 March · 2015

Introduction. Settings. Notations. Case $\mathbf{K} = \mathbf{R}$.

Topological facts. $\mathbf{K}(m)$. Matrix Lie groups. Connectedness and compactness. Classical groups.

Classification of Clifford algebras. The basic ingredients. A corner of the Clifford chessboard. Induction formulas. The full chessboard. Periodicity mod 8. The classification theorem. The complex case.

In this chapter the field K will be $\mathbf{R}$ or $\mathbf{C}$ and we will assume that E is endowed with a non-degenerate metric $g : E \to E^*$. In the real case, the metric g is said to be *positive*, or *positive definite*, if $g(e, e) > 0$ for all $e \in E$ and it is said to be *negative*, or *negative definite*, if $-g$ is positive. A metric which is neither positive nor negative is said to be *indefinite*.

For $K = \mathbf{C}$, (E, g) is uniquely determined, up to isometry, by $n = \dim_{\mathbf{C}}(E)$. Indeed, if $\boldsymbol{e} = e_1, \ldots, e_n \in E$ is an orthogonal basis, and we choose $r_k \in \mathbf{C}$ such that $r_k^2 = g(e_k, e_k)$ $(k = 1, \ldots, n)$, then the $\hat{e}_k = r_k^{-1} e_k$ satisfy $g(\hat{e}_k, \hat{e}_k) = 1$ and hence (E, g) is isometric to $\mathbf{C}^n$ with the standard metric $(g(z, z') = z{z'}^T = z_1 z'_1 + \cdots + z_n z'_n)$. An orthogonal basis $\boldsymbol{e}$ such that $g(e_k, e_k) = 1$ for all k is said to be *orthonormal*.

For $K = \mathbf{R}$, (E, g) is uniquely determined, up to isometry, by its *signature* (r, s), where r counts, given any orthogonal basis $\boldsymbol{e} = e_1, \ldots, e_n \in E$, the number of k such that $g(e_k, e_k) > 0$ and $s = n - r$ (so s counts the the number of k such that $g(e_k, e_k) < 0$). It is an easy exercise to see that this definition does not depend on the basis used to compute (r, s). If we choose $r_k \in \mathbf{R}$ such that $r_k^2 = g(e_k, e_k)$ or $r_k^2 = -g(e_k, e_k)$, depending on the sign of $g(e_k, e_k)$, and define $\hat{e}_k = r_k^{-1} e_k$, then $g(\hat{e}_k, \hat{e}_k) = \pm 1$, with 1 and -1 appearing r and s times, respectively. Reordering this normalized basis, we can achieve that 1 occurs for $k = 1, \ldots, r$ and that -1 occurs for $k = r + 1, \ldots, r + s = n$. Orthogonal bases satisfying this condition will be said to be *orthonormal* (some authors say *pseudo-orthonormal*).

We write $E_{r,s}$ to denote a real vector space with a metric of signature (r,s). Instead of $E_{n,0}$ we will simply write E_n, a symbol that will also be used for the complex case. Instead of $E_{0,n}$ we will write $\bar{E}_n$.

We adapt the general notations and conventions of the preceding chapters to the present context as follows:

1) $\mathrm{O}_{r,s}$: The orthogonal group of $E_{r,s}$. In terms of matrices, it is isomorphic to the subgroup of the group GL_n of invertible real matrices of orden n formed by the matrices A such that $A^T I_{r,s} A = I_{r,s}$, where $I_{r,s} = \mathrm{diag}(\mathbf{1}_r, -\mathbf{1}_s)$. Note that this relation implies that $\det(A) = \pm 1$.
2) $\mathrm{SO}_{r,s} = \mathrm{O}^+_{r,s}$: The subgroup of $\mathrm{O}_{r,s}$ of *rotations*, that is, of the isometries whose determinant is $+1$.
3) $\mathcal{G}_{r,s} = \Lambda_g(E_{r,s})$: The geometric algebra of $E_{r,s}$.
4) $\mathcal{G}^\times_{r,s}$: The multiplicative group of invertible elements of $\mathcal{G}_{r,s}$.

5) $\widetilde{\Gamma}_{r,s}$: The twisted Lipschitz group of $\mathcal{G}_{r,s}$. It is the subgroup formed by the *even* and *odd* $u \in \mathcal{G}_{r,s}^{\times}$ such that $uE_{r,s}u^{-1} = E_{r,s}$.
6) $\widetilde{\Gamma}_{r,s}^{+}$: the subgroup of even elements of $\widetilde{\Gamma}_{r,s}$.
7) $\Gamma_{r,s}$: The Lipschitz group of $\mathcal{G}_{r,s}$. It is the subgroup of $\mathcal{G}_{r,s}^{\times}$ consisting of the elements $u = u_1 \cdots u_m$ with $u_k \in E_{r,s}^{\times}$ $(k = 1, \ldots, m)$. It is a normal subgroup of $\widetilde{\Gamma}_{r,s}$.
8) $\Gamma_{r,s}^{+}$: the subgroup of even elements of $\Gamma_{r,s}$.
9) $\mathrm{Pin}_{r,s}$: The group $\mathrm{Pin}_g(E_{r,s})$, which is the subgroup of $\mathcal{G}_{r,s}^{\times}$ whose elements u have the form $u = u_1 \cdots u_m$, with $u_k \in E^{\times}$ and $g(u_k, u_k) = \pm 1$ $(k = 1, \ldots, m)$.
10) $\mathrm{Spin}_{r,s}$: The subgroup of even elements of $\mathrm{Pin}_{r,s}$. Its elements u have the form $u = u_1 \cdots u_m$, with $u_k \in E^{\times}$, $g(u_k, u_k) = \pm 1$ $(k = 1, \ldots, m)$ and m even.

Remark

In all cases, we set $X_n = X_{n,0}$, $\bar{X}_n = X_{0,n}$ ($X_n(\mathbf{C})$ in the complex case), where X stands any of the symbols define above:

$$\mathrm{O},\ \mathrm{SO} = \mathrm{O}^+,\ \mathcal{G},\ \mathcal{G}^\times,\ \widetilde{\Gamma},\ \widetilde{\Gamma}^+,\ \Gamma,\ \Gamma^+,\ \mathrm{Pin} \text{ and } \mathrm{Spin}.$$

Note X_n and $\bar{X}_n$ point to difference structures, as for example $\mathcal{G}_n$ and $\bar{\mathcal{G}}_n$. The exceptions are O and SO, for it is plain that $\mathrm{O}_n = \bar{\mathrm{O}}_n$ and $\mathrm{SO}_n = \bar{\mathrm{SO}}_n$.

