ON HALPHEN'S FIRST FORMULA

C. PROCESI† AND S. XAMBÓ DESCAMPS††

Abstract. We generalize Halphen's first formula (for the number of plane conics in a one dimensional system that satisfy a simple condition) to quadratic varieties in projective n-space. In fact the arguments are valid on any algebraic homogeneous space that does not have compactifications with infinitely many orbits (spherical varieties).

1. Introduction

Halphen (1844-1889) devoted several papers to explain his ideas and results about enumerative geometry. He obtained particularly nice results for plane conics. The reader is referred to pages 1-12 of the short survey Halphen [1985], written on the occasion of his candidacy to the French Academy of Sciences, for an overview of his work; in it he underlined very neatly the key concepts of his enumerative theory and the main results he had discovered, taking pains to stress, with compelling reasons, the radical advance brought in by his theory of characteristics, as compared to previous works by several authors (included himself). His progress was not only not readily understood, but kindled a long and bitter and unfortunate polemic involving several authors; the reader is referred to Kleiman [1980] for a very detailed and masterful historical account, especially pages 131-134.

Roughly speaking, the problems considered by Halphen were to find the number of (smooth) plane conics in a 1-dimensional system satisfying a simple condition, and also the number of plane conics satisfying five independent simple conditions.

The analysis of these problems, which shows a deep understanding of the issues involved, led Halphen to his first and second formulas (Halphen [1878], § III, and [1879], Th. I). His ideas were analyzed, using contemporary language, in Casas–Xambó [1986]. In this work Halphen's theory of characteristics was also extended to deal with the problem of finding the number of (smooth) plane conics in a 2-dimensional system that satisfy a double condition. Independently, De

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Concini and Procesi [1983, 1985] developed a group-theoretic framework that seems to be well suited for analyzing the kinds of problems that Halphen's theory points at (see Section 2).

The goal of this note is to outline a generalization of Halphen's first formula to quadratic varieties in $\mathbb{P}^n$ (the latter stands for projective space of dimension n over an algebraically closed field $\boldsymbol{k}$). As the attentive reader will notice, our arguments are valid for any spherical variety, in particular for symmetric varieties, but here, in order to phrase the relevant ideas in the simplest terms, we will focus on the concrete example of quadratic varieties; in this way we also remain closer to the geometric spirit of Halphen [1878],

Halphen (and Casas–Xambó) worked in the framework of projective geometry and used, as one of the main tools, the analysis of the singularities of the system and the condition along the variety of Halphen conics, that is, the variety whose closed points are double lines with a double focus (= a double dual line). Here the formula will be derived using the technique of symmetric varieties, as developed by De Concini and Procesi (*loc. cit.*)

We end this introduction explainig some notations and conventions.

Notations. Given an abelian group N, we will often consider the $\mathbb{Q}$-vector space $N \otimes \mathbb{Q}$. This space sometimes will be denoted $N_{\mathbb{Q}}$.

Cycles. Given a smooth variety X, we will set Z_dX (Z^dX) to denote the group of cycles of dimension (codimension) d on X. The quotient of Z_dX by the the subgroup of cycles rationally equivalent to 0 will be denoted A_dX (Chow group of dimension d). The rational class of a cycle z (the image of z in A_dX under the canonical projection) will be denoted $[z]$. A^cX is, by definition, $A_{n-c}X$, where $n = \mathsf{dim}(X)$.

Compactifications. Given a variety U, by a *compactification* of U we understand a *complete* variety X which contains U as an open set. A *partial compactification* of U is a variety X' that can be obtained from a compactification X of U by removing a finite number of subvarieties of codimension 2 or bigger.

Polyhedral cones and $\mathbf{D}$*-functions* (See Oda [1988]). Let N be a finitely generated free abelian group of rank n and set $V = \mathbb{Q} \otimes N$, so that V is a $\mathbb{Q}$-vector space of dimension n. A subset C of V is said to be a *(rational) polyhedral cone* if there exist $v_1, \ldots, v_r \in C$ (respectively $\in C \cap N$) such that $C = [v_1, \ldots, v_r]$, where $[v_1, \ldots, v_r]$ is the set of rational linear combinations of $v_1, \ldots, v_r$ with non-negative coefficients. We will say that C is *generated* by $v_1, \ldots, v_r$. A cone C is said to be *simplicial* if it is generated by vectors that are part of a free basis of N. The dimension of a cone C is the dimension of the linear space $\langle C\rangle$ spanned by C.

Given a cone $C = [v_1, \ldots, v_r]$ and a linear map $f : V \to \mathbb{Q}$ which is non-negative on C, the set $C' = C \cap \mathsf{ker}(f)$ is the cone spanned by the v_i such that $f(v_i) = 0$. Such cones C' are said to be the *faces* of C. The faces of dimension 1 are called *edges* of C. Any edge is of the form $[v]$, $v \in C \cap N$ *primitive* (that is, not divisible by integers other than ± 1). Such primitive vectors will also be referred to as edges of C.

