Geometry
of
Complete Cuspidal Plane Cubics

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Abstract. We show how to compute *all* fundamental numbers for plane cuspidal cubics. This updates and extends the work of Schubert on this subject. In our approach we need a far more precise description of the first order degenerations (13 in all) than that given by Schubert and this is obtained by proving a number of key geometric relations that are satisfied by cuspidal cubics. Moreover, our procedure does not require using coincidence formulas to derive the basic degeneration relations.

Introduction

The enumerative theory of cuspidal cubics was first considered by Maillard (doctoral thesis, 1871) and Zeuthen [1872]. Subsequently they were extensively studied by Schubert. For an exposition of his (and others) results, see Schubert [1879], § 23, pp. 106-143. Schubert also considers cuspidal cubics in $\mathbf{P}^3$, but here for simplicity we will study only cuspidal cubics in a fixed projective plane $\mathbf{P}^2$ over an algebraically closed ground field $\mathbf{k}$. In case the characteristic p of $\mathbf{k}$ is positive we will assume that $p \neq 2, 3$.

Let S be the space of plane non degenerate cuspidal cubics, so that S is an orbit under the action of the group $\mathrm{G} = \mathrm{PGL}(\mathbf{P}^2)$ on the space of plane cubics. Each cuspidal cubic determines a triangle, called *singular triangle* (*Singularitätendreieck*, Schubert [1879], p. 106), whose verteces c, v, y are, respectively, the cusp, the inflexion and the intersection point of the cuspidal and inflexional tangents. The sides of this triangle, denoted q, w, z are, respectively, the cuspidal tangent, the inflexional tangent and the line cv (see Fig. 1 at the end).

The conditions that were first considered in the enumerative theory of cuspidal cubics were the *characteristic conditions* μ, ν (i.e., going through a point and being tangent to a line, respectively). Schubert also considers conditions imposing that a given vertex (side) of the singular triangle lies on a line (goes through a point), and denotes any of these six conditions with the same symbol used to denote the corresponding element

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of the singular triangle. Altogether we have eight conditions, which will be called *fundamental conditions* for the cuspidal cubics.

By transversality of general translates (Kleiman [1974]), the cubics satisfying seven (possibly repeated) fundamental conditions whose data are in general position are finite in number and at least in characteristic zero they count with multiplicity 1. In characteristic $p > 0$ each solution may have to be weighted with a multiplicity that is a power of p. The numbers so obtained are called *fundamental numbers* for the cuspidal cubics. The fundamental numbers involving only μ and ν are the *characteristic numbers*.

It turns out that there are 620 *non-zero* fundamental numbers for the cuspidal cubics (discounting those that may be obtained by duality), and of these Schubert gives explicit tables for 391 (*loc. cit.*, pp. 140–142). Of the remaining 229, a few (actually 21) can be deduced from related entries in tables he gives for space cuspidal cubics. As we explain below, Schubert's work is also incomplete on other (more fundamental) counts. The general problem of verifying and understanding all the geometric numbers computed by 19th century geometers, which is the main motivation of this and related works, was stated by Hilbert [1902] as Problem 15 of his list.

Schubert's calculations rely on the method of degenerations, which in turn requires to know, if we want to compute all fundamental numbers,

i) *that the space S^* of complete cubics* (see Section 1) *is smooth in codimension one,*

ii) *how many boundary components* (called *degenerations*) *there are in S^** (see Section 2),

iii) *how to solve a number of related enumerative problems on each of the degenerations* (see Sections 4–7 and 9), and

iv) *to express, on S^*, the fundamental conditions in terms of the degenerations* (*degeneration relations*, see Section 10) *and to establish that a number may be computed by substituting one of its conditions by its expression in terms of the degenerations.*

For a given subset of fundamental numbers much less may be needed. Thus, in order to compute the 8 characteristic numbers, it is enough to know a single degeneration (degeneration σ, whose points consist of a conic and one of its tangent lines), but for this one it is nevertheless still necessary to take care of the points i)-iv) to verify them. This was done recently, in different ways, by Sacchiero [1984] and by Kleiman – Speiser [1986].

Question i) is not considered by Schubert. As far as ii) goes, Schubert constructs, in addition to σ, 12 degenerations, by means of the so called homolography process, but he does not provide any formal verifications, nor does he prove that they are all possible degenerations. These questions were clarified in Miret – Xambó [1987] (see Section 2 below).

Question iii) is rather involved. Since the building elements of some of the degenerations exceed in number what would be allowed by their dimension, they cannot be independent and so *there must exist relations among those elements.* Schubert gives lists of such relations, expressed in enumerative terms (tables of "Stammzahlen", loc. cit., pp. 120-127), and asserts that they were obtained by an indirect process ("a posteriori erschlossen", ibid., p. 119). Now in Miret – Xambó [1987] the Stammzahlen that are needed to describe the degenerations were studied and were showed to be related to basic projective geometry properties of the cuspidal cubics. In this paper we continue the study of this topic and give a detailed *geometric description* of all the degenerations.

Another difference with Schubert arises in the treatment of question iv). Schubert derived degeneration relations by means of coincidence formulas (loc. cit., p. 107 and ff.). This procedure leads, however, to computations of multiplicities that seem very difficult to handle, and which have been verified, as far as the authors know, only in very special cases, like some that arise in the verification of the characteristic numbers. Instead, one may work on the idea, already used by Schubert to cross-check his results, that most geometric numbers can be computed in several different ways. When used systematically, this observation allows to establish, if we already have assembled suitable enumerative information on the various degenerations, the required degeneration relations by simple algebra. This version of the method of degenerations is explained in Section 8.

The organization of this paper is as follows. Section 1 is devoted to the determination of the Picard group of S. At the end we define the space of complete cuspidal cubics. In Section 2 we briefly recall the description of the 13 first order degenerations of the cuspidal cubics. Then in Section 3 we prove a few geometric properties of cuspidal cubics that supplement and refine those given in Miret-Xambó [1987]. In Sections 4–7 we carry out systematic enumerative computations on the various degenerations (Stammzahlen) based on the properties inherited by the degenerations from corresponding properties of the cuspidal cubics. Then in Section 8 we outline, as we said above, a setup for the method of degenerations. In Section 9 we include a number of tables of degeneration numbers; they are obtained from the elementary numbers by direct arithmetic calculation. In Section 10 we determine the degeneration relations for the cuspidal cubics, that is, the expressions of the first order conditions in terms of the degenerations and of the condition that the cusp of the cubic be on a line. Section 11 contains examples that show how to put together the information gathered before to effectively compute the fundamental numbers of cuspidal cubics. Finally in Section 12 we give the tables of all the fundamental numbers.

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1. Spaces of cuspidal cubics

1.1. Let $\mathbf{P}^2$ be the *complex* projective plane. The homogeneous coordinates of $\mathbf{P}^2$ will be denoted (x_0, x_1, x_2). The point $P_0 = (1,0,0)$ will be called the origin of coordinates. The space parametrizing plane cubics is isomorphic to $\mathbf{P}^9$ and we will identify these spaces. We shall let S denote the 7 dimensional locally closed subset whose points represent non-degenerate cuspidal cubics. Thus S is an orbit of the natural action of the group $\mathrm{G} = \mathrm{PGL}(\mathbf{P}^2)$ on $\mathbf{P}^9$. In particular S is a smooth variety.

1.2. If X is a point or a line, we shall set S_X to denote the subvariety of S whose points are cuspidal cubics with its cusp on X. Similarly, if P is a point and L is a line, $P \in L$, then $S_{P,L}$ will denote the cycle of cuspidal cubics that have the cusp at P with cuspidal tangent L. The cycle $S_{P,L}$ is irreducible, because it is an open set of a linear space. From this it follows that the cycle S_X is also irreducible. The class of S_L in Pic(S) will be denoted c and the class of the cycle of cuspidal cubics whose cuspidal tangent goes through a point will be denoted q.

1.3. Theorem. *Pic(S)* $= \mathbf{Z} \oplus \mathbf{Z}/(5)$. *The free generator of this group is c and the generator corresponding to $\mathbf{Z}/(5)$ is the projection of q.*

Proof: Let L be a given line, and let U be the open set of S whose points are cuspidal cubics with the cusp not on L. Thus $S - U = S_L$ and hence we have an exact sequence

$$A^0(S_L) \to A^1(S) \to A^1(U) \to 0.$$

From this we see that $\mathrm{Pic}(S) = A^1(S)$ is generated by c and $A^1(U)$. Now we have an isomorphism $U \simeq \mathbf{A}^2 \times S_{P_0}$, induced by translations in $\mathbf{A}^2 \simeq \mathbf{P}^2 - L$, and so $A^1(U) \simeq A^1(S_{P_0})$.

To study the last group, let T denote the space of cubics that have a double point, and let T_P denote the 6 dimension linear space of cubics that have a double point at P. Thus cubics in T_{P_0} have equations of the form

$$x_0 f_2 + f_3 = 0, \tag{1}$$

where f_i, $i = 2,3$, is a homogeneous polynomial of degree i in the variables x_1, x_2. It is clear that $\overline{S} \subset T$, where $\overline{S}$ is the closure of S in T. Now $\overline{S}_{P_0}$ is a quadratic cone of rank 3 in T_{P_0}, for it is clear that (1) has a double tangent at P_0 if and only if $\mathrm{Disc}(f_2) = 0$. Moreover, if $\tilde{F}$ is the quintic hypersurface of T_{P_0} given by the equation $\mathrm{Res}(f_2, f_3) = 0$, and $\mathrm{F} = \tilde{F} \cap \overline{S}_{P_0}$, then points in F_{red} represent degenerate cuspidal cubics and conversely. Indeed, if in (1) $f_2 = w^2$, where w is a linear form in x_1, x_2, then the cubic $x_0 w^2 + f_3 = 0$ is a degenerate cuspidal cubic if and only if w divides f_3.