Now we can proceed to specialize the main results of Lecture 3 to the present context.

Remark

Let $\mathbf{i}_{r,s}$ be the pseudoscalar of $\mathcal{G}_{r,s}$. Then $\mathbf{i}_{r,s}^2 = (-1)^{s+n//2}$, where $n = r + s$. Indeed, we know that the value is $(-1)^{n//2} Q(\mathbf{i}_{r,s})$ and it is clear that $Q(\mathbf{i}_{r,s}) = (-1)^s$.

Case $K = \mathbf{C}$. $\widetilde{\Gamma}_n(\mathbf{C}) = \Gamma_n(\mathbf{C})$ and the following sequences are exact:

$$1 \to \mathbf{C}^\times \to \Gamma_n(\mathbf{C}) \xrightarrow{\widetilde{\rho}} \mathrm{O}_n(\mathbf{C}) \to 1$$

$$1 \to \mathbf{C}^\times \to \Gamma_n^+(\mathbf{C}) \xrightarrow{\widetilde{\rho}} \mathrm{SO}_n(\mathbf{C}) \to 1$$

$$1 \to \{\pm 1, \pm i\} \to \mathrm{Pin}_n(\mathbf{C}) \xrightarrow{\widetilde{\rho}} \mathrm{O}_n(\mathbf{C}) \to 1$$

$$1 \to \{\pm 1, \pm i\} \to \mathrm{Spin}_n(\mathbf{C}) \xrightarrow{\widetilde{\rho}} \mathrm{SO}_n(\mathbf{C}) \to 1$$

Proof. The first assertion is a direct consequence of the Corollary on slide 50 of Lecture 3. Indeed, every element of $\mathbf{C}^\times$ is a square, hence $\mathbf{C}^\times = \mathbf{C}^{\times 2} \subseteq \mathbf{C}_0^\times \subseteq \mathbf{C}^\times$, and hence $\widetilde{\Gamma}_n(\mathbf{C})/\Gamma_n(\mathbf{C}) \simeq \mathbf{C}^\times/\mathbf{C}_0^\times = \{1\}$.

Now the first and second exact sequences are special cases of the sequences established in Lecture 3: Theorem 5 and second Corollary on slide 49, respectively. The third and fourth sequences are special cases of the exact sequences of Theorem 6. □

Case $K = \mathbf{R}$. For any signature (r,s), $\mathbf{R}_0^\times = \mathbf{R}^\times$ ($\mathbf{R}_0^\times = \mathbf{R}^\times \cap \Gamma_{r,s}$), $\widetilde{\Gamma}_{r,s} = \Gamma_{r,s}$ and the following sequences are exact:

$$1 \to \mathbf{R}^\times \to \Gamma_{r,s} \xrightarrow{\widetilde{\rho}} \mathrm{O}_{r,s} \to 1$$

$$1 \to \mathbf{R}^\times \to \Gamma_{r,s}^+ \xrightarrow{\widetilde{\rho}} \mathrm{SO}_{r,s} \to 1$$

$$1 \to \{\pm 1\} \to \mathrm{Pin}_{r,s} \xrightarrow{\widetilde{\rho}} \mathrm{O}_{r,s} \to 1$$

$$1 \to \{\pm 1\} \to \mathrm{Spin}_{r,s} \xrightarrow{\widetilde{\rho}} \mathrm{SO}_{r,s} \to 1$$

Proof. For the first assertion, we know that $\mathbf{R}_0^\times$ contains $\mathbf{R}^{\times 2} = \mathbf{R}_{>0}$. If we show that $-1 \in \Gamma_{r,s}$, then $\mathbf{R}_0^\times$ also contains $-\mathbf{R}^{\times 2} = \mathbf{R}_{<0}$ and so $\mathbf{R}_0^\times = \mathbf{R}^\times$, as wanted. To see that $-1 \in \Gamma_{r,s}$, pick any $u \in E_{r,s}^\times$ and normalize it so that $g(u,u) = \pm 1$. If the sign is $-$, then $-1 = g(u,u) = u^2 \in \Gamma_{r,s}$ and if the sign is $+$, then $-1 = -u^2 = u(-u) \in \Gamma_{r,s}$.

The other assertions are derived as in the complex case from the results given in Lecture 3. □

Proposition. The 2 to 1 surjection $\mathrm{Spin}_{r,s} \to \mathrm{SO}_{r,s}$ is non-trivial if $r \geqslant 2$ or $s \geqslant 2$.

Proof. It will be enough to construct a path on $\mathrm{Spin}_{r,s}$ connecting 1 and -1. To that end, let u_1, u_2 be an orthonormal pair of positive $(\epsilon = 1)$ or negative $(\epsilon = -1)$ vectors. Now define $s(t) \in \mathrm{Spin}_{r,s}$, $t \in [0, \pi/2]$, as follows:

$$
\begin{aligned}
s(t) &= (u_1 \cos(t) + u_2 \sin(t))(u_1 \cos(t) - u_2 \sin(t)) \\
&= \epsilon \cos^2(t) - \epsilon \sin^2(t) - u_1 u_2 \sin(t) \cos(t) + u_2 u_1 \sin(t) \cos(t) \\
&= \epsilon \cos(2t) - u_1 u_2 \sin(2t).
\end{aligned}
$$

Now it is clear that $s(0) = \epsilon$ and $s(\pi/2) = -\epsilon$. □

The K-algebra of square matrices of orden n with entries in K will be denoted $K(n)$. The group $K(n)^{\times}$ of invertible matrices will be denoted $\mathrm{GL}_n(K)$. We also set

$$\mathrm{SL}_n(K) = \{A \in \mathrm{GL}_n(K) \mid \det(A) = 1\}.$$

Note that these objects are defined when K is a commutative ring, as for example the ring of integers $\mathbf{Z}$. For $K = \mathbf{R}$ we simply write GL_n and SL_n.

