ON SCHUBERT'S DEGENERATIONS OF CUSPIDAL PLANE CUBICS

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Abstract. In this paper we prove that the variety of complete cuspidal cubics is smooth in codimension one and that there are no first order degenerations other than Schubert's 13. We also establish a number of properties of cuspidal cubics that give a geometric understanding of the "Stammzahlen" tables of Schubert.

Introduction

Hilbert's 15th problem asks for a justification, and for a delimitation of their validity, of the geometric numbers computed by 19th century geometers, especially those obtained by Schubert and included in his book Schubert [1879].

A key step in this direction, given some sort of figures, is a detailed study of the degenerations that those figures can undergo, for then a modified version of the classical method of degenerations (see Xambó [1987]) allows in principle to compute several sorts of numbers concerning the given figures, the significance of the numbers being built-in in the method (see Miret - Xambó [1987]). The main feature of this approach is that it does not rely on coincidence formulas, which often lead to computation of multiplicities that are very difficult to control. Instead it relies on the idea, already used by Schubert to cross-check his computations, that most geometric numbers can be computed in several different ways. This circumstance, used sistematically, allows to establish the required degeneration relations by simple algebra.

One of the non-trivial cases studied by Schubert is that of cuspidal cubics. These objects were recently considered by Sacchiero [1984] and Kleiman - Speiser [1986]. Essentially these works are concerned with the verification of the 8 characteristic numbers of the family of cuspidal cubics and, as it turns out, a single degeneration

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suffices for this purpose, namely, the so called degeneration σ, consisting of a conic and one of its tangent lines.

In Schubert's book, however, we find a list of 13 degenerations for the (complete) cuspidal cubics, with no formal verifications. In fact the process whereby they are obtained (sometimes called a *homolography*) turns out to be not hard to justify in itself, but nevertheless *it does not guarantee that the degenerations produced are all possible degenerations*, nor *that the degenerations so obtained are simple* on the variety of complete cuspidal cubics.

On the other hand, the constitutive elements of some of the degenerations cannot be independent, since their number exceeds what is allowed for by its dimension. Therefore *there must exist relations among those elements*. The knowledge of these relations is important because it plays a *key role in the determination of the fundamental numbers of cuspidal cubics*. Schubert gives lists of such relations, expressed in enumerative terms (tables of "Stammzahlen", loc. cit., pp. 120-127), and asserts that they were obtained by an indirect process ("a posteriori erschlossen", ibid., p. 119).

In this paper we carry out an analytical investigation of the possible degenerations of cuspidal cubics that allows, aside from verifying Schubert's results, to settle the questions raised above. In particular it turns out that a careful study of the homolography process uncovers not only what the relations mentioned above are, but also *leads in a natural way to discover a number geometric properties of cuspidal cubics* that as far as we know have hitherto not been noticed. These properties then allow to determine the relevant Stammzahlen following straigtforward enumerative procedures.

The differences we find with Schubert are a few misprints and the fact that although he uses a correct version of the degeneration η_1 (as it may be inferred from the results of his computations, or from duality), nevertheless his description of that degeneration (loc. cit., p. 112) is incorrect, for, as we shall see, the double focus is v, not c.

The organization of the paper is as follows. Section 1 is for notations and preliminaries. Sections 2 and 3 are devoted to introduce the space of complete cuspidal cubics, and the analytic means to study them. Section 4 gives a simple analytic account of the homolography process for plane curves, which then is applied in Section 5 to obtain Schubert's degenerations for cuspidal cubics (other than σ). Section 5 also considers the question, given some degeneration, of "normalizing" the cubics from which it can be obtained. In Section 6 we prove that there are no other first order degenerations in the space of complete cuspidal cubics and that this space is regular in codimension one. Finally in Section 7 we prove a number of geometric properties of cuspidal cubics which then are applied in Section 8 to describe in enumerative terms the relations that exist between the building elements of the degenerations.

1. Notations and preliminaries

Given a smooth variety X, an invertible sheaf $\mathcal{L}$ on X and sections

$$\sigma = (s_0, \dots, s_k)$$

of $\mathcal{L}$, we shall denote by X_σ the blow up of X with respect to the sheaf of ideals defined by the sections s_i after removing their common codimension one components. This means that if D is the maximal effective divisor on which the sections σ vanish, then we take the blow up of X with respect to the sheaf of ideals defined by the sections of

$$\mathcal{O}_X(-D) \otimes \mathcal{L}$$

corresponding to the sections σ. We will say that X_σ is the blow up of X with respect to $s_0, \dots, s_k$. More generally, if

$$f: X' \to X$$

is a dominant map between smooth complete varieties the blow up of X' with respect to the sections $f^*(\sigma)$ of $f^*(\mathcal{L})$ will be referred to as the blowing up of X' with respect to the sections σ.

Let Γ_σ be the closure of the graph of the map

$$X \to \mathbf{P}^k$$

defined by the sections σ.

PROPOSITION 1. *There exists a natural isomorphism* $X_\sigma \xrightarrow{\sim} \Gamma_\sigma$.

PROOF: There exists a *regular* map $X_\sigma \to \mathbf{P}^k$ that coincides, as a rational map, with the composition

$$X_\sigma \to X \to \mathbf{P}^k$$

(Harshorne [1977], Ex. 7.17.3). This map, together with the natural map $X_\sigma \to X$, defines a map

$$X_\sigma \to X \times \mathbf{P}^k$$

whose image is clearly contained in Γ_σ. On the other hand, Γ_σ satisfies the relations

$$X_i s_j - X_j s_i = 0,$$

from which it follows that the inverse image of the ideal defined by the sections σ on Γ_σ is locally principal. Hence, by the universality of the blow up, there exists a map

$$\Gamma_\sigma \to X_\sigma$$

that composed with the map of X_σ to X coincides with the natural projection

$$\Gamma_\sigma \to X.$$

Now it is clear that the maps

$$X_\sigma \to \Gamma_\sigma \quad \text{and } \Gamma_\sigma \to X_\sigma$$

are inverse isomorphisms. ◇

In the rest of this paper we shall deal with the variety S of non-degenerate plane cuspidal cubics. It is an open subvariety of the variety T of plane cuspidal cubics, which itself is an irreducible closed subvariety of codimension 2 of the projective space of all plane cubics. If P is a point and L a line through P, the closed subvariety of T of cuspidal cubics with cusp at P and cuspidal tangent L will be denoted by $T_{P,L}$. It is a 4 dimension linear space. The open subset of $T_{P,L}$ whose points are the non-degenerate cuspidal cubics will be denoted by $S_{P,L}$. We shall write G := PGL(3) and will denote G′ the subgroup of G that leaves the point P and the line L fixed, or equivalently, that leaves $S_{P,L}$ invariant (in order that an element of G belongs to G′ it is enough that it transforms some cubic in $S_{P,L}$ into another cubic of $S_{P,L}$).

2. Point cuspidal cubics

Let $\mathbf{P}^2$ be the *complex* projective plane. The homogeneous coordinates of $\mathbf{P}^2$ will be denoted (x_0, x_1, x_2).

Cuspidal cubics in $\mathbf{P}^2$ with cusp at the point $c = (1,0,0)$ and cuspidal tangent the line $q = \{x_2 = 0\}$ form a 4 dimensional linear space $T_{c,q}$ in the $\mathbf{P}^9$ of plane cubics. We shall parametrize this linear space by homogeneous coordinates in such a way that a point $(a_0, \ldots, a_4) \in \mathbf{P}^4$ corresponds to the cuspidal cubic whose equation is

$$(*) \qquad a_0 x_0 x_2^2 = a_1 x_1^3 + a_2 x_1^2 x_2 + a_3 x_1 x_2^2 + a_4 x_2^3.$$

For subsequent references, the right hand side of $(*)$ will be denoted $p(x_1, x_2)$. The discriminant d of $p(x_1, x_2)$ is given by the relation

$$d = 27a_1^2a_4^2 - 18a_1a_2a_3a_4 + 4a_1a_3^3 + 4a_2^3a_4 - a_2^2a_3^2.$$

We will also use the expression

$$d' = 27a_1^2a_4 - 9a_1a_2a_3 + 2a_2^3.$$

Notice that $2d' = \partial d/\partial a_4$. The relation $d = 0$ is equivalent to say that p has a double factor and it is not hard to see that this factor is not triple iff either $d' \neq 0$ or else $a_1 = a_2 = 0$ and $a_3 \neq 0$.