A function $f : C \to \mathbb{Q}$ is said to be *linear* on a cone C if it is the restriction to C of a $\mathbb{Q}$-linear function $V \to \mathbb{Q}$. If in addition f takes integral values on $C \cap N$, then we shall say that f is an *integral linear* function on C. Notice that if $v_1, \ldots, v_r \in N$ are linearly independent and $a_1, \ldots, a_r \in \mathbb{Q}$, then there exists a unique linear map $f : [v_1, \ldots, v_r] \to \mathbb{Q}$ such that $f(v_i) = a_i$. This map is integral if and only if the a_i are integers. If $C = [v_1, \ldots, v_r]$ has dimension s and, say, $v_1, \ldots, v_s$ are linearly independent, then to give a linear map $f : C \to \mathbb{Q}$ is equivalent to give its values $a_i = f(v_i)$ for $i = 1, \ldots, s$.

Given a cone C in V, a polyhedral decomposition of C is a set of cones $\mathcal{C} = \{C_1, \ldots, C_k\}$ such that (a) Any face of a cone in $\mathcal{C}$ is a cone in $\mathcal{C}$, (b) for all i and j, $C_i \cap C_j$ is a face of C_i and C_j, and (c) $C = C_1 \cup \ldots \cup C_k$. If in addition the C_i are simplicial, then we we will speak of a *simplicial* polyhedral decomposition of C.

We will say that a function $f : C \to \mathbb{Q}$ is a **D**-function (**D** for divisor and also for Demazure) if f is continuous and there exists a polyhedral decomposition of C, $C = \bigcup_i C_i$, such that the restrictions $f|C_i$ are linear. We will write $\mathbf{D}(C, \mathbb{Q})$ to denote the $\mathbb{Q}$-vector space of all **D**-functions of C. The subgroup of $\mathbf{D}(C, \mathbb{Q})$ whose elements are functions that take integral values on $C \cap N$ will be denoted $\mathbf{D}(C, \mathbb{Z})$. **D**-functions are called "C-linear support functions" in Oda [1988].

Given a cone C, the set of all primitive vectors in $C \cap N$ will be denoted Π_C and will be called the set of *primitive vectors* of C.

Chains of quadratic varieties. If Q is a quadratic variety in $\mathbb{P}^n$ (quadratic for short), we shall write $L(Q)$ to denote the linear space of its double points, and $\ell(Q)$ to denote the dimension of $L(Q)$. Thus $\ell(Q) = -1$ if and only if Q is smooth. Moreover, $r(Q) = n - \ell(Q)$ is the *rank* of Q. Notice that on $\mathbb{P}^0$ there is a unique quadratic, which is empty.

Given any strictly decreasing sequence $I = \{i_1 > \ldots > i_k\}$ in $\{0, 1, \ldots, n-1\}$ (for $k = 0$, I is the empty sequence), a *quadratic chain of type* I is a sequence $Q = (Q_0, \ldots, Q_k)$ with the following properties:

(1) Q_0 is a quadratic in $\mathbb{P}^n$.
(2) If $k > 0$, $\ell(Q_{j-1}) = i_j$ and Q_j is a quadratic variety in $L(Q_{j-1})$, $j = 1, \ldots, k$.
(3) Q_k is smooth.

Thus a quadratic chain of type $\emptyset$ (empty set) is just a smooth quadratic variety. A quadratic chain of type $\{j\}$, $j \in \{0, 1, \ldots, n-1\}$, is a pair (Q_0, Q_1) consisting of a quadratic variety Q_0 of rank $n-j$ together with a smooth quadratic variety Q_1 in $L(Q_0)$. We shall write Ω_I to denote the set of all quadratic chains of type I, and Ω_j instead of $\Omega_{\{j\}}$. Regarding Q_j as a smooth quadratic variety of L_j/L_{j+1}, where $L_j = L(Q_{j-1})$ for $j = 1, \ldots, k$, $L_0 = \mathbb{P}^n$ and $L_{k+1} = \emptyset$, we see that Ω_I is in one-to-one correspondence with a bundle over the variety of flags of type I with fiber a product of open sets of suitable projective spaces. Hence Ω_I has a natural structure of smooth algebraic variety. It is also not hard to see that the projective group of $\mathbb{P}^n$ acts transitively on Ω_I. For example, $X_0 := \Omega_\emptyset$ parametrizes smooth quadratic varieties, which is an open set of a suitable projective space (see the beginning of Section 2). At the other end, $\Omega_{\{n-1,\ldots,1,0\}}$ is isomorphic to the variety of maximal flags.

