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Computing some fundamental numbers of the variety of nodal cubics in $\mathbb{P}^3$

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ABSTRACT

In this paper we obtain some fundamental numbers of the family U_{nod} of non-degenerate nodal cubics in $\mathbb{P}^3$ involving, in addition to the characteristic conditions, other fundamental conditions, as for example that the node lies on a plane. Some of these numbers were first obtained by Schubert in his *Kalkül der abzählenden Geometrie*. In our approach we construct several compactifications of U_{nod}, which can be obtained as a sequence of blow-ups of a suitable projective bundle K_{nod}. We also provide geometric interpretations of the degenerations that appear as exceptional divisors. The computations have been carried out with the Wɪᴛ system.

0. Introduction

One of the goals of enumerative geometry is to find the number of curves of a given irreducible n-dimensional family in $\mathbb{P}^3$ that satisfy n geometric conditions. This includes the numbers involving the *characteristic conditions*, which require that the curve goes through a given point, intersects a given line or is tangent to a given plane.

The 8-dimensional family of nodal cubics in $\mathbb{P}^2$ is one of the varieties which has received more attention from an enumerative geometric point of view. Its characteristic numbers (and many other intersection numbers) were calculated first by Maillard (1871) and Zeuthen (1872) and later by Schubert, who devoted to them a part (pp. 144–163) of his masterpiece (Schubert, 1879). Nevertheless, Hilbert's Fifteenth Problem (Hilbert, 1902) is asking for a justification and a verification of all geometric numbers computed by the 19th century geometers. In this sense, some of these intersection

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numbers were verified, using different approaches, by Sacchiero (1984), Kleiman and Speiser (1988), Aluffi (1991), Miret and Xambó (1991) and Ernström and Kennedy (1998).

Concerning the 11-dimensional family U_{nod} of nodal cubics in $\mathbb{P}^3$, the number of those that intersect 11 lines was computed in Kleiman et al. (1987), and in Hernández et al. (2007) all the characteristic numbers given by Schubert (1879) were verified. Besides, Pandharipande (1999) gave an algorithm to compute the characteristic numbers of rational curves in $\mathbb{P}^r$ introducing techniques of quantum cohomology.

This paper continues the determination of the fundamental numbers of the variety of nodal cubics in $\mathbb{P}^3$ begun in Hernández et al. (2007). More precisely, we study and compute those fundamental numbers involving the characteristic conditions and the condition that the node lies on a plane. These conditions are denoted as follows:

- μ, the plane determined by the nodal cubic goes through a point;
- ν, the nodal cubic intersects a line;
- ρ, the nodal cubic is tangent to a given plane;
- b, the node lies on a plane.

We also consider the codimension 2 condition P that the nodal cubic goes through a point.

Thus, while in Hernández et al. (2007) we focused on the numbers of the form $\mu^i\nu^j\rho^{11-i-j}$, in this paper we aim at the geometrical and computational problems involved in the determination of all the numbers of the form $\mu^i b^j \nu^k \rho^{11-i-j-k}$. We complete Schubert's work not only because our tables include all the numbers (Schubert computed a little less than half the numbers), but also because of a deeper geometric and enumerative understanding of the two degenerations (ε and ϑ) involved.

We remark that the numbers we compute have 'enumerative significance' if the characteristic of the ground field is 0. This is because the compactifications are chosen to make sure that the intersection of the conditions for a given fundamental number has no points on the boundary (the union of the degenerations), so that the degree of that intersection agrees with the definition of the fundamental number in question. Here we invoke Kleiman's transversality theorem (see Kleiman (1974)) that guarantees that in characteristic 0 the intersection multiplicities are 1. The detailed argument can be found in Miret and Xambó (1991), where it is also recalled that in characteristic $p > 0$ the intersection multiplicities (for a given number) are equal (say q) and that q is a power of p. In particular it follows that an intersection number has 'enumerative significance' ($q = 1$) if p does not divide the intersection number.

The material is organized as follows. In Section 1 we construct a compactification K_{nod} of the variety U_{nod} of non-degenerate nodal cubics of $\mathbb{P}^3$ via the projectivization of a suitable vector bundle. From this we get that the Picard group $\mathsf{Pic}(K_{\text{nod}})$ is a rank 4 free group generated by the classes of the closures in K_{nod} of the hypersurfaces of U_{nod} determined by the conditions μ, ν, b and p (that the nodal tangents intersect a line). Then we show that the boundary $K_{\text{nod}} - U_{\text{nod}}$ consists of three irreducible components of codimension 1.

In Section 2 we introduce some other conditions related to the distinguished elements of the nodal cubics, and we construct new compactifications of the variety U_{nod} taking successive blow-ups of K_{nod}. Finally, Section 3 is devoted to the study of the tangency condition ρ. We compute the intersection numbers of the form $\mu^i b^j \nu^k \rho^{11-i-j-k}$, some of which were already found by Schubert (1879).

One aspect of this paper is the structure and functionality of the symbolic computations. They have been carried out with the online system Wit (see Xambó-Descamps (2008)) using the scripts collected in Section 4.

1. The variety K_{nod} of nodal cubics

Throughout this paper, $\mathbb{P}^3$ will denote the projective space associated to a 4-dimensional vector space over an algebraically closed ground field **k** of characteristic 0, and the term *variety* will mean a quasi-projective **k**-variety. Moreover, we will also write z to indicate the degree of a 0-cycle z, if the underlying variety can be understood from the context.

Let $\mathbb{U}$ denote the rank 3 tautological bundle over the grassmannian variety Γ of planes of $\mathbb{P}^3$. Therefore, the projective bundle $\mathbb{P}(\mathbb{U})$ is the nonsingular incidence variety defined by

$$\mathbb{P}(\mathbb{U}) = \{(\pi, x) \in \Gamma \times \mathbb{P}^3 \mid x \in \pi\}.$$

Let $\mathbb{L}$ be the tautological line subbundle of the rank 3 bundle $\mathbb{U}|_{\mathbb{P}(\mathbb{U})}$ over $\mathbb{P}(\mathbb{U})$ and let $\mathbb{Q}$ be the tautological quotient bundle. Then the projective bundle $\mathbb{F} = \mathbb{P}(S^2\mathbb{Q}^*)$ parameterizes triples (π, x, u) such that u is a pair of lines (possibly coincident) contained in the plane π and both passing through the point x.

We will denote by ξ the hyperplane class of the projective bundle $\mathbb{F}$. The pullback to $\mathbb{F}$ of the hyperplane class $c_1(\mathcal{O}_{\mathbb{P}(\mathbb{U})}(1))$ under the natural projection $\mathbb{F} \to \mathbb{P}(\mathbb{U})$ will be denoted by b and the pullback to $\mathbb{F}$ of the hyperplane class $c_1\,(\mathcal{O}_\Gamma(1))$ under the natural projection $\mathbb{F} \to \Gamma$ will be denoted by μ, so that μ is the class of the hypersurface of $\mathbb{F}$ consisting of the triples (π, x, u) such that the plane π goes through a given point and b coincides with the class of the hypersurface of $\mathbb{F}$ consisting of the triples (π, x, u) such that the point x is on a given plane. Moreover, we will denote by p the class of the hypersurface of $\mathbb{F}$ consisting of the triples (π, x, u) such that u intersects a given line.

Lemma 1.1. *Let E be a vector bundle over a nonsingular variety X and let H be a subbundle of E such that the quotient E/H is a line bundle. Then $\mathbb{P}(H)$ is a divisor of $\mathbb{P}(E)$ and*

$$[\mathbb{P}(H)] = \xi + \pi^* c_1(E/H)$$

in $\operatorname{Pic}(\mathbb{P}(E))$*, where ξ is the hyperplane class of $\mathbb{P}(E)$ and $\pi : \mathbb{P}(E) \to X$ is the natural projection.*

Proof. See Fulton (1984), ex. 3.2.17, or Ilori et al. (1974). □

In the next Lemma we determine the relation between the classes ξ and p.

Lemma 1.2. *In* $\operatorname{Pic}(\mathbb{F})$ *the following relation holds:*

$$\xi = p - 2\mu.$$

Proof. Let r be a line in $\mathbb{P}^3$ and let

$$H_r = \{(\pi, x, u) \in \mathbb{F} :\ u \cap r \neq \emptyset\},$$

so that $p = [H_r] \in \operatorname{Pic}(\mathbb{F})$. Over the open set $U_r = \{(\pi, x) \in \mathbb{P}(\mathbb{U}) :\ \ x \notin r\} \subseteq \mathbb{P}(\mathbb{U})$ there exists a monomorphism of vector bundles

$$\iota : \mathbb{L}|U_r \to \mathbb{Q}^*|U_r$$

with the property that $\iota(v)$ vanishes at x and at $r \cap \pi$, for any $(\pi, x) \in U_r$ and $v \in \mathbb{L}_{(\pi,x)}$. Therefore, if $\bar{\iota} : \mathbb{L}|U_r \otimes \mathbb{Q}^*|U_r \to S^2\mathbb{Q}^*|U_r$ is the morphism induced by ι, it turns out

$$\mathbb{P}\big(\bar{\iota}(\mathbb{L}|U_r \otimes \mathbb{Q}^*|U_r)\big) = H_r.$$

Consequently, if we take $V_r = \pi^{-1}(U_r)$, where $\pi : \mathbb{F} \to \mathbb{P}(\mathbb{U})$ is the natural projection, by Lemma 1.1 we have

$$p|V_r = \xi|V_r + \pi^* c_1\big(S^2\mathbb{Q}^*/\mathbb{L} \otimes \mathbb{Q}^*\big).$$

Thus, by using the expression of $c_1\big(S^2\mathbb{Q}^*\big)$, we have

$$c_1\big(S^2\mathbb{Q}^*/\mathbb{L} \otimes \mathbb{Q}^*\big) = c_1(S^2\mathbb{Q}^*) - c_1(\mathbb{Q}^*) - 2c_1(\mathbb{L}) = 2(\mu - b) + 2b = 2\mu,$$

from which we obtain that $p|V_r = \xi|V_r + 2\mu|V_r$. Finally, since $\mathbb{F} - V_r$ has codimension 2, the claimed relation (of divisor classes) follows. □

Proposition 1.1. *The intersection ring $A^*(\mathbb{F})$ is isomorphic to the quotient of the polynomial ring $\mathbb{Z}[\mu, b, p]$ by the ideal*

$$\langle \mu^4, b^3 - \mu b^2 + \mu^2 b - \mu^3, p^3 - 3(\mu + b)p^2 + 2(3\mu^2 + 2\mu b + 3b^2)p - 8\mu^3 - 8\mu b^2\rangle.$$

In particular, the group $\operatorname{Pic}(\mathbb{F})$ *is a rank* 3 *free group generated by μ, b and p.*

Proof. The ring $A^*(\mathbb{F})$ is (Fulton, 1984, ex. 8.3.4) isomorphic to

$$A^*(\mathbb{P}(\mathbb{U}))[\xi]/\sum \pi^* c_i(S^2\mathbb{Q}^*)\xi^{3-i},$$

where $\pi : S^2\mathbb{Q}^* \to \mathbb{P}(\mathbb{U})$ is the natural projection. From the formula of the total Chern class of $S^2\mathbb{Q}^*$ we have

$$\sum \pi^* c_i(S^2\mathbb{Q}^*)\xi^{3-i} = \xi^3 + 3(\mu - b)\xi^2 + (6\mu^2 - 8\mu b + 6b^2)\xi + (4\mu^3 - 8\mu^2 b + 8\mu b^2 - 4b^3).$$

By substituting ξ for the expression $p - 2\mu$ given in Lemma 1.2, we obtain the third relation of the ideal. The remaining relations come from the intersection ring $A^*(\mathbb{P}(\mathbb{U}))$. □

We define $\mathbb{E}_{\text{nod}}$ as the subbundle of $S^3\mathbb{U}^*|_{\mathbb{F}}$ whose fiber over $(\pi, x_b, u_p) \in \mathbb{F}$ is the linear subspace of forms $\varphi \in S^3\mathbb{U}^*$ defined over π that have multiplicity at least 2 at x_b and for which u_p is a pair of tangents (possibly coincident) at x_b. In fact, given a point $(\pi, x_b, u_p) \in \mathbb{F}$ and taking projective coordinates x_0, x_1, x_2, x_3 so that $\pi = \{x_3 = 0\}$, $x_b = [1, 0, 0, 0]$ and $u_p = \{x_1(b_1x_1 + b_2x_2) = 0\}$, $b_1, b_2 \in \mathbf{k}$, we can express the elements φ of the fiber of $\mathbb{E}_{\text{nod}}$ over (π, x_b, u_p) as follows:

$$\varphi = a_0x_0x_1(b_1x_1 + b_2x_2) + a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3, \tag{1}$$

where a_i for $i = 0, \ldots, 4$ are in $\mathbf{k}$. Thus, $\mathbb{E}_{\text{nod}}$ is a rank 5 subbundle of $S^3\mathbb{U}^*|_{\mathbb{F}}$.