Now we shall describe a partition of $T_{c,q}$ into locally closed subsets $T_i, i = 0, \ldots, 9$.

T_0. The subset $S_{c,q}$ of $T_{c,q}$ is the open orbit under the action of the group G' and it is not hard to see that $(*)$ is in $S_{c,q}$ iff $a_0a_1 \neq 0$. We shall set $T_0 = S_{c,q}$.

T_1. This is the set defined by the conditions $a_1 = 0$ and $a_0 a_2 \neq 0$. Thus cuspidal cubics in T_1 have the form

$$x_2(a_2x_1^2 + a_3x_1x_2 + a_4x_2^2 - a_0x_0x_2) = 0.$$

So the point cubic is the union of the line $x_2 = 0$ (line q) and the conic K whose equation is

$$a_2x_1^2 + a_3x_1x_2 + a_4x_2^2 - a_0x_0x_2 = 0.$$

It is easy to see that q is tangent to K at the point (1,0,0), and that the determinant of the matrix of K is $2a_0^2a_2 \neq 0$. Therefore cuspidal cubics in T_1 consist of line q together with a non-degenerate conic that is tangent to it at the point c.

T_2. This set is defined by the relations $a_0 = 0$ and $a_1 d \neq 0$. Cubics in T_2 are given by the equation $p(x_1, x_2) = 0$, which consist of three distinct lines concurring at the point $c = (1,0,0)$. Moreover, none of the lines coincides with q.

T_3. The relations that define T_3 are $a_0 = a_1 = 0$ and $d \neq 0$. This set consists of three distinct lines concurrent at $c = (1,0,0)$, one of which coincides with q.

T_4. The conditions defining this set are $a_1 = a_2 = 0$ and $a_0 \neq 0$. Under this assumption the point equation of the cubic is

$$x_2^2(a_0x_0 - a_3x_1 - a_4x_2) = 0,$$

which consists of the double line q ($x_2 = 0$) and a simple line different from q and meeting it at the point $P = (a_3, a_0, 0)$. Notice that P does not coincide with c.

T_5. This is the set satisfying the relations $a_0 = d = 0$ and $a_1 d' \neq 0$. The cubics in this set consist of a double line and a simple line, both different from q, meeting at $c = (1,0,0)$.

T_6. Take $a_0 = a_1 = d = 0$ and $a_2 \neq 0$. Such cubics consist of the line q and a double line through $c = (1,0,0)$ different from q.

T_7. $a_0 = a_1 = a_2 = 0$ and $a_3 \neq 0$. They consist of the line q counted twice and another line through c, different from q.

T_8. $a_0 = d = d' = 0$ and $a_1 \neq 0$. A triple line through c which does not coincide with q.

T_9. $a_0 = a_1 = a_2 = a_3 = 0$. It consists of line q counted three times.

Proposition 2

(1) *The sets T_i are irreducible. T_0 has dimension 4, T_1, T_2 have dimension 3, T_3, T_4 and T_5 have dimension 2, T_6, T_7 and T_8 have dimension 1 and T_9 has dimension 0.*
(2) *The locally closed sets T_i are the orbits of G' acting on T.*
(3) *The sets GT_i are orbits of G. More precisely we have:*

- $GT_0 = S$, *which is the dense orbit of non–degenerate cuspidal cubics.*
- GT_1 *is the 6– dimensional orbit whose points correspond to cubics consisting of a conic and one of its tangent lines.*
- $GT_2 = GT_3$ *is the 5–dimensional orbit whose points correspond to triples of distinct concurrent lines.*
- $GT_4 = GT_5 = GT_6 = GT_7$ *is the 3–dimensional orbit whose points correspond to a double line and a simple line different from the former.*
- $GT_8 = GT_9$ *is the 2–dimensional orbit whose points correspond to triple lines.* ◇

3. Complete cuspidal cubics

Let $\mathbf{P}^{9^*}$ be the dual space of the $\mathbf{P}^9$ of plane cubics. We shall make the convention that a point

$$(b_0, \dots, b_9) \in \mathbf{P}^{9^*}$$

corresponds to the dual cubic

$$f_0 u_0^3 + f_1 u_0^2 + f_2 u_0 + f_3 = 0$$

where

$$\begin{aligned} &f_0 = b_0, f_1 = b_1 u_1 + b_2 u_2, \\ &f_2 = b_3 u_1^2 + b_4 u_1 u_2 + b_5 u_2^2, \text{and} \\ &f_3 = b_6 u_1^3 + b_7 u_1^2 u_2 + b_8 u_1 u_2^2 + b_9 u_2^3. \end{aligned}$$

We will put, following Schubert,

$$\begin{aligned} &b: S \to \mathbf{P}^{9^*} \\ &c: S \to \mathbf{P}^2 \\ &v: S \to \mathbf{P}^2 \\ &q: S \to \mathbf{P}^{2^*} \\ &w: S \to \mathbf{P}^{2^*} \\ &y: S \to \mathbf{P}^2 \\ &z: S \to \mathbf{P}^{2^*} \end{aligned}$$

to denote the maps that transform a given cubic C in S into, respectively, the dual cubic C^*, the cusp, the inflexion point, the cuspidal tangent, the inflexional tangent, the intersection point of w with q and the line joining v and c.

The expression of these maps for cubics in $S_{c,q}$ using the homogeneous coordinates of C given by $(*)$, Section 2, is as follows.

$$
\begin{aligned}
b_0 &= 27a_1^2a_4^2 - 18a_1a_2a_3a_4 + 4a_1a_3^3 + 4a_2^3a_4 - a_2^2a_3^2\\
b_1 &= 2a_0(-9a_1a_2a_4 + 6a_1a_3^2 - a_2^2a_3)\\
b_2 &= 2a_0(27a_1^2a_4 - 9a_1a_2a_3 + 2a_2^3)\\
b_3 &= a_0^2(12a_1a_3 - a_2^2)\\
b_4 &= -18a_0^2a_1a_2\\
b_5 &= 27a_0^2a_1^2\\
b_6 &= 4a_0^3a_1\\
b_7 &= b_8 = b_9 = 0\\
v_0 &= 27a_1^2a_4 - 9a_1a_2a_3 + 2a_2^3\\
v_1 &= -9a_0a_1a_2\\
v_2 &= 27a_0a_1^2\\
w_0 &= -27a_0a_1^2\\
w_1 &= 9a_1(3a_1a_3 - a_2^2)\\
w_2 &= 27a_1^2a_4 - a_2^3\\
y_0 &= 3a_1a_3 - a_2^2\\
y_1 &= 3a_0a_1\\
y_2 &= 0\\
z_0 &= 0\\
z_1 &= 3a_1\\
z_2 &= a_2
\end{aligned}
$$

Notice that b_0 is the discriminant d of the polynomial $p(x_1, x_2)$.

Some algebraic computations in this paper have been checked using MACAULAY (Bayer - Stillman [1986]), sometimes also using REDUCE. An example to illustrate this is the calculation of the formulas for the b's, which give the dual curve C^*. In fact these formulas can be obtained eliminating x_0, x_1, x_2 from the relations

$$u_i = \partial_i f$$

and this operation is easily handled with MACAULAY.