Here are some more specific examples. Since all elements in Ω_I are projectively equivalent, it is enough to give a projective description of one element $\omega_I \in \Omega_I$. For $n = 2$ we have: ω_0, a pair of disctinct lines (its partial flag is the intersection point); ω_1, a line with a distinguished pair of points (its flag is the line); and ω_{10}, a line with a distinguished point on it (its flag consists of the line and the point). If $n = 3$, the description is as follows: ω_0, a quadric cone (the flag is the vertex); ω_1, a pair of distinct planes with a pair of distinct points on the double line (the flag is this last line); ω_2, a plane with a distinguished smooth conic (the flag is the plane); ω_{10}, a pair of distinct planes with a distinguished point on the double line (the flag is the line and its distinguished point); ω_{20}, a plane containing a pair of distinct lines (the flag is the plane and the point of intersection of the two lines); ω_{21}, a plane containing a line with a distinguished pair of distinct points on it (the flag is the plane and the line); and ω_{210}, a complete flag of $\mathbb{P}^3$.

2. Halphen groups of X_0

Let X_0 be the variety of smooth quadratic varieties in $\mathbb{P}^n$, so that X_0 is the open set $\mathsf{det}(a_{ij}) \neq 0$ of the projective space $\mathbb{P}^{n(n+3)/2}$ with homogeneous coordinates $[a_{ij}]$, $0 \le i \le j \le n$ (henceforth, as usual, we set $a_{ij} = a_{ji}$ when $i > j$). Here the point $[a] = [a_{ij}]$ corresponds, if $x_0, \ldots, x_n$ are projective coordinates for $\mathbb{P}^n$, to the quadratic variety given by the equation

$$\sum_{i,j} a_{ij} x_i x_j = 0 \ .$$

Using matrix notation, with $x = (x_0, \ldots, x_n)$ and $a = (a_{ij})$, the equation may be written as $xax^t = 0$.

The group $\mathsf{GL}(n+1, \boldsymbol{k})$ acts in $\mathbb{P}^n$,

$$\mathsf{GL}(n+1, \boldsymbol{k}) \times \mathbb{P}^n \to \mathbb{P}^n \ , \quad \alpha \cdot [x] = [x(\alpha^t)^{-1}] \ .$$

This action induces an action on X_0, which is easily seen to be

$$\mathsf{GL}(n+1,\boldsymbol{k}) \times X_0 \to X_0 \ , \quad \alpha \cdot [a] = [\alpha^t a \alpha] \ .$$

Since for any $[a] \in X_0$ there exists $\alpha \in \mathsf{GL}(n+1,\boldsymbol{k})$ such that $\alpha^t a \alpha$ is the identity matrix I, $\mathsf{GL}(n+1,\boldsymbol{k})$ acts transitively on X_0. The isotropy group of $[a]$ is the subgroup

$$\left\{\alpha \in \mathsf{GL}(n+1,\boldsymbol{k}) \mid \text{ there exists } \lambda \in \boldsymbol{k}^* \text{ such that } \alpha^t a \alpha = \lambda a\right\} \ .$$

If we let $G = \mathsf{SL}(n+1,\boldsymbol{k})$, then G also acts transitively on X_0 and the isotropy group of $[a]$ is

$$G_{[a]} = \left\{\alpha \in G' \mid \alpha^t a \alpha = a\right\} \ ,$$

which is the group $\mathsf{SO}(a)$. Thus

$$X_0 \simeq G/\mathsf{SO}(a) \ .$$

Notice that

$$\alpha \mapsto (\alpha^t)^{-1}$$

is an involution σ of G such that $G^\sigma = \mathsf{SO}(a)$. Thus X_0 is an example of a symmetric variety.

The fact that X_0 is homogeneous under the action of G allows us to define a *Halphen pairing*

$$Z^d X_0 \times Z_d X_0 \to \mathbb{Z} \ , \quad (\delta, \gamma) \mapsto \langle \delta | \gamma \rangle \ ,$$

where

$$\langle \delta | \gamma \rangle := \mathsf{deg}(\delta \cdot \alpha\gamma) = \mathsf{deg}(\alpha\delta \cdot \gamma) \ ,$$

α generic in G. This definition may be justified using the transversality theorem of general translates on a homogeneous variety (see Kleiman [1974]). It means that given δ and γ there exists a non-empty open set U in G such that the intersection cycles $\delta \cdot \alpha\gamma$ and $\alpha\delta \cdot \gamma$ are defined for all $\alpha \in U$ and have a common degree which is idependent of α.

Here γ plays the role of a d-dimensional system of (smooth) quadratic varieties and δ of a d-fold condition. Given a d-dimensional system γ and a d-fold condition δ, Halphen observed that even when the intersection cycle $\delta \cdot \gamma$ is defined, $\mathsf{deg}(\delta \cdot \gamma)$ need not be the right number, but that it is correct if δ is "independent" of γ. His notion of independence amounts to allowing a general translate of δ (cf. Kleiman [1980]), which has the efect of placing the data used to define the condition in a general position (cf. Casas–Xambó [1986]). Halphen explicitely ruled out of his inquiries (cf. Halphen [1985], p. 7) the study of numbers $\langle \delta | \gamma \rangle$ with δ "special" with respect to γ, although he informs us in a brief remark that his point of view in relation to these questions is a dynamical one, in the sense that to him the difficulty is to determine in which ways the solutions to the corresponding general problem can coalesce when δ is specialized, say by letting the data used to define δ come to satisfy some non-trivial aditional relation.