We will show that $[F] = 2[F_{\text{red}}]$, and that F_{red} is irreducible. If this is so, from the exact sequence

$$A^0(F_{\text{red}}) \to A^1(\overline{S}_{P_0}) \to A^1(S_{P_0}) \to 0$$

and the fact, also proved below, that

$$A^1(\overline{S}_{P_0}) \simeq \mathbf{Z},$$

generated by a ruling of the cone, we deduce that $A^1(S_{P_0}) = \mathbf{Z}/(5)$, because a quintic hypersurface section is rationally equivalent to 10 rulings and so F_{red} is equivalent to 5 rulings. Now observe that the rulings of the cone are the subspaces of cuspidal cubics that have a given cuspidal tangent, and that one of these rulings generates, by translations, the cycle of cuspidal cubics whose cuspidal tangent goes through a fixed point.

To prove that $[F] = 2[F_{\text{red}}]$, consider an affine space $\mathbf{A}^5$ and define a map

$$f\colon \mathbf{A}^5 \longrightarrow \overline{S}_{P_0}$$

by transforming (s, b_0, b_1, b_2, b_3) into the cubic

$$x_0(x_1 + sx_2)^2 = b_0x_1^3 + b_1x_1^2x_2 + b_2x_1x_2^2 + b_3x_2^3.$$

This induces an isomorphism of $\mathbf{A}^5$ with $\overline{S}_{P_0} - R$, where R is the ruling of $\overline{S}_{P_0}$, given by the cuspidal cubics whose cuspidal tangent is the line $\{x_2 = 0\}$. The pull-back under f of the subscheme F is the subscheme given by the equation $\operatorname{Res}((x_1 + sx_2)^2, f_3) = 0$. Now using Fulton [1984], Example A.2.1, p. 410, it is easy to see that

$$\operatorname{Res}((x_1 + sx_2)^2, f_3) = \operatorname{Res}(x_1 + sx_2, f_3)^2$$

and so on the open set $\overline{S}_{P_0} - R$ we see that F is divisible by 2, and that the restriction of $\frac{1}{2}F$ to each ruling is a hyperplane of the ruling. Hence the equality $[F] = 2[F_{\text{red}}]$ is correct on the complementary set of any ruling, and therefore it holds globally.

To end the proof we have to see that a rank three projective quadratic cone K satisfies $A^1(K) = \mathbf{Z}$, generated by a ruling. To see this notice that in order to compute $A^1(K)$ we may throw away the vertex of the cone, because its codimension is 2. Having done that, K is a fibre bundle over a smooth conic C with fibre $\mathbf{A}^1$. Hence $A^1(K)$ is isomorphic to $A^1(C)$. But $A^1(C) \simeq \mathbf{Z}$, generated by the class of a point of C, and from this the claim follows. ◇

1.4. Corollary. *The Picard group of the space of non degenerate nodal cubics is generated by the class of the cycle of nodal curves with node on a fixed line.*

PROOF: Let T_L be the cycle of nodal curves that have its node on a line L. This cycle is irreducible and we have an exact sequence

$$A^0(T_L) \to A^1(T) \to A^1(V) \to 0, \quad V = T - T_L.$$

So Pic$(T) = A^1(T)$ is generated by the class $b = [T_L]$ and by $A^1(V)$. Now $V \simeq \mathbf{A}^2 \times T_P$, so $A^1(V) = A^1(T_P)$. Now cubics that have a double point at P form a 6 dimensional linear space, which is nothing but $\overline{T}_P$. In this space we have the quadratic cone $D = \overline{S}_P$ and the hypersurface E whose points consist of cubics that split in a conic and a line, and, up to subvarieties of codimension 2 or higher, $\overline{T}_P - T_P = D \cup E$. Thus we have an exact sequence

$$A^0(D \cup E) \to A^1(\overline{T}_P) \to A^1(T_P) \to 0$$

So it is clear that $A^1(T_P) = \mathbf{Z}/(m)$, where $m = \gcd(d, e)$, d and e the degrees of D and E in $\overline{T}_P$, respectively. Now D has degree 2, as we noticed above, and E is a Segre variety, which has degree 5. So we conclude that $A^1(T_P) = 0$ and so our statement follows. ◇

1.5. Complete cuspidal cubics. We will use the letters b, c, v, y, z, q, w also to denote the maps that transform a given cubic C in S into, respectively, the dual cubic C^*, the cusp, the inflexion point, the intersection of the cuspidal and inflexional tangents, the line joining the inflexion and the cusp. the cuspidal tangent, and the inflexion tangent.

Set

$$\mathbf{P} = \mathbf{P}^{9^*} \times (\mathbf{P}^2)^3 \times (\mathbf{P}^{2^*})^3$$

and consider the map

$$h: S \to \mathbf{P}, \quad h = (b, c, v, y, z, q, w).$$

Let S^* be the closure of the graph of h in $Z = \overline{S} \times \mathbf{P}$. The space S^* will be referred to as the space of *complete cuspidal cubics.* The points in $S^* - S$ will be called *degenerate* cuspidal cubics, where the inclusion of S in S^* is given by id$\times h$. Since the composition of h with the projection of $\mathbf{P}$ onto its first factor is $b: S \to \mathbf{P}^{9^*}$, it is natural to define $b: S^* \to \mathbf{P}^{9^*}$ as the restriction to S^* of the projection onto $\mathbf{P}^{9^*}$. Given a point C' of S^*, we shall say that $b(C')$ is the *tangential cubic* associated to the complete cubic C'. In the same way we can define morphisms c, v, y, z, q and w from S^* to the corresponding factors of Z. Given $C' \in S^*$, $c(C')$ will be called *the cusp* of C' and similarly with the other maps.

For a non-degenerate cuspidal cubic, the triangle whose vertexes are c, v, and y, and whose sides are z, q, w, is called *singular triangle.* The same notion can now be defined for degenerate cuspidal cubics in S^*. In other words, given a degenerate complete cuspidal cubic C', the six-tuple

$$(c(C'), v(C'), y(C'), z(C'), q(C'), w(C'))$$

will be called singular triangle of C', the first three elements being the vertices and the last three the sides. The cubic is degenerate if and only if its singular triangle is a degenerate triangle.

The projection of a point $C' \in S^*$ to $\overline{S}$ will be referred to as the *point cubic* associated to C'.

1.6. Theorem (see Miret-Xambó [1987]). *The variety S^* of complete cuspidal cubics is non-singular in codimension 1.*

In next section we give a description of the boundary components of S^*.

1.7. Conventions. Henceforth we will say that a point P is general with respect to a cuspidal cubic C if it does not lie on C nor on any side of the singular triangle. A point P of C will be said to be general if it is different from the cusp and the inflexion.

Given four colinear points A, B, C, D we shall write $\rho(A, B, C, D)$ to denote their cross ratio.

We also recall here that given a cuspidal cubic of the form $x_0x_2^2 = x_1^3$ then the dual cubic has equation $27u_0u_2^2 + 4u_1^3 = 0$.

2. Degenerations

The boundary $S^* - S$ has 13 irreducible components D_i, all of dimension 6 (see Miret-Xambó [1987]). The brief descriptions given below are intended to outline the structure of the general points of D_i, $i = 0, \ldots, 12$ (see the drawings at the end). In each case we indicate what the corresponding point and line cycles are, as well as the sides and verteces of the singular triangle. The degenerations $D_1, \ldots, D_{12}$ can be obtained by applying a homolography to a non-degenerate cuspidal cubic with suitable choices of its center P and axis L. This means that points on D_i, $i = 1, \ldots, 12$, are the limit cycles for $t = 0$ or $t = \infty$ of the cycles obtained transforming the given cuspidal cubic by a homology of modulus t with center at P and axis L.

In what follows instead of saying "the pencil of lines through point P is a component of the dual cubic" we will say that "P is a focus of the cubic". Thus, if three points are declared as foci, this means that the dual cubic decomposes into the three pencils of lines through the given points.

2.1. D_0. General points in D_0 consist of a smooth conic K together with a distinguished tangent line L of K. The three sides of the singular triangle of such a pair coincide with L, while the three verteces coincide with the contact point, say P. The tangential cubic consists of the dual conic K^* and the pencil of lines through P.

2.2. D_1 and D_{12}. Points in D_{12} consist of a triple line L with three distinct foci on it. The sides of the singular triangle coincide with L and its three vertices are three distinct points on L disjoint from the foci. The degeneration D_1 is dual of D_{12}.

2.3. D_2 and D_{11}. Points in D_{11} consist of a triple line L with three distinct foci on it. The vertices c and y fall together on a focus, and the vertex v is a point on L which is not a focus. The sides w and z coincide with L and q is a line through $c = y$ different from L. The degenerations D_2 is dual of D_{11}.

2.4. D_3 and D_{10}. Points in D_{10} consist of a triple line L with three distinct foci on it. The sides q and w coincide with L and z is a line different from L that does not go through a focus. The verteces c and v fall together on the intersection of z and L and y is a point on L different from $c = v$ and which is not a focus. The degeneration D_3 is dual of D_{10}.

2.5. D_4 and D_9. Points on D_9 consist of a triple line L with a simple focus and a double focus. The sides q and z coincide with L, while w is a line through the double focus distinct from L. The verteces $v = y$ fall on the double focus and c is a point on L different from the foci. The degeneration D_4 is dual of D_9.

2.6. D_5 and D_8. Points in D_8 consist of a triple line L with a simple focus and a double focus. The side z coincides with L, while q and w are lines different from L that go through the simple and the double focus, respectively. The intersection of q and w is the vertex y, while c falls on the simple focus and v on the double focus.

2.7. D_6 and D_7. Degenerations of type D_7 consist of a double line L and a simple line L', with a simple focus Q on L and a double focus R that falls on $L \cap L'$. The three sides of the singular triangle coincide with L, while the verteces are three distinct points of L disjoint from the foci. The degeneration D_6 is dual of D_7.