The formulas for the v's and w's can be obtained easily imposing that a line meets the cuspidal cubic C at a point with multiplicity 3. The formulas for the y's and z's are obtained, taking into account their definition, by simple algebraic manipulations.

Set

$$\mathbf{P} = \mathbf{P}^{9^*} \times (\mathbf{P}^2)^3 \times (\mathbf{P}^{2^*})^3$$

and consider the map

$$h: S \to \mathbf{P}, \quad h = (b, c, v, y, z, q, w).$$

Let S^* be the closure of the graph of h in $Z = \overline{S} \times \mathbf{P}$. The space S^* will be referred to as the space of *complete cuspidal cubics*. The points in $S^* - S$ will be called *degenerate* cuspidal cubics, where the inclusion of S in S^* is given by $\mathrm{id}\times h$. Since the composition of h with the projection of $\mathbf{P}$ onto its first factor is $b: S \to \mathbf{P}^{9^*}$, it is natural to define $b: S^* \to \mathbf{P}^{9^*}$ as the restriction to S^* of the projection onto $\mathbf{P}^{9^*}$. Given a point C' of S^*, we shall say that $b(C')$ is the *tangential cubic* associated to the complete cubic C'. In the same way we can define morphisms c, v, y, z, q and w from S^* to the corresponding factors of Z. Given $C' \in S^*$, $c(C')$ will be called *the cusp* of C' and similarly with the other maps.

Given a point P and a line L through P, we will write by $S^*_{P,L}$ to denote the space of complete cuspidal cubics that have cusp at P and cuspidal tangent L. We shall say that the points in $S^*_{P,L}$ are (P, L)*−complete cuspidal cubics*.

For a non-degenerate cuspidal cubic, the triangle whose vertexes are c, v, and y, and whose sides are z, q, w, is called *singular triangle*. The same notion can now be defined for degenerate cuspidal cubics in S^*. In other words, given a degenerate complete cuspidal cubic C', the six-tuple

$$(c(C'), v(C'), y(C'), z(C'), q(C'), w(C'))$$

will be called singular triangle of C', the first three elements being the vertices and the last three the sides. The cubic is degenerate if and only if its singular triangle is a degenerate triangle.

The projection of a point $C' \in S^*$ to $\overline{S}$ will be referred to as the *point cubic* associated to C'.

4. Homolographic degenerations of plane curves

In this Section we study the degeneration of contact structure of an irreducible plane curve by means of the so called *homolographic* process (cf. Fulton [1984]). The information gathered here will be used in next Section to describe Schubert's 13 degenerations for plane cuspidal cubics. That these are the only possible degenerations will be established in Section 6.

Let C be an irreducible plane curve (that is, a curve in $\mathbf{P}^2$) of degree $d \geq 2$. Given a point $P \in \mathbf{P}^2$, let $m_P = m_P(C)$ be the multiplicity of P on C. If $P \notin C$, then $m_P = 0$, and conversely. We shall write $T_P(C)$ to denote the *cone of tangents to C at the point P*. Thus $T_P(C)$ is the union of m_P lines through P, each counted with its natural multiplicity. It is empty iff $P \notin C$.

Dually, let L be a line in $\mathbf{P}^2$, and let $T^*_L(C)$ denote the *contact locus of L with C*. So if L has multiplicity μ_L on the dual curve C^* then $T^*_L(C)$ consists of μ_L points on L, each counted with its natural multiplicity. It is empty iff L does not touch C.

Assume that $P \notin L$. We shall also consider the cone whose vertex is P and whose directrix is the 0-dimensional scheme $L \cap C$. It consists of d lines through P, each counted with its natural multiplicity. We shall denote this scheme by $C_{P,L}$. Dually, let $C^*_{L,P}$ denote the 0 dimensional scheme of points obtained intersecting L with the tangents to C from P. So it consists of a finite number of points on L, each counted with its natural multiplicity. The total number of points is δ, the class of C.

Now we have:

THEOREM 1. *The pairs*

$$\left(T_P(C) + (d - m_P)L, C^*_{L,P}\right)$$

and

$$\left(C_{P,L}, T^*_L(C) + (\delta - \mu_L)P\right)$$

are flat degenerations of the cycle (C, C^*), *dual of each other.*

PROOF: We take coordinates in such a way that $P = (1,0,0)$, $L = \{x_0 = 0\}$.

Let F be a homogeneous equation of C. Then

$$F = F_m x_0^{d-m} + F_{m+1} x_0^{d-m-1} + \cdots + F_d,$$

where F_j is a degree j homogeneous polynomial in x_1, x_2 and $m = m_P(C)$. In terms of this equation we see that

$$T_P(C) = V(F_m)$$

and so

$$T_P(C) + (d - m)L = V(F_m x_0^{d-m}).$$

Similarly

$$C_{P,L} = V(F_d).$$

Now consider the polynomials

$$\Phi(x_0, x_1, x_2, t) = F_m x_0^{d-m} + tF_{m+1} x_0^{d-m-1} + \cdots + t^{d-m} F_d.$$

and

$$\Psi(x_0, x_1, x_2, t) = t^{d-m} F_m x_0^{d-m} + \cdots + F_d.$$

Let

$$V = V(\Phi), W = V(\Psi) \subset \mathbf{P}^2 \times \mathbf{A}^1.$$

It is clear that V and W are flat families of closed subschemes of $\mathbf{P}^2$. Moreover,

$$V_1 = W_1 = C$$

and the special fibers V_0 and W_0 are given by

$$V_0 = V(F_m x_0^{d-m}) = T_P(C) + (d - m)L,$$
$$W_0 = V(F_d) = C_{P,L}.$$

To see how these degenerations behave with respect to the dual curve, consider, for $t \neq 0$, the map $h_t\colon \mathbf{P}^2 \to \mathbf{P}^2$ such that $(x_0, x_1, x_2) \mapsto (tx_0, x_1, x_2)$. Since

$$x \in h_t(C) \Longleftrightarrow F(h_{1/t}x) = 0$$

we see that $h_t(C)$ is given by the relation

$$F(x_0/t, x_1, x_2) = 0,$$

which is equivalent to

$$\Phi(x_0, x_1, x_2, t) = 0,$$

and so

$$h_t(C) = V_t.$$

Similarly,

$$h_{1/t}(C) = W_t.$$

Now given a line of coordinates (u_0, u_1, u_2), $h_t(u)$ has coordinates $(u_0/t, u_1, u_2)$. From this it follows that if

$$G = G_\mu u_0^{\delta-\mu} + \cdots + G_\delta$$

is the equation of C^*, then the equation of V_t^* is

$$t^{\delta-\mu} G_\mu u_0^{\delta-\mu} + \cdots + G_\delta = 0$$

and that the equation of W_t^* is

$$G_\mu u_0^{\delta-\mu} + tG_{\mu+1} u_0^{\delta-\mu-1} + \cdots + t^{\delta-\mu} G_\delta.$$

From this we see that under the degeneration $C = V_1 \to V_0$ the dual curve C^* degenerates into $V(G_\delta) = C^*_{L,P}$. Similarly, under the degeneration $C = W_1 \to W_0$ the dual curve C^* degenerates into

$$V(G_\mu u_0^{\delta-\mu}) = T_L^*(C) + (\delta - \mu)P.$$

This completes the proof. ◇

The tangential aspect of the degenerations given in the theorem consist of pencils of lines through points, each counted with its natural multiplicity. Such points will be called *foci* of the degeneration.

5. Schubert's 13 degenerations

In this Section we shall study Schubert's degenerations of plane cuspidal cubics. For computational purposes we will denote the resulting degenerations in the form

$D_i, i = 0, \ldots, 12$, instead of the greek letters with suffixes used by Schubert. (See the table of illustrations at the end.) The degenerations $D_1, \ldots, D_{12}$ are obtained by the homolography process (see next table below). The degeneration D_0 has been studied by Sacchiero [1984] and Kleiman – Speiser [1986].