Let $B^dX_0 \subseteq Z^dX_0$ be the subgroup of all those cycles δ such that $\langle\delta|\gamma\rangle = 0$ for all $\gamma \in Z_dX_0$. We set Hal^dX_0 to denote the quotient group Z^dX_0/B^dX_0 and we will say that it is the Halphen group (of codimension d) of X_0. The class of $\delta \in Z^dX_0$ will be denoted $\langle\delta|$. We define the subgroup B_dX_0 of Z_dX_0 in the same way and let $\mathsf{Hal}_dX_0 = Z_dX_0/B_dX_0$ (Halphen group of dimension d), so that $\mathsf{Hal}_dX_0 = \mathsf{Hal}^{m-d}X_0$ if $m = \mathsf{dim}(X_0)$. We shall write $|\gamma\rangle$ to denote the class of $\gamma \in Z_dX_0$ in Hal_dX_0. If z is a cycle, $\langle z|$ and $|z\rangle$ denote the same element, the only difference being that in the first notation we declare the codimension of z while in the second we declare its dimension. From the definitions it follows that we have a non-degenerate pairing

$$\mathsf{Hal}^dX_0 \times \mathsf{Hal}_dX_0 \to \mathbb{Z}$$

given by $(\langle\delta|, |\gamma\rangle) \mapsto \langle\delta|\gamma\rangle$. Thus we shall write

$$\langle\delta| \cdot |\gamma\rangle = \langle\delta|\gamma\rangle \ .$$

The main questions to ask about these groups are to describe explicitly:

- The groups Hal^dX_0;
- The pairings $\mathsf{Hal}^dX_0 \times \mathsf{Hal}_dX_0 \to \mathbb{Z}$; and
- The maps $Z^dX_0 \to \mathsf{Hal}^dX_0$.

Here is a quite trivial illustration: it is easy to check that $\mathsf{Hal}^0X_0 = \mathbb{Z}$ (generated by $\langle X_0|$) and $\mathsf{Hal}_0X_0 = \mathbb{Z}$ (generated by $|a\rangle$, for any point $a \in X_0$). Moreover, under these isomorphisms the pairing $\mathsf{Hal}^0X_0 \times \mathsf{Hal}_0X_0 \to \mathbb{Z}$ is just the product map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

In Section 4 of this note we give a description of Hal^1X_0, as some group of **D**-functions, and of the funtion that corresponds to a given divisor on X_0. Then, in the two Sections that follow, we will give an explicit description of Hal_1X_0, of the map $Z_1 \to \mathsf{Hal}_1X_0$ and of the pairing $\mathsf{Hal}^dX_0 \times \mathsf{Hal}_dX_0 \to \mathbb{Z}$. A similar description of Hal^dX_0 for the remaining d seems to be not yet known, except for the case $n = 2$, which is worked out in Casas–Xambó [1986].

To end this section we will define the fundamental conditions on X_0. Consider the flag $F : L_0 \subset L_1 \subset ... \subset L_{n-1} \subset L_n = \mathbb{P}^n$, where L_k is the linear space $x_{k+1} = \ldots = x_n = 0$. A quadratic variety $[a]$ is tangent to L_k if and only if the restriction of the quadratic form $\sum a_{ij}x_ix_j$ to L_k is degenerate, which happens if and only if the determinant of (a_{ij}), $0 \leq i, j \leq k$, vanishes. Thus the quadratic varieties which are tangent to L_k form a hypersurface Λ_k of X_0. Since the determinant of a generic symmetric matrix is irreducible, the varieties Λ_k are themselves irreducible. The elements $\lambda_k = \langle\Lambda_k| \in \mathsf{Hal}^1X_0$ do not depend on the flag, because any two flags are conjugate under the action of G. We will say that $\lambda_0, \ldots, \lambda_{n-1}$ are the *fundamental conditions* of X_0. The varieties Λ_k will be called *fundamental varieties* (with respect to the flag F).

Notice that if we let Λ denote the union of the varieties Λ_k, then the open set $X_0 - \Lambda$ is an orbit of the isotropy group $B \subset G$ of F (thus B is the group of upper-triangular matrices). Moreover, if we let $S \subset X_0$ denote the diagonal quadrics, then S is an n-dimensional algebraic torus and $X_0 - \Lambda \simeq S \times \mathbb{A}^{n(n+1)/2}$. Here $\mathbb{A}^{n(n+1)/2}$ is an affine space with coordinates u_{ij}, $0 \leq i < j \leq n$, and a pair $([q], u) \in S \times \mathbb{A}^{n(n+1)/2}$ corresponds to the quadratic $[a]$, $a = (I+u)^t q(I+u)$, where we identify u with the matrix that has zeroes in the entries with $i \geq j$, and u_{ij} in the entry (i, j).