It is to be remarked that the elements with which a degeneration is built up need not be independent. Take, for instance, D_{12}. We have six points on a line. Such configurations fill a space $\overline{D}_{12}$ of dimension 8. Since D_{12} has dimension 6 we see that D_{12} is a codimension 2 subvariety of $\overline{D}_{12}$. Similarly we can define varieties $\overline{D}_{11}$, $\overline{D}_{10}$ and $\overline{D}_7$ of dimensions 7, 8 and 8 that contain the degenerations D_{11}, D_{10} and D_7 as subvarieties of codimensions 1, 2 and 2, respectively. Thus $\overline{D}_{11}$ may be described as the variety whose points are ordered pairs of lines with three distinguished points on the first, and $\overline{D}_{10}$ and $\overline{D}_7$ as varieties whose points are ordered pairs of lines with four distinguished points on the first line. Of course, similar remarks can be made for the dual degenerations D_1, D_2, D_3 and D_5.

The enumerative geometry of D_7, D_{10}, D_{11} and D_{12} will be studied in Sections 4, 5, 6 and 7, respectively.

3. Projective properties of cuspidal cubics

3.1. Proposition. *Let C be a non-degenerate cuspidal cubic and P a general point with respect to C. Let L_1, L_2, L_3 be the tangent lines to C through P and set $\rho_i = \rho(Pc, Pv, Py, L_i)$. Then*

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = 3$$
$$\rho_1\rho_2\rho_3 = 1.$$

Conversely, given non-zero scalars ρ_i, $i = 1,2,3$, satisfying the two equations above, three distinct concurrent lines L_1, L_2, L_3, say at the point P, and a triangle c, v, y with no vertex on the lines such that $\rho_i = \rho(Pc, Pv, Py, L_i)$, then there exists a cuspidal cubic C with singular triangle c, v, y which is tangent to the lines L_i, $(i = 1,2,3)$. (The proof actually shows that C is unique.)

Proof: Take the singular triangle of C as the reference triangle and take a general point of C as the unit point. Let $P = (a, b, 1)$. The projection of y from P on the line $z = cv$ is $y' = (a, 0, 1)$. Let $M = (m, 0, 1)$ be the point where a tangent to C through P meets the line $z = cv$. Then imposing that the line PM satisfies the dual equation we find that m has to satisfy the relation

$$m^3 + (27b^3 - 3a)m^2 + 3a^2m - a^3 = 0.$$

Let m_i, $i = 1,2,3$, be the roots of this equation and M_i the corresponding points. One computes that $\rho(c, v, y', M_i) = m_i/a$ and from this the first part of the proposition follows easily.

To see the converse, take $(c, y, v; P)$ as a reference. With respect to this reference the line L_i has coordinates $(1, \rho_i - 1, -\rho_i)$. We know that the cuspidal cubics with singular triangle c, v, y are of the form $\alpha x_1^3 = x_0 x_2^2$, $\alpha \neq 0$. Using the line equation of this cubic we see that it is tangent to the line L_i if and only if

$$\rho_i^3 + (\frac{27}{4}\alpha - 3)\rho_i^2 + 3\rho_i - 1 = 0.$$

Thus if the ρ_i satisfy the conditions in the first part of the statement, then in order that the cubic be tangent to the three lines it is necessary and sufficient that $\frac{27}{4}\alpha - 3 = -(\rho_1 + \rho_2 + \rho_3)$. Since this equation has a unique solution with respect to α, which is non-zero, this ends the proof. ◇

The preceeding result still holds if P is a point on C not on the singular triangle, taking the tangent to C at P twice. In this case, however, we have a more precise statement:

3.2. Proposition. *Given a point P of C, let L be the tangent to C at P and L' the tangent to C through P other than L. Then the cross-ratio of any four of the lines Pc, Pv, Py, L, L' is independent of P. In fact we have that*

$$\begin{aligned} \rho(Pc, Pv, Py, L) &= -\,2 \\ \rho(Pc, Pv, Py, L') &= \ \tfrac{1}{4} \\ \rho(Pc, Pv, L, L') &= -\tfrac{1}{8} \\ \rho(Pc, Py, L, L') &= \ \tfrac{1}{4} \\ \rho(Pv, Py, L, L') &= -\tfrac{1}{2}. \end{aligned}$$

Notice that any two of these relations imply the other three.

Conversely, given a triangle c, v, y and two lines L and L' meeting at a point P not on the sides of the triangle and in such a way that two (and hence all) of the equations above are satisfied, then there exists a cuspidal cubic C with singular triangle c, v, y that is tangent to L at P and also tangent to L' (necessarily at a point different from P).

Proof: A straightforward computation as in the proof of 3.1. ◇

3.3. Proposition. *Given a point P of the cuspidal tangent q of a non-degenerate cuspidal cubic C, different from c, then the pair of lines q, Pv is harmonic with respect to the pair of tangents to C through P other than q. Conversely, given a harmonic tetrad of concurrent lines q, L, L' and L'' (say at P), and points c on q and v on L, both different from P, there exists a cuspidal cubic C with cusp at c and inflexion at v such that the tangent lines to C from P are q, L' and L''.*

Proof: Taking (c, v, y) as reference triangle and the unit point on C then the equation of C has the form $x_1^3 = x_0 x_2^2$ and the point P is of the form $(a, 1, 0)$. Let u, u' be the tangents to C, other than q, through P. Let $Q = (m, 0, 1)$ and $Q' = (m', 0, 1)$ be the intersections of u and u' with the line cv. It suffices to show that the pairs of points (c, v) and (Q, Q') are harmonic. Imposing that the lines $u = PQ$ and $u' = PQ'$ are tangent to C (using the dual equation) we find that $m + m' = 0$, and this ends the first part of the proof. The converse part can be seen in the same way as the converse part of 3.1. ◇

3.4. Proposition.

- **(a)** *Given a point P of the line z of a non-degenerate cuspidal cubic C, different from c and v, then the cross ratio of the lines z, Py and any pair of tangents to C from P is a primitive cube root of unity.*
- **(b)** *The line z and the three tangents to C from P form an equianharmonic tetrad, that is, its cross-ratio is a primitive cube root of -1.*
- **(c)** *The line Py and the three tangents to C from P form also an equianharmonic tetrad.*
- **(e)** *Conversely, given a triple of concurrent lines $\{L, L', L''\}$, say at a point P, and a pair of points c, y not on those lines, there is a cuspidal cubic with singular triangle c, y, v, where v is a point on the line cP, and which is tangent to the lines L, L' and L'' if either the cross ratio of Pc, Py and any pair of L's is a primitive cube root of unity or the tetrads Pc, L, L', L'' and Py, L, L', L'' are equianharmonic.*

Proof: Take the same reference as in the proof of **3.1** Let $P = (a, 0, 1)$. Then the line joining P and the point $M = (m, 1, 0)$ on the line q is given by the equation $-x_0 + mx_1 + ax_2 = 0$. Imposing that it satisfies the dual equation we get the relation

$$4m^3 = 27a^2,$$

whose solutions are of the form $m_i = \xi^k m_0$, $k = 0, 1, 2$, where ξ is a primitive cube root of unity and $m_0/3$ is a fixed cube root of $a^2/4$. Computation shows that

$\rho(c, y, M_i, M_j) = \xi^{j-i}$, which proves part **(a)**. Similarly, $\rho(c, M_0, M_1, M_2) = \xi + 1$, which proves **(b)**. The proof of **(c)** is similar. The converse part can be seen in the same way as the converse part of 3.1. ◇

We also collect here a three lemmas about cross ratios because we do not know a reference for them. The proofs are obtained by straightforward analytic computations.

3.5. Lemma. *Given three non-concurrent lines L_1, L_2, L_3, a point P not lying on any of them and a scalar $k \neq 1$, there exists a unique line L through P such that $\rho(P, L \cap L_1, L \cap L_2, L \cap L_3) = k$.*

3.6. Lemma. *Given a four lines L_1, L_2, L_3, L_4 such that no three of them are concurrent, a point P not lying on any of them and a scalar $k \neq 1$, there exist exactly two lines L through P such that the $\rho(L \cap L_1, L \cap L_2, L \cap L_3, L \cap L_4) = k$.*

3.7. Lemma. *Given five lines $L_1, \ldots, L_5$ in general position and two scalars k_1 and k_2 different from 1, there exists a unique line L such that*

$$\rho(L \cap L_1, L \cap L_2, L \cap L_3, L \cap L_4) = k_1$$
$$\rho(L \cap L_1, L \cap L_2, L \cap L_3, L \cap L_5) = k_2.$$

We also need a few cycle identities for ordered and unordered triples of collinear points. First recall that for flags "point-line" in the projective plane, $\{p, g\}$, we have the relation $gp = g^2 + p^2$, where g is the condition that the line goes through a point and p the condition that the point be on a line. Now consider configurations $(L; c, v, y)$ consisting of a line L and three distinguished points c, v, y on L. The variety V parametrizing such configurations is smooth and complete. Moreover, it follows easily from the relation just recalled that on V we have the following relations:

3.8. Lemma.

$$L^2 + c^2 = Lc,$$
$$L^2 + v^2 = Lv,$$
$$L^2 + y^2 = Ly.$$

Now consider configurations consisting of a line L together with a zero cycle Z of degree r on L. The points in the support of Z will be called foci of the configuration. The variety V' of such configurations is smooth and complete. In fact, V' can be defined as the projective bundle associated to the vector bundle $S^r(E^*)$, where E is the tautological rang 2 bundle on $\check{\mathbf{P}}^2$. Given j lines in general position, and a point $(L; Z)$ of V', write $Z = Z' + Z''$, where the support of Z' lies on the union of the lines and the support of Z'' is disjoint from them. We shall write Q_j to denote the subvariety of V' whose points $(L; Z)$ satisfy that on each of the lines there is at least a point of Z (hence of Z') and that $\deg(Z'') \leq r - j$. It is not hard to see that Q_j is irreducible of codimension j. Now let Σ be the set of $\binom{j}{2}$ points of intersection of the j lines. For each $P \in \Sigma$, let Q_j^P denote the subvariety of V' whose points $(L; Z)$ satisfy that $P \leq Z'$, that on each of the j lines there is at least a point of Z, and that $\deg Z'' \leq r - j + 1$. It is also easy to see that Q_j^P is an irreducible subvariety of codimension j. For each pair of points $P, Q \in \Sigma$, $P \neq Q$, let $Q_j^{P,Q}$ denote the subvariety of V' whose points $(L; Z)$ satisfy that $P + Q \leq Z'$, that on each of the j lines there is at least a point of Z, and that $\deg Z'' \leq r - j + 2$. It is also easy to see that $Q_j^{P,Q}$ is an irreducible subvariety of codimension j.