As a complex (real) vector space, $\mathbf{C}(n)$ is isomorphic to $\mathbf{C}^{n^2}$ ($\mathbf{R}^{2n^2}$). The topology so induced in $\mathbf{C}(n)$ is equivalent to the one defined by the *hermitian metric* (this means that it is linear in B and complex conjugate linear in A) given by

$$\langle A|B\rangle = \mathrm{Tr}(A^{\dagger}B),$$

where $A^{\dagger} = \bar{A}^T$ is the *hermitian adjoint* of A (the transpose of the complex conjugate of A). Note that $\langle A|A\rangle = \sum_{j,k} |A_{jk}|^2$.

The group $\mathrm{GL}_n(\mathbf{C})$ is an open set of $\mathbf{C}(n)$ and any subgroup $G \subseteq \mathrm{GL}_n(\mathbf{C})$ that is closed (in $\mathrm{GL}_n(\mathbf{C})$, but not necessarily in $\mathbf{C}(n)$) is said to be *matrix Lie group*. It is a basic fact that a matrix group is automatically a *Lie group* (see, for example, Hall-2003 or Goodman-Wallach-2009).

Since the topology of $\mathbf{C}(n)$ is Euclidean, a matrix group is *compact* if and only if it is bounded.

Henceforth, all Lie groups that we consider will be matrix Lie groups unless we indicate otherwise explicitly.

A closed subgroup of a compact Lie group is a compact Lie group.

Since a Lie group is *locally arc-connected*, it is connected if and only if it is *arc-connected*. A group $G \subseteq \mathrm{GL}_n(\mathbf{C})$ is arc-connected when for any $A, B \in G$ there is a continuous path $X(t) \in G$, $0 \leqslant t \leqslant 1$, such that $X(0) = A$ and $X(1) = B$ (in this case we say that B is *reachable* from A on G, or that X connects A and B on G). Notice that it is sufficient to check that any $A \in G$ is reachable from I_n on G.

Connected component of the identity. Let G be a Lie group and let G^0 be the connected component of $I_n \in G$. Then G^0 is a (closed) subgroup of G.

Proof. If $A, B \in G^0$, there are continuous paths $X(t), Y(t) \in G$, $t \in [0,1]$, connecting I_n to A and B, respectively. Then $Z(t) = A(t)B(t)$ is a continuous path on G connecting $I_n = Z(0)$ to $AB = Z(1)$. This proves that G^0 is closed under multiplication. Since $X(t)^{-1}$ is a continuous path on G that connects I_n to A^{-1}, it also follows that $A^{-1} \in G^0$. □

As commonly understood, the following families of Lie groups fall under the label of *classical groups*:

1) $\mathrm{GL}_n(\mathbf{C})$, $\mathrm{SL}_n(\mathbf{C})$, GL_n, and SL_n ($n \geqslant 1$). $\mathrm{GL}_n(\mathbf{C})$ is connected and GL_n has two connected components that are distinguished by the sign of the determinant. The connected component of I_n in GL_n is $\mathrm{GL}_n^+ = \{A \in \mathrm{GL}_n \mid \det(A) > 0\}$. The groups $\mathrm{SL}_n(\mathbf{C})$ and SL_n are both connected. Since SL_n has matrices with unbounded elements, like $\begin{pmatrix} m+1 & 1 \\ m & 1 \end{pmatrix}$ in SL_2 (any m), none of these groups is compact.
2) $\mathrm{O}_n(\mathbf{C})$, O_n, $\mathrm{SO}_n(\mathbf{C})$, SO_n ($n \geqslant 1$). The group O_n has two connected components: $O_n^+ = \mathrm{SO}_n$ (the connected component of I_n) and $O_n^- = J_n\mathrm{SO}_n$, where $J_n = \mathrm{diag}(-1, \mathbf{1}_{n-1})$. Similar statements are valid for the complex case. The group O_n, and hence also SO_n, are compact, whereas $\mathrm{SO}_n(\mathbf{C})$, and hence also $\mathrm{O}_n(\mathbf{C})$, are non-compact.

3) $\mathrm{O}_{r,s}$, $\mathrm{SO}_{r,s}$. Since $\mathrm{O}_{s,r} = \mathrm{O}_{r,s}$ and $\mathrm{SO}_{s,r} = \mathrm{SO}_{r,s}$, we can assume $0 < r \leqslant s$ (the case $r = 0$ is included in the previous list). If we set $J = \mathrm{diag}(-1, \mathbf{1}_{n-1})$ ($n = r + s$), then $J_n \in \mathrm{O}_{r,s}$ and $\det(J_n) = -1$. This implies that $\mathrm{O}_{r,s} = \mathrm{SO}_{r,s} \sqcup J_n\mathrm{SO}_{r,s}$ (where $\sqcup$ denotes disjoint union) and hence we are reduced to study $\mathrm{SO}_{r,s}$. This group has two connected components, $\mathrm{SO}^{\pm}_{r,s}$, where $\mathrm{SO}^{+}_{r,s}$ is the subgroup of the $f \in \mathrm{SO}_{r,s}$ such that $\det(f^{+}) = 1$, where $f^{+} : E^{+}_{r,s} \to E^{+}_{r,s}$ is the composition of f with the orthogonal projection of $E_{r,s}$ to $E^{+}_{r,s}$.

4) U_n and SU_n. The *unitary group* U_n is formed by the *unitary* matrices $A \in \mathbf{C}(n)$ ($AA^{\dagger} = I_n$). The *special unitary group* SU_n is the subgroup of U_n of matrices A such that $\det(A) = 1$.

5) Sp_n, $\mathrm{Sp}_n(\mathbf{C})$ and USp_n. The *symplectic group* Sp_n is the subgroup of *symplectic* matrices $A \in \mathrm{GL}_{2n}$, i.e. $A^T\Omega A = \Omega$, where $\Omega = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$. $\mathrm{USp}_n = \mathrm{Sp}_n(\mathbf{C}) \cap \mathrm{U}_{2n}$.

Notations. $\mathbf{K}$ will denote one of the fields $\mathbf{R}$ (real field), $\mathbf{C}$ (complex field) and $\mathbf{H}$ (quaternion field). For any integer $n \geq 2$, $\mathbf{K}(n)$ will denote the ring of $n \times n$ matrices with coefficients in $\mathbf{K}$. Since $\mathbf{K}(n) = \mathbf{K} \otimes \mathbf{R}(n)$, its real dimension is $d_{\mathbf{K}} n^2$, where $d_{\mathbf{K}} = \dim_{\mathbf{R}} \mathbf{K} = 1, 2, 4$, respectively. *Note*: $\mathbf{K}(m) \otimes \mathbf{R}(n) \simeq \mathbf{K}(mn)$.