In the next proposition we give a resolution of the vector bundle $\mathbb{E}_{\text{nod}}$ over $\mathbb{F}$. To do this, we consider the natural inclusion map $i : \mathbb{Q}^* \to \mathbb{U}^*$, the product map $\kappa : \mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-1) \to S^3\mathbb{Q}^*|_{\mathbb{F}}$, and the maps

$$h : \mathbb{U}^* \otimes \mathcal{O}_{\mathbb{F}}(-1) \to S^3\mathbb{U}^*|_{\mathbb{F}} \quad \text{and} \quad j : S^3\mathbb{Q}^*|_{\mathbb{F}} \to S^3\mathbb{U}^*|_{\mathbb{F}}$$

whose images are clearly contained in $\mathbb{E}_{\text{nod}}$.

Proposition 1.2. *The sequence*

$$0 \longrightarrow \mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-1) \xrightarrow{\alpha} (\mathbb{U}^* \otimes \mathcal{O}_{\mathbb{F}}(-1)) \oplus S^3\mathbb{Q}^*|_{\mathbb{F}} \xrightarrow{\beta} \mathbb{E}_{\text{nod}} \longrightarrow 0,$$

where $\alpha = \binom{i\otimes 1}{-\kappa}$ *and* $\beta = h + j$, *is an exact sequence of vector bundles over* $\mathbb{F}$.

Proof. It is similar to the one given in Proposition 1.1 of Hernández et al. (2007). □

Let K_{nod} be the projective bundle $\mathbb{P}(\mathbb{E}_{\text{nod}})$ over $\mathbb{F}$. Then K_{nod} is a non-singular variety of dimension

$$\dim(K_{\text{nod}}) = \dim(\mathbb{F}) + \text{rk}(\mathbb{E}_{\text{nod}}) - 1 = 11$$

whose points are pairs $(f, (\pi, x_b, u_p)) \in \mathbb{P}(S^3\mathbb{U}^*) \times_{\Gamma} \mathbb{F}$ such that the cubic f is contained in the plane π, has a point of multiplicity at least 2 at x_b and has u_p as a pair of tangents (possibly coincident) at x_b. The generic points are those such that f is a non-degenerate cubic with a node at x_b and nodal tangents u_p.

Notice that the variety K_{nod} can be obtained as a blow up of the variety X_{nod}, introduced in Hernández et al. (2007), along the subvariety consisting of pairs $(f, (\pi, x_b)) \in X_{\text{nod}}$ whose nodal cubic f degenerates into three concurrent lines in π meeting at x_b, so that its exceptional divisor is the variety K_{trip} given in Section 1.1.

Indeed, X_{nod} is the projective bundle $\mathbb{P}(\mathbb{E}'_{\text{nod}})$, where $\mathbb{E}'_{\text{nod}}$ is the subbundle of $S^3\mathbb{U}|_{\mathbb{P}(\mathbb{U})}$ whose fiber over $(\pi, x_b) \in \mathbb{P}(\mathbb{U})$ is the linear subspace of forms $\varphi \in S^3\mathbb{U}$ defined over π that have multiplicity al least 2 at x_b. Then, the natural projection $K_{\text{nod}} \to X_{\text{nod}}$ is isomorphic to the blow-up of X_{nod} along $\mathbb{P}(S^3\mathbb{Q}^*)$. It can be deduced from Proposition 4.1 in Hernández and Miret (2003), where the description of the blow-up of a projective bundle $\mathbb{P}(\mathbb{E})$ along a projective subbundle $\mathbb{P}(\mathbb{F})$ is given in terms of the quotient vector bundle $\mathbb{E}/\mathbb{F}$.

We will continue denoting by b and p the pullbacks to $\text{Pic}(K_{\text{nod}})$ of the classes b and p in $\text{Pic}(\mathbb{F})$ under the natural projection $K_{\text{nod}} \to \mathbb{F}$. Since this projection is flat, b and p are the classes of the hypersurfaces of K_{nod} whose points $(f, (\pi, x_b, u_p))$ satisfy that x_b is on a given plane and that u_p intersects a given line, respectively. Furthermore, by Lemma 1.1, the relation

$$\zeta = \nu - 3\mu$$

holds in $\text{Pic}(K_{\text{nod}})$, where ζ is the hyperplane class of K_{nod} and ν the class of the hypersurface of K_{nod} whose points $(f, (\pi, x_b, u_p))$ satisfy that f intersects a given line.

Proposition 1.3. *The intersection ring* $A^*(K_{\text{nod}})$ *is isomorphic to the quotient of the polynomial ring* $\mathbb{Z}[\mu, b, p, \nu]$ *by the ideal*

$$\begin{aligned}\langle \mu^4, \quad & b^3 - \mu b^2 + \mu^2 b - \mu^3, \\ & p^3 - 3(\mu + b)p^2 + 2(3\mu^2 + 2\mu b + 3b^2)p - 8\mu^3 - 8\mu b^2, \\ & \nu^5 - (7\mu + 5b + p)\nu^4 + (27\mu^2 + 22\mu b + 6\mu p + 15b^2 + 6bp)\nu^3 \\ & -(57\mu^3 + 49\mu^2 b + 21\mu^2 p + 47\mu b^2 + 22\mu bp + 15b^3 + 21b^2 p)\nu^2 \\ & +(48\mu^3 b + 36\mu^3 p + 54\mu^2 b^2 + 48\mu^2 bp + 24\mu b^3 + 48\mu b^2 p + 36b^3 p)\nu \\ & -18\mu^3 b^2 - 36\mu^3 bp + 18\mu^2 b^3 - 54\mu^2 b^2 p - 36\mu b^3 p\rangle.\end{aligned}$$

In particular, the Picard group $\text{Pic}(K_{\text{nod}})$ *is a rank 4 free group generated by* μ, b, p *and* ν.

Proof. Since $\zeta = \nu - 3\mu$, the intersection ring $A^*(K_{\text{nod}})$ is isomorphic to

$$A^*(\mathbb{F})[\nu]/\sum \bar{\pi}^* c_i(\mathbb{E}_{\text{nod}} \otimes \mathcal{O}_\Gamma(-3))\nu^{5-i},$$

where $\bar{\pi} : \mathbb{E}_{\text{nod}} \to \mathbb{F}$ is the natural projection (see Fulton, 1984, ex. 8.3.4). Now, using Proposition 1.2, the total Chern class of $\mathbb{E}_{\text{nod}}$ can be computed:

$$c(\mathbb{E}_{\text{nod}}) = c(\mathbb{L}^* \otimes \mathcal{O}_{\mathbb{F}}(-1)) \cdot c(S^3\mathbb{Q}^*).$$

From this and Lemma 2.27 in Xambó-Descamps (1996) the fourth claimed relation follows.

The remaining relations come from the intersection ring of $\mathbb{F}$ described in Proposition 1.1. □

Thus we have, using the projection formula, that

$$\int_{K_{\text{nod}}} \mu^i b^j p^h \nu^{11-i-j-h} = \int_{\mathbb{F}} \mu^i b^j p^h s_{7-i-j-h}(\mathbb{E}_{\text{nod}} \otimes \mathcal{O}_\Gamma(-3)),$$

where $s_t(\mathbb{E}_{\text{nod}} \otimes \mathcal{O}_\Gamma(-3))$ denotes the t-th Segre class of the vector bundle $\mathbb{E}_{\text{nod}} \otimes \mathcal{O}_\Gamma(-3)$, which can be calculated from the resolution of Proposition 1.2.

This allows us to compute all the intersection numbers of K_{nod} in the conditions μ, b, p and ν (see the Eq. (2)). The Wit script for the computation of these numbers is included in the Section 4.1.

$\mu^3b^2p^2\nu^4 = 1$	$\mu^2b^3p^2\nu^4 = 1$	$\mu b^3p^3\nu^4 = 3$	$b^3p^4\nu^4 = 3$
$\mu^3bp^3\nu^4 = 3$	$\mu^2b^2p^3\nu^4 = 6$	$\mu b^2p^4\nu^4 = 17$	$b^2p^5\nu^4 = 20$
$\mu^3p^4\nu^4 = 3$	$\mu^2bp^4\nu^4 = 17$	$\mu bp^5\nu^4 = 50$	$bp^6\nu^4 = 70$
$\mu^3b^2p\nu^5 = 1$	$\mu^2p^5\nu^4 = 20$	$\mu p^6\nu^4 = 70$	$p^7\nu^4 = 140$
$\mu^3bp^2\nu^5 = 8$	$\mu^2b^3p\nu^5 = 1$	$\mu b^3p^2\nu^5 = 10$	$b^3p^3\nu^5 = 24$
$\mu^3p^3\nu^5 = 18$	$\mu^2b^2p^2\nu^5 = 18$	$\mu b^2p^3\nu^5 = 74$	$b^2p^4\nu^5 = 154$
$\mu^3b^2\nu^6 = 1$	$\mu^2bp^3\nu^5 = 68$	$\mu bp^4\nu^5 = 254$	$bp^5\nu^5 = 520$
$\mu^3bp\nu^6 = 7$	$\mu^2p^4\nu^5 = 126$	$\mu p^5\nu^5 = 460$	$p^6\nu^5 = 980$
$\mu^3p^2\nu^6 = 25$	$\mu^2b^3\nu^6 = 1$	$\mu b^3p\nu^6 = 11$	$b^3p^2\nu^6 = 49$
$\mu^3b\nu^7 = 6$	$\mu^2b^2p\nu^6 = 18$	$\mu b^2p^2\nu^6 = 147$	$b^2p^3\nu^6 = 444$
$\mu^3p\nu^7 = 18$	$\mu^2bp^2\nu^6 = 123$	$\mu bp^3\nu^6 = 638$	$bp^4\nu^6 = 1770$
$\mu^3\nu^8 = 12$	$\mu^2p^3\nu^6 = 316$	$\mu p^4\nu^6 = 1482$	$p^5\nu^6 = 4020$
	$\mu^2b^2\nu^7 = 18$	$\mu b^3\nu^7 = 12$	$b^3p\nu^7 = 60$
	$\mu^2bp\nu^7 = 112$	$\mu b^2p\nu^7 = 154$	$b^2p^2\nu^7 = 722$
	$\mu^2p^2\nu^7 = 398$	$\mu bp^2\nu^7 = 974$	$bp^3\nu^7 = 3584$
	$\mu^2b\nu^8 = 100$	$\mu p^3\nu^7 = 2780$	$p^4\nu^7 = 9852$
	$\mu^2p\nu^8 = 304$	$\mu b^2\nu^8 = 160$	$b^3\nu^8 = 72$
	$\mu^2\nu^9 = 216$	$\mu bp\nu^8 = 932$	$b^2p\nu^8 = 816$
		$\mu p^2\nu^8 = 3324$	$bp^2\nu^8 = 4956$
		$\mu b\nu^9 = 872$	$p^3\nu^8 = 15768$
		$\mu p\nu^9 = 2696$	$b^2\nu^9 = 904$
		$\mu\nu^{10} = 2040$	$bp\nu^9 = 5072$
			$p^2\nu^9 = 18336$
			$b\nu^{10} = 5040$
			$p\nu^{10} = 15960$
			$\nu^{11} = 12960$

(2)

1.1. *Degenerations of K_{nod}*

Let U_{nod} be the subvariety of K_{nod} whose points are pairs $(f, (\pi, x_b, u_p)) \in K_{\text{nod}}$ such that f is an irreducible nodal cubic contained in the plane π, with a node at x_b and u_p as a pair of nodal tangents.