D_0 (Schubert's σ)

Consider the family of plane cuspidal cubics

$$x_2^2 x_0 = t x_1^3 + x_1^2 x_2. \tag{1}$$

The fiber over $t = 0$ consists of the smooth conic K

$$x_2 x_0 = x_1^2,$$

and the line $L = \{x_2 = 0\}$, which is tangent to K at $P = (1, 0, 0)$. The curve dual of (1) can be computed by means of the formulas in Section 3. The resulting family has the fiber

$$u_0(4u_0u_2 - u_1^2)$$

over $t = 0$, which consists of the dual conic K^* and the pencil of vertex P. Finally the singular triangle of (1) can also be computed by means of the formulas in Section 3, and it turns out that in the limit all vertices come to coincide with P, while the three sides come to coincide with the line L.

In what follows we shall let C denote a fixed cuspidal cubic, P a point in the plane, and L a line not through P. Choosing P to lie at special positions with respect to C and applying THEOREM 1 we get the 12 degenerations $D_i, i = 1, \ldots, 12$. The correspondence with Schubert's notation and the homolography involved in each case is summarized in the following table.

Homolography	*Schubert's notation*	D_i
$P \notin C \cup q \cup z \cup w$	ϵ_2	D_{12}
$P \in q - \{c, y\}$	ϵ_1	D_{11}
$P \in z - \{c, v\}$	ϵ_3	D_{10}
$P \in w - \{v, y\}$	η_2	D_9
$P = y$	η_1	D_8
$P \in C - \{c, v\}$	δ_2	D_7
dual D_7	δ_1	D_6
dual D_8	θ_1	D_5
dual D_9	θ_2	D_4
dual D_{10}	τ_3	D_3
dual D_{11}	τ_1	D_2
dual D_{12}	τ_2	D_1

D_1 and D_{12} (Schubert's τ_2 and ϵ_2)

Assume that P is general with respect to C, that is, that P does not lie on C nor on any side of the singular triangle. In that case T_PC is empty. On the other hand the three tangent lines to C from P are distinct and so $C^*_{L,P}$ consists of three distinct points on L. We have therefore a triple line (L counted three times) with three distinct foci on it. It is also clear that the sides of the singular triangle coincide with L and that its three vertices are three distinct points on L disjoint from the foci. This degeneration will be denoted D_{12}. The degeneration D_1 is dual of D_{12}, and hence pointwise it consists of three concurrent lines at P, and tangentwise of the point P as a triple focus. The three sides of the singular triangle are three distinct lines through P, disjoint from the three lines of the point cubic, and so the three vertices coincide at P.

D_2 and D_{11} (Schubert's τ_1 and ϵ_1)

Now take $P \in q - \{c, y\}$. Again T_PC is empty, and so pointwise we have a triple line, L. The three tangent lines to C are distinct, but one coincides with q, so we have three distinct foci on L, one equal to c. The vertex y coincides with c, but they are different from v, which moreover is not a foci. Sides w and z coincide with L, but q is a line through c different from L, namely, the line joining P and c. We will call this degeneration D_{11}. The dual degeneration is D_2, which therefore consists of three concurrent lines at P, itself a triple foci. Vertices c and y coincide with P and sides w and z coincide with one of the lines. Finally q is a forth line through P and v is a point on $w = z$, namely, the intersection of L with $w = z$.

D_3 and D_{10} (Schubert's τ_3 and ϵ_3)

Assume now that $P \in z - \{c, v\}$. Pointwise we again have line L counted three times. In this case we have also three foci on L. The sides q and w coincide with L and the vertices c and v coincide with a point of L which is not a foci. Finally z is the line joining P and $c = v$ and y is a point on L distinct from $c = v$ and the foci. This degeneration will be denoted D_{10}. Its dual degeneration is denoted D_3. Thus D_5 consists of three distinct lines through P, which is a triple focus. The sides of the singular triangle go through P and are different from the lines of the point cubic, but $q = w$. The vertices v and c coincide with P and y is the intersection of L with the line $q = w$.

D_4 and D_9 (Schubert's θ_2 and η_2)

This time take $P \in w - \{v, y\}$. Then L becomes a triple line. Since from a point on w other than v and y there is one simple tangent (different from q) and the tangent w counted twice, we see that on L we have a simple and a double focus. The vertices v and y coincide with the double focus, while $c \in L$ is different from both foci. The sides z and q coincide with L and w is the line joining P and $v = y$. This degeneration will be denoted D_9. The degeneration dual of D_9 will be denoted by D_4. It thus consists of a double line and a simple line meeting it at P, which is a triple focus. We also have q and z coincide with the double line and that $v = y = P$. Finally w is a third line through P and c is the intersection point of L with the double line.

D_5 and D_8 (Schubert's θ_1 and η_1)

Choose P to be point y of C. Then pointwise we get L as a triple line. Since the tangents to C from y are w counted twice and q, tangentwise we get a double and a simple focus. Side z of the triangle coincides with L, side w is the line joining P and the double focus, and side q is the line joining P and the simple focus. Finally v coincides with the double focus, c coincides with the simple focus and y is the intersection of w and q, that is, $y = P$. This type of degeneration will be denoted D_8. Its dual degeneration, D_5, may be described as follows. Pointwise it consists of a double and a simple line meeting at P and tangentwise it consists of P as a triple focus. Side q coincides with the double line, side w with the simple line and side z is L. Therefore c and v are the intersections of L with the double and the simple line, respectively, and y coincides with P.

D_6 and D_7 (Schubert's δ_1 and δ_2)

To obtain D_7, let P lie on $C - \{c, v\}$. Then we obtain L counted twice and a simple line through P, the tangent L' to C at P. Since L' counts as a double tangent from P, we have another tangent L'', and so the point $L' \cap L$ is a double focus and $L'' \cap L$ is a simple focus. The three sides of the singular triangle coincide with L, while the three vertices are three distinct points on L disjoint from the foci. The degeneration dual of D_7 is D_6. It therefore consists of a double line and a simple line meeting at P. Tangentwise P is a double focus and the intersection of L with the double line is a simple focus. The three vertices of the singular triangle coincide with P, while its sides are three lines through P different from the lines of the point cubic.

Normalized homolographies

Given a homolographic degeneration of a cuspidal cubic, it is useful to be able to exhibit the same degeneration as a homolographic degeneration of another cuspidal cubic that is somehow normalized with respect to the given degeneration. The rest of this Section is devoted to prove a few statements that will be used in Section 7 to obtain several key numbers related to some of the degenerations.

Let C be a cuspidal cubic, P a general point with respect to C and L a line not through P. Consider the degeneration of type D_{12} associated to such data, and let P_1, P_2, P_3, denote the foci of the degeneration and c_0, v_0, y_0 the projections on L from P of the verteces c, v, y of the singular triangle of C. Let P' be any point not on L, and choose non colinear points c', v', y', different from P', on the lines $P'c_0, P'v_0, P'y_0$, respectively. Then we have the following

PROPOSITION 3 (1). *The given degeneration of type D_{12} can be obtained as a homolographic degeneration, with center P', of a cuspidal cubic whose singular triangle* is c', v', y'.