Lemma

(1) $\mathbf{C} \otimes \mathbf{C} \simeq \mathbf{C} \oplus \mathbf{C}$

(2) $\mathbf{C} \otimes \mathbf{H} \simeq \mathbf{C}(2)$

(3) $\mathbf{H} \otimes \mathbf{H} \simeq \mathbf{R}(4)$

Proof. (1) Since $(i \otimes i)^2 = 1 \otimes 1$, the elements $e_\pm = \frac{1}{2}(1 \otimes 1 \pm i \otimes i)$ are idempotents with $e_+ + e_- = 1 \otimes 1$ and $e_+ e_- = e_- e_+ = 0 \otimes 0$. Then the map $\mathbf{C} \oplus \mathbf{C} \to \mathbf{C} \otimes \mathbf{C}$, $(x, y) \mapsto x e_+ + y e_-$, satisfies $(x e_+ + y e_-)(x' e_+ + y' e_-) = x x' e_+ + y y' e_-$ and with this it is easy to prove that it is an isomorphism.

(2) If z is a complex number and q a quaternion, let $f_{z,q} : \mathbf{H} \to \mathbf{H}$ be defined by $f_{z,q}(h) = zh\bar{q}$. Then $f_{z,q}$ is $\mathbf{C}$-linear, so that we have a map $\mathbf{C} \times \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H})$, $(x, q) \mapsto f_{z,q}$. The map is clearly bilinear and hence induces a linear map $\mathbf{C} \otimes \mathbf{H} \to \mathrm{End}_{\mathbf{C}}(\mathbf{H})$. This map is an algebra homomorphism, for

$$z_2 z_1 h \bar{q}_1 \bar{q}_2 = (z_1 z_2) h \, \overline{q_1 q_2}.$$

It can be checqued that this map sends the basis $\{1, i\} \otimes \{1, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$ into linearly independent endomorphisms, and hence the map is an isomorphism, for both sides have dimension 8. Finally note that $\mathrm{End}_{\mathbf{C}}(\mathbf{H}) \simeq \mathrm{End}_{\mathbf{C}}(\mathbf{C}^2) \simeq \mathbf{C}(2)$.

(3) If $q_1, q_2 \in \mathbf{H}$, define $f_{q_1,q_2} : \mathbf{H} \to \mathbf{H}$ by $f_{q_1,q_2}(h) = q_1 h \bar{q}_2$. In this way we we get, as in 2), an algebra homomorphism $\mathbf{H} \otimes \mathbf{H} \to \mathrm{End}(\mathbf{H})$ which can be shown to be an isomorphism (both sides have dimension 16). Finally $\mathrm{End}(\mathbf{H}) \simeq \mathrm{End}(\mathbf{R}^4) \simeq \mathbf{R}(4)$.

We are aiming at giving isomorphic descriptions of $C_{r,s}$ and $C^+_{r,s}$ in terms of basic algebra forms. It will turn out that it is enough to achieve this for $0 \leq r, s \leq 7$. So we will first look at how to fill in the slots in this 8×8 *chessboard*.

The main tools will be the explicit description of $C_{r,s}$ for slots close to the corner $(0, 0)$, which contains $C_{0,0} = \mathbf{R}$, and three *inductive formulas*.

Let us begin with the slots near $(0, 0)$, For row 0, $C_{0,s} = \bar{C}_s$, and we know that $\bar{C}_1 \simeq \mathbf{C}$ and $\bar{C}_2 \simeq \mathbf{H}$. Then $C_{1,0} = C_1 \simeq \mathbf{R} \oplus \mathbf{R}$, $C_{1,1} \simeq \mathbf{R}(2)$ and $C_{2,0} = C_2 \simeq \mathbf{R}(2)$. In sum,

$r \backslash s$	0	1	2
0	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$
1	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{R}(2)$	
2	$\mathbf{R}(2)$		

Proposition

(1) $C_{r+2} \simeq \bar{C}_r \otimes C_2 \simeq \bar{C}_r \otimes \mathbf{R}(2)$.

(2) $\bar{C}_{r+2} \simeq C_r \otimes \bar{C}_2 \simeq C_r \otimes \mathbf{H}$

(3) $C_{r+1,s+1} \simeq C_{r,s} \otimes \mathbf{R}(2)$.

Proof. (1) Let $\bar{\gamma}_1, \ldots, \bar{\gamma}_r$ be standard generators of $\bar{C}_r$, so $\bar{\gamma}_k^2 = -1$, and γ_1, γ_2 standard generators of C_2, so $\gamma_1^2 = \gamma_2^2 = 1$. Let $\boldsymbol{i}_2 = \gamma_1\gamma_2$, so that $\boldsymbol{i}_2^2 = -1$.

Consider the elements $\Gamma_k \in \bar{C}_r \otimes C_2$ defined by $\Gamma_k = \bar{\gamma}_k \otimes \boldsymbol{i}_2$ $(k = 1, \ldots, r)$, and $\Gamma_{r+\ell} = 1 \otimes \gamma_\ell$ $(\ell = 1, 2)$.

The Γ_j $(j = 1, \ldots, r+2)$ are linearly independent and satisfy the relations of a standard basis of C_{r+2}.

So we have an injective homomorphism $C_{r+2} \to \bar{C}_r \otimes C_2$, which must be an isomorphism because both algebras have dimension 2^{r+2}.

(2) Let $\gamma_1, \ldots, \gamma_r$ be standard generators of C_r, so $\gamma_k^2 = 1$, and $\bar{\gamma}_1, \bar{\gamma}_2$ standard generators of $\bar{C}_2$, so $\bar{\gamma}_1^2 = \bar{\gamma}_2^2 = -1$. Let $\boldsymbol{i}_2 = \bar{\gamma}_1\bar{\gamma}_2$, so that $\boldsymbol{i}_2^2 = -1$.

Consider the elements $\bar{\Gamma}_k \in C_r \otimes \bar{C}_2$ defined by $\bar{\Gamma}_k = \gamma_k \otimes \boldsymbol{i}_2$ $(k = 1, \ldots, r)$, and $\bar{\Gamma}_{r+\ell} = 1 \otimes \bar{\gamma}_\ell$ $(\ell = 1, 2)$.