3. The variety of complete quadratics

Studied by many authors, it is a projective variety X, with a G-action, which has the following properties (cf. Thorup–Kleiman [1988] and the references therein):

(1) X has an open orbit isomorphic to X_0 (in what follows we will identify X_0 with this open orbit).
(2) There is a ono-to-one correspondence $I \mapsto O_I$ between subsets I of $\{0, 1, \ldots, n-1\}$ and the G-orbits O_I in X such that $|I| = \mathsf{codim}(O_I)$. In particular $O_\emptyset = X_0$.
(3) For all I, the closure D_I of O_I is smooth. In particular, X is smooth, since X is the closure of X_0. If $0 \leq i \leq n-1$, we shall write O_i and D_i instead of $O_{\{i\}}$ and $D_{\{i\}}$. Thus $D_0, \ldots, D_{n-1}$ are smooth divisors on X.
(4) For all $I = \{i_1, \ldots, i_k\}$, $D_{i_1}, \ldots, D_{i_k}$ intersect transversally and its intersection is D_I. In particular we have that $D_I \subseteq D_J$ if and only if $J \subseteq I$.
(5) For all I, $O_I = D_I - \bigcup_{j \notin I} D_j$. In particular
$$X_0 = X - \bigcup_i D_i \quad \text{or} \quad X - X_0 = \bigcup_i D_i \ .$$
(6) The variety D_I is isomorphic to the variety Ω_I of quadratic chains of type I. To conform to the conventions set up in Section 1, each I is to be arranged in decreasing order.

The Picard group of X_0 is isomorphic to $\mathbb{Z}/(n+1)$, because its the complement in $\mathbb{P}^{n(n+3)/2}$ of a hypersurface of degree $n+1$. So $\mathsf{Pic}(X_0)_\mathbb{Q} = 0$. This implies that $\mathsf{Pic}(X)_\mathbb{Q}$ is $\mathbb{Q}$ vector space of dimension n with basis the classes $\delta_j = [D_j]$, $j = 0, \ldots, n-1$. It is well known, and not too hard to establish directly (cf. Xambó [1988]), that if we set $\mu_j = [\overline{\Lambda}_j]$ ($\Lambda_0, \ldots, \Lambda_{n-1}$ the fundamental varieties, $\overline{\Lambda}_k$ the closure of Λ_k in X), then
$$\delta_{n-j-1} = -\mu_{j-1} + 2\mu_j - \mu_{j+1} \ ,$$

with the convention that $\mu_0 = \mu_n = 0$. These relations show that $\mu_0, \ldots, \mu_{n-1}$ is also a basis of $\mathsf{Pic}(X)_{\mathbb{Q}}$. Actually we have the expressions (*loc. cit.*)

$$(n+1)\mu_j = \sum_{0\le k\le j} (k+1)(n-j)\delta_{n-k-1} + \sum_{j<k<n} (j+1)(n-k)\delta_{n-k-1} ,$$

which are useful for recursive explicit computations.

4. The group $\text{Hal}^1 X_0$

We are going to see that $\mathsf{Hal}^1(X_0)_{\mathbb{Q}}$ is isomorfic to a $\mathbb{Q}$-vector space of **D**-functions. Our arguments here are closely related to those that Bifet introduced in his thesis (see Bifet [1990]) and use in an essential way the technology introduced in De Concini–Procesi [1983, 1985], which in turn is based on toric geometry. In this section we summarize some facts in a form that is useful for our purposes. We refer to De Concini–Procesi for the basic terminology concerning toric varieties, equivariant and wonderful compactifications, etc. For a thorough treatment of toric varieties, see Oda [1988], or Fulton [1989].

Let $e_0, \ldots, e_n$ be the standard basis of $\mathbb{Z}^{n+1}$. The group $W = S_{n+1}$ acts in $\mathbb{Z}^{n+1}$ permuting coordinates and this action leaves invariant the subgroup M whose equation is $m_0 + m_1 + \ldots + m_n = 0$, thus inducing an action of W on M. The group M is free of rank n. Notice that the vectors $\alpha_i = e_{i-1} - e_i$, $i = 1, \ldots, n$, form a free basis of M. We shall put $N = M^*$ (the dual group) and $V = N \otimes \mathbb{Q}$, so that V is a $\mathbb{Q}$-vector space of dimension n. The group W still acts, by the dual action, on N and V. The pairing between an element m of M and an element ψ of N will be denoted $\langle m, \psi\rangle$.

We will identify the character group $\hat{S}$ of S with M: an $m = (m_0, \ldots, m_n) \in M$ corresponds to the character χ^m defined by $\chi^m(a) = a_0^{m_0} \ldots a_n^{m_n}$, $a = \mathsf{diag}(a_0, \ldots, a_n)$. So the elements χ of $\hat{S}$ can be evaluated over elements ψ of the group $N := M^*$ (or over elements of the space $V = \mathbb{Q} \otimes N$). We will still write $\langle \chi, \psi\rangle$ to denote the evaluation; hence $\langle \chi^m, \psi\rangle = \langle m, \psi\rangle$.