PROOF: Let σ be the (unique) homography of the plane that transforms the points P, c, v, y into P', c', v', y', respectively. Let C' be the image of C under σ. Then the singular triangle of C' is c', v', y'. It is clear that the homolographic degeneration of C' with center P' and axis L transforms c', v', y' into c_0, v_0, y_0, respectively. Notice that σ transforms the pencil of lines through P into the pencil of lines through P', and that the induced trasformation is a perspectivity with axis L. This means that a line through P and its transform meet on L. Now the three tangents to C from P are transformed into the three tangents to C' from P'. Hence the foci of the homolographic degeneration of C' (with center P', axis L) coincide with the foci of the initial degeneration. ◇

Let C be a cuspidal cubic, P a point on the cuspidal tangent different from c and y. Consider a degeneration of type D_{11} obtained as a homolography of C with center P and axis a line L not through P. So we have four points on L — $c_0 = y_0, v_0$ (the projections of c, y and v, respectively), and foci P_1, P_2, P_3, the third coinciding with $c_0 = y_0$. Let P' be any point not on L, choose two points c', y' on the line $P'c_0$ (different from c_0 and P') and one point v' on the line $P'v_0$ (different from P' and v_0). Then we have:

PROPOSITION 3 (2). *The given degeneration of type D_{11} can be obtained as a homolographic degeneration, with center P', of a cuspidal cubic whose singular triangle is c', v', y'.*

PROOF: Let U be the point where the lines PP_1 and vy meet, and let U' be the point where the lines $P'P_1$ and $v'y'$ meet. There is a unique homography transforming the points P, c, v, U into the points P', c', v', U'. Let C' be the image of C under this homography. Then the singular triangle of C' is c', v', y' and the homolography of C' with center P' and axis L agrees with the given degeneration. This is clear because the lines PP_1, Pc_0, Pv_0 are transformed into the lines $P'P_1, P'c_0, P'v_0$, and so the homography induces a perspectivity (whose axis is L) between the pencils of lines through P and P'. It follows that the line PP_2 is transformed into the line $P'P_2$, ans so this line is tangent to C'. ◇

Let C be a cuspidal cubic, P a point on the line z different from c and v. Consider a degeneration of type D_{10} obtained as a homolography of C with center P and axis a line L not through P. So we have five points on L — $c_0 = v_0, y_0$ (the projections of c, v and y, respectively), and foci P_1, P_2, P_3. Let P' be any point not on L, choose two points c', v' on the line $P'c_0$ (different from c_0 and P') and one point y' on the line $P'y_0$ (different from P' and y_0). Then we have:

PROPOSITION 3 (3). *The given degeneration of type D_{10} can be obtained as a homolographic degeneration, with center P', of a cuspidal cubic whose singular triangle is c', v', y'.*

The proof is similar to the preceeding one and so will be omited.

Let C be a cuspidal cubic, P a point on C different from c and v, and L a line not through P. Consider the degeneration of type D_7 associated to such data, and let Q, R denote the foci of the degeneration (simple and double, respectively). Let c_0, v_0, y_0 be the projections on L, with center P, of the verteces c, v, y of the singular triangle of C. Let P' be any point not on L, and choose non colinear points c', v', y' different from P' and on the lines $P'c_0, P'v_0, P'y_0$, respectively. Then we have the following

PROPOSITION 3 (4). *The given degeneration of type D_7 can be obtained as a homographic degeneration, with center P', of a cuspidal cubic going through P' and whose singular triangle is c', v', y'.*

Since the proof is similar to the proof of PROPOSITION 3 (1), it will be omited.

6. And there are no more

The goal in this Section is to prove that there are no more degenerations of cuspidal cubics other than the D_i.

THEOREM 2. *S^* is smooth in codimension one and $S^* - S$ has exactly 13 components. All these components have codimension one and therefore coincide with the hypersurfaces D_i.*

PROOF: We shall give a proof for the space $S^*_{c,q}$. The statement follows from this case using the natural fibration $(c,q)\colon S^* \to I$, where $I \subset \mathbf{P}^2 \times \mathbf{P}^{2^*}$ is the incidence variety. According to the Proposition in Section 1,$S^*_{c,q}$is the result of blowing up five successive times the space of point cuspidal cubics. The method of the proof is to look carefully at the centers of these blow ups to keep track of how many components the final exceptional divisor will have.

Let $Z \subset \mathbf{P}^4 \times \mathbf{P}^1$ be the blow up of T with respect to the sections $z_1 = 3a_1$ and $z_2 = a_2$. The scheme of zeroes of z_1 and z_2 coincides with the closure of T_4 (that is, $T_4 \cup T_6 \cup T_9$). The exceptional divisor on Z, which we will denote $E(Z)$, is given by the equations

$$a_1 = 0 \text{ if } z_1 \neq 0 \text{ and } a_2 = 0 \text{ if } z_2 \neq 0.$$

The points of $E(Z)$ are obtained adding a line z through c to points in the closure of T_4. The subvariety of $E(Z)$ whose points satisfy that z coincides with q will be denoted $E'(Z)$. The strict transform on Z of those orbits T_i not contained in T_4 will be denoted by T_i^Z. Similar notations will be used henceforth to denote strict transforms on the successive blow ups we will consider.

Now let $Y \subset Z \times \mathbf{P}^1$ be the blow up of Z with respect to the sections

$$y_0 = a_2 z_2 - a_3 z_1 \text{ and } y_1 = a_0 z_1.$$

The scheme of zeroes of these sections on Z is the reducible subvariety defined by the equations

$$a_0 = a_2 z_2 - a_3 z_1 = 0 \text{ or } a_2 = z_1 = 0.$$

The exceptional divisor $E(Y)$ has therefore two components. The component lying on the variety $V(a_0, a_2z_2 - a_3z_1)$ consists of cuspidal cubics in the closure of T_2^Z whose three lines satisfy the relation $a_2^2 = 3a_1a_3$, together with a distinguished point y on the line q, while the component lying on the variety $V(a_2, z_1)$ consists of cubics in the closure of $E'(Z)$ together with a point y on the double line. These two components will be denoted $E_1(Y)$ and $E_2(Y)$, respectively. We will denote by $E'(Y)$ the 2-dimension subvariety of $E_1(Y)$ whose points represent cubics such that the three lines and z are coincident. The subvariety of $E'(Y)$ for whose points y coincides with c will be denoted by $E''(Y)$.

To look at next blow up notice that the expressions for v_0, v_1, v_2 on Z have a fixed component, since they vanish along the center of the first blow up. Once this component is removed, the expression for v_0, v_1, v_2 is as follows:

$$\begin{aligned} v_0 &= 2a_2z_2^2 - 3a_3z_1z_2 + 3a_4z_1^2, \\ v_1 &= -3a_0z_1z_2, \\ v_2 &= 3a_0z_1^2. \end{aligned}$$

Let us consider the variety V obtained by blowing up Y with respect to v_0, v_1, v_2. It is not hard to see that the scheme of zeros of v_0, v_1, v_2 is given by the relations

$$a_0 = 0,\ 2a_2z_2^2 - 3a_3z_1z_2 + 3a_4z_1^2 = 0.$$

The first of these relations says that the cubic splits into three lines and the second that line z coincides with one of them. It turns out that the exceptional divisor $E(V)$ splits into two components $E_1(V)$ and $E_2(V)$. These components may be described as follows. Looking at the intersection

$$E_1(V) \cap \{d \neq 0\}$$

one sees that $E_1(V)$ consists of cubics in the closure of T_2^Y whose three lines satisfy the relation

$$27a_1^2a_4 - 9a_1a_2a_3 + 2a_2^3 = 0$$

together with a distinguished point v on line z (which is one of the three lines). Similarly, looking at the intersection

$$E_2(V) \cap \{y_1 \neq 0\}$$

we see that $E_2(V)$ consists of cubics in the closure of $E'(Y)$ together with a distinguished point v on the triple line.