3.9. Lemma.

$$Q^j = Q_j + \sum_P Q_j^P + \sum_{P,Q} Q_j^{P,Q}$$

Proof: That the left hand sides are equal to the right hand sides up to multiplicities follows from simple combinatoric arguments. That the multiplicities are equal to 1 in all cases can be seen by the principle of general translates (see Kleiman [1974] and Laksov-Speiser [1987]). ◇

With the same notations, let Q and P denote the conditions that a configuration has, respectively, a focus on a given line and a focus at a given point. If the number of foci is 2 or 3, from the preceeding lemma we conclude:

3.10. Lemma.

$$\begin{aligned} [Q^2] &= [Q_2] + [P] \\ [Q^3] &= 3[PQ] \\ [Q^4] &= 3[P^2] \\ [Q^5] &= 0 \end{aligned} \qquad \text{resp.} \qquad \begin{aligned} [Q^2] &= [Q_2] + [P] \\ [Q^3] &= [Q_3] + 3[PQ] \\ [Q^4] &= 6[PQ_2] + 3[P^2] \\ [Q^5] &= 15[P^2Q] \end{aligned}$$

◇

4. Stammzahlen for D_7

We shall use the notations introduced in **2.7**.

4.1. Proposition. *The singular triangle c, v, y of a degeneration of type D_7 may be any triple of distinct collinear points. The simple focus Q and the double focus R are collinear with c, v, y and are uniquely determined by the relations $\rho(c, v, y, Q) = 1/4$ and $\rho(c, v, y, R) = -2$. The simple line may be any line through R.*

Proof: It is a direct consequence of **3.2** and the way the degeneration is obtained by a homolography. ◇

Let $\overline{D}'_7$ be the variety of ordered 5-tuples of distinct collinear points c, v, y, Q, R. Let D'_7 be the subvariety of $\overline{D}'_7$ given by the relations in **4.1**. Let $\overline{\pi}\colon \overline{D}_7 \to \overline{D}'_7$ be the map that forgets the simple line L' and $\pi\colon D_7 \to D'_7$ the restriction of $\overline{\pi}$ to D_7. Next lemma shows that the computation of the Stammzahlen for D_7 is equivalent to the computation of Stammzahlen for D'_7.

4.2. Lemma. *Let N be a fundamental number for D_7.*

(a) *If the exponent of L' in N is 0 or at least 3, then $N = 0$.*

(b) *If the condition L' appears just once in N, then $N = N'$, where N' is the number on D'_7 obtained dropping the condition L' from N.*

(c) *If the condition L' appears just twice in N, say $N = L'^2 x$, then $N = R'x'$, where the product x' on D'_7 corresponds to the product x on D_7 (that is, $x = \pi^*(x')$) and where R' is the condition on D'_7 that the double focus be on a line.*

Proof: Follows easily using the projection formula and we omit it. ◇

4.3. Theorem. *The number $L^{i_1}Q^{i_2}R^{i_3}c^{i_4}v^{i_5}y^{i_6}$, $i_1 + \ldots + i_6 = 5$, is equal to 1 if one exponent is 2 and the others are at most 1 or if $i_1 = 0$ and the other exponents are at most 2; is equal to 2 if $i_1 = 1$ and the remaning are at most 1; otherwise is 0.*

Proof: If $i_1 = 2$, then the line is fixed and so by **4.1** the number must be one if the remaining exponents are at most 1 and 0 otherwise. The similar reasoning works if $i_1 = 1$ and some other exponent is 2 or if two exponents are 2. If $i_1 = 0$ and there is a single square, then the conclusion follows from **3.5** and **4.1**. If $i_1 = 1$ and the remaining exponents are at most one, then the value is 2 by **3.6** and **4.1**. If $i_1 = 0$ and the others are at most 1 (and hence all equal to 1), then we can apply **3.7**. ◇

4.4. Remark The expression of $[D_7]$ in the Chow ring of $\overline{D}_7$ is as follows:

$$[D_7] = L^2 - 2Lc - 2Lv - 2Ly - 2LQ - 2LR + cv + cy + cQ + cR + vy + vQ + vR + yQ + yR + QR.$$

The proof of this relation and of the similar relations for D_{10}, D_{11} and D_{12} (see 5.4, 6.4 and 7.4) are similar and we will give details only for the case of D_{12}. The method of proof consists in writing the corresponding D_k as a linear combination of a basis of the corresponding Chow group, with undetermined coefficients, and then to establish enough linear relations among the coefficients by multiplying with suitable monomials in the fundamental conditions, using the tables of Stammzahlen in each case. One reason for bothering only about D_{12} is that in this case the expression is actually used to complete the computation of the Stammzahlen , while in the remaining three cases we do not need the expression for such a purpose.

5. Stammzahlen for D_{10}

5.1. Proposition. *The three foci of a degeneration of type D_{10} may be any unordered triple of collinear points. For each such triple there are two possible pairs $\{c, y\}$ and z is any line through c. More precisely,*

(a) *The cross ratio of c, y and any two foci is a primitive cube root of unity.*

(b) *The point c and the three foci form an equianharmonic tetrad.*

(c) *The point y and the three foci form also an equianharmonic tetrad.*

Proof: It is a direct consequence of **3.4** and the definition of D_{10} by the homolography process. ◇

Let $\overline{D}'_{10}$ be the variety whose points are unordered triples Q_1, Q_2, Q_3 of colinear points (that will be called foci) together with two distinguished points $c = v$ and y of the line defined by the foci. Let D'_{10} be the subvariety of $\overline{D}'_{10}$ given by the relations in **5.1**. Let $\overline{\pi}\colon \overline{D}_{10} \to \overline{D}'_{10}$ be the map that forgets the line z and $\pi\colon D_{10} \to D'_{10}$ the restriction of $\overline{\pi}$ to D_{10}. Next lemma shows that the computation of the Stammzahlen for D_{10} is equivalent to the computation of Stammzahlen for D'_{10}.

5.2. Lemma. *Let N be a fundamental number for D_{10}.*

(a) *If the exponent of z in N is 0 or at least 3, then $N = 0$.*

(b) *If the condition z appears just once in N, then $N = N'$, where N' is the number on D'_{10} obtained dropping the condition z from N.*

(c) *If the condition z appears just twice in N, say $N = z^2x$, then $N = c'x'$, where the product x' on D'_{10} corresponds to the product x on D_{10} (that is, $x = \pi^*(x')$) and where c' is the condition on D'_{10} that the cusp be on a line.*

Proof: Projection formula. ◇

5.3. Theorem. *The fundamental numbers of D'_{10} are given in the following table:*

$L^2Q^3 = \mathbf{2}$	$LQ^2c^2 = \mathbf{2}$	$Q^4y = 6 \cdot \mathbf{2} + 3 \cdot \mathbf{2}$
$L^2Q^2c = \mathbf{2}$	$LQ^2cy = \mathbf{4} + \mathbf{1}$	$Q^3c^2 = \mathbf{2} + 3 \cdot \mathbf{2}$
$L^2Q^2y = \mathbf{2}$	$LQ^2y^2 = \mathbf{2}$	$Q^3cy = \mathbf{2} + 3 \cdot \mathbf{2}$
$L^2Qcy = \mathbf{1}$	$LQc^2y = \mathbf{1}$	$Q^3y^2 = \mathbf{2} + 3 \cdot \mathbf{2}$
$LQ^4 = 6 \cdot \mathbf{2}$	$LQcy^2 = \mathbf{1}$	$Q^2c^2y = \mathbf{2} + \mathbf{1}$
$LQ^3c = \mathbf{4} + 3 \cdot \mathbf{2}$	$Q^5 = 15 \cdot \mathbf{2}$	$Q^2cy^2 = \mathbf{2} + \mathbf{1}$
$LQ^3y = \mathbf{4} + 3 \cdot \mathbf{2}$	$Q^4c = 6 \cdot \mathbf{2} + 3 \cdot \mathbf{2}$	$Qc^2y^2 = \mathbf{1}$

In this table an expression of the form $m \cdot \mathbf{n}$ on the right hand side means that the factor m has a combinatorial origin and that $\mathbf{n}$ is due to the nature of the relations that exist among the elements of the degeneration. On the other hand, the reason why we decompose some of the numbers as the sum of two expressions comes from using lemma 3.10, as will be seen along the proof (cf. 7.4).