The basic fact (cf. De Concini–Procesi [1985], Thm. 5.3) is that there exists a rational (with respect to N) simplicial cone $C = [v_1, \ldots, v_n]$ in V such that the cones $\{\sigma C \mid \sigma \in W\}$, together with all their faces, form a complete (simplicial) rational fan F_X of V and that there is a one-to-one natural correspondence between G-equivariant wonderful compactifications Y of X_0 lying over X and W-invariant complete simplicial rational fans F_Y of V which are subdivisions of F_X. Moreover, under this correspondence the set $\Psi = \Psi_Y$, whose elements are the primitive vectors ψ in $C \cap M$ that span edges of cones in F_Y, is in one-to-one correspondence with the boundary divisors of Y, that is, the irreducible components of $Y - X_0$, or equivalently, the codimension one orbits of Y. For example, the boundary divisors D_i of X correspond bijectively to the primitive vectors $v_1, \ldots, v_n$ of C. It is not hard to see that any primitive vector in C defines a codimension one orbit in some compactification Y. The open set in Y obtained by removing all orbits with codimension greater or equal than 2 is a partial compactification of X_0 that depends only on Ψ; we will denote it by X_Ψ. For example, if $\Delta = \{v_1, \ldots, v_n\}$, then X_Δ is the union of X_0 and the codimension one orbits $O_1, \ldots, O_n$ of X.

Remark. If $\Psi_1 \subseteq \Psi_2$, then $X_{\Psi_1} \subseteq X_{\Psi_2}$. In particular, any primitive vector ψ in $C \cap N$ defines a codimension one orbit of any Y_Ψ such that $\psi \in \Psi$. This orbit will be denoted D_ψ.

Now from the theorem of classification of equivariant compactifications (De Concini–Procesi [1985], Thm. 5.2) and the determination of the Picard group of toric varieties (Oda [1988], Chapter II) it follows (see Bifet [1990]) that the group $\mathsf{Pic}(Y)_\mathbb{Q} = \mathsf{Pic}(X_\Psi)_\mathbb{Q}$ can be canonically identified with the space $\mathbf{D}_Y(C, \mathbb{Q})$ of rational functions defined on C which are continuous and linear on the cones of F_Y contained in C, and that the map $\mathsf{Pic}(Y)_\mathbb{Q} \to \mathsf{Pic}(Y')_\mathbb{Q}$ induced by a G-equivariant map $Y' \to Y$ of wonderful compactifications is just the canonical inclusion $\mathbf{D}_Y(C, \mathbb{Q}) \to \mathbf{D}_{Y'}(C, \mathbb{Q})$.

Since $\mathsf{Hal}^1(X_0)$ is the direct limit of $\mathsf{Pic}(Y)$, when Y ranges over the wonderful compactifications of X_0 that lie over X (see De Concini–Procesi [1985], Thm. 6.3), we conclude that there is a canonical isomorphism $\mathsf{Hal}^1(X_0)_\mathbb{Q} \simeq \mathbf{D}(C, \mathbb{Q})$. Thus we can identify $\mathsf{Hal}^1(X_0)_\mathbb{Q}$ with the space of $\mathbf{D}$-functions on C.

Computation of φ_D. Let D be a divisor in X_0. The $\langle D| \in \mathsf{Hal}^1(X_0)_\mathbb{Q}$ corresponds to a function $f_D \in \mathbf{D}(C, \mathbb{Q})$. We want to describe f_D. To this end we may assume that D does not contain any of the fundamental varieties $\Lambda_0, \ldots, \Lambda_{n-1}$. We know that

$$X - (\Lambda_0 \cup ... \cup \Lambda_{n-1}) = S \times U \ ,$$

U an affine space of dimension $n(n+1)/2$. So $D \cap (S \times U)$ has an equation of the form

$$f_D = \sum \chi f_\chi \ ,$$

where the sum runs over the character group $M = \hat{S}$ of S and f_χ are polynomial functions on U, almost all zero. Let $|f_D|$ denote the finite set of characters χ such that $f_\chi \neq 0$. Let Y be any wonderful compactification of X, with $\{D_\psi\}$, $\psi \in \Psi \subset \Pi_C$, the set of boundary divisors. On Y

$$\mathsf{div}(f_D) = \overline{D} + \sum n_\psi D_\psi \ ,$$

where $\overline{D}$ is the closure of D in Y and $n_\psi = \mathsf{ord}_{D_\psi}(f_D)$. If f_D were a single character ψ, then $n_\psi = \langle \chi, \psi \rangle$ (Fulton [1989], Lemma 3.3). In general

$$n_\psi = \mathsf{min}_\chi \langle \chi, \psi \rangle$$

where χ runs in the set $|f_D|$.

Let us sketch a proof. In the rational simplicial cone decomposition of C corresponding to Y, there is a cone with edges $\psi = \psi_1, ..., \psi_n \in N$, which are a basis for N. Let $\xi_1, \ldots, \xi_n \in M$ be the dual basis. This basis determines a torus embedding $S \hookrightarrow \mathbb{A}^r$, to which it corresponds an open set $\mathbb{A}^r \times U$ in Y that contains $S \times U$. Since $D_\psi \cap S \times U$ is the subspace $\chi_1 = 0$, n_ψ will be the order of f_D with respect to χ_1, that is, the minimum exponent of χ_1 in the expression of f_D obtained substituting every character $\chi \in |f_D|$ by the corresponding monomial in $\chi_1, \ldots, \chi_r$. But the exponent of χ_1 in the monomial associated to χ is just $\langle \chi, \psi_1 \rangle = \langle \chi, \psi \rangle$.