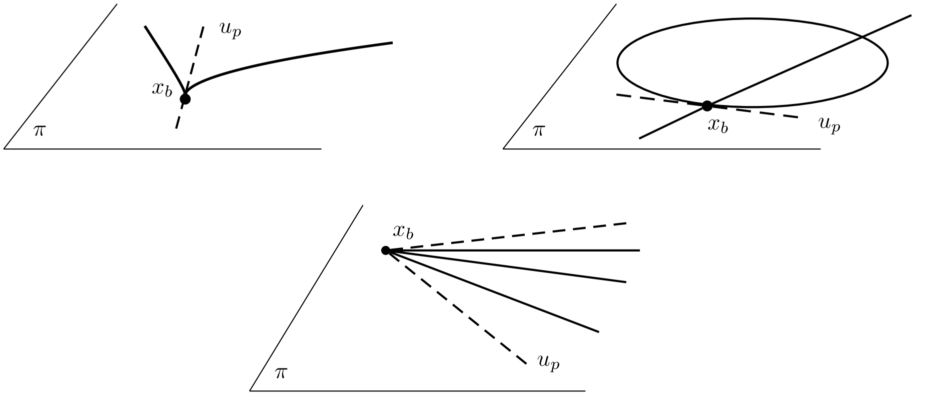

Fig. 1. A closed point of K_{ncusp}, K_{consec} and K_{trip} (degenerations γ, χ and τ).

In fact, K_{nod} is a compactification of U_{nod} whose boundary $K_{\text{nod}} - U_{\text{nod}}$ consists of the following three codimension 1 irreducible components, called *degenerations* of first order of K_{nod} (see Fig. 1):

- $K_{\text{ncusp}} = \mathbb{P}(v_2^*(\mathbb{E}_{\text{nod}}))$, where $v_2 : \mathbb{P}(\mathbb{Q}^*) \to \mathbb{F}$ is the Veronese map that assigns $(\pi, x_c, u_q) \mapsto (\pi, x_c, u_q^2)$, parameterizes pairs $(f, (\pi, x_c, u_q^2)) \in K_{\text{nod}}$ such that f is a cuspidal cubic with cusp x_c and cuspidal tangent u_q at x_c.

  The cubics f of K_{ncusp} over the fiber (π, x_c, u_q^2), with $\pi = \{x_3 = 0\}$, $x_c = [1, 0, 0, 0]$ and $u_q = \{x_1 = 0\}$, that is to say, where u_p is a double line u_q^2, correspond to the forms (1) which satisfy $b_2 = 0$. Hence, the equation of f is given by

  $$a_0x_0x_1^2 + a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3 = 0.$$

- K_{consec} parameterizes pairs $(f, (\pi, x_b, u_p)) \in K_{\text{nod}}$ such that f is a cubic consisting of a line $u_{\ell'}$ which goes through x_b and a conic f' tangent to a line u_ℓ at x_b, where $u_p = u_\ell \cdot u_{\ell'}$.

  The cubics f of K_{consec} over the fiber (π, x_b, u_p), with $\pi = \{x_3 = 0\}$, $x_b = [1, 0, 0, 0]$ and $u_p = \{x_1(b_1x_1 + b_2x_2) = 0\}$, where $u_\ell = \{b_1x_1 + b_2x_2 = 0\}$ and $u_{\ell'} = \{x_1 = 0\}$, correspond to the forms (1) which satisfy $a_4 = 0$, that is to say, forms which have two nodes. Hence, the equation of f is given by

  $$a_0x_0x_1(b_1x_1 + b_2x_2) + a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 = 0.$$

- $K_{\text{trip}} = \mathbb{P}(S^3\mathbb{Q}^*|_{\mathbb{F}})$ parameterizes pairs $(f, (\pi, x_b, u_p)) \in K_{\text{nod}}$ such that f is a cubic consisting of three lines concurrent at x_b. Notice that K_{trip} is a projective subbundle of K_{nod}.

  The cubics f of K_{trip} over the fiber (π, x_b, u_p), with $\pi = \{x_3 = 0\}$, $x_b = [1, 0, 0, 0]$ and $u_p = \{x_1(b_1x_1 + b_2x_2) = 0\}$ correspond to the forms (1) which satisfy $a_0 = 0$, that is to say, forms which have multiplicity three at x_b. Hence, the equation of f is given by

  $$a_1x_1^3 + a_2x_1^2x_2 + a_3x_1x_2^2 + a_4x_2^3 = 0.$$

  We will denote the classes in $\text{Pic}(K_{\text{nod}})$ of the degenerations K_{ncusp}, K_{consec} and K_{trip} by γ, χ and τ, respectively.

Proposition 1.4. *In* $\text{Pic}(K_{\text{nod}})$ *the following relations hold:*

$$\gamma = -2\mu - 2b + 2p,$$
$$\chi = -6\mu - 6b + 3p + 2\nu,$$
$$\tau = -\mu + b - p + \nu.$$

Proof. To get the first expression notice that $\gamma = [\pi^*D]$, where D is the image of the Veronese map $v_2 : \mathbb{P}(\mathbb{Q}^*) \to \mathbb{F}$ and $\pi : K_{\text{nod}} \to \mathbb{F}$ is the natural projection. Then the expression of γ turns out of the relation $[D] = -2\mu - 2b + 2p$ in $\text{Pic}(\mathbb{F})$. To obtain the second expression note that we can write χ as a linear combination of the basis $\{\mu, b, p, \nu\}$ of $\mathrm{Pic}(K_{\mathrm{nod}})$, $\chi = \alpha_0\mu + \alpha_1 b + \alpha_2 p + \alpha_3\nu$. Multiplying this relation by $\mu^3 b^2 p^2 \nu^3$, and taking into account Proposition 1.3 and relations (2), we get that

$$\alpha_3 = \int_{K_{\mathrm{nod}}} \mu^3 b^2 p^2 \nu^3 \chi = \int_{K_{\mathrm{consec}}} \mu^3 b^2 (\ell + \ell')^2 (\nu' + \ell')^3 = 2 \int_{K_{\mathrm{consec}}} \mu^3 b^2 \ell\ell' \nu'^3 = 2,$$

where ν', ℓ' and ℓ are the conditions that the conic f', the lines $u_{\ell'}$ and u_ℓ intersect, respectively, a given line. Now, multiplying by $\mu^3 b^2 p \nu^4$, $\mu^3 b p \nu^5$ and $\mu^2 b^2 p^2 \nu^4$ we obtain $\alpha_2 = 3$, $\alpha_1 = -6$ and $\alpha_0 = -6$. In order to calculate the expression of $\tau = \mathbb{P}(S^3\mathbb{Q}^*|_{\mathbb{F}})$ notice that

$$\tau = \nu - 3\mu + c_1\left(\frac{\mathbb{E}_{\mathrm{nod}}}{S^3\mathbb{Q}^*|_{\mathbb{F}}}\right) = \nu - 3\mu + c_1\big((\mathbb{U}^*/\mathbb{Q}^*) \otimes \mathcal{O}_{\mathbb{F}}(-1)\big),$$

from which we obtain the third relation of the proposition. □

Notice that from the Proposition it is straightforward to obtain the relations

$$6b = -2\mu + 5\gamma - 2\chi + 4\tau,$$
$$3p = 4\mu + 4\gamma - \chi + 2\tau,$$
$$2\nu = 4\mu + \gamma + 2\tau.$$

2. More fundamental conditions

If we consider a nodal cubic given by an equation of the type (1), the analytical expressions of the inflection points x_v, the line u_s which goes through them, and the triple of lines u_z joining the node with each of the inflection points are as follows:

$$u_s = \{a_0x_0 + a_2x_1 + a_3x_2 = 0,\ x_3 = 0\},$$
$$u_z = \{a_1x_1^3 + a_4x_2^3 = 0,\ x_3 = 0\},$$
$$x_v = \left\{\left(a_2\rho^i\sqrt[3]{a_4} + a_3\sqrt[3]{a_1},\ a_0\rho^i\sqrt[3]{a_4},\ a_0\sqrt[3]{a_1}\right)\right\}_{i\in\{0,1,2\}},$$

where ρ is a primitive cube root of -1.

Notice that there exist nodal cubics such that some of these distinguished elements are not well defined. The aim of this section is to construct a compactification of the variety of nodal cubics in $\mathbb{P}^3$ where all the elements u_z, u_s and x_v are well defined.

2.1. *The condition z*

In this subsection we will construct first a new compactification K^z_{nod} as the blow-up of K_{nod} along the projective subbundle $\mathbb{P}(\mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-1))$ (consisting of those cubics of (1) with $a_1 = a_4 = 0$), in such a way that it parameterizes the family of nodal cubics that have a point of multiplicity at least 2 at x_b, for which u_p is a pair of tangents (possibly coincident) at x_b, and where the triple of lines u_z joining x_b with each of the inflection points is always well defined.

In order to construct this new variety we consider, as in Proposition 2.1 of Hernández and Miret (2003), the subbundle $F = \mathbb{U}^*_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-1)$ of $\mathbb{E}_{\mathrm{nod}}$ and the quotient bundle

$$G_z = \mathbb{E}_{\mathrm{nod}}/F = S^3\mathbb{Q}^*/(\mathbb{Q}^* \otimes \mathcal{O}_{\mathbb{F}}(-1)).$$

The generic point of the projective bundle $\mathbb{G}_z = \mathbb{P}(G_z)$ is a tuple (π, x_b, u_p, u_z) such that x_b is a point on the plane π, u_p is a pair of distinct lines which meet at x_b, and u_z is a triple of lines which meet at x_b that satisfy the following relations of cross-ratios with respect to u_p:

$$\rho(u_{z_1}, u_{z_2}, u_{z_3}, u_{p_i}) = \rho_i, \quad i = 1, 2, \tag{3}$$

where ρ_i is a primitive cube root of -1. Notice that when u_p is a double line, then two of the lines u_z coincide with u_p. Moreover, by Lemma 1.1, the relation

$$\zeta = z - \mu - 2p + 2b$$

holds in $\mathrm{Pic}(\mathbb{G}_z)$, where ζ is the hyperplane class of $\mathbb{G}_z$ and z is the class of the hypersurface whose points (π, x_b, u_p, u_z) satisfy that u_z intersects a given line.

We will denote by $\mathbb{E}^z_{\mathrm{nod}}$ the subbundle of $\mathbb{E}_{\mathrm{nod}}|_{\mathbb{G}_z}$ whose fiber $\left(\mathbb{E}^z_{\mathrm{nod}}\right)_{[u_z]}$ over $(\mathbb{G}_z)_{(\pi,x_b,u_p)}$, $(\pi, x_b, u_p) \in \mathbb{F}$, consists of those $f \in (\mathbb{E}_{\mathrm{nod}})_{(\pi,x_b,u_p)}$ such that $k(f) \in \langle u_z \rangle$, where $k : \mathbb{E}_{\mathrm{nod}} \to \mathbb{G}_z$ is the canonical projection. Then, the sequence of vector bundles over $\mathbb{G}_z$

$$0 \to \mathbb{U}^*|_{\mathbb{G}_z} \otimes \mathcal{O}_{\mathbb{F}}(-1)|_{\mathbb{G}_z} \to \mathbb{E}^z_{\mathrm{nod}} \to \mathcal{O}_{\mathbb{G}_z}(-1) \to 0, \tag{4}$$

is exact and allows us to compute all the Chern and Segre classes of $\mathbb{E}^z_{\mathrm{nod}}$.