The $\bar{\Gamma}_j$ $(j = 1, \ldots, r+2)$ are linearly independent and satisfy the relations of a standard basis of $\bar{C}_{r+2}$.

So we have an injective homomorphism, $\bar{C}_{r+2} \to C_r \otimes \bar{C}_2$, which must be an isomorphism because both algebras have dimension 2^{r+2}.

(3) Let $\gamma_1, \ldots, \gamma_r, \bar{\gamma}_1, \ldots, \bar{\gamma}_s$ be standard generators of $C_{r,s}$: $\gamma_j^2 = 1$ ($j = 1, \ldots, r$) and $\bar{\gamma}_k^2 = -1$ ($k = 1, \ldots, s$). Let $\gamma, \bar{\gamma}$ be standard generators of $C_{1,1}$ ($\gamma^2 = 1$, $\bar{\gamma}^2 = -1$) and let $\boldsymbol{i}_2 = \gamma\bar{\gamma}$, so that $\boldsymbol{i}_2^2 = 1$.

Consider the elements Γ_j and $\bar{\Gamma}_k$ of $C_{r,s} \otimes C_{1,1}$, $j = 1, \ldots, r+1$, $k = 1, \ldots, s+1$, defined as $\Gamma_j = \gamma_j \otimes \boldsymbol{i}_2$ ($j = 1, \ldots, r$), $\Gamma_{r+1} = 1 \otimes \gamma$, $\bar{\Gamma}_k = \bar{\gamma}_k \otimes \boldsymbol{i}_2$ ($k = 1, \ldots, r$) and $\bar{\Gamma}_{s+1} = 1 \otimes \bar{\gamma}$.

The $\Gamma_1, \ldots, \Gamma_{r+1}$ are linearly independent and satisfy the relations of a standard basis of $C_{r+1,s+1}$.

Now argue as in the previous cases. □

Remark. The C_r and $\bar{C}_r$, $r = 0, \ldots, 7$, fill the chessboard 0-th column and 0-th row, respectively, and the Proposition, (1) and (2), says that if for either one we know the values up to r, then we can know the values of the other up to $r+2$. Since we know the values up to $r = 2$ for both of them, the determination of the other values can be carried out, for example, as follows:

$C_3 \simeq \bar{C}_1 \otimes \mathbf{R}(2) \simeq \mathbf{C} \otimes \mathbf{R}(2) \simeq \mathbf{C}(2)$; $C_4 \simeq \bar{C}_2 \otimes \mathbf{R}(2) \simeq \mathbf{H}(2)$;

$\bar{C}_3 \simeq C_1 \otimes \mathbf{H} \simeq \mathbf{H} \oplus \mathbf{H}$; $\bar{C}_4 \simeq C_2 \otimes \mathbf{H} \simeq \mathbf{H}(2)$;

$\bar{C}_5 \simeq C_3 \otimes \mathbf{H} \simeq \mathbf{C}(2) \otimes \mathbf{H} \simeq \mathbf{C}(4)$ (use the Lemma);

$\bar{C}_6 \simeq C_4 \otimes \mathbf{H} \simeq \mathbf{H}(2) \otimes \mathbf{H} \simeq \mathbf{R}(8)$ (use the lemma again);

$C_5 \simeq \bar{C}_3 \otimes \mathbf{R}(2) \simeq \mathbf{H}(2) \oplus \mathbf{H}(2)$; $C_6 \simeq \bar{C}_4 \otimes \mathbf{R}(2) \simeq \mathbf{H}(4)$;

$C_7 \simeq \bar{C}_5 \otimes \mathbf{R}(2) \simeq \mathbf{C}(8)$; $\bar{C}_7 \simeq C_5 \otimes \mathbf{H} \simeq \mathbf{R}(8) \oplus \mathbf{R}(8)$.

Now use the Proposition (3) to fill in the rest:

$r\backslash s$	0	1	2	3
0	$\mathbf{R}$	$\mathbf{C}$	$\mathbf{H}$	$\mathbf{H}\oplus\mathbf{H}$
1	$\mathbf{R}\oplus\mathbf{R}$	$\mathbf{R}(2)$	$\mathbf{C}(2)$	$\mathbf{H}(2)$
2	$\mathbf{R}(2)$	$\mathbf{R}(2)\oplus\mathbf{R}(2)$	$\mathbf{R}(4)$	$\mathbf{C}(4)$
3	$\mathbf{C}(2)$	$\mathbf{R}(4)$	$\mathbf{R}(4)\oplus\mathbf{R}(4)$	$\mathbf{R}(8)$
4	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8)\oplus\mathbf{R}(8)$
5	$\mathbf{H}(2)\oplus\mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$	$\mathbf{R}(16)$
6	$\mathbf{H}(4)$	$\mathbf{H}(4)\oplus\mathbf{H}(4)$	$\mathbf{H}(8)$	$\mathbf{C}(16)$
7	$\mathbf{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8)\oplus\mathbf{H}(8)$	$\mathbf{H}(16)$

$r\backslash s$	4	5	6	7
0	$\mathbf{H}(2)$	$\mathbf{C}(4)$	$\mathbf{R}(8)$	$\mathbf{R}(8)\oplus\mathbf{R}(8)$
1	$\mathbf{H}(2)\oplus\mathbf{H}(2)$	$\mathbf{H}(4)$	$\mathbf{C}(8)$	$\mathbf{R}(16)$
2	$\mathbf{H}(4)$	$\mathbf{H}(4)\oplus\mathbf{H}(4)$	$\mathbf{H}(8)$	$\mathbf{C}(16)$
3	$\mathbf{C}(8)$	$\mathbf{H}(8)$	$\mathbf{H}(8)\oplus\mathbf{H}(8)$	$\mathbf{H}(16)$
4	$\mathbf{R}(16)$	$\mathbf{C}(16)$	$\mathbf{H}(16)$	$\mathbf{H}(16)\oplus\mathbf{H}(16)$
5	$\mathbf{R}(16)\oplus\mathbf{R}(16)$	$\mathbf{R}(32)$	$\mathbf{C}(32)$	$\mathbf{H}(32)$
6	$\mathbf{R}(32)$	$\mathbf{R}(32)\oplus\mathbf{R}(32)$	$\mathbf{R}(64)$	$\mathbf{C}(64)$
7	$\mathbf{C}(32)$	$\mathbf{R}(64)$	$\mathbf{R}(64)\oplus\mathbf{R}(64)$	$\mathbf{R}(128)$