Proof: From 5.1 we immediately get the relations

$$L^2Q_3 = 2, L^2Q_2c = 2, L^2Q_2y = 2, L^2Qcy = 1.$$

From 5.1 and 3.5 we get

$$\begin{array}{rr} PQ_2c = 2 & Q_3c^2 = 2 \\ PQ_2y = 2 & Q_3y^2 = 2 \\ PQcy = 2 & Q_2c^2y = 2 \\ & Q_2cy^2 = 2 \end{array}$$

Similarly, from 5.1 and 3.6 we get

$$LQ_3c = 4, LQ_3y = 4, LQ_2cy = 4.$$

Finally from 5.1 and 3.7 we get

$$Q_3cy = 2.$$

Now using 3.10 we see that the proof is reduced to computations. ◇

5.4. Remark The expression of $[D_{10}]$ in terms of the fundamental conditions of $\overline{D}_{10}$ is the following (cf. 4.4):

$$[D_{10}] = 5L^2 - 4Lc - 4Ly + Q^2 - 5QL + 2Qc + 2Qy + 2cy.$$

6. Stammzahlen for D_{11}

6.1. Proposition. *For D_{11} the point $c = y$ and the two foci Q, Q' other than c can be any triple of collinear points and q can be any line through c. The point v is uniquely determined from Q, Q' and c by the relation that the pair (Q, Q') is harmonic with respect to (c, v).*

Proof: This is a direct consequence of **3.1** and the description of D_{11} by homolographies. ◇

Given that the only relation among the elements of the degeneration D_{11} is the one given in **6.1**, we may work, in order to find the Stammzahlen of D_{11}, on the variety D'_{11} whose points parametrize unordered pairs of distinct points $\{Q, Q'\}$ together with two distinguished points c, v on the line QQ' that are harmonic with respect to the pair $\{Q, Q'\}$. In fact, if $\pi\colon D_{11} \to D'_{11}$ is the map which forgets the line q, then next lemma reduces the computation of the Stammzahlen for D_{11} to the computation of certain numbers on D'_{11}.

6.2. Lemma. *Let N be a fundamental number for D_{11}.*

- **(a)** *If the exponent of q in N is 0 or at least 3, then $N = 0$.*
- **(b)** *If the condition q appears just once in N, then $N = N'$, where N' is the number on D'_{11} obtained dropping the condition q from N.*
- **(c)** *If the condition z appears just twice in N, say $N = q^2x$, then $N = c'x'$, where the product x' on D'_{11} corresponds to the product x on D_{11} (that is, $x = \pi^*(x')$) and where c' is the condition on D'_{11} that the cusp be on a line.*

Proof: Projection formula. ◇

6.3. Theorem. *The fundamental numbers of D'_{11} are given in the following table:*

$L^2Q^2c = \mathbf{1}$	$LQ^2cv = \mathbf{2+1}$	$Q^3c^2 = \mathbf{3\cdot 1}$
$L^2Q^2v = \mathbf{1}$	$LQ^2v^2 = \mathbf{1}$	$Q^3cv = \mathbf{3\cdot 1}$
$L^2Qcv = \mathbf{1}$	$LQc^2v = \mathbf{1}$	$Q^3v^2 = \mathbf{3\cdot 1}$
$LQ^3c = \mathbf{2+3\cdot 1}$	$LQcv^2 = \mathbf{1}$	$Q^2c^2v = \mathbf{1+1}$
$LQ^3v = \mathbf{2+3\cdot 1}$	$Q^4c = \mathbf{3\cdot 1}$	$Q^2cv^2 = \mathbf{2+1}$
$LQ^2c^2 = \mathbf{1}$	$Q^4v = \mathbf{3\cdot 1}$	$Qc^2v^2 = \mathbf{1}$

Proof: If the number contains L^2 then line is fixed. The three remaining conditions fix three points and **6.1** fixes the last one. Hence all numbers containing L^2 are equal to 1.

The same reasoning is valid if the number contains Lc^2, Lv^2, LP, c^2v^2, Pc^2, Pv^2 or P^2.

From 6.1 and 3.5 one sees that

$$PQcv = 1, Q_2c^2v = 1, Q_2cv^2 = 1.$$

From 6.1 and 3.6 we see that $LQ_2cv = 2$.

Using now 3.10 it a simple computation to find the values in the table. ◇

6.4. Remark Let $\overline{D}_{11}$ be the variety parametrizing configurations consisting of an unordered pair Q, Q' of points together with two distinguished points c, y on the line QQ' and a line q through c. Then the expression of D_{11} in terms of the first order fundamental conditions of $\overline{D}_{11}$ (with the obvious notations) is the following (cf. 4.4).

$$[D_{11}] = c + v + Q - 2L.$$

7. Stammzahlen for D_{12}

7.1. Proposition. *Given six distinct collinear points c, v, y and Q_1, Q_2, Q_3, let $\rho_i = \rho(c, v, y, Q_i)$. Then in order that c, v, y is the singular triangle and $\{Q_1, Q_2, Q_3\}$ the foci of a degeneration of type D_{12} it is necessary and sufficient that*

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3} = 3$$
$$\rho_1\rho_2\rho_3 = 1.$$

Proof: It is a direct consequence of **3.1** and the way the degeneration is obtained by a homolography. ◇

7.2. Theorem. *The fundamental numbers of D_{12} are given by the following table:*

$L^2Q^3c = 4$	$LQ^3cy = 6 + 3 \cdot 2$	$LQcv^2y = 1$	$Q^3c^2y = 4 + 3 \cdot 2$
$L^2Q^3v = 1$	$LQ^3v^2 = 1$	$LQcvy^2 = 1$	$Q^3cv^2 = 3 + 3 \cdot 3$
$L^2Q^3y = 2$	$LQ^3vy = 3 + 3 \cdot 1$	$Q^5c = 15 \cdot 4$	$Q^3cvy = 4 + 3 \cdot 3$
$L^2Q^2cv = 3$	$LQ^3y^2 = 2$	$Q^5v = 15 \cdot 1$	$Q^3cy^2 = 2 + 3 \cdot 2$
$L^2Q^2cy = 2$	$LQ^2c^2v = 3$	$Q^5y = 15 \cdot 2$	$Q^3v^2y = 1 + 3 \cdot 1$
$L^2Q^2vy = 1$	$LQ^2c^2y = 2$	$Q^4c^2 = 6 \cdot 4$	$Q^3vy^2 = 2 + 3 \cdot 1$
$L^2Qcvy = 1$	$LQ^2cv^2 = 3$	$Q^4cv = 6 \cdot 4 + 3 \cdot 3$	$Q^2c^2vy = 4 + 1$
$LQ^4c = 6 \cdot 4$	$LQ^2cvy = 5 + 1$	$Q^4cy = 6 \cdot 4 + 3 \cdot 2$	$Q^2cv^2y = 3 + 1$
$LQ^4v = 6 \cdot 1$	$LQ^2cy^2 = 2$	$Q^4v^2 = 6 \cdot 1$	$Q^2cvy^2 = 2 + 1$
$LQ^4y = 6 \cdot 2$	$LQ^2v^2y = 1$	$Q^4vy = 6 \cdot 2 + 3 \cdot 1$	$Qc^2v^2y = 1$
$LQ^3c^2 = 4$	$LQ^2vy^2 = 1$	$Q^4y^2 = 6 \cdot 2$	$Qc^2vy^2 = 1$
$LQ^3cv = 7 + 3 \cdot 3$	$LQc^2vy = 1$	$Q^3c^2v = 6 + 3 \cdot 3$	$Qcv^2y^2 = 1$

Proof: The numbers that contain L^2 have been determined in Miret-Xambó [1987] (Theorem 4, Table 1).

The computation of the remaining numbers of the table will be based on lemma **7.4**, in which we first compute six auxiliary numbers; on lemma **3.10**, which allows to relate the auxiliary numbers to those we need, and on lemma **7.5**, in which we give an expression of the class $[D_{12}]$ in terms of a basis of the codimension 2 Chow group of $\overline{D}_{12}$.

Given j lines in general position ($j = 2, 3$), we shall write Q_j to denote the condition that there is exactly one focus on each of the j lines. We will also write P to denote the codimension 2 condition that one focus coincides with a given point. With these notations we have:

7.4. Lemma.

(1) $Q_3cv^2 = 3$.
(2) $Q_3cvy = 4$.
(3) $Q_3cy^2 = 2$.
(4) $Q_3v^2y = 1$.
(5) $Q_2c^2vy = 4$.
(6) $QPcvy = 3$.

Proof: The proofs can be done, in more or less straightforward manner, choosing a suitable reference and imposing the conditions **7.1** that a degeneration of type D_{12} must satisfy. We will only give details of **(1)**.

To establish **(1)** the reference we choose is the following. Let L_1, L_2, L_3 be the lines in general position required to define Q^3, M the line required to define the condition c and A the point v^2. Then we take the points $M \cap L_1, L_2 \cap L_3, A$ as the vertices of the reference triangle and $L_1 \cap L_2$ as unit point. Thus we have that

$$\begin{aligned} L_1 &: \quad x_1 = x_2, \\ L_2 &: \quad x_0 = x_2, \\ L_3 &: \quad ax_0 = x_2, \\ M &: \quad x_1 = mx_2, \end{aligned}$$

where $a, m \neq 0, 1$.