Now, we consider K^z_{nod} the projective bundle $\mathbb{P}(\mathbb{E}^z_{\mathrm{nod}})$, whose points are pairs $(f, (\pi, x_b, u_p, u_z))$ in $\mathbb{P}(S^3\mathbb{U}^*) \times_\Gamma \mathbb{G}_z$ satisfying the following conditions: f is a cubic contained in the plane π and has a point of multiplicity at least 2 at x_b; u_p is a pair of tangents lines (possibly coincident) at x_b; and u_z as a triple of lines through x_b satisfying (3). Notice that when f is non-degenerate then x_b, u_p and u_z are its node, nodal tangents and lines joining the node with the inflections, respectively.

It turns out that K^z_{nod} is isomorphic to the blow-up of K_{nod} along the projective subbundle $\mathbb{P}(\mathbb{U}^*|_{\mathbb{F}} \otimes \mathcal{O}_{\mathbb{F}}(-1))$. Moreover, the exceptional divisor coincides with $\mathbb{P}(\mathbb{U}^*|_{\mathbb{G}_z} \otimes \mathcal{O}_{\mathbb{F}}(-1)|_{\mathbb{G}_z})$, whose points $(f, (\pi, x_b, u_p, u_z))$ satisfy that f consists of three lines, two of them coinciding with the pair of tangents u_p. We will denote the class in $\mathrm{Pic}(K^z_{\mathrm{nod}})$ of the exceptional divisor by ψ (see Fig. 2).

Hence, K^z_{nod} is a compactification of U_{nod} whose boundary $K^z_{\mathrm{nod}} - U_{\mathrm{nod}}$ consists of four codimension 1 irreducible components which correspond to γ, χ, τ and ψ. The description of the triple of lines u_z for each degeneration is as follows:

- γ: One line of u_z coincides with the line joining the cusp and the inflection of the cuspidal curve and the two remaining lines coincide with the cuspidal tangent.
- χ: The three lines of u_z coincide with the line $u_{\ell'}$ of the pair u_p different from the tangent line to the conic at x_b.
- τ, ψ: The triple of lines u_z satisfy the relations (3) with respect to the pair of lines u_p. For the degeneration τ, there is an analogous dependence relation between the lines of u_z and the triple of lines through x_b that determine the nodal cubic.

We will continue denoting by μ, b, p and ν the pullbacks to $\mathrm{Pic}(K^z_{\mathrm{nod}})$ of the classes μ, b, p and ν in $\mathrm{Pic}(K_{\mathrm{nod}})$ under the natural projection $K^z_{\mathrm{nod}} \to K_{\mathrm{nod}}$. And we will denote by z the pullback to $\mathrm{Pic}(K^z_{\mathrm{nod}})$ of the class z in $\mathrm{Pic}(\mathbb{G}_z)$ under the natural projection $K^z_{\mathrm{nod}} \to \mathbb{G}_z$. Thus, using the projection formula,

$$\int_{K^z_{\mathrm{nod}}} \mu^i b^j p^h z^r \nu^k = \int_{\mathbb{G}_z} \mu^i b^j p^h z^r s_{6-i-j-h-r}(\mathbb{E}^z_{\mathrm{nod}} \otimes \mathcal{O}_\Gamma(-3)),$$

and taking into account the sequence (4), from which the Segre classes of $\mathbb{E}^z_{\mathrm{nod}}$ are obtained, we can compute all the intersection numbers of K^z_{nod} in the conditions μ, b, p, z and ν. The actual computations have been done with the script included in 4.2. Here is a sample of the result:

$$\begin{array}{llll}
\mu^3 z\nu^7 = 36 & \mu^2 z\nu^8 = 600 & \mu z\nu^9 = 5256 & z\nu^{10} = 30720 \\
\mu^3 z^2\nu^6 = 63 & \mu^2 z^2\nu^7 = 1026 & \mu z^2\nu^8 = 8748 & z^2\nu^9 = 49296 \\
\mu^3 z^3\nu^5 = 69 & \mu^2 z^3\nu^6 = 1206 & \mu z^3\nu^7 = 10660 & z^3\nu^8 = 60816 \\
\mu^3 z^4\nu^4 = 45 & \mu^2 z^4\nu^5 = 1002 & \mu z^4\nu^6 = 9966 & z^4\nu^7 = 60428 \\
\mu^3 z^5\nu^3 = 15 & \mu^2 z^5\nu^4 = 570 & \mu z^5\nu^5 = 7150 & z^5\nu^6 = 48640 \\
 & \mu^2 z^6\nu^3 = 180 & \mu z^6\nu^4 = 3720 & z^6\nu^5 = 31120 \\
 & & \mu z^7\nu^3 = 1120 & z^7\nu^4 = 14840 \\
 & & & z^8\nu^3 = 4200
\end{array} \tag{5}$$

Condition z was not considered by Schubert, so the numbers involving this condition do not appear in Schubert (1879).

2.2. The condition s

Now, in order to add to each nodal cubic the line u_s that goes through the three inflection points, we construct a new variety blowing-up K^z_{nod} along the subvariety $\mathbb{P}(\mathcal{O}_{\mathbb{G}_z}(-1))$. Again, as in Proposition 2.1

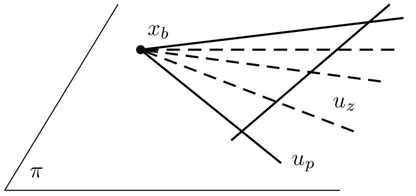

Fig. 2. A closed point of the exceptional divisor of K^z_{nod} (degeneration ψ).

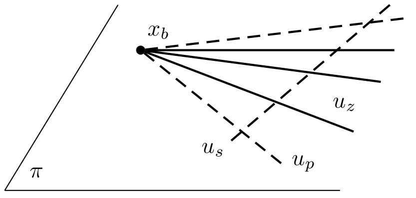

Fig. 3. A closed point of the exceptional divisor of $K^{z,s}_{\text{nod}}$ (degeneration τ').

of Hernández and Miret (2003), we consider the subbundle $F_z = \mathcal{O}_{\mathbb{G}_z}(-1)$ of $\mathbb{E}^z_{\text{nod}}$ and the quotient bundle

$$G_{z,s} = \mathbb{E}^z_{\text{nod}}/F_z = \mathbb{U}^*|_{\mathbb{G}_z} \otimes \mathcal{O}_{\mathbb{F}}(-1)|_{\mathbb{G}_z}.$$

The projective bundle $\mathbb{G}_{z,s} = \mathbb{P}(G_{z,s})$ parameterizes tuples $(\pi, x_b, u_p, u_z, u_s)$ such that x_b is a point on the plane π, u_p is a pair of lines in π which meet at x_b, u_z is a triple of lines in π which meet at x_b that satisfy the relations (3), and u_s is a line on π. Moreover, by Lemma 1.1, the relation

$$s = g - p + \gamma \tag{6}$$

holds in $\text{Pic}(\mathbb{G}_{z,s})$, where g is the hyperplane class of $\mathbb{G}_{z,s}$, s the class of the hypersurface whose points $(\pi, x_b, u_p, u_z, u_s)$ satisfy that u_s intersects a given line, and γ is the class of the degeneration of $\mathbb{G}_{z,s}$ whose points $(\pi, x_b, u_p, u_z, u_s)$ satisfy that the pair of lines u_p is a double line which coincides with two lines of u_z and the remaining line of u_z coincides with the line u_s.

We will denote by $\mathbb{E}^{z,s}_{\text{nod}}$ the vector subbundle of $\mathbb{E}^z_{\text{nod}}|_{\mathbb{G}_{z,s}}$ whose fiber $\left(\mathbb{E}^{z,s}_{\text{nod}}\right)_{[u_s]}$ over $\left(\mathbb{G}_{z,s}\right)_{(\pi,x_b,u_p,u_z)}$, $(\pi, x_b, u_p, u_z) \in \mathbb{G}_{z,s}$, consists of those $f \in \left(\mathbb{E}^z_{\text{nod}}\right)_{(\pi,x_b,u_p,u_z)}$ such that $k'(f) \in \langle u_s \rangle$, where $k' : \mathbb{E}^z_{\text{nod}} \to \mathbb{G}_{z,s}$ is the canonical projection. From the definition, it follows that the sequence

$$0 \to \mathcal{O}_{\mathbb{G}_z}(-1)|_{\mathbb{G}_{z,s}} \to \mathbb{E}^{z,s}_{\text{nod}} \to \mathcal{O}_{\mathbb{G}_{z,s}}(-1) \to 0, \tag{7}$$

is an exact sequence of vector bundles over $\mathbb{G}_{z,s}$.

The projective bundle $K^{z,s}_{\text{nod}} = \mathbb{P}(\mathbb{E}^{z,s}_{\text{nod}})$ parameterizes pairs $(f, (\pi, x_b, u_p, u_z, u_s))$ in $\mathbb{P}(S^3\mathbb{U}^*) \times_\Gamma \mathbb{G}_{z,s}$ such that $(f, (\pi, x_b, u_p, u_z)) \in K^z_{\text{nod}}$. Notice that when f is non-degenerate then x_b, u_p, u_z and u_s are their node, nodal tangents, lines joining the node with the inflections, and line through the inflection points, respectively.

The variety $K^{z,s}_{\text{nod}}$ is isomorphic to the blow-up of K^z_{nod} along the projective subbundle $\mathbb{P}(\mathcal{O}_{\mathbb{G}_z}(-1))$. Furthermore, the exceptional divisor coincides with $\mathbb{P}(\mathcal{O}_{\mathbb{G}_z}(-1)|_{\mathbb{G}_{z,s}})$, whose points $(f, (\pi, x_b, u_p, u_z, u_s))$ satisfy that f degenerates into three concurrent lines which coincide with u_z (see Fig. 3). We will denote the class in $\text{Pic}(K^{z,s}_{\text{nod}})$ of the exceptional divisor by τ' (this degeneration was not considered by Schubert (1879)).

The variety $K^{z,s}_{\text{nod}}$ is a compactification of U_{nod} whose boundary $K^{z,s}_{\text{nod}} - U_{\text{nod}}$ consists of five codimension 1 irreducible components which correspond to γ, χ, τ, ψ and τ'. The description of the line u_s for each degeneration is as follows:

– γ: u_s coincides with the line joining the cusp and the inflection of the cuspidal curve.
– χ: u_s coincides with the tangent line to the conic at one of the intersection points with the pair of nodal tangents different from x_b.

– τ: u_s is a line through x_b determined by the dependence relations with the remaining lines that form such a degeneration.
– ψ: u_s coincides with one of the lines that constitute the nodal cubic, different from the nodal tangents.
– τ': u_s is a new line in the plane π.

Once again, using the projection formula and denoting by g the pullback to $K^{z,s}_{\mathrm{nod}}$ of the hyperplane class g of $\mathbb{G}_{z,s}$, we get:

$$\int_{K^{z,s}_{\mathrm{nod}}} \mu^i b^j p^h z^r g^t \nu^k = \int_{\mathbb{G}_{z,s}} \mu^i b^j p^h z^r g^t s_{6-i-j-h-r}(\mathbb{E}^{z,s}_{\mathrm{nod}} \otimes \mathcal{O}_\Gamma(-3)).$$

Then, computing the Segre classes $\mathbb{E}^{z,s}_{\mathrm{nod}}$ from (7), we can calculate all the intersection numbers of $K^{z,s}_{\mathrm{nod}}$ in the conditions μ, b, p, z, g and ν. The script for this computation is included in 4.3.