Corollary

(1) $C_{n+8} \simeq C_n \otimes \mathbf{R}(16)$

(2) $\bar{C}_{n+8} \simeq \bar{C}_n \otimes \mathbf{R}(16)$

(3) $C_{r+4,s+4} \simeq C_{r,s} \otimes \mathbf{R}(16)$

Proof. The Proposition, (1) and (2), allows us to write:

$$\begin{aligned} C_{n+8} &\simeq \bar{C}_{n+6} \otimes C_2 \simeq C_{n+4} \otimes \bar{C}_2 \otimes C_2 \\ &\simeq \bar{C}_{n+2} \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \\ &\simeq C_n \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \end{aligned}$$

Now we have, using the chessboard and part (3) of the Lemma,

$$\begin{aligned} \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \otimes C_2 &\simeq \mathbf{H} \otimes \mathbf{R}(2) \otimes \mathbf{H} \otimes \mathbf{R}(2) \\ &\simeq \mathbf{H} \otimes \mathbf{H} \otimes \mathbf{R}(4) \\ &\simeq \mathbf{R}(4) \otimes \mathbf{R}(4) \simeq \mathbf{R}(16). \end{aligned}$$

With this we conclude the proof of (1).

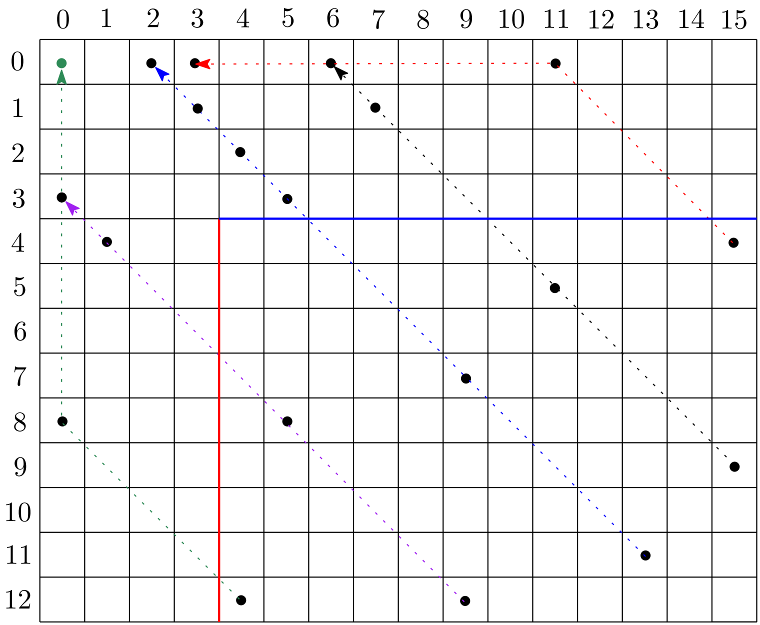
The proof of (2) follows the same pattern as the proof of (1):

$$\begin{aligned}\bar{C}_{n+8} &\simeq C_{n+6} \otimes \bar{C}_2 \simeq \bar{C}_{n+4} \otimes C_2 \otimes \bar{C}_2 \\ &\simeq C_{n+2} \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \\ &\simeq \bar{C}_n \otimes C_2 \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2\end{aligned}$$

and clearly $C_2 \otimes \bar{C}_2 \otimes C_2 \otimes \bar{C}_2 \simeq \mathbf{R}(16)$.

The proof of (3) is simpler: it suffices to apply rule (3) of the Proposition four successive times to conlude that

$$C_{r+4,s+4} \simeq C_{r,s} \otimes \mathbf{R}(2)^{\otimes 4} \simeq C_{r,s} \otimes \mathbf{R}(16).$$ □


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
0
1
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11
12

Remark (Reduction to the chessboard). Given r, s, let $m = \min(r, s)$ and k the greatest non-negative integer such that $4k \leq m$. Let $r' = r - 4k$, $s' = s - 4k$ and $m' = m - 4k = \min(r', s')$. Then part (3) of the Corollary tells us that $C_{r,s} \simeq C_{r',s'} \otimes \mathbf{R}(16^k)$ and by part (3) of the Proposition $C_{r',s'} \simeq C_{r'',s''} \otimes \mathbf{R}(2^{m'})$, with $r'' = r' - m'$, $s'' = s' - m'$, or $C_{r.s} \simeq C_{r'',s''} \otimes \mathbf{R}(2^{m'}16^k)$. Since either $s'' = 0$ (when $s \leqslant r$) or $r'' = 0$ (when $r \leqslant s$), we see that $C_{r,s} \simeq C_{r''}$ (when $s \leqslant r$) or $C_{r,s} \simeq \bar{C}_{s''}$ (when $r \leqslant s$).

Algorithm. While $r, s \geqslant 4$, we jump to $r-4, s-4$ and update the matrix factor by $\mathbf{R}(16)$. At some point we will cross either the red line (case $r \geqslant s$) or the blue line (the case $r \leqslant s$). At this moment, and while $r, s \geqslant 1$, we jump to $r-1, s-1$ and update the matrix factor by $\mathbf{R}(2)$. After at most three steps, we are going to hit the 'red boundary' (the C_n) or the 'blue boundary' (the $\bar{C}_n$). Now, while $n \geqslant 8$, we jump along the boundary to $n-8$ and update the matrix factor by $\mathbf{R}(16)$. See illustration on next slide.