Now consider the expressions of w_0, w_1, w_2. These sections do not have a fixed component up to Z, where they take the form

$$\begin{aligned} w_0 &= -3a_0z_1^2 \\ w_1 &= 3z_1(a_2z_2 - a_3z_1) \\ w_2 &= a_2z_2^2 - 3a_4z_1^2. \end{aligned}$$

The scheme of zeroes of w_0, w_1, w_2 on Z intersects the closed set $\{a_0 = 0\}$ precisely along the scheme of zeroes of v_0, v_1, v_2 on Z. The expressions of w_0, w_1, w_2 on V, after removing the fixed component, are

$$\begin{aligned} w_0 &= -3a_0z_1y_1 \\ w_1 &= 3a_0z_1y_0 \\ w_2 &= 3a_0z_1y_1 + (a_0y_0 - a_3y_1)z_2. \end{aligned}$$

From this is follows that the scheme of zeroes of w_0, w_1, w_2 on V is given by the relations

$$z_1 = 0, \;\; a_0y_0 - a_3y_1 = 0.$$

The cubics in the exceptional divisor $E(W)$ consist of cubics that are in the closure of T_4 together with a line w through point $P = (a_3, a_0, 0)$. In this case z coincides with q and the points y and v with P.

Finally we blow up W with respect to the sections $b_0, \ldots, b_6$. The expression of these sections on Z is the following

$$\begin{aligned} b_0 &= 9a_1a_4^2z_1 - 6z_1a_2a_3a_4 + \tfrac{4}{3}z_1a_3^3 + 4a_2^2a_4z_2 - a_2a_3^2z_2 \\ b_1 &= 2a_0(-3z_1a_2a_4 + 2z_1a_3^2 - a_2a_3z_2) \\ b_2 &= 2a_0(9a_1a_4z_1 - 3z_1a_2a_3 + 2a_2^2z_2) \\ b_3 &= a_0^2(4z_1a_3 - a_2z_2) \\ b_4 &= -6a_0^2z_1a_2 \\ b_5 &= 9a_0^2a_1z_1 \\ b_6 &= \tfrac{4}{3}a_0^3z_1 \end{aligned}$$

The scheme of zeroes of the sections $b_0, \ldots, b_6$ on W is contained in $\{a_0 = 0\}$. Indeed, if $a_0 \neq 0$, then on Y the sections b_3 and b_6 can be expressed as follows:

$$\begin{aligned} b_3 &= a_0^3y_0 + 3a_0^2a_3y_1 \\ b_6 &= \tfrac{4}{3}a_0^3y_1, \end{aligned}$$

which do not vanish simultaneously.

Therefore the scheme of zeroes of the sections $b_0, \ldots, b_6$ on W is given by the relations

$$\begin{aligned} a_0 &= 0, \\ b_0 &= 9a_1a_4^2z_1 - 6z_1a_2a_3a_4 + \tfrac{4}{3}z_1a_3^3 + 4a_2^2a_4z_2 - a_2a_3^2z_2 = 0. \end{aligned}$$

This scheme, say X, has 6 irreducible components, namely $E_2(V)$, which has codimension one, and five more of codimension two not contained in $E_2(V)$. To see this we shall first study the intersection of the subscheme X with five open sets $U_1, \ldots, U_5$ that we define presently.

$$U_1 := \{(a_2 z_2 - a_3 z_1)(a_2 z_2^2 - 3a_3 z_1 z_2 + 3a_4 z_1^2) \neq 0\}.$$

The intersection $X \cap U_1$ coincides with T_5^W, and so its points are cubics consisting of a double and a simple line meeting at $c = v = y$ together with three lines q, z and w through that point.

$$U_2 := \{z_1 v_1 \neq 0\}.$$

So $X \cap U_2 = E''(Y)^W$.

$$U_3 := \{v_1 w_0 \neq 0\}.$$

Here $X \cap U_3$ is contained in the closure of T_9^W and its points are cubics that consist of line q counted three times, $z = q$, and a line w meeting q at the point $v = y$.

$$U_4 := \{v_1 y_1 \neq 0\}.$$

$X \cap U_4$ is contained in the closure of T_9^W and its points are cubics that consist of line q counted three times, $z = w = q$, and v, y are two points on q.

$$U_5 := \{z_1 y_1 \neq 0\}.$$

$X \cap U_5$ is contained in the closure of T_9^W and its points are cubics that consist of line q counted three times, $w = q$, and a line z meeting q at the point $v = c$.

7. Projective geometry of cuspidal cubics

There are a number of questions which one is lead to investigate when attempting to find out how to translate properties of the degenerations to properties of the curves themselves. The theorem below summarizes those properties of cuspidal cubics that have been found to underlie the computation of several basic numbers (*Stammzahlen*) related to some of the degenerations of cuspidal cubics.

THEOREM 3.

(1) *Let P_1, P_2, P_3 be three points on line z, different from v and P_1, P_2 different from c, let Q_1, Q_2, Q_3 be three points aligned with y, and assume that the lines $P_1 Q_i$, and $Q_1 P_i$ are tangent to C. Then the lines $P_i Q_j$ are tangent to C for $i, j = 1, 2, 3$.*

(2) *Let P_1, P_2, P_3 be three points on line q, and Q_1, Q_2 two points aligned with v and not on line q. Assume that the lines $P_1 Q_i$ and $P_j Q_1$ are tangent to C. Then the lines $P_i Q_j$ are tangent to C for $i = 1, 2, 3, j = 1, 2$.*

(3) *Let P_1 and P_2 be two points on line z. Then the nine points of intersection of the tangents to C from P_1 and P_2 lie on three lines through y, three on each line.*

(4) *Let P_1, P_2 be two points on q. Then the two diagonals (other than q) of the quadrilateral formed by the tangents from P_1 and P_2 (other than q) pass through v.*

(5) *Let P be a point on z and p_1, p_2, p_3 the tangent lines to C from P. Then the three contact points are aligned with y, and so do the three additional intersection points.*

(6) *Let P be a point on q and let p_1, p_2 be the two tangent lines to C from P that are different from q. Then the contact points of p_1, p_2 with C are aligned with v, and so do the two additional points where those tangents meet C.*

By duality we have

THEOREM 3*.

(1*) *Let p_1, p_2, p_3 be three lines through y different from q, p_1, p_2 different from w, and q_1, q_2, q_3 three lines through a given point on z, and assume that the intersection points $p_1 q_i, p_j q_1$ lie on C. Then the points $p_i q_j$ lie on C for $i, j = 1, 2, 3$.*

(2*) *Let p_1, p_2, p_3 be three lines through point v and q_1, q_2 be two lines that do not pass through z and meeting at a point on q. Assume that the points $p_1 q_i$ and $p_j q_1$ lie on C. Then the points $p_i q_j$ lie on C for $i = 1, 2, 3, j = 1, 2$.*

(3*) *Let p_1, p_2 be two lines through y. Then the nine lines joining the points of intersection of p_1 and C with the points of intersection of p_2 and C go through three points on z, three through each point.*

(4*) *Let p_1, p_2 be two lines through v. Then the two vertexes (other than v) of the diagonal triangle of the quadrangle formed by the intersection points (other than v) of p_1, p_2 with C lie on q.*

(5*) *Let L be a line through y and P_1, P_2, P_3 the points where L meets C. Then the tangent lines at P_1, P_2, P_3 meet at a point on z, and so do the three additional tangents to C from P_1, P_2, P_3.*

(6*) *Let L be a line through v and let P_1, P_2 be the points different from v where L meets C. Then the tangents to C at the points P_i meet at a point on q, and so do the two additional tangents to C from P_1, P_2.*

PROOF OF (1): Let L be a line through y different from w. Its equation has the form $x_2 = a x_0$. Let $P_i = (1, 0, t_i)$, and $Q_j = (1, b_j, a)$. The line $P_i Q_j$ has equation

$$b_j t_i x_0 + (a - t_i) x_1 - b_j x_0.$$

This line belongs to the dual curve if and only if

$$4(t_i - a)^3 = 27b_j^3 t_i$$

and from this the assertion follows easily.