Together with Miret et al. (2003), where the fundamental numbers of cuspidal cubics were computed, and taking into account relation (6), all intersection numbers of $K^{z,s}_{\mathrm{nod}}$ in the conditions μ, b, p, z, s and ν can be obtained in the following way:

$$\int_{K^{z,s}_{\mathrm{nod}}} \mu^i b^j p^h z^r s^t \nu^k = \int_{K^{z,s}_{\mathrm{nod}}} \mu^i b^j p^h z^r s^{t-1}(g-p)\nu^k + 2^h \int_{K_{\mathrm{ncusp}}} \mu^i b^j q^h z^{r+t-1} \nu^k,$$

where q is the condition on K_{ncusp} that the cuspidal tangent intersects a given line. The Wit script for the calculation of the numbers of K_{ncusp} is included in 4.4.

Now, from the intersection numbers of the varieties $K^{z,s}_{\mathrm{nod}}$ and K_{ncusp}, we can compute the numbers of nodal plane curves with the condition μ, b, p, z, s and ν. In order to obtain them we can use the Wit script 4.5.

A sample of them is given below. In this case, we include those numbers involving the conditions μ, b, s and ν.

$$\begin{array}{llll}
\mu^3 s\nu^7 = 18 & \mu^2 s\nu^8 = 296 & \mu s\nu^9 = 2560 & s\nu^{10} = 14760 \\
\mu^3 bs\nu^6 = 11 & \mu^2 bs\nu^7 = 164 & \mu bs\nu^8 = 1284 & bs\nu^9 = 6560 \\
\mu^3 b^2 s\nu^5 = 2 & \mu^2 b^2 s\nu^6 = 32 & \mu b^2 s\nu^7 = 254 & b^2 s\nu^8 = 1256 \\
 & \mu^2 b^3 s\nu^5 = 2 & \mu b^3 s\nu^6 = 21 & b^3 s\nu^7 = 108
\end{array} \tag{8}$$

$$\begin{array}{llll}
\mu^3 s^2\nu^6 = 25 & \mu^2 s^2\nu^7 = 374 & \mu s^2\nu^8 = 2948 & s^2\nu^9 = 15280 \\
\mu^3 bs^2\nu^5 = 20 & \mu^2 bs^2\nu^6 = 263 & \mu bs^2\nu^7 = 1822 & bs^2\nu^8 = 8012 \\
\mu^3 b^2 s^2\nu^4 = 4 & \mu^2 b^2 s^2\nu^5 = 56 & \mu b^2 s^2\nu^6 = 391 & b^2 s^2\nu^7 = 1642 \\
 & \mu^2 b^3 s^2\nu^4 = 4 & \mu b^3 s^2\nu^5 = 36 & b^3 s^2\nu^6 = 153
\end{array}$$

$$\begin{array}{lll}
\mu^2 s^3\nu^6 = 50 & \mu s^3\nu^7 = 712 & s^3\nu^8 = 5304 \\
\mu^2 bs^3\nu^5 = 40 & \mu bs^3\nu^6 = 504 & bs^3\nu^7 = 3316 \\
\mu^2 b^2 s^3\nu^4 = 8 & \mu b^2 s^3\nu^5 = 108 & b^2 s^3\nu^6 = 718 \\
 & \mu b^3 s^3\nu^4 = 8 & b^3 s^3\nu^5 = 68
\end{array}$$

$$\begin{array}{ll}
\mu s^4\nu^6 = 60 & s^4\nu^7 = 676 \\
\mu bs^4\nu^5 = 40 & bs^4\nu^6 = 482 \\
\mu b^2 s^4\nu^4 = 8 & b^2 s^4\nu^5 = 104 \\
 & b^3 s^4\nu^4 = 8.
\end{array}$$

Among these numbers, only two of them, $\mu^3 s\nu^7 = 18$ and $\mu^3 s_e \nu^6 = \mu^3 s^2 \nu^6 = 25$, were given by Schubert (see Schubert (1879), p. 160).

2.3. The condition v

We construct now another compactification of U_{nod} by considering the closure K^{v}_{nod} of the graph of the rational map that assigns the triplet of flexes x_v to a given nodal cubic $(f, (\pi, x_b, u_p, u_z, u_s))$ of $K^{z,s}_{\mathrm{nod}}$. Notice that the generic points of this new variety K^{v}_{nod} consist of pairs $(f, (\pi, x_b, u_p, u_z, u_s, x_v))$

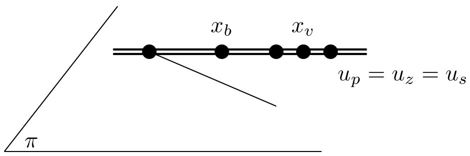

Fig. 4. An irreducible component of $K^{v}_{\text{nod}} - K^{z,s}_{\text{nod}}$ (degeneration ϑ).

where x_v is the triple of flexes of the nodal cubic f. The projection map $h_v : K^{v}_{\text{nod}} \to K^{z,s}_{\text{nod}}$ is just the blow-up of $K^{z,s}_{\text{nod}}$ along the subvariety of $K^{z,s}_{\text{nod}}$ where the triplet x_v is not well-defined.

Notice as well that the tuples $(\pi, x_b, u_p, u_z, u_s, x_v)$ belong to the closure $\mathbb{G}_v$ of the graph of the rational map that assigns to each $(\pi, x_b, u_p, u_z, u_s) \in \mathbb{G}_{z,s}$ the triplet x_v of points intersecting the line u_s with the triple of lines u_z. Then, there exists a natural projection $\overline{\pi} : K^{v}_{\text{nod}} \to \mathbb{G}_v$ that assigns to each $(f, (\pi, x_b, u_p, u_z, u_s, x_v)) \in K^{v}_{\text{nod}}$ the tuple $(\pi, x_b, u_p, u_z, u_s, x_v) \in \mathbb{G}_v$.

In Fig. 4 above we show one of the irreducible components of the exceptional divisor of K^{v}_{nod}, called degeneration ϑ by Schubert. This degeneration can be obtained by means of a homolography process (see Fulton (1984, p. 190), Kleiman (1984, p. 17), and Kleiman (1986, p. 53)). As we can see, there exist dependence relations among the distinguished elements of this degeneration, otherwise the dimension of this variety would be greater than 10. More precisely, there exists a relation among the five points on the double line, in the sense that given four of them, the fifth point is determined.

Moreover, the description of the triplet of flexes x_v for the remaining degenerations of $K^{z,s}_{\text{nod}}$ is as follows:

- γ: Two points of the triplet x_v coincide with the inflection point of the cuspidal cubic, whereas the third one coincides with the cusp.
- χ: The three points of the triplet x_v coincide with the point x_a, which is the intersection point, other than the node x_b, of the conic f' and the line $u_{\ell'}$.
- τ: The three points of the triplet x_v coincide with the point where the three lines of the nodal cubic meet.
- ψ, τ': The points of the triplet x_v are the intersection points of the line u_s with the triple of lines u_z.

Now, we express the condition v in $\mathsf{Pic}(X_{\text{nod}})$ in terms of s and the degenerations γ and χ. This formula, which we justify below was given by Schubert (1879, p. 150).

Proposition 2.1. *The following relation holds in* $\mathsf{Pic}(X_{\text{nod}})$*:*

$$v = \frac{3}{2}s + \frac{3}{4}\gamma + \frac{1}{2}\chi. \tag{9}$$

Proof. Consider the projective bundle $\mathbb{F}_{s,v}$ which parameterizes triples (π, u_s, x_v), where u_s is a line contained on π and x_v is a triple of points over u_s. It is easy to see that its degeneration $D_{s,v}$, consisting of triples (π, u_s, x_v) such that two of the three points in x_v coincide, satisfy the relation $4v = 6s + [D_{s,v}]$ in $\mathsf{Pic}(\mathbb{F}_{s,v})$. Hence, in $\mathsf{Pic}(X_{\text{nod}})$ the following holds: there exist integers α_0 and α_1 such that $4v = 6s + \alpha_0\gamma + \alpha_1\chi$. Multiplying this relation by $\mu^3 b^2 p^2 s^2 \nu$ and $\mu^3 bps^2\nu^3$ we get $\alpha_0 = 3$ and $\alpha_1 = 2$. □

In particular, using formula (9) or its equivalent expression $v = -\frac{15}{2}\mu - \frac{3}{2}b + \frac{3}{2}s + 4\nu$, and taking enough of the numbers given in (2) and (8), we obtain the following intersection numbers:

$$\begin{array}{llll}
\mu^3 v\nu^7 = 66, & \mu^2 v\nu^8 = 1068, & \mu v\nu^9 = 9072, & v\nu^{10} = 51120 \\
\mu^3 bv\nu^6 = 39, & \mu^2 bv\nu^7 = 574, & \mu bv\nu^8 = 4424, & bv\nu^9 = 22104 \\
\mu^3 b^2 v\nu^5 = 7, & \mu^2 b^2 v\nu^6 = 111, & \mu b^2 v\nu^7 = 868, & b^2 v\nu^8 = 4192 \\
 & \mu^2 b^3 v\nu^5 = 7, & \mu b^3 v\nu^6 = 72, & b^3 v\nu^8 = 360
\end{array}$$

Only the numbers involving μ^3 were given by Schubert (1879). The Wit script for computing these numbers is given in 4.6.

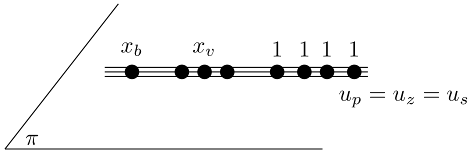

Fig. 5. An irreducible component of $\overline{K}_{\text{nod}} - K^v_{\text{nod}}$ (degeneration ε).

3. The condition ρ

We denote by ρ the class of the hypersurface of the variety K^v_{nod} whose points $(f, (\pi, x_b, u_p, u_z, u_s, x_v))$ satisfy that f is tangent to a given plane. In this section we will consider the tangential structure of the figures of K^v_{nod} in order to introduce ρ and we will compute the fundamental numbers involving conditions μ, b, ν and ρ.

Recall that the dual curve f^* of an irreducible nodal cubic f on π is a quartic curve on the dual plane π^* with three cusps and a bitangent. Furthermore, the map $f \mapsto f^*$ is a rational map whose indeterminacy locus is the 2-codimensional closed set of K^v_{nod} consisting of points $(f, (\pi, x_b, u_p, u_z, u_s, x_v))$ such that f degenerates and contains a double line.

In order to compute intersection numbers involving the ρ condition, we consider the closure $\overline{K}_{\text{nod}}$ of the graph of the rational map $K^v_{\text{nod}} \to \mathbb{P}(S^4\mathbb{U}|_{\mathbb{G}_v})$ that assigns the quartic curve f^* of tangents of f, that is,

$$\overline{K}_{\text{nod}} = \{(f,\ f^*, (\pi, x_b, u_p, u_z, u_s, x_v)) \mid f \text{ and } f^* \text{ dual to each other}\}.$$

The variety $\overline{K}_{\text{nod}}$ is a compactification of K^v_{nod} where the dual nodal cubic is always well defined.

Given a degenerate nodal cubic, we say that a point P is a focus (of multiplicity m) of f if f^* contains the pencil of lines through P over π as a component (of multiplicity m). With this convention, the description of the dual structure for the degenerations γ, χ, ψ, ϑ, τ and τ' is as follows:

- γ: The dual cuspidal cubic together with the cusp as a simple focus.
- χ: The dual conic and the point x_a as a double focus, where x_a is the intersection point, other than the node x_b, of the conic f' and the line $u_{\ell'}$.
- ψ: Two double foci corresponding to the intersection points of each pair of lines different from x_b.
- ϑ: The intersection point of the double line with the simple line as a double focus and two other simple foci on the double line.
- τ, τ': The point where the three lines meet as a focus of multiplicity 4.