Now we can describe f_D:

$$f_D(\psi) = -\mathsf{min}_\psi \langle \chi, \psi \rangle \ ,$$

where χ ranges over $|f_D|$. To prove this, notice that the formula for $\mathsf{div}(f_D)$ implies that

$$[\overline{D}] = -\sum_\psi \mathsf{min}\,\langle \chi, \psi \rangle [D_\psi]$$

in $\mathsf{Pic}(Y)$. If we take now Y in such a way that $\overline{D}$ intersects properly all orbits of Y (De Concini–Procesi [1985], Thm. 4.7), we see that the function f_D corresponding to D has the claimed form.

Remark. For a particular Y, the boundary divisors form a basis of $\mathsf{Pic}(Y)_\mathbb{Q}$, and it is tempting to believe that all possible boundary divisors (in one-to-one correspondence with Π_C) form a basis of $\mathsf{Hal}^1(X_0)_\mathbb{Q}$. However this is wrong, inasmuch as the maps involved in the direct limit of $\mathsf{Pic}(Y)_\mathbb{Q}$ are *total transforms*, and not strict transforms. In other words, a given boundary divisor D_ψ of a given Y certainly defines, through the isomorphism of $\mathsf{Pic}(Y)_\mathbb{Q} \simeq \mathbf{D}_Y(C, \mathbb{Q})$, a $\mathbf{D}$-function on C, namely the (unique) $\mathbf{D}$-function $f_{Y,\psi}$ that has value 1 at ψ and 0 at all other edges of the fan F_Y. But this function depends on the fan (that is, it depends on Y) and hence it does not define an element of $\mathsf{Hal}^1(X_0)_\mathbb{Q}$. If, on the other hand, we take the element of $\mathsf{Hal}^1(X_0)_\mathbb{Q}$ defined by $f_{Y,\psi}$, this element is the one that corresponds to the *total* transforms of D_ψ in all compactifications Y' that dominate Y.

Example. Let $t_1, \ldots, t_n$ be the basis of M that is dual of the basis $v_1, \ldots, v_n$ of N. Let $p = (p_1, \ldots, p_n) \in \mathbb{N}^r$ be non-zero and consider the divisor S_p in X_0 defined by $t_1^{p_1} + \ldots + t_n^{p_n}$). Let $f_p = f_{S_p}$. Then

$$f_p(q_1 \cdot v_1 + \ \ldots + q_n \cdot v_n) = \mathsf{min}\,(p_1 q_1, \ldots, p_n q_n) \ .$$

In particular we have functions $f_p \in \mathbf{D}(C, \mathbb{Q})$ for all primitive p. In the case $n = 2$, these functions form a basis, because the classes $\langle S_p|$ form a basis of $\mathsf{Hal}^1(X_0)_\mathbb{Q}$ (see Casas-Xambó [1986]). But for $n > 2$ this is no longer true. For example (Kleiman–Xambó, June 1990), the function $\mathsf{min}(3q_1, 4q_2, 5q_3, q_1 + q_2 + q_3)$ is a $\mathbf{D}$-function but it is not a linear combination of functions of the form $f_{(p_1,p_2,p_3)}$.

5. Halphen's first formula

Let $H_1 = \mathbb{Q}\Pi_C$ denote the rational vector space with basis Π_C. The basis element corresponding to a primitive vector v will be denoted $[v]$. More generally, given an integral non-zero vector v in C, there exists a unique positive integer m such that $v = mv'$ with v' a primitive vector in C and hence we may identify v with the element $[v] := m[v']$ of H_1.

Now let us define a canonical map

$$v : Z_1X \to H_1 \ .$$

By linearity it is enough to define $v(\Gamma)$ for any irreducible curve Γ on X_0. Let Σ be the complete smooth model of Γ. Given any compactification Y of X_0, there exists a unique map $\nu : \Sigma \to Y$ which extends the normalization of Γ. Take a wonderful Y with the property that $\nu(\Sigma)$ meets the boundary $Y - X_0$ of X_0 in Y only along codimension one orbits. This implies that there is a finite set Ψ of primitive vectors in C such that $\nu(\Sigma) \subseteq X_\Psi$. For $\psi \in \Psi$, let D_ψ be the corresponding codimension one orbit in X_Ψ. Then we set

$$v_\Gamma = \sum \mathsf{ord}_P(\nu^* D_\psi)[\psi] \ ,$$

where the sum runs through all points P of Σ. Because of the first Remark in the previous section, the expression on the right is independent of Ψ, for no additional codimension one orbit meets $\nu(\Sigma)$.