Theorem . Let $n = r + s = \dim(E_{r,s})$ and set $d_k = 2^{\frac{n-k}{2}}$ $(k = 0, \ldots, 4)$. Let $\nu = r - s \mod 8$. Then the isomorphism classes of $C_{r,s}$ and $C^+_{r,s}$ are determined according to the following tables:

ν	$C_{r,s}$
0, 2	$\mathbf{R}(d_0)$
1	$\mathbf{R}(d_1) \oplus \mathbf{R}(d_1)$
3, 7	$\mathbf{C}(d_1)$
4, 6	$\mathbf{H}(d_2)$
5	$\mathbf{H}(d_3) \oplus \mathbf{H}(d_3)$

ν	$C^+_{r,s}$
1, 7	$\mathbf{R}(d_1)$
0	$\mathbf{R}(d_2) \oplus \mathbf{R}(d_2)$
2, 6	$\mathbf{C}(d_2)$
3, 5	$H(d_3)$
4	$H(d_4) \oplus H(d_4)$

Proof. The integer $r - s \mod 8$ is clearly invariant in the reduction process. It follows that $C_{r,s} \simeq C_\nu \otimes \mathbf{R}(d)$ if $r \geqslant s$ and $C_{r,s} \simeq \bar{C}_{8-\nu} \otimes \mathbf{R}(d')$ if $r < s$, where d and d' are positive integers. Now in the 15 algebras C_ν $(\nu = 0, \ldots, 7)$ and $\bar{C}_{8-\nu}$ $(\nu = 1, \ldots, 7)$ there appear exactly 5 forms (up to tensoring by $\mathbf{R}(2^m)$, for some m):

Form	R	R+R	C	H	H+H
ν	0, 2	1	3, 7	4, 6	5

So the classification in terms of ν has indeed the form of the first table in the statement. That the d's are as claimed follows by counting dimensions. The dimension of $C_{r,s}$ is 2^n, and the dimensions of the five forms are

Form	$\mathbf{R}(m)$	$\mathbf{R}(m) \oplus \mathbf{R}(m)$	$\mathbf{C}(m)$	$\mathbf{H}(m)$	$\mathbf{H}(m) \oplus \mathbf{H}(m)$
$d(m)$	m^2	$2m^2$	$2m^2$	$4m^2$	$8m^2$

Solving for m in the equation $2^n = d(m)$ we get the claimed expressions. For example, if $2^n = 8m^2$, then $m^2 = 2^{n-3}$ and hence $m = 2^{(n-3)/2} = d_3$.

To prove the second part, we first establish the following

Lemma. For any signature (r, s), we have

$$C_{r,s} \simeq C^+_{r,s+1} \simeq C^+_{s+1,r}.$$

Proof. Take a standard basis of $C_{r,s+1}$ of the form γ_j ($j = 1, \ldots, r$), $\bar{\gamma}_k$ ($k = 1, \ldots, s+1$) and write $\bar{\gamma} = \bar{\gamma}_{s+1}$. Now consider the elements $\Gamma_j = \bar{\gamma}\gamma_j$ ($j = 1, \ldots, r$) and $\bar{\Gamma}_k = \bar{\gamma}\bar{\gamma}_k$ ($k = 1, \ldots, s$). These elements belong to $C^+_{r,s+1}$, are linearly independent, anticommute and satisfy the standard relations for the signature (r, s): $\Gamma_j^2 = 1$ ($j = 1, \ldots, r$) and $\bar{\Gamma}_k^2 = -1$ ($k = 1, \ldots, s$). This implies the isomorphism
$C_{r,s} \simeq C^+_{r,s+1}$.

For the other isomorphism, take a standard basis of $C_{s+1,r}$ of the form γ_k ($k = 1, \ldots, s+1$), $\bar{\gamma}_j$ ($j = 1, \ldots, r$) and write $\gamma = \gamma_{r+1}$. Now consider the elements $\Gamma_j = \gamma\bar{\gamma}_j$ ($j = 1, \ldots, r$) and $\bar{\Gamma}_k = \gamma\gamma_k$ ($k = 1, \ldots, s$). These elements belong to $C^+_{s+1,r}$, are linearly independent, anticommute and satisfy the standard relations for the signature (r, s): $\Gamma_j^2 = 1$ ($j = 1, \ldots, r$) and $\bar{\Gamma}_k^2 = -1$ ($k = 1, \ldots, s$). This implies the isomorphism $C_{r,s} \simeq C^+_{s+1,r}$. □

Corollary. If $s > 0$, then $C^+_{r,s} \simeq C_{r,s-1}$, and $C^+_n \simeq \bar{C}_{n-1}$. □

Now $\nu(r, s-1) = \nu(r,s) + 1 = \nu + 1$, and so the class of $C^+_{r,s}$ has the same form as the class of $C_{r,s}$ corresponding to $\nu + 1$. And this covers all the cases, because the type of $\bar{C}_{n-1}$ coincides with the type of $C_{9-\nu}$. The orders of the $\mathbf{R}(m)$ involved are determined with the same procedure as before, that is, solving $2^{n-1} = d(m)$ for m. For example, for $\nu = 0$ we have to solve $2^{n-1} = 2m^2$, or $m^2 = 2^{n-2}$, which gives $m = d_2$. □□

$n \mod 2$	C_n	C_n^+
0	$\mathbf{C}(d_0)$	$\mathbf{C}(d_2) \oplus \mathbf{C}(d_2)$
1	$\mathbf{C}(d_1) \oplus \mathbf{C}(d_1)$	$\mathbf{C}(d_3)$

Assume, as we may, that $0 < r < r + s$.

Let $e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s}$ be orthonormal basis.

Timelike and spacelike vectors e: $e^2 > 0$, $e^2 < 0$.

$E^+ = \langle e_1, \ldots, e_r \rangle$, $E^- = \langle e_{r+1}, \ldots, e_{r+s} \rangle$.

If $f \in \mathrm{O}_{r,s}$, its matrix has the form $f \sim \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, according to decompostion $E = E^+ \perp E^-$.

Lemma. $\det(A) \neq 0$.

Proof. Indeed, A is the matrix of the composition $\pi^+ \circ f : E^+ \to E^+$, where π^+ is the orthogonal projection $E \to E^+$. The kernel of this map is formed by the vectors $e \in E^+$ such that $f(e) \in E^-$, which implies that $f(e) \cdot f(e) \leq 0$. But $f(e) \cdot f(e) = e \cdot e \geqslant 0$, which implies that $e = 0$. □

Let $\mathrm{SO}^{\pm}_{r,s}$ be defined according to the sign of $\det(A)$.

Lemma

$\mathrm{SO}^{\pm}_{r,s}$ are the connected components of $\mathrm{SO}_{r,s}$

Proof

We will assume $r \geqslant 2$ (the case $r = 1$ requires a little extra work).

Any element of $\mathrm{SO}_{r,s}$ can be written as $m_{u_1} \cdots m_{u_{2m}}$, where each u_j is either timelike of spacelike.

Claim: $m_u m_{u'} = m_{m_u(u')} m_u$, or $m_u m_{u'} m_u = m_{m_u(u')}$. Indeed, both sides of the second relation map $m_u(u')$ to $-m_u(u')$, and both sides leave invariant any vector v orthogonal to $m_u(u')$ (note that $v = m_u(v')$, with v' orthogonal to u').