On the other hand, the projection map $h_\rho : \overline{K}_{\text{nod}} \to K^v_{\text{nod}}$ is just the blow-up of K^v_{nod} along a subvariety D_ρ of codimension 2. The geometric description of one of the irreducible components of the exceptional divisor $h_\rho^{-1}(D_\rho)$, whose class in $\text{Pic}(\overline{K}_{\text{nod}})$ we call ε as denoted by Schubert, is given below (see Fig. 5).

To compute the intersection numbers with conditions μ, ν, ρ and b the unique degenerations of the 1-dimensional systems $\mu^i\nu^j\rho^h b^k$ are γ, χ, ϑ and ε. We will study the geometry of the degenerations ϑ and ε first. Before giving the intersection numbers over ϑ and ε, we need to know the number of triples of flexes that can be present on a given degeneration of these types. These numbers were called by Schubert *Stammzahlen* (Schubert, 1879).

Lemma 3.1. (i) *Given a degenerate nodal cubic of type ϑ on $\overline{K}_{\text{nod}}$, the three flexes over its double line are completely determined by the node, the double focus and the two simple foci.*

(ii) *Given a degenerate nodal cubic of type ε on $\overline{K}_{\text{nod}}$, there are exactly 12 different possible positions for the triples of flexes over its triple line once the node and the four simple foci are fixed.*

Proof. The degenerations ϑ and ε can be obtained by means of a homolography process (see Fulton (1984, p. 190), Kleiman (1984, p. 17), Kleiman (1986, p. 53)) which consists of projecting a non-degenerate nodal cubic C from a point P to a line L. Since the proof of the claim is quite similar in both cases, we will focus on the degeneration ε. Taking the plane π with equation $x_0 = 0$, and the node x_b together with the intersection points of the pair of nodal tangents with the line u_s as the reference triangle, the equation of C has the form

$$x_0x_1x_2 = \alpha x_1^3 + \beta x_2^3. \tag{10}$$

Besides, we choose the line L on π as $x_1 = x_2$, the point $P = (a, b, 1)$ and the four foci $P_i = (t_i, 1, 1)$, $i = 1, 2, 3, 4$, so that $s_i(t_1, t_2, t_3, t_4) = 0$, $i = 1, 2, 3$, and $s_4(t_1, t_2, t_3, t_4) = t$, where s_i is the elementary i-th symmetric function. We know that the tangential structure of a nodal cubic of (10) is given by the equation

$$u_1^2u_2^2 - 4\alpha u_0u_1^3 - 4\beta u_0u_2^3 + 18\alpha\beta u_0^2u_1u_2 - 27\alpha^2\beta^2u_0^4 = 0,$$

where (u_0, u_1, u_2, u_3) are the dual coordinates of (x_0, x_1, x_2, x_3). Now, given a point $X = (x, 1, 1)$ on L, the line PX is tangent to C if

$$(x-a)^2(xb-a)^2 - 4\alpha(x-a)^3 - 4\beta(xb-a)^3 + 18\alpha\beta(x-a)(xb-a) - 27\alpha^2\beta^2 = 0. \tag{11}$$

Since the lines PP_1, PP_2, PP_3 and PP_4 are to be tangent to C, Eq. (11) above will have $t_i, i = 1, 2, 3, 4$, as roots. Writing the coefficients of (11) in terms of the symmetric functions of the roots, we get

$$\begin{aligned}
&f_1(a, b, \alpha, \beta) = -2ab(1+b) - 4\alpha - 4b^3\beta = 0,\\
&f_2(a, b, \alpha, \beta) = a^2(1+4b+b^2) + 12a\alpha + 12ab^2\beta + 18b\alpha\beta = 0,\\
&f_3(a, b, \alpha, \beta) = -2a^3(1+b) - 12a^2\alpha - 12a^2b\beta - 18a(1+b)\alpha\beta = 0,\\
&f_4(a, b, \alpha, \beta) = a^4 + 4a^3\alpha + 4a^3\beta + 18a^2\alpha\beta - 27\alpha^2\beta^2 - tb^2 = 0.
\end{aligned}$$

Since $a \neq 0$, from $3a^2f_1 + a(1+b)f_2 + bf_3 = 0$ it follows that

$$(1+b)^3 - 6b(1+b) = 0.$$

Hence b takes exactly three different values. On the other hand, equating $\alpha = -b^3\beta - \frac{1}{2}ab(1+b)$ from $f_1 = 0$, the equations $f_2 = 0$ and $f_3 = 0$ become

$$\begin{aligned}
&18z^2 + 3(3(1+b) - 4(1-b))z + (1 - 2b - 5b^2) = 0,\\
&18(1+b)z^2 + (11b^2 + 18b - 3)z + b(1+b)(b-2) = 0,
\end{aligned}$$

where $z = \frac{b^2\beta}{a}$. Eliminating the quadratic term in the system above, clearly z remains a function of b alone, and therefore there are three possible values for z as well, each of them corresponding to one value of b. Now, if one substitutes $\alpha = -b^3\beta - \frac{1}{2}ab(1+b)$ and $\beta = \frac{az}{b^2}$ into equation $f_4 = 0$, a^4 becomes a rational expression which depends only on b and z. Hence, there are 12 possible values for a and also for α and β. Finally, taking into account the three cube roots of $\frac{\alpha}{\beta}$, the projection of the inflection points of C from P to the line L gives us a triplet of points of the type

$$\left(a\left(\sqrt[3]{\frac{\alpha}{\beta}} - 1\right), b\sqrt[3]{\frac{\alpha}{\beta}} - 1, b\sqrt[3]{\frac{\alpha}{\beta}} - 1\right).$$

Because of the number of possible values for a, b, α and β obtained above, it follows that there are 12 possible positions for such a triplet of flexes. □

Here we list all numbers with the conditions μ, ν, ρ and b over the degenerations ϑ and ε:

Proposition 3.1. *In $A^*(\overline{K}_{\mathrm{nod}})$ we have:*

$$\begin{aligned}
\mu^3b\vartheta &= 0,\ 0,\ 24,\ 126,\ 219,\ 150,\ 0\\
\mu^2b\vartheta &= 0,\ 0,\ 240,\ 1104,\ 1986,\ 2060,\ 1200,\ 0\\
\mu b\vartheta &= 0,\ 0,\ 1240,\ 4980,\ 8710,\ 9400,\ 6550,\ 2940,\ 0\\
b\vartheta &= 0,\ 0,\ 3360,\ 10080,\ 14920,\ 13920,\ 8300,\ 2940,\ 0,\ 0\\
\mu^3b^2\vartheta &= 0,\ 0,\ 6,\ 33,\ 48,\ 0\\
\mu^2b^2\vartheta &= 0,\ 0,\ 64,\ 306,\ 514,\ 420,\ 0\\
\mu b^2\vartheta &= 0,\ 0,\ 340,\ 1410,\ 2384,\ 2330,\ 1200,\ 0\\
b^2\vartheta &= 0,\ 0,\ 880,\ 2640,\ 3684,\ 2980,\ 1200,\ 0,\ 0\\
\mu b^3\vartheta &= 0,\ 0,\ 40,\ 180,\ 295,\ 270,\ 0\\
b^3\vartheta &= 0,\ 0,\ 100,\ 306,\ 398,\ 270,\ 0,\ 0
\end{aligned}$$

where the numbers listed to the right of a given $\mu^ib^j\vartheta$ correspond to the intersection numbers $\mu^ib^j\nu^k\rho^{10-i-j-k}\vartheta$, for $k = 10-i-j, \ldots, 0$.

Proof. According to Lemma 3.1(i), the points of ϑ can be seen as tuples $(\pi, u_\ell, u_{\ell'}, x_a, x_b, x_d)$, where u_ℓ is a double line and $u_{\ell'}$ is a simple line on π meeting u_ℓ at x_a, x_b is the node which is on u_ℓ and x_d is a pair of points on u_ℓ corresponding to the two simple foci. Therefore we can parameterize these tuples as a projective bundle over the variety of flags $\mathbb{P}(\mathbb{Q}^*)$ consisting of triples (π, x_a, u_ℓ). □

Proposition 3.2. *In $A^*(\overline{K}_{\text{nod}})$ we have:*

$$\begin{aligned}
\mu^3 b\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.9,\ 12.30,\ 12.45\\
\mu^2 b\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.54,\ 12.180,\ 12.330,\ 12.420\\
\mu b\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.162,\ 12.540,\ 12.990,\ 12.1260,\ 12.1330\\
\mu^3 b^2\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.3,\ 12.10\\
\mu^2 b^2\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.18,\ 12.60,\ 12.110\\
\mu b^2\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.54,\ 12.180,\ 12.330,\ 12.420\\
\mu b^3\varepsilon &= 0,\ 0,\ 0,\ 0,\ 12.9,\ 12.30,\ 12.65
\end{aligned}$$

where the numbers listed to the right of a given $\mu^i b^j \varepsilon$ correspond to the intersection numbers $\mu^i b^j \nu^k \rho^{10-i-j-k}\varepsilon$, for $k = 10 - i - j, \ldots, 0$.

Proof. Similarly to degeneration ϑ above, the points of ε can be seen as tuples (π, u_ℓ, x_b, x_d), where u_ℓ is a triple line on π, x_b is the node which is on u_ℓ and x_d are four points on u_ℓ corresponding to the four simple foci. Once again these tuples can be parameterized as a projective bundle over the variety of flags $\mathbb{P}(\mathbb{Q}^*)$ consisting of triples (π, x_b, u_ℓ). The intersection numbers we obtain using this parameterization must be multiplied by 12, according to Lemma 3.1(ii), that is the number of possibilities there are for the flexes x_v. □

The numbers of the first row on the tables in Propositions 3.1 and 3.2 coincide with the ones given by Schubert (1879) in *Tabelle von sonstigen Ausartungszahlen* in pp. 156–157. The Wit script for computing the intersection numbers concerning ϑ and ε are included in 4.7 and 4.8, respectively.

With these results, we can generalize Zeuthen's degeneration formula $3\rho = \gamma + 2\chi$ given in Zeuthen (1872) and later rewritten over $\mathbb{P}^3$ in Hernández et al. (2007), with the contribution of the degenerations ϑ and ε.

Proposition 3.3. *In $\mathrm{Pic}(\overline{K}_{\text{nod}})$ the following relation holds*

$$3\rho = 4\mu + \gamma + 2\chi + 4\vartheta + 4\varepsilon$$

modulo the remainder degenerations of $\overline{K}_{\text{nod}}$ different from γ, χ, ϑ and ε.

Proof. Zeuthen's degeneration formula was verified by Kleiman and Speiser (1988) and its generalization to $\mathbb{P}^3$, $3\rho = 4\mu + \gamma + 2\chi$, was given in Hernández et al. (2007). Therefore, we know that there exist integers n and m such that $3\rho = 4\mu + \gamma + 2\chi + n\vartheta + m\varepsilon$. In order to determine n and m we can compute the intersection numbers $\mu^2 b\nu^6\rho^2$ and $\mu^2 b\nu^4\rho^4$ in two different ways. First, by substituting the expression of ρ we have $3\mu^2 b\nu^6\rho^2 = 4\mu^3 b\nu^6\rho + \mu^2 b\nu^6\rho\gamma + 2\mu^2 b\nu^6\rho\chi + n\mu^2 b\nu^6\rho\vartheta$. From Table 5 and Proposition 2.1 in Hernández et al. (2007) we get $\mu^2 b\nu^6\rho\gamma = 568$ and from Table 6 and Proposition 2.3 in Hernández et al. (2007) we get $\mu^2 b\nu^6\rho\chi = 770$. On the other hand, using relation $6b = -2\mu + 5\gamma - 2\chi + 4\tau$ given by the three relations at the end of Section 1, we obtain $\mu^2 b\nu^6\rho^2 = 1052$ and $\mu^3 b\nu^6\rho = 22$. From this, and taking into account that $\mu^2 b\nu^6\rho\vartheta = 240$, it turns out that $n = 4$. Proceeding in a similar way with the intersection number $\mu^2 b\nu^4\rho^4$, it follows that $m = 4$. □

Corollary 3.1. *In $\mathrm{Pic}(\overline{K}_{\text{nod}})$ the following relation holds*

$$\rho = -6\mu + 4\nu - 2b - 2\vartheta - 6\varepsilon$$

modulo the remainder degenerations of $\overline{K}_{\text{nod}}$ different from γ, χ, ϑ and ε.