Let L the axis of the degeneration, so that L goes through A and hence $L:\quad x_1 = \lambda x_0$. Let $Q_i = L \cap L_i$ be the foci of the degeneration. A simple computation shows that

$$Q_1 = (1, \lambda, \lambda), Q_2 = (1, \lambda, 1), Q_3 = (1, \lambda, a).$$

Let $y = (1, \lambda, \mu)$. Then a computation of cross ratios shows that if we put $\rho_i = \rho(c, v, y, Q_i)$ then $\rho_1 = \mu/\lambda$, $\rho_2 = \mu$ and $\rho_3 = \mu/a$. The equations **7.1** are equivalent to the conditions $\lambda = 3\mu - a - 1$ and $\mu^3 = a(3\mu - a - 1)$, and hence there are exactly 3 degenerations of type D_{12} that satisfy the conditions Q_3cv^2. $\diamond$

7.5. Lemma.

$$[D_{12}] = 7L^2 - 3Lc - 6Lv - 7Ly - 6LQ + Qc + 2Qv + 3Qy + Q^2 + 2cv + cy + 4vy.$$

Proof: From the fact that $\overline{D}_{12}$ is a projective bundle over $\check{\mathbf{P}}^2$ it follows that the Chow group $A^2(\overline{D}_{12})$ is freely generated by the degree 2 monomials in $\overline{L}, \overline{c}, \overline{v}, \overline{y}, \overline{Q}$. Hence there exist integers $m_1, \ldots, m_4$, $n_1, \ldots, n_4$, $r_1, \ldots, r_4$ and $s_1, \ldots, s_3$ such that

$$(*)\quad [D_{12}] = m_1\overline{c}^2 + m_2\overline{v}^2 + m_3\overline{y}^2 + m_4\overline{L}^2 + n_1\overline{Lc} + n_2\overline{Lv} + n_3\overline{Ly} + n_4\overline{LQ} +$$
$$r_1\overline{Qc} + r_2\overline{Qv} + r_3\overline{Qy} + r_4\overline{Q}^2 + s_1\overline{cv} + s_2\overline{cy} + s_3\overline{vy}.$$

Now from the values of the three numbers that contain L^2 which are equal to 1 we see that if $\overline{u}$ is any of the first order conditions on $\overline{D}_{12}$ then $\overline{u}|_{D_{12}} = u$. More generally, given a monomial $\overline{x}$ on the first order conditions on $\overline{D}_{12}$, let x denote its restriction to D_{12}, so that x is obtained replacing the first order conditions in $\overline{x}$ by the corresponding conditions on D_{12}. It turns out that $x = \overline{x} \cdot D_{12}$. Using this relation with the 7 numbers that contain L^2 it is easy to find the values of the r_i and s_j, $i = 1, \ldots, 4$, $j = 1, \ldots, 3$.

Notice that from **3.8** we may compute the following values:

(1) $Q^3cv^2 = \mathbf{3} + 3 \cdot \mathbf{3} = 12$.
(2) $Q^3cvy = \mathbf{4} + 3 \cdot \mathbf{3} = 13$.
(3) $Q^3cy^2 = \mathbf{2} + 3 \cdot \mathbf{2} = 8$.
(4) $Q^3v^2y = \mathbf{1} + 3 \cdot \mathbf{1} = 4$.
(5) $Q^2c^2vy = \mathbf{4} + \mathbf{1} = 5$.

Now we have:

$5 = Q^2c^2vy = n_4 + r_2 + r_3 + 6r_4 = n_4 + 11$, so $n_4 = -6$.

$4 = Q^3v^2y = m_1 + n_1 + 6r_1 + s_2$, and so $n_1 = -(m_1 + 3)$.

$8 = Q^3cy^2 = m_2 + n_2 + 6r_2 + s_1 = m_2 + n_2 + 14$, and so $n_2 = -(m_2 + 6)$.

$12 = Q^3cv^2 = m_3 + n_3 + 6r_3 + s_2 = m_3 + n_3 + 19$, and so $n_3 = -(m_3 + 7)$.

$13 = Q^3cvy = m_4 + n_1 + n_2 + n_3 + 6n_4 + 6r_1 + 6r_3 + 15r_4 + s_1 + s_2 + s_3$, and so $m_4 = m_1 + m_2 + m_3 + 7$.

The conclusion follows from the relations **3.6**. $\diamond$

With the expression (*) and the knowledge of the fundamental numbers of $\overline{D}_{12}$ (which can be obtained by combinatorial arguments and so here will be assumed to be known) we can now obtain the values of the table **7.3**. We omit the details. There is, however, one aspect of the table which we want to comment, namely, the boldfaced numbers. We will do this by looking at an example. Take the number Q^4cv. Its value can be obtained as follows:

$$Q^4cv = D_{12}\cdot\overline{Q}^4\overline{cv} = -7\overline{LQ}^4\overline{cvy}+3\overline{Q}^5\overline{cvy}+\overline{Q}^4\overline{c}^2\overline{vy}+4\overline{Q}^4\overline{cv}^2 = -7\cdot6+3\cdot15+6+4\cdot6 = 33.$$

Now by **3.8**

$$Q^4cv = 6PQ_2cv + 3P^2cv = 6PQ_2cv + 3\cdot L^2Q^2cv = 6PQ_2cv + 3\cdot 3,$$

from which it follows that

$$PQ_2cv = 4.$$

This has been taken into account in the form we write the value of Q^4cv in the table decomposed as $6\cdot\mathbf{4}+3\cdot 3$. ◇

8. On the method of degenerations

In this section we introduce a version of the method of degenerations, especially as used by Schubert, which does not rely on coincidence formulas. Then in next section we indicate how we have used it to derive the degeneration relations (**9.1**) for the plane cuspidal cubics. To see how conditions arise in practice, and also for additional terminology, see **8.11**.

8.1. Let S be a smooth variety and let $d = \dim S$. Let

(8.1.1) $$X_1, \ldots, X_p, Z_1, \ldots, Z_s\ (p \geq 1, s \geq 0)$$

be subvarieties of S, where the X_i are *hypersurfaces* and the Z_j have at least codimension 2. The varieties (**8.1.1**) will be referred to as *conditions*. The codimension of a condition will also be called *order* of the condition. Conditions of order one are said to be *simple* conditions. We shall assume that the given list of conditions satisfies the conditions **A1** and **A2** below. In this paper we will not use higher order conditions (the Z's); they are included here because they are needed in other cases, like in twisted cubics.

A1. The sum of the codimensions of the Z_j ($j = 1, \ldots, s$) is $d-p$, and the intersection of all the varieties $X_1, \ldots, X_p, Z_1, \ldots, Z_s$ is a finite set.

A2. The intersection of all the varieties

$$X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_p, Z_1, \ldots, Z_s$$

is a reduced curve C_i, $(i = 1, \ldots, p)$.

We shall let N denote the number of points in this set, counting multiplicities if they are present and we will write

$$N = X_1 \cdots X_p \cdot Z_1 \cdots Z_s$$

We shall say that N is the number of figures of type S that satisfy the conditions $X_1, \ldots, X_p, Z_1, \ldots, Z_s$.

We shall also assume that we have hypersurfaces $Y_1, \ldots, Y_q$ of S that satisfy the following condition:

A3. The classes $[Y_1], \cdots, [Y_q]$ generate $\mathrm{Pic}(S)_{\mathbf{Q}}$ (as a $\mathbf{Q}$-vector space).

8.2. In order to explain how we will approach the computation of N, let us first remark that if S were *complete*, then we would have

$$N = \deg_S [X_1] \cdots [X_p] \cdot [Z_1] \cdots [Z_s],$$

where $[Z]$ denotes the rational class of the cycle Z, which often is an affordable computation, inasmuch as under the completeness assumption one sometimes knows the rational intersection ring of S. This is the case, for example, if S is a Grassmannian, or a flag manifold, in which case the computation is just "Schubert calculus", but it is not the case for, say, smooth conics and quadrics or plane cuspidal cubics. So to end the description of our setup we need a modified procedure, with respect to the complete case, that is sufficient for the the computation of N.

8.3. To that end we shall assume that there exists a smooth variety S' (not necessarily complete) that satisfies the conditions **D1-D3** below (*axioms for degenerations*). Given any subset A of S, we shall write A' to denote its closure of A in S'.

D1. $S \subsetneq S'$ and $D := S' - S = D_1 \cup \ldots \cup D_r$, where $D_1, \ldots, D_r$ are smooth irreducible hypersurfaces of S' and $D_i \cap D_j = \phi$. The varieties D_i will be called *degenerations*.

D2. Let

$$D_i \cdot X'_j = \textstyle\sum_k m_{ijk} X_{ijk},$$
$$D_i \cdot Z'_j = \textstyle\sum_k n_{ijk} Z_{ijk},$$

where the X_{ijk}, Z_{ijk} are the irreducible components of $D_i \cap X'_j$ and $D_i \cap Z'_j$, so that they have the same codimension in D_i as X_j, Z_j in S, respectively, and m_{ijk}, n_{ijk} are the corresponding multiplicities. Then we assume that for any choice of integers $k_1, \ldots, k_p$, $h_1, \ldots, h_s$, each in its appropriate range, the varieties

$$X_{i1k_1}, \ldots, X_{ipk_p},\ Z_{i1h_1}, \ldots, X_{ish_s},$$

have empty intersection, and that omiting any of the X's, say X_{ijk_j}, the remaining have finite intersection. The number of points in this intersection, counting multiplicities if present (computed on D_i), will be denoted by

$$N_{ij}[k_1, \ldots, k_p, h_1, \ldots, h_s] = N_{ij}[k, h].$$

These numbers will be called *elementary numbers* with respect to the problem of computing N.

D3. Let C'_j be the intersection of the varieties $X'_1, \ldots, X'_p$, $Z'_1, \ldots, Z'_s$, except X'_j; by assumptions **A2** and **C2**, C'_j is a curve. We shall assume that this curve is *complete* and that the inclusion

$$u_j \colon C'_j \to S'$$

is a regular embedding.

8.4. Lemma. *The classes*

$$[D_1], \ldots, [D_r], [Y'_1], \ldots, [Y'_q]$$

generate $Pic(S')_{\mathbf{Q}}$.