Given $\gamma \in Z_1X_0$, instead of $v(\gamma)$ we shall also write v_γ or $\hat{\gamma}$.

Notice also that an element $\varphi \in \mathbf{D}(C, \mathbb{Q})$ can be extended to a unique linear function $H_1 \to \mathbb{Q}$, which we shall still denote by φ. Now let Γ be a curve and D a divisor on X. Let $\varphi_D \in \mathbf{D}(C, \mathbb{Q})$ be the $\mathbf{D}$-function corresponding to D in $\mathbf{D}(C, Q)$, as explained in the previous section. Then

$$\langle D|\Gamma\rangle = \varphi_D(\hat{\Gamma}) \ . \qquad (*)$$

Indeed, choose a wonderful compactification Y of X such that the closures $\overline{\Gamma}$ and $\overline{D}$ of Γ and D properly meet all orbits in Y. Then it is clear that $\langle D|\Gamma\rangle$ equals the degree of the 0-dimensional rational class $[\overline{D}] \cdot [\overline{\Gamma}]$. On the other hand, we know that $[\overline{D}] = \sum \varphi_D(\psi)[D_\psi]$, where the sum runs through all primitive vectors ψ corresponding to the codimension one orbits in Y. Thus we have

$$\begin{aligned}\langle D|\Gamma\rangle &= \mathsf{deg}([\overline{D}]\cdot[\overline{\Gamma}]) \\ &= \sum_{\psi}\varphi_D(\psi)\mathsf{deg}([D_\psi]\cdot[\overline{\Gamma}]) \\ &= \sum_{\psi}\varphi_D(\psi)\sum_{P}\mathsf{ord}_P(\nu^* D_\psi) \\ &= \varphi_D(v_\Gamma)\,.\end{aligned}$$

We will refer to (∗) as *Halphen's first formula.*

6. The group $\mathrm{Hal}_1 X_0$

In this section we shall prove the following:

Theorem. *The map v induces a canonical isomorphism*

$$v : \mathsf{Hal}_1 X \to H_1\ .$$

Moreover, under this isomorphism and the isomorphism $\mathsf{Hal}^1(X_0)_{\mathbb{Q}} \simeq \mathbf{D}(C,\mathbb{Q})$ described in Section 4, the intersection pairing

$$\mathsf{Hal}^1 X_{\mathbb{Q}} \times \mathsf{Hal}_1 X_{\mathbb{Q}} \to \mathbb{Q}$$

is represented, due to Halphen's first formula, by the pairing

$$\mathbf{D}(C,\mathbb{Q}) \times H_1 \to \mathbb{Q}$$

given by evaluation of functions, $(f,\psi) \mapsto f(\psi)$.

The kernel of v. From Halphen's first formula it follows that a 1-cycle γ for which $v_\gamma = 0$ is in $B^{n-1}X$. Now we shall see that the converse is also true. In order to do this, it is enough to see that if γ is a 1-cycle on X and $v_\gamma \neq 0$, then there exists D such that $\langle D|\gamma\rangle \neq 0$. To see this, take any $\mathbf{D}$-function φ such that $\varphi(v_\gamma) \neq 0$. Take any compactification Y on which φ is represented as a divisor D on X_0 whose closure in Y has proper intersection with all the orbits of Y. We also can arrange that the closure of γ on Y has proper intersection with the orbits. Then, by Halphen's first formula again, $\langle D|\gamma\rangle \neq 0$.

The map v is surjective. It is enough to see that a given primitive vector ψ in C is in the image of v. To see this, choose a wonderful compactification Y such that $\psi = \psi_1$ is among the primitive vectors $\psi_1, \ldots, \psi_r$ indexing boundary divisors in Y. We know that $[D_{\psi_1}], \ldots, [D_{\psi_r}]$ are a basis of $A^1 Y_{\mathbb{Q}}$. Let $\alpha_1, \ldots, \alpha_r \in A_1 Y_{\mathbb{Q}}$ be a dual basis, that is, such that $\mathsf{deg}(\alpha_i \cdot [D_{\psi_j}]) = \delta_{ij}$. By Chow's moving lemma, there exists a 1-cycle γ in X such that its closure $\bar{\gamma}$ in Y has proper intersection with the orbits and such that $[\bar{\gamma}] = \alpha_1$. By construction it is clear that $v_\gamma = \psi$. □

Using the canonical isomorphism v to identify

$\mathsf{Hal}_1 X_0$ and H_1, we see that $|\Gamma\rangle = \hat{\Gamma}$, for any curve Γ on X_0. In the classical spirit, we can say that $\hat{\Gamma}$ is the *characteristic* of Γ, or better, that the ψ appearing in the expression of $\hat{\Gamma}$ are the *characteristics* of Γ. The *multiplicity* of a characteristic ψ of Γ is the coefficient of ψ in $\hat{\Gamma}$.

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• Dip. di Matematica, G. Castelnuovo, Univ. di Roma, Piazzale A. Moro 00185, Roma, Italy

• Dept. Algebra, Fac. Matemàticas, Univ. Complutense, 28040 Madrid, Spain