Proof. On the variety X_{nod} described in Hernández et al. (2007), the degeneration relation given in Proposition 3.3 can be written as $3\rho = 4\mu + \gamma + 2\chi$. Substituting the expressions $\gamma = -4\mu + 2\nu$ and $\chi = -9\mu - 3b + 5\nu$ given in Hernández et al. (2007) we obtain $\rho = -6\mu + 4\nu - 2b$. Therefore in our variety $\overline{K}_{\text{nod}}$ this relation modulo other degenerations different from γ, χ, ϑ and ε should be expressed as $\rho = -6\mu + 4\nu - 2b + r\vartheta + s\varepsilon$ for certain integers r and s. Now in order to obtain the values of r and s we can proceed as in the proof of Proposition 3.3 above. □

From the intersection numbers given in Propositions 3.1 and 3.2 and using the formula in Corollary 3.1 we can compute all intersection numbers of nodal cubics with conditions μ, ν, ρ and b:

Proposition 3.4. *In $A^*(\overline{K}_{\text{nod}})$ we have:*

$$
\begin{aligned}
\mu^3 &= 12,\ 36,\ 100,\ 240,\ 480,\ 712,\ 756,\ 600,\ 400\\
\mu^2 &= 216,\ 592,\ 1496,\ 3280,\ 6080,\ 8896,\ 10232,\ 9456,\ 7200,\ 4800\\
\mu &= 2040,\ 5120,\ 11792,\ 23616,\ 40320,\ 56240,\ 64040,\ 60672,\ 49416,\\
&\quad 35760,\ 23840\\
1 &= 12960,\ 29520,\ 61120,\ 109632,\ 167616,\ 214400,\ 230240,\ 211200,\\
&\quad 170192,\ 124176,\ 85440,\ 56960\\
\mu^3 b &= 6,\ 22,\ 80,\ 240,\ 604,\ 1046,\ 1212,\ 1000\\
{}^*\mu^2 b &= 100,\ 328,\ 1052,\ 2800,\ 6272,\ 10540,\ 13468,\ 13512,\ 10800\\
{}^*\mu b &= 872,\ 2568,\ 7288,\ 17232,\ 34280,\ 53772,\ 67048,\ 68268,\ 59352,\ 45200\\
{}^*b &= 5040,\ 13120,\ 32048,\ 64608,\ 107072,\ 144960,\ 162760,\ 155288,\ 132048,\\
&\quad 98352,\ 70880
\end{aligned}
$$

$$
\begin{aligned}
\mu^3 b^2 = \mu^2 b^3 &= 1,\ 4,\ 16,\ 52,\ 142,\ 256,\ 304\\
{}^*\mu^2 b^2 &= 18,\ 64,\ 224,\ 640,\ 1532,\ 2668,\ 3464,\ 3504\\
{}^*\mu b^2 &= 160,\ 508,\ 1564,\ 3944,\ 8316,\ 13560,\ 17368,\ 18024,\ 15824\\
{}^*b^2 &= 904,\ 2512,\ 6568,\ 13904,\ 23904,\ 33304,\ 38432,\ 36808,\ 28864,\ 25664\\
{}^*\mu b^3 &= 12,\ 42,\ 144,\ 400,\ 928,\ 1622,\ 2252,\ 2504\\
{}^*b^3 &= 72,\ 216,\ 612,\ 1384,\ 2524,\ 3732,\ 4656,\ 5112,\ 5424
\end{aligned}
$$

where the numbers listed to the right of a given $\mu^i b^j$ (1 for $\mu^0 b^0$) correspond to the intersection numbers $\mu^i b^j \nu^k \rho^{11-i-j-k}$, for $k = 11 - i - j, \ldots, 0$.

The Wit script for computing these numbers is included in 4.9.

The rows marked with a * in Proposition 3.4 contain numbers not listed in Schubert (1879). Of the remaining numbers, those in the first four rows, and the numbers with μ, ν, ρ and the 2-codimensional condition $P = \nu\mu - 3\mu^2$ (that the nodal cubic goes through a point), were verified in Hernández et al. (2007). Notice also that our results verify the values given by Schubert (1879) in Tabelle I and II in pages 157–160 involving μ, ν, ρ and b.

4. Wit scripts

In this Section we collect the Wit scripts used for the computations.

4.1. Intersection numbers on K_{nod}

```
# PU is the incidence variety of point-plane in P3.
# It has dimension 5 and generating classes b and m.
PU=variety(5, {gcs=[b,m], monomial_values={m^3*b^2->1, m^2*b^3->1}});

# Ud is tautological vector bundle on PU relative to the planes.
# Its chern vector is [m,m^2,m^3]
Ud=sheaf(3,[m,m^2,m^3],PU);

# Qd is the quotient of Ud by the line bundle corresponding to b.
Qd = Ud / o_(b,PU);

# EP is the second symmetric power of Qd
EP = symm(2,Qd);

# DP is the vector bundle EP "shifted" by the line bundle of -2m
DP = EP * o_(-2*m,PU);
```

```
# PP is a projective bundle of planes over PU
# It has dimension 7 and its generating classes are b, m and p
# the later corresponds to the tautological
# "hyperplane" class of the bundle
PP=variety(7, {gcs=[b,m,p]});

# The table of monomial values of PP is computed as follows:
PP(monomial_values)=
   { m^i*b^j*p^(7-i-j) -> integral(m^i*b^j*segre(5-i-j,DP),PU)
     with (i,j) in (0..3,0..min(3,5-i)) };

# Udp is the bundle Ud lifted to PP
Udp=sheaf(3,[m,m^2,m^3],PP);

# Qdp is the quotient of Udp by the line bundle of b on PP.
Qdp= Udp / o_(b,PP);

# Qqp is the quotient of Udp by Qdp
Qqp=Udp / Qdp;

# Enodp is Qqp twisted by 2m-p direct sum the third symmetric
# power of Qdp
Enodp=Qqp * o_(2*m-p,PP) + symm(3,Qdp);

# Dnodp is Enodp twisted by -3m
Dnodp= Enodp * o_(-3*m,PP);

# The table of monomial values of Knod, which is the table
# we were aiming at.
Knod(monomial_values)=
    { m^i*b^j*p^h*n^(11-i-j-h) ->
       integral(m^i*b^j*p^h*segre(7-i-j-h,Dnodp),PP)
        with (i,j,h) in (0..3,0..min(3,5-i),0..7-i-j) }
```

4.2. Intersection numbers on K_{nod}^{z}

```
# PP is a projective bundle of planes over PU
# It has dimension 7 and its generating classes are b, m and p

# Ud1 is the bundle Ud lifted to PP
Ud1=sheaf(3,[m,m^2,m^3],PP);

# Qd1 is the quotient of Ud1 by the line bundle of b on PP.
Qd1= Ud1 / o_(b,PP);

# Equot is the quotient of the third symmetric power of Qd1
# by Qd1 twisted by 2m-p
Equot = symm(3, Qd1) / (Qd1 * o_(2*m-p,PP));

# Dquot is Equot twisted by 2b-2p-m
Dquot= Equot * o_(2*b-2*p-m,PP);

# PZ is a projective bundle of planes over PP
```

```
# It has dimension 8 and its generating classes are  b, m, p and z
# (the later corresponds to the tautological "hyperplane" class)
PZ=variety(8, {gcs=[b,m,p,z]});

# The table of monomial values of PZ is computed as follows:
PZ(monomial_values)=
   { m^i*b^j*p^h*z^(8-i-j-h) ->
      integral(m^i*b^j*p^h*segre(7-i-j-h,Dquot),PP)
       with (i,j,h) in (0..3,0..min(3,5-i),0..(7-i-j))};

# Udz is the bundle Ud lifted to PZ
Udz=sheaf(3,[m,m^2,m^3],PZ);

# Enodz is Qqp twisted by 2m-p direct sum the third symmetric
# power of Qdp
Enodz=Udz * o_(2*m-p,PZ) + o_(m-z+2*p-2*b,PZ);

# Dnodz is Enodz twisted by -3m
Dnodz= Enodz * o_(-3*m,PZ);

# The table of monomial values of Knodz is computed as follows:
Knodz(monomial_values)=
{ m^i*b^j*p^h*z^k*n^(11-i-j-h-k) ->
   integral(m^i*b^j*p^h*z^k*segre(8-i-j-h-k,Enodz),PZ)
    with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h)) };

#tabulate(Knodz(monomial_values),"Knodz.res");
```

4.3. Intersection numbers on $K_{\text{nod}}^{z,s}$

```
# PZ is a projective bundle of planes over PP
# It has dimension 8 and its generating classes are b, m, p and z

# Ud2 is the bundle Ud lifted to PZ
Ud2=sheaf(3,[m,m^2,m^3],PZ);

# Eg is Ud2 twisted by 2m-p
Eg=Ud2 * o_(2*m-p,PZ);

# Egg is Eg twisted by -3m
Egg= Eg * o_(-3*m,PZ);

# Pg is a projective bundle of planes over PZ # It has dimension 10
and its generating classes are b, m, p, z and g # the later
corresponds to the tautological # "hyperplane" class of the bundle
Pg=variety(10, {gcs=[b,m,p,z,g]});

# The table of monomial values of Pg is computed as follows:
Pg(monomial_values)=
   { m^i*b^j*p^h*z^k*g^(10-i-j-h-k) ->
      integral(m^i*b^j*p^h*z^k*segre(8-i-j-h-k,Egg),PZ)
       with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h)) };

# Enods is a direct sum of two line bundles
```

```
Enods= o_(m-z+2*p-2*b,Pg) + o_(3*m-g,Pg);

# Dnods is Enods twisted by -3m
Dnods= Enods * o_(-3*m,Pg);

# Knods is a projective bundle of planes over Pg # It has dimension
11 and its generating classes are b, m, p, z, g and n # the later
corresponds to the tautological # "hyperplane" class of the bundle
Knods=variety(11, {gcs=[b,m,p,z,g,n]});

# The table of monomial values of Knods is computed as follows:
Knods(monomial_values)=
   { m^i*b^j*p^h*z^k*g^r*n^(11-i-j-h-k-r) ->
      integral(m^i*b^j*p^h*z^k*g^r*segre(10-i-j-h-k-r,Dnods),Pg)
       with (i,j,h,k,r) in
       (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h),0..(10-i-j-h-k)) };
```