Proof: We have an exact sequence (Fulton [1984], Prop. 1.8)

$$\textbf{(8.4.1)} \qquad \to A^0(D)_{\mathbf{Q}} \to A^1(S')_{\mathbf{Q}} \to A^1(S)_{\mathbf{Q}} \to 0.$$

By **A3**, $A^1(S)_{\mathbf{Q}}$ is generated by $[Y_1], \cdots, [Y_q]$. On the other hand, the classes of the components of D form a free $\mathbf{Q}$-basis of $A^0(D)_{\mathbf{Q}}$. The conclusion follows readily. ◇

8.5. We may inparticular express the classes $[X'_j]$ as rational linear combinations of $[D_1], \ldots, [D_r], [Y'_1], \ldots, [Y'_q]$,

$$\textbf{(DR)} \qquad [X'_j] = a_{1j}[D_1] + \ldots + a_{rj}[D_r] + b_{1j}[Y'_1] + \ldots + b_{qj}[Y'_q].$$

Any such equation will be called a *degeneration relation* for X'_j. The rational numbers a_{kj}, b_{kj} will be called coefficients of the degeneration relation. A priori they need not be uniquely determined, but in concrete applications they will. Notice that they are uniquely determined if $[D_1], \ldots, [D_r], [Y'_1], \ldots, [Y'_q]$ are $\mathbf{Q}$-linearly independent. Conversely, if the coefficients in a degeneration relation are all non-zero and unique, then $[D_1], \ldots, [D_r], [Y'_1], \ldots, [Y'_q]$ are $\mathbf{Q}$-linearly independent. This is the criterion we shall use to determine $\mathrm{Pic}(S')_{\mathbf{Q}}$ in our examples. We could also proceed observing that the sequence **(8.4.1)** is exact to the left if and only if the map

$$cl_S \colon \mathrm{Pic}(S)_{\mathbf{Q}} \to \mathrm{H}_2(S)_{\mathbf{Q}}$$

is an isomorphism and using the fact that the latter holds, for instance, if S has a cellular decomposition, or even in more general cases (see Rosselló-Xambó [1987]).

8.6. Let

$$d_i\colon D_i \to S'$$

be the inclusions. Then we will write $N_{ij} = \deg(D_i \cdot C_j')$ and we will say that the N_{ij}, $i = 1, \ldots, r$, are the *degeneration numbers* of C_j. Since C_j' is a complete curve, we also have

$$N_{ij} = \deg_{C_j'}[D_i \cdot C_j'] = \deg_{C_j'}(u_j^*[D_i]).$$

8.7. Degeneration lemma.

(a) $N = \deg_{C_j'}(u_j^*[X_j'])$ *for all* $j = 1, \ldots, p$.

(b) *Given a degeneration relation* **DR** *for* X_j', *then*

$$N = \sum_i a_{ij} N_{ij} + N',$$

for any $i = 1, \ldots, p$, *where*

$$N' = \sum_i b_{ij} \deg_{C_j'}(u_j^*[Y_j'])$$

(so N' does not involve X_j).

(c) *If we let*

$$M_{ij}(k,h) = (\prod_{l \neq j} m_{ilk_l}) \cdot (\prod_l n_{ilh_l})$$

then we have

$$N_{ij} = \sum_{k,h} M_{ij}(k,h) N_{ij}[k,h].$$

Proof:

(a) By definition $N = \deg(X_j \cdot C_j)$, and $N = \deg_{C_j'}(u_j^*(X_j'))$ by **D2**. Now the fact that C_j' is complete implies that $N = \deg_{C_j'}([u_j^* X_j']) = \deg_{C_j'}(u_j^*[X_j'])$.

(b) It is a direct consequence of **(a)** and the definitions.

(c)
$$\begin{aligned} N_{ij} &= \deg(D_i \cdot C_j') = \deg d_i^*(C_j') \\ &= \deg d_i^*(X_1' \cdots X_{j-1}' \cdot X_{j+1}' \cdots X_p' \cdot Z_1' \cdots Z_s'), \\ &= \deg d_i^*(X_1') \cdots d_i^*(X_{j-1}') \cdot d_i^*(X_{j+1}') \cdots d_i^*(X_p') \cdot d_i^*(Z_1') \cdots d_i^*(Z_s'). \end{aligned}$$

From this, the expression of **D2** and the definitions of $N_{ij}[k,h]$ and $M_{ij}(k,h)$, the stated expression for N_{ij} follows immediately. ◇

8.8. The degeneration lemmma gives a foundation to the "method of degenerations", especially as used by Schubert. The expression of N given in **(b)** breaks up the problem of computing N into (i) the determination of the degeneration coefficients, (ii) the computation of the degeneration numbers N_{ij} and (iii) the computation of the numbers N'. Part **(c)** of the lemma reduces the computation of degeneration numbers into the determination of the varieties X_{ijk} and Z_{ijk}, the multiplicities m_{ijk} and n_{ijk} with which they appear, and the computation of the elementary numbers $N_{ij}[k,h]$. The latter are enumerative problems in a space of dimension $d-1$ and for their determination usually the same method can be applied, so that the whole procedure has a recursive quality. As far as (*iii*) goes, in practice the numbers N' will be easier to compute than the number N itself.

8.9. Part **(a)** of the degeneration lemma *gives p expressions for the number N*. So in particular we have equalities

$$\deg_{C'_j}(u_j^*[X'_j]) = \deg_{C'_{j'}}(u_{j'}^*[X'_{j'}])$$

for any j, j' in $\{1, \ldots, p\}$. Thus if we know degeneration relations **DR** for X'_j and $X'_{j'}$, then we get an equation of the form

$$\textbf{(8.9.1)} \qquad a_{1j}N_{1j} + \ldots a_{rj}N_{rj} + N' = a_{1j'}N_{1j'} + \ldots a_{rj'}N_{rj'} + N''.$$

This yields a necessary condition that the coefficients of the degeneration relations must satisfy. It turns out that in interesting enumerative situations a suitable selection of equations of the form **(8.9.1)** is enough to determine them. If some of the multiplicities m, n that appear in the definition of the degeneration numbers were also unknown, they may as well be left in **(8.9.1)** as integer unknowns.

8.10. Classically degeneration relations were established through the use of "coincidence formulas", which often lead to elusive computations of multiplicities. For example, Schubert's derivation of the 4 degeneration relations for twisted cubics (Schubert [1879], p. 168) has not been made rigorous because of his application of the coincidence formulas (or rather the way he suggests to apply them) leaves undetermined certain fundamental multiplicities. The approach advanced here suffices to determine those degeneration formulas without needing coincidence formulas. Below we will show how to find suitable degeneration relations for the cuspidal cubics.

8.11. Let us discuss how conditions arise. A common way to describe cycles on a variety S which parametrizes a certain kind of figures is by means of geometric relations imposed to the figures ("räumliche Bedingungen" in Schubert's terminology; see Schubert [1879], p. 5). The geometric relations will involve some other kind of figure. When we allow the latter to move we obtain an *algebraic family of cycles* on S. Such algebraic families of cycles are the usual source for supplying conditions in the sense given above.

In order to simplify notations, we shall use the conventions, which go back to Schubert and before, that we explain presently. Suppose S is a smooth variety of dimension d and that X is an algebraic family of cycles on S. Then given an integer n, X^n will mean that we take n (independent) general values of the parameter space of the family and that we consider as conditions the cycles $X_1, \ldots, X_n$ corresponding to those values. Given families

$$X, X', \ldots, Z, Z', \ldots$$

$(X, X', \ldots$ of codimension 1, $Z, Z', \ldots$ of codimension at least 2) and integers

$$n, n', \ldots, m, m', \ldots$$

the expression

$$N = X^n X'^{n'} \cdots Z^m Z'^{m'} \cdots$$

will mean the enumerative problem whose conditions are n general cycles of the family X, n' general cycles of the family X', and so on. In order for the problem to be well posed we need that the sum of the codimensions be equal to d. In the explicit examples the assumptions **A1**, **A2** and **D2** can be ascertained from general principles such as the transversality of the general translates (Kleiman [1974]), or a generalized version in which it is not required that the group acts transitively on S (Casas [1987], Laksov-Speiser [1988]).

In specific examples, the conditions in the list $X, X', \ldots, Z, Z', \ldots$ will be selected so that they express basic geometric relationships that our figures satisfy and will be referred to as *fundamental conditions*. The numbers formed with fundamental conditions will be called *fundamental numbers*. If the only conditions involved are (simple) contact conditions with linear varieties then the numbers are referred to as *characteristic numbers*.

9. Tables of degeneration numbers

In Sections 4-7 we have studied the elementary numbers with respect to the fundamental conditions for cuspidal cubics. With the elementary numbers we can compute the degeneration numbers. In this section we assemble the tables of all degeneration numbers that are needed to compute all fundamental numbers. Each table is labled with a monomial α in the variables c, v, y, q, w, z and the monomials are ordered lexicographically. The numbers to the right of a given D_j are the degeneration numbers of the form $D_j \cdot (X_0^{6-d-i} X_1^i \alpha)$, $i = 0, \ldots, 6-d$, where d is the degree of α, X_0 the condition of going through a point and X_1 of being tangent to a line. Thus there are $7-d$ numbers in each row. A row corresponding to a degeneration is omited if it turns out to be identically 0.