(*) As a result, we assume that in the product $m_{u_1} \cdots m_{u_{2m}}$ the $u_1, \ldots, u_k$ are timelike and $u_{k+1}, \ldots, u_{2m}$ spacelike.

Let $x \in E$ such that $x \cdot x = 1$, and write $x = x^+ + x^-$, $x^+ \in E^+$ and $x^- \in E^-$. Then $x^+ \cdot x^+ \geqslant 1$, so $x^+ = \alpha u^+$ with $u^+ \cdot u^+ = 1$. If $x^- \neq 0$, we can write $x^- = \beta u^-$, $u^- \cdot u^- = -1$. And if $x^- = 0$, set $u^- = e_{r+1}$ and $\beta = 0$. So we have $x = \alpha u^+ + \beta u^-$ and $1 = x \cdot x = \alpha^2 - \beta^2$. This implies that there exists $t \in \mathbf{R}$ such that $x = \cosh(t) u^+ + \sinh(t) u^-$. Letting $t \to 0$, we see that x can be continuously deformed to e_1.

This shows that the timelike vectors form a connected domain. A similar argument shows that a spacelike vectors can be connected to e_{r+1} and so the spacelike vectors also form a connected domain. Since the map $u \mapsto m_u$ is continuous, (*) implies that any $f \in \mathrm{SO}_{r,s}$ can be deformed to $m_{e_1}^{\epsilon} m_{r+1}^{\epsilon}$, with $\epsilon = 0, 1$. If $\epsilon = 0$, $f \in \mathrm{SO}_{r,s}^+$, otherwise $f \in \mathrm{SO}_{r,s}^-$. □

Postnikov 1982

Gallier 2001

Hall 2003

Varadarajan 2004

Figueroa O'Farrill 2010

Garling 2011

Geometric Algebra Techniques in Mathematics and Physics

S. Xambó

UPC

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Let $n = r + s$ and $\nu = r - s \mod 8$.

We define $d_k = 2^{(n-k)/2}$ (it will be used for $k = 0, 1, \ldots, 4$ and in cases that will guarantee that $(n - k)/2$ is an integer).

Let $\mathbf{i} = \mathbf{i}_{r,s}$ be the pseudoscalar (volume element) of $C_{r,s}$.

Lemma

(1) $\mathbf{i}^2 = (-1)^{s+n//2} = (-1)^{(r-s+1)//2}$. Thus

$$\mathbf{i}^2 = 1 \text{ if } \nu \equiv 0, 3 \mod 4$$

$$\mathbf{i}^2 = -1 \text{ if } \nu \equiv 1, 2 \mod 4$$

(2) For any vector e, $e\mathbf{i} = (-1)^{n-1}\mathbf{i}e$. Therefore, $\mathbf{i}$ is central if n is odd and anticommutes with vectors if n is even (so it anticommutes with odd multivectors and commutes with even multivectors). Since $n \equiv \nu \mod 2$, we can use ν instead of n.

Let **K** be one of the fields **R**, **C**, **H**.

A **K**-representation of a *real* algebra A is an **R**-linear homomorphism $\rho : A \to \mathrm{End}_{\mathbf{K}}(E)$ for some **K**-vector space E.

Equivalent **K**-representations are defined as usual: isomorphic under a **K**-linear isomorphism. Note that ρ defines an A-module structure on E.

A representation ρ is irreducible if the only there are no non-trivial submodules.

Similar definitions can be phrased for groups instead of algebras.

Facts

(1) Every irreducible $\mathbf{R}$-representation of the real algebra $\mathbf{R}(n)$ is isomorphic to $\mathbf{R}^n$

(2) Every irreducible $\mathbf{H}$-representation of the real algebra $\mathbf{H}(n)$ is isomorphic to $\mathbf{H}^n$ (as a right $\mathbf{H}$-vector space).

(3) Every irreducible $\mathbf{C}$-representation of the real algebra $\mathbf{C}(n)$ is isomorphic either to $\mathbf{C}^n$ or to $\bar{\mathbf{C}}^n$.

A *pinor representation* of $\mathrm{Pin}_{r,s}$ is the restriction to $\mathrm{Pin}_{r,s}$ of an *irreducible* representation of $C_{r,s}$.

Theorem. The type of the pinor representations depends only on ν.

ν **even**. Unique pinor respresentation $P_{s,t}$.

$\nu = 0, 6$: real of dimension d_0 (*Majorana*)

$\nu = 2, 4$: quaternionic of dimension d_2 (*symplectic Majorana*).

ν **odd**. Two pinor representations.

$\nu = 3, 7$, so $\mathbf{i}^2 = 1$. There are two pinor representations $P^{\pm}_{r,s}$, distinguished by the action ($+1$ or -1) of $\mathbf{i}$.

$\nu = 7$: real of dimension d_1 (*Majorana*).

$\nu = 3$: quaternionic of dimension d_3 (*symplectic Majorana*).

$\nu = 1, 5$, so $\mathbf{i}^2 = -1$: complex $P_{r,s}$ and $\bar{P}_{r,s}$ of complex dimension d_1, distinguished by the action ($+i$ or $-i$) of $\mathbf{i}$ (*Dirac*).

A *spinor representation* of $\mathrm{Spin}_{r,s}$ is the restriction to $\mathrm{Spin}_{r,s}$ of an *irreducible* representation of $C^+_{r,s}$.

Theorem. The type of the spinor representations depends only on ν.

ν odd. There is a unique spinor representation $S_{r,s}$.

$\nu = 1,7$: real of dimension d_1 (*Majorana*).

$\nu = 3,5$: quaternionic of dimension d_3 (*symplectic Majorana*).

ν even. Two representations (*Weyl spinors*).

$\nu = 2,6$ ($\mathbf{i}^2 = -1$): S and $\bar{S}$ of complex dimension d_2, distinguished by the action of $\mathbf{i}$: i and $-i$.

$\nu = 0,4$ ($\mathbf{i}^2 = 1$): $S^{\pm}$, distinguished by the action of $\mathbf{i}$: $+1$ and -1.

- $\nu = 0$: real, dimension d_2 (*Majorana-Weyl*).
- $\nu = 4$: quaternionic, dimension d_4 (*symplectic Majorana-Weyl*).