4.4. Intersection numbers on K_{ncusp}

```
# PUc is the incidence variety of point-plane in P3.
# It has dimension 5 and generating classes c and m
PUc=variety(5, {gcs=[c,m], monomial_values={m^3*c^2->1, m^2*c^3->1}});

# Udc is tautological vector bundle on PUc relative to the planes.
Udc=sheaf(3,[m,m^2,m^3],PUc);

# Qdc is the quotient of Udc by the line bundle corresponding to c.
Qdc = Ud / o_(c,PUc);

# DQ is Qdc twisted by -m
DQ=Qdc*o_(-m,PUc);

# PQ is a projective bundle of planes over PUc
# It has dimension 6 and its generating classes are c, m and q
PQ=variety(6,{gcs=[c,m,q]});

PQ(monomial_values)={m^i*c^j*q^(6-i-j)->
integral(m^i*c^j*segre(5-i-j,DQ),PUc) with i,j in 0..3,0..min(3,5-i)};

# UdQ is the bundle Ud lifted to PQ
UdQ=sheaf(3,[m,m^2,m^3],PQ);

# QdQ is the quotient of UdQ by the line bundle of b on PQ.
QdQ = UdQ / o_(c,PQ);

# EquotQ is the quotient of the third symmetric power of QdQ
# by QdQ twisted by 2m-2q
EquotQ = symm(3, QdQ) / (QdQ * o_(2*m-2*q,PQ));

# DquotQ is Equot twisted by 2b-2q-m
DquotQ= EquotQ* o_(-m+2*c-2*q,PQ);

# PQZ is a projective bundle of planes over PQ
```

```
# It has dimension 7 and its generating classes are c, m, q and z
PQZ=variety(7,{gcs=[c,m,q,z]});

PQZ(monomial_values)=
   {m^i*c^j*q^h*z^(7-i-j-h)->
     integral(m^i*c^j*q^h*segre(6-i-j-h,DquotQ),PQ)
      with i,j,h in 0..3,0..min(3,5-i),0..6-i-j};

# Ud2Q is the bundle Ud lifted to PQZ
Ud2Q=sheaf(3,[m,m^2,m^3],PQZ);

# EgQ is Ud2Q twisted by 2m-2q
EgQ=Ud2Q * o_(2*m-2*q,PQZ);

# EggQ is Eg twisted by -3m
EggQ= EgQ * o_(-3*m,PQZ);

# PQg is a projective bundle of planes over PQZ
# It has dimension 9 and its generating classes are c, m, q, z and g
PQg=variety(9, {gcs=[c,m,q,z,g]});

# The table of monomial values of PQg
PQg(monomial_values)={ m^i*c^j*q^h*z^k*g^(9-i-j-h-k) ->
integral(m^i*c^j*q^h*z^k*segre(7-i-j-h-k,EggQ),PQZ)
with (i,j,h,k) in (0..3,0..min(3,5-i),0..(6-i-j),0..(7-i-j-h)) };

# Ecusp is a direct sum of two line bundles
Ecusp= o_(m-z+2*q-2*c,PQg) + o_(3*m-g,PQg);

# Dcusp is Ecusp twisted by -3m
Dcusp= Ecusp * o_(-3*m,PQg);

# Kcusp is a projective bundle of planes over PQg
# It has dimension 10 and its generating classes are c, m, q, z, g and n
Kcusp=variety(10, {gcs=[c,m,q,z,g,n]});

# The table of monomial values of Kcusp
Kcusp(monomial_values)=
   { m^i*c^j*q^h*z^k*g^r*n^(10-i-j-h-k-r) ->
      integral(m^i*c^j*q^h*z^k*g^r*segre(9-i-j-h-k-r,Dcusp),PQg)
       with (i,j,h,k,r) in
       (0..3,0..min(3,5-i),0..(6-i-j),0..(7-i-j-h),0..(9-i-j-h-k)) };
```

4.5. Intersection numbers involving μ, b, p, z, s and ν

```
#Numbers involving a single s:
Ns1={m^i*b^j*p^h*z^k*s*n^(10-i-j-h-k) ->
   integral(m^i*b^j*p^h*z^k*(g-p)*n^(10-i-j-h-k),Knods) +
    (2^h)*integral(m^i*c^j*q^h*(z+2*q)^k*n^(10-i-j-h-k),Kcusp)
      with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h))};

# Numbers involving s^2:
Ns2={m^i*b^j*p^h*z^k*s^2*n^(9-i-j-h-k) ->
```

```
  integral(m^i*b^j*p^h*z^k*(g-p)^2*n^(9-i-j-h-k),Knods) +
  (2^h)*integral(m^i*c^j*q^h*z*(z+2*q)^k*n^(9-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*(z+2*q)^k*(g-2*q)*n^(9-i-j-h-k),Kcusp)
  with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h)) };

# Numbers involving s^3:
Ns3={m^i*b^j*p^h*z^k*s^3*n^(8-i-j-h-k) ->
  integral(m^i*b^j*p^h*z^k*(g-p)^3*n^(8-i-j-h-k),Knods) +
  (2^h)*integral(m^i*c^j*q^h*z^2*(z+2*q)^k*n^(8-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*z*(z+2*q)^k*(g-2*q)*n^(8-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*(z+2*q)^k*(g-2*q)^2*n^(8-i-j-h-k),Kcusp)
  with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(8-i-j-h)) };

# Numbers involving s^4:
Ns4={m^i*b^j*p^h*z^k*s^4*n^(7-i-j-h-k) ->
  integral(m^i*b^j*p^h*z^k*(g-p)^4*n^(7-i-j-h-k),Knods) +
  (2^h)*integral(m^i*c^j*q^h*z^3*(z+2*q)^k*n^(7-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*z^2*(z+2*q)^k*(g-2*q)*n^(7-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*z*(z+2*q)^k*(g-2*q)^2*n^(7-i-j-h-k),Kcusp) +
  (2^h)*integral(m^i*c^j*q^h*(z+2*q)^k*(g-2*q)^3*n^(7-i-j-h-k),Kcusp)
  with (i,j,h,k) in (0..3,0..min(3,5-i),0..(7-i-j),0..(7-i-j-h)) };
```

4.6. Intersection numbers involving μ, b, ν and a single v

```
Nv={m^i*b^j*v*n^(10-i-j) ->
   -(15/2)*integral(m^(i+1)*b^j*n^(10-i-j),Knods)
   -(3/2)* integral(m^i*b^(j+1)*n^(10-i-j),Knods)
   +(3/2)*integral(m^i*b^j*s*n^(10-i-j),Knods)
   + 4*integral(m^i*b^j*n^(11-i-j),Knods)
     with (i,j) in (0..3,0..min(3,5-i)) };
```

4.7. Intersection numbers on ϑ

```
# Determination of number of cubics of degeneration \vartheta
# involving the conditions m,b,n,r

PU=variety(5, {gcs=[a,m], monomial_values={m^3*a^2->1, m^2*a^3->1}});

Ud=sheaf(3,[m,m^2,m^3],PU);

Qd = Ud / o_(a,PU);

Qda = Qd * o_(-m,PU);

# Fa is a projective bundle of planes over PU
# It has dimension 6 and its generating classes are a, m and l
Fa=variety(6, {gcs=[a,m,l]});

# The table of monomial values of Fa is computed as follows:
Fa(monomial_values)=
 {m^i*a^j*l^(6-i-j) ->
```

```
  integral(m^i*a^j*segre(5-i-j,Qda),PU)
  with (i,j) in (0..3,0..min(3,5-i)) };

Ud1=sheaf(3,[m,m^2,m^3],Fa);

Udd=dual(Ud1);

Qla= Udd / o_(l-m,Fa);

# Fab is a projective bundle of planes over Fa
# It has dimension 7 and its generating classes are a, b, m, l
Fab=variety(7, {gcs=[a,b,m,l]});

# The table of monomial values of Fab is computed as follows:
Fab(monomial_values)=
 {m^i*a^j*l^h*b^(7-i-j-h) ->
  integral(m^i*a^j*l^h*segre(6-i-j-h,Qla),Fa)
  with (i,j,h) in (0..3,0..min(3,5-i),0..min(4,6-i-j)) }\trim;

Ud2=sheaf(3,[m,m^2,m^3],Fab);

Udd2=dual(Ud2);

Qld=Udd2 / o_(l-m,Fab);

Qldd= symm(2,Qld);

# Fabd is a projective bundle of planes over Fab
# It has dimension 9 and its generating classes are a, b, m, l and d
Fabd=variety(9, {gcs=[a,b,m,l,d]});

Ud3=sheaf(3,[m,m^2,m^3],Fabd);

Qldf=Ud3 / o_(a,Fabd);

Qldff= Qldf *o_(-m,Fabd);

# DegTh is a projective bundle of planes over Fabd
# It has dimension 10 and its generating classes are a, b, m, l, d and f
DegTh=variety(10, {gcs=[a,b,m,l,d,f]});

# The table of monomial values of Fabdf, which is the table we were aiming at.
DegTh(monomial_values)=
 {m^i*a^j*l^h*b^k*d^x*f^(10-i-j-h-k-x) ->
   integral(m^i*a^j*l^h*b^k*d^x*segre(9-i-j-h-k-x,Qldff),Fabd)
   with (i,j,h,k,x) in
   (0..3,0..min(3,5-i),0..min(4,6-i-j),0..min(3,7-i-j-h),
    0..min(6,9-i-j-h-k)) };

# Numbers involving condition r

NdegTh_r={ m^i*b^j*n^h*r^(10-i-j-h) ->
  integral(m^i*b^j*(2*l+f)^h*(2*a+d)^(10-i-j-h),DegTh)
```

```
   with (i,j,h) in (0..3,0..min(3,5-i),0..10-i-j ) };
```

4.8. Intersection numbers on ε

```
# Determination of number of cubics of degeneration \varepsilon
# involving the conditions m,b,n,r

PU=variety(5,{gcs=[b,m], monomial_values={m^3*b^2->1,m^2*b^3->1}});

Ud=sheaf(3,[m,m^2,m^3],PU);

Qd = Ud / o_(b,PU);

Qdb = Qd * o_(-m,PU);

# Fb is a projective bundle of planes over PU
# It has dimension 6 and its generating classes are a, m and l
Fb=variety(6, {gcs=[b,m,l]});

# The table of monomial values of Fb is computed as follows:
Fb(monomial_values)=
   {m^i*b^j*l^(6-i-j) ->
     integral(m^i*b^j*segre(5-i-j,Qdb),PU)
      with (i,j) in (0..3,0..min(3,5-i))};

Ud1=sheaf(3,[m,m^2,m^3],Fb);

Udd=dual(Ud1);

Qlb= Udd / o_(l-m,Fb);

Qlbb=symm(4,Qlb);

# DegEps is a projective bundle of planes over Fb
# It has dimension 10 and its generating classes are b, m, l and r
DegEps=variety(10, {gcs=[b,m,l,r]});

# The table of monomial values of DegEps is computed as follows:
DegEps(monomial_values)=
   { m^i*b^j*n^h*r^(10-i-j-h) ->
      integral(m^i*b^j*(3*l)^h*segre(6-i-j-h,Qlbb),Fb)
       with (i,j,h) in (0..3,0..min(3,5-i),0..min(4,6-i-j)) };
```

4.9. Intersection numbers involving μ, b, ν and ρ

```
# Numbers of nodal cubics involving conditions m,b,n,r

Nr1={m^i*b^j*n^(11-i-j-h)*r ->
      -6*integral(m^(i+1)*b^j*n^(11-i-j-h),Knods)
      + 4*integral(m^i*b^j*n^(12-i-j-h),Knods)
      - 2*integral(m^i*b^(j+1)*n^(11-i-j-h),Knods)
      - 2*integral(m^i*b^j*n^(11-i-j-h),NdegTh)
      - 6*integral(m^i*b^j*n^(11-i-j-h),NdegEps)
```

```
     with (h,i,j) in (1..11,0..min(3,11-h),0..min(3,5-i,11-h-i)) };

Nrx={Nr1}

for h in 2..11 do
X={m^i*b^j*n^(11-i-j-h)*r^h ->
    - 6*integral(m^(i+1)*b^j*n^(11-i-j-h)*r^(h-1),Nr)
    + 4*integral(m^i*b^j*n^(12-i-j-h)*r^(h-1),Nr)
    - 2*integral(m^i*b^(j+1)*n^(11-i-j-h)*r^(h-1),Nr)
    - 2*integral(m^i*b^j*n^(11-i-j-h)*r^(h-1),DegTh)
    - 6*integral(m^i*b^j*n^(11-i-j-h)*r^(h-1),DegEps)
    with (i,j) in (0..min(3,11-h),0..min(3,5-i,11-h-i)) };

  Nrx=Nrx|{X}
end;
```

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