PROOF OF (2): Let $P_i = (t_i, 1, 0)$ and $Q_j = (1, a, b_j)$. Then the line P_iQ_j is given by the equation

$$b_j x_0 - t_i b_j x_1 + (at_i - 1)x_2 = 0.$$

The condition for this line to be tangent to C is

$$4b_j^2 t_i^3 = 27(at_i - 1)^2,$$

and from this the claim follows readily.

PROOF OF (3): Let $P_i = (1, 0, t_i)$. The equations of the tangent lines to C from P_i are

$$L_{ik}: \quad -t_i x_0 + \rho^k a_i x_1 + x_2 = 0, \quad k = 0, 1, 2,$$

where ρ is a primitive cubic root of 1 and a_i satisfies the equation

$$4a_i^3 = 27t_i.$$

Let Q_{kh} denote the intersection point of L_{1k} and L_{2h}. Then the triples

$$\begin{array}{lll} Q_{00}, & Q_{11}, & Q_{22}, \\ Q_{01}, & Q_{12}, & Q_{20}, \\ Q_{02}, & Q_{21}, & Q_{10} \end{array}$$

lie on the lines L_0, L_1, L_2, respectively, where L_k is given by the equation

$$x_2(a_j - \rho^k a_i) = x_0(t_i a_j - t_j \rho^k a_i).$$

This ends the proof of (3).

PROOF OF (4): Let $P_i = (t_i, 1, 0)$. The equation of the two tangents other than q to C from P_i is

$$-x_0 + t_i x_1 \pm a_i x_2 = 0,$$

where a_i is a solution of the equation

$$27a_i^2 = 4t_i^3.$$

The equations of the two diagonals (other than q) of the quadrilateral formed by the tangents from P_1 and P_2 (other than q) are the following:

$$x_0(a_j \pm a_i) = x_1(t_i a_j \pm t_j a_i).$$

So they clearly go through v.

PROOF OF (5*): Let L be the line $x_0 = ax_2$. The points P_i have coordinates $(a\alpha_i, \alpha_i, 1)$, where α_i are the square roots of a. Let $Q = (t, 1, 0)$ be a point on line q. Then in order that the line joining Q and a point $P = (a\alpha, \alpha, 1)$, α a square root of a, be tangent to C it is necessary and sufficient that

$$4t^3 = 27a(t-a)^2.$$

From this the claim follows immediately.

PROOF OF (6*): Let L be the line $x_0 = ax_2$. The points P_i of intersection of L with C have coordinates $(a, \alpha_i, 1)$, where α_i are the cubic roots of a. Let $Q = (t, 0, 1)$ be a point on line z. Then in order that the line joining Q and a point $P = (a, \alpha, 1)$, α a cubic root of a, be tangent to C it is necessary and sufficient that

$$4(a-t)^3 = 27at^2.$$

From this the claim follows immediately. This completes the proof of the Theorem. ◇

8. Stammzahlen

For some of the degenerations there must exist relations among the elements from which it is built up. Thus in degeneration D_{12} we have a triple line with 6 distinguished points on it, so these points cannot vary independently. In fact, since D_{12} has dimension 6, given any 4 of the 6 points there must be only a finite number of possibilities for the other 2. The numbers expressing such possibilities were called *Stammzahlen* by Schubert. The goal in this Section is to study these numbers.

The degenerations for which there must exist relations among its distinguished elements are D_{12}, D_{11}, D_{10} and D_7 (ϵ's and δ_2) and, by duality, D_1, D_2, D_3 and D_6 (τ's and δ_1). For the former the relations that may exist are relations among its distinguished points (on the multiple line), and hence by duality for the latter the relations involve only the distinguished lines through the multiple focus.

The number of distinguished points in these degenerations is 6 for D_{12}, 4 for D_{11}, and 5 for D_{10} and D_7. These points will be denoted as in Section 5. The notation we shall use to denote that we fix some of the points agrees with the monomial notation used for expressing fundamental numbers. So for instance the notation $QRc = 1$ for degeneration D_7 means that there is a single determination for the pair of points v and y when we fix the the simple focus Q, the double focus R and the cusp c. For D_{10}, D_{11} and D_{12}, condition P means that a simple focus be on a line.

Now we first give the tables of Stammzahlen and afterwards we will show how to establish them using the results of Sections 5 and 7.

THEOREM 4. *The Stammzahlen are given by the following tables:*

Table 1. Stammzahlen for D_{12}

$P^3c = 4$	$P^3v = 1$	$P^3y = 2$
$P^2cv = 3$	$P^2cy = 2$	$P^2vy = 1$
$Pcvy = 1$		

Table 2. Stammzahlen for D_{11}

$P^2c = 1$	$P^2v = 1$	$Pcv = 1$

Table 3. Stammzahlen for D_{10}

$P^3 = 2$	$P^2c = 2$	$P^2y = 2$	$Pcy = 1$

Table 4. Stammzahlen for D_7

$QRc = 1$	$QRv = 1$	$QRy = 1$
$Qcv = 1$	$Qcy = 1$	$Qvy = 1$
$Rcv = 1$	$Rcy = 1$	$Rvy = 1$
$cvy = 1$		

The computation of some of these numbers can be done directly, and others by means of THEOREM 3 together with PROPOSITION 3. Here we will not show how to compute them all, but only a sample that will be representative of the ideas involved.

Examples for D_{12}

(1) $P^3c = 4$
(2) $P^2cy = 2$
(3) $Pcvy = 1$

The relation (1) will follow directly from the following:

PROPOSITION 4. *Let c, v, y be three non colinear points and let L be a line through c different from cv and cy. Let P_1 be the point where L and vy meet, and let P_2 and P_3 be two additional points of L different from c. Then there exist exactly four points P such that the three lines PP_1, PP_2, PP_3 are tangent to a cubic whose singular triangle is c, v, y.*

PROOF: Choose coordinates so that c, v, y is the triangle of reference and that the unit point is P_2. Thus we have $P_1 = (0,1,1)$ and $P_3 = (t,1,1)$, where $t \neq 0,1$. Since a point P that satisfies the conditions of the statement cannot be on the cuspidal tangent (the line cy), we will have $P = (a,b,1)$. The tangential equation of a cuspidal cubic C with singular triangle c, v, y has the form

$$u_1^3 + \alpha u_0 u_2^2 = 0.$$

Now given a point $X = (x, 1, 1)$ it is a simple computation to show that the line XP is tangent to C iff

$$x^3 + [\alpha(b-1)b^2 - 3a]x^2 + [3a^2 - 2\alpha ab(b-1)]x - a^2[a - \alpha(b-1)] = 0.$$

Therefore, if the lines PP_1, PP_2, PP_3 are to be tangent to C, this equation will have $0, 1, t$ as roots. Since $a \neq 0$, this is equivalent to the relations

$$a = \alpha(b-1), \quad a[3a - 2\alpha b(b-1)] = t, \quad 3a - \alpha(b-1)b^2 = 1 + t.$$

Substituting $\alpha(b-1)$ by a in the second and third relations we get the relations

$$a^2(3 - 2b) = t, \quad a(3 - b^2) = 1 + t.$$

From these we get the relation

$$(1+t)^2(3-2b) = t(3-b^2)^2.$$

It can easily be seen that the only possible double root of this equation is $b = -1$, which only can occur when

$$5(1+t)^2 = 4t.$$

Therefore for general values of t the 4 solutions are indeed different.

Relation (2) can be derived from THEOREM 3 as follows. We take degenerations of type D_{12} with two foci P_1, P_2 and points c, y fixed. We want to show that there are exactly 2 possibilities for the pair P_3, v. In order to do this, normalize the cubic from which the degeneration is obtained so that its singular triangle has the c-vertex at c and the y-vertex at y. Then the possible centers of the homolography are the 4 intersection points of the two tangents to the cubic other than q from P_1 with the similar two tangents from P_2. The possibilities for the pair P_3, v will be the points of intersection with the line L of the third tangent through each of the four points and the lines joining them with the v-vertex. Now according to THEOREM 3 (4) and (2) there are only 2 possibilities.

Relation (3), $Pcvy = 1$, can be shown similarly. We want to show that if we fix a focus P_1 and the points c, v, y then there is only one possibility for the other two foci. To see this, normalize the cubic so that its singular triangle has verteces at v and y. Then the center of the homolography has to be on the line joining the cusp c' of the cubic and point c. On the other hand, from P_1 there is only one tangent to the cubic other than the line w (which counts as a double tangent). So this proves the claim. Let us remark that the same relation can be proved by normalizing so that two verteces of the singular triangle are c and v (in which case it suffices to apply THEOREM 3(1)), or c, y (in which case it suffices to apply THEOREM 3(2)).

Examples for D_{11}

(1) $P^2v = 1$
(2) $Pcv = 1$

To see (1), let v_0, P_1, P_2 be three colinear points, and set L to denote the line containing them. We want to see that there is only one point $c_0 = y_0$ on L such that the data $\{P_1, P_2, c_0, v_0\}$ is a degeneration of type D_{11}. Take two points c and Z outside L and colinear with P_1. Let y be a point on the vZ and let P be the point of intersection of yc with the line ZP_2. By PROPOSITION 3(2), we may assume that the degeneration is obtained by a homolography of a cuspidal cubic C with singular triangle c, v, y with center P and axis L. So we want to see that there is only one possible position for y, and that for such y there is a single cuspidal cubic C that is tangent to the lines PP_1 and PP_2.

Choose c, Z, v as triangle of coordinates and P_2 as unit point. Then it is easy to see that $P_1 = (1,1,0)$, and that $y = (0,1,x)$, $P = (x,1,x)$. The tangent equation of the cubics whose singular triangle is c, v, y has the form

$$(u_1 - xu_2)^3 = \alpha u_0 u_2^2.$$

The relations obtained when imposing that the lines PP_1, PP_2 are tangent to C are

$$x^6 = \alpha x(1-x)^2 \text{ and } x^3 = \alpha$$

which have only one solution, namely $x = 1/2, \alpha = 1/8$.

To see (2), we want to show that if we fix a focus P_1 and the points $c_0 = y_0, v_0$ of a degeneration of type D_{11}, then there is only one possibility for the other focus P_2. To see this, normalize the cubic so that its singular triangle has verteces at c and v (use PROPOSITION 3(2)). Then the center of the homolography has to be one of the points of intersection with q of the tangents to C from P_1. The claim is a direct consequence of THEOREM 3(1).

Examples for D_{10}

(1) $P^3 = 2$
(2) $P^2c = 2$
(3) $Pcy = 1$

To see (1) consider a degeneration of type D_{10} with the three foci P_1, P_2, P_3 fixed. We want to see that there are exactly two solutions for the pair $c_0 = v_0$ and y_0. Choose two points c, y outside the line L of the degeneration and colinear with P_3, a point v on yP_1 (different from y and P_1), and a point v on the line cv (different from c and v). According to PROPOSITION 3(3) the degeneration may be obtained by means of a homolography with center P and axis L applied to a cuspidal cubic whose singular triangle is c, v, y. Thus what we want to see is that there are exactly two positions for the pair $\{v, P\}$ for which there exists such a cuspidal cubic with the lines PP_1, PP_2, PP_3 tangent to it.

Choose as coordinate triangle c, y, P_1, and let P_2 be the unit point. Then we will have $P_3 = (1,1,0)$, $v = (0,x,1)$, $P = (\gamma, x, 1)$, where $x \neq 0,1$, $\gamma \neq 0, x$. It is easy to see that the tangential equation of the cuspidal cubics with singular triangle c, v, y has the form

$$u_1^3 = \alpha u_0(xu_1 + u_2)^2.$$

When we impose that the lines PP_1, PP_2, PP_3 belong to such a dual curve, we get the relations

$$\gamma = -\alpha x^3, \alpha\gamma^2 = -1, (1-\gamma)^3 = \alpha\gamma^2(x-1)^3.$$

After some transformations these relations are seen to be equivalent to the relations

$$\begin{aligned}
&\gamma = \rho x, \text{ where } \rho^3 = 1, \rho \neq 1;\\
&\gamma(\gamma - 1) = x(x-1);\\
&\gamma = -\alpha x^3.
\end{aligned}$$

Since $x \neq 0$, the second of these relations gives that $x = \dfrac{1}{1+\rho}$ (two possible values), for each of which there is a single value for γ and α.

Now let us consider (2). Assume given a degeneration of type D_{10}, with $c_0 = v_0$ and two foci P_1, P_2 fixed. We want to see that there are exactly two possible positions for the pair (y_0, P_3). Choose two points v, y outside the line L of the degeneration and colinear with P_1, and a point P colinear with v and $c_0 = v_0$. By PROPOSITION 3(3), the degeneration can be obtained by means of a homolography of center P and axis L applied to a cuspidal cubic whose singular triangle is $c_0 = v_0, v, y$. So what we want to see is that there are exactly two positions of P such that the lines PP_1, PP_2 are tangent to such a cuspidal cubic. Take as coordinate triangle $c_0 = v_0, y, v$ and unit point P_2, so that $P_1 = (0,1,1), P = (1,0,x), x \neq 0$. Then the tangential equation of a cuspidal cubic with singular triangle $c_0 = v_0, v, y$ has the form

$$u_1^3 = \alpha u_0 u_2^2.$$

The relations obtained when we impose that the lines PP_1, PP_2 belong to that dual cubic are the following

$$\alpha x = 1, (x-1)^3 = -\alpha x,$$

from which the claim follows.

Finally let us consider the relation (3). So assume that we have a degeneration of type D_{10} with a focus, say P_1, and the points $c_0 = v_0$ and y_0 fixed. We want to see that there is only one position for the remainig two foci of the degeneration. Take a point v outside the line L of the degeneration. By the PROPOSITION 3(3) we can assume that the degeneration is obtained by a homolography with axis L applied to a cuspidal cubic C whose singular triangle is $c_0 = v_0, v, y_0$. So the possible centers of such a homolography are the intersection with the line z with the tangents to C from P other than q. So the claim follows from THEOREM 3(2).

Examples for D_7

$$(1) \quad Rcv = 1$$
$$(2) \quad cvy = 1$$

To see (1), assume we have a degeneration of type D_7 with fixed c_0, v_0 and R (the double focus). We want to see that there is a unique position for Q (the simple focus) and y_0. Now choose a point y not on the line L of the degeneration. By PROPOSITION 3(4), we may assume that the given degeneration can be obtained by a homolography of a cuspidal cubic C with singular triangle c_0, v_0 and y. The possible centers are the contact points Q_1, Q_2, Q_3 of the three tangents to C from R. These three points are colinear with y, by THEOREM 3(5). Let L_i be the tangent to C going through Q_i that is different from the tangent to C at Q_i. Then our claim follows because the three lines L_i are concurring at a point on z, by THEOREM 3* (5*).

To see (2), assume we have a degeneration of type D_7 with fixed c_0, v_0 and y_0. Our goal is to see that there is a unique position for the foci R and Q (double and simple, respectively). Take a point c not on the line L of the degeneration. By PROPOSITION 3(4) we can assume that the degeneration is obtained by a homolography of a cuspidal cubic C whose singular triangle is c, v_0 and y_0. Since the center of the homolgraphy must be on the line cc_0 and on $C - \{c, v\}$, we see that there is a unique solution.

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