Table 1

D_0	42	87	141	168	141	87	42

Table c

D_0	27	45	54	45	27	12	D_{12}	0	0	0	36	72	60
D_7	0	24	78	78	24	0							

Table v

D_0	27	45	54	45	27	12	D_7	0	24	78	78	24	0
D_2	45	54	27	0	0	0	D_{12}	0	0	0	9	18	15

Table y

D_0	27	45	54	45	27	12	D_7	0	24	78	78	24	0
D_3	30	36	18	0	0	0	D_{12}	0	0	0	18	36	30

Table c^2

D_0	5	8	8	5	2	D_{12}	0	0	0	12	24
D_7	0	6	21	18	0						

Table cv

D_0	5	8	8	5	2	D_7	24	60	57	18	0
D_2	18	9	0	0	0	D_{12}	0	0	27	48	33
D_5	24	54	36	0	0						

Table cy

D_0	5	8	8	5	2	D_7	24	60	57	18	0
D_3	12	6	0	0	0	D_{12}	0	0	18	36	30

Table cz

D_0	7	13	16	13	7	D_7	0	6	21	18	0
D_1	12	6	0	0	0	D_{10}	0	0	18	30	18
D_5	24	54	36	0	0	D_{12}	0	0	0	12	24
D_6	0	18	21	6	0						

Table cq

D_0	7	13	16	13	7	D_7	0	6	21	18	0
D_1	6	3	0	0	0	D_{11}	0	0	9	18	15
D_6	0	18	21	6	0	D_{12}	0	0	0	12	24

Table cw

D_0	7	13	16	13	7	D_7	0	6	21	18	0
D_1	24	12	0	0	0	D_9	0	0	36	54	24
D_4	24	54	36	0	0	D_{12}	0	0	0	12	24
D_6	0	18	21	6	0						

Table v^2

D_0	5	8	8	5	2	D_7	0	6	21	18	0
D_2	15	18	9	0	0	D_{12}	0	0	0	3	6

Table vy

D_0	5	8	8	5	2	D_7	24	60	57	18	0
D_2	18	9	0	0	0	D_{12}	0	0	9	18	15
D_3	12	6	0	0	0						

Table y^2

D_0	5	8	8	5	2	D_8	0	0	36	54	24
D_3	18	30	18	0	0	D_{12}	0	0	0	6	12
D_7	0	6	21	18	0						

Table c^2v

D_2	3	0	0	0	D_7	6	15	9	0
D_5	6	15	9	0	D_{12}	0	0	9	15

Table c^2y

D_3	2	0	0	0	D_7	6	15	9	0	D_{12}	0	0	6	10

Table c^2z

D_0	1	2	2	1	D_7	0	1	4	0
D_1	2	0	0	0	D_{10}	0	0	6	8
D_5	6	15	9	0	D_{12}	0	0	0	4
D_6	0	4	1	0					

Table c^2q

D_0	1	2	2	1	D_7	0	1	4	0
D_1	1	0	0	0	D_{11}	0	0	3	3
D_6	0	4	1	0	D_{12}	0	0	0	4

Table c^2w

D_0	1	2	2	1	D_7	0	1	4	0
D_1	4	0	0	0	D_9	0	0	12	18
D_4	6	15	9	0	D_{12}	0	0	0	4
D_6	0	4	1	0					

Table cv^2

D_2	6	3	0	0	D_7	6	15	9	0
D_5	12	24	18	0	D_{12}	0	0	9	12

Table cvy

D_2	3	0	0	0	D_7	24	27	9	0
D_3	2	0	0	0	D_{12}	0	9	18	13
D_5	18	12	0	0					

Table cvz

D_0	1	2	2	1	D_6	0	4	1	0
D_1	2	0	0	0	D_7	6	16	13	0
D_2	6	3	0	0	D_{10}	0	0	6	8
D_5	18	39	27	0	D_{12}	0	0	9	16

Table cvq

D_0	1	2	2	1	D_6	0	4	1	0
D_1	1	0	0	0	D_7	6	16	13	0
D_2	15	9	0	0	D_{11}	0	9	15	9
D_5	6	15	9	0	D_{12}	0	0	9	16

Table cy^2

D_3	10	6	0	0	D_8	0	18	24	12
D_7	6	15	9	0	D_{12}	0	0	6	8

Table cyz

D_0	1	2	2	1	D_6	0	4	1	0
D_1	2	0	0	0	D_7	6	16	13	0
D_3	10	6	0	0	D_{10}	0	9	15	8
D_5	18	12	0	0	D_{12}	0	0	6	12

Table v^2y

D_2	6	3	0	0	D_7	6	15	9	0
D_3	2	0	0	0	D_{12}	0	0	3	4

Table v^2q

D_0	1	2	2	1	D_7	0	1	4	0
D_1	1	0	0	0	D_{11}	0	0	9	12
D_2	9	15	9	0	D_{12}	0	0	0	1
D_6	0	4	1	0					

Table vy^2

D_2	3	0	0	0	D_8	0	9	15	6
D_3	10	6	0	0	D_{12}	0	0	3	5
D_7	6	15	9	0					

Table vyz

D_0	1	2	2	1	D_6	0	4	1	0
D_1	2	0	0	0	D_7	6	16	13	0
D_2	6	3	0	0	D_{10}	0	9	15	8
D_3	10	6	0	0	D_{12}	0	0	3	6
D_5	18	12	0	0					

Table y^2z

D_0	1	2	2	1	D_7	0	1	4	0
D_1	2	0	0	0	D_8	0	0	12	18
D_3	8	15	9	0	D_{10}	0	0	6	8
D_6	0	4	1	0	D_{12}	0	0	0	2

Table $c^2v^2 = c^2vz$

D_2	1	0	0	D_7	1	3	0
D_5	4	9	9	D_{12}	0	0	3

Table c^2vy

D_5	5	3	0	D_7	5	3	0	D_{12}	0	3	5

Table c^2vq

D_2	3	0	0	D_7	1	3	0	D_{12}	0	0	3
D_5	1	3	0	D_{11}	0	3	2				

Table c^2vw

D_2	1	0	0	D_5	4	9	9	D_9	0	3	5
D_4	5	3	0	D_7	1	3	0	D_{12}	0	0	3

Table c^2y^2

D_3	2	0	0	D_8	0	6	4
D_7	1	3	0	D_{12}	0	0	2

Table c^2yz

D_3	2	0	0	D_7	1	3	0	D_{12}	0	0	2
D_5	5	3	0	D_{10}	0	3	3				

Table c^2yw

D_3	2	0	0	D_7	1	3	0	D_9	0	3	5
D_4	5	3	0	D_8	0	6	4	D_{12}	0	0	2

Table c^2zq

D_1	1	0	0	D_6	0	3	1	D_{11}	0	0	1
D_5	1	3	0	D_{10}	0	0	2				

Table c^2zw

D_1	2	0	0	D_5	4	9	9	D_9	0	0	4
D_4	1	3	0	D_6	0	3	1	D_{10}	0	0	2

Table c^2qw

D_1	3	0	0	D_6	0	3	1	D_9	0	0	4
D_4	1	3	0	D_8	0	6	4	D_{11}	0	0	1

Table c^2w^2

D_4	5	3	0	D_9	0	3	5

Table cv^2y

D_2	1	0	0	D_7	5	3	0
D_5	8	6	0	D_{12}	0	3	4

Table cvy^2

D_3	2	0	0	D_7	5	3	0	D_{12}	0	3	3
D_5	4	0	0	D_8	9	9	4				

Table $cvyz$

D_2	1	0	0	D_5	13	9	0	D_{10}	0	3	3
D_3	2	0	0	D_7	6	6	0	D_{12}	0	3	6

Table $cvyq$

D_2	3	0	0	D_7	6	6	0	D_{11}	0	3	2
D_3	2	0	0	D_8	9	9	4	D_{12}	0	3	6
D_5	5	3	0								

Table v^2y^2

D_2	1	0	0	D_7	1	3	0	D_{12}	0	0	1
D_3	2	0	0	D_8	0	3	1				

Table v^2yz

D_2	1	0	0	D_5	8	6	0	D_{10}	0	3	3
D_3	2	0	0	D_7	1	3	0	D_{12}	0	0	1

Table vy^2z

D_2	1	0	0	D_7	1	3	0	D_{10}	0	3	3
D_3	5	3	0	D_8	0	3	5	D_{12}	0	0	1
D_5	4	0	0								

Table c^2v^2y

D_5	3	3	D_7	1	0	D_{12}	0	1

10. Degeneration relations

In next theorem we state the degeneration expressions of the first order conditions for cuspidal cubics and then we indicate how they can be obtained by application of the procedure explained in section 8. Here we see that $\mathrm{Pic}(S)_{\mathbf{Q}}$ is generated by c (see **1.3**) and hence $\mathrm{Pic}(S')_{\mathbf{Q}}$ is generated by c and the 13 degenerations.

10.1. Theorem. *Let $D = D_1 + D_2 + D_3$ and $D' = D_{10} + D_{11} + D_{12}$. Then the expressions on S' of the first order conditions in terms of c and the first order degenerations is as follows:*

1) $5X_0 = 3c + 2D_0 + 3D + 6D_4 + 2D_5 + 3D_6 + 4D_7 + 3D_8 + 9D_9 + 9D'$.
2) $5X_1 = -3c + 8D_0 + 12D + 9D_4 + 3D_5 + 7D_6 + 6D_7 + 2D_8 + 6D_9 + 6D'$.
3) $5v = -4c + 9D_0 + 6D_1 + D_2 + 6D_3 + 2D_4 - D_5 + 6D_6 + 3D_7 + D_8 + 3D_9 + 3D'$.
4) $5y = -c + 6D_0 + 4D_1 + 4D_2 - D_3 + 3D_4 + D_5 + 4D_6 + 2D_7 - D_8 + 2D_9 + 2D'$.
5) $5z = c + 4D_0 + D + 2D_4 - D_5 + D_6 + 3D_7 + D_8 + 3D_9 - 2D_{10} + 3D_{11} + 3D_{12}$
6) $5q = 4c + D_0 - D + 3D_4 + D_5 - D_6 + 2D_7 - D_8 + 2D_9 + 2D_{10} - 3D_{11} + 2D_{12}$.
7) $w = -c + 2D_0 + D + D_6 + D_7 + D'$.

Here is the same information in matrix form:

	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	D_8	D_9	D_{10}	D_{11}	D_{12}	c
$5X_0$	2	3	3	3	6	2	3	4	3	9	9	9	9	3
$5X_1$	8	12	12	12	9	3	7	6	2	6	6	6	6	−3
$5v$	9	6	1	6	2	−1	6	3	1	3	3	3	3	−4
$5y$	6	4	4	−1	3	1	4	2	−1	2	2	2	2	−1
$5z$	4	1	1	1	2	−1	1	3	1	3	−2	3	3	1
$5q$	1	−1	−1	−1	3	1	−1	2	−1	2	2	−3	2	4
w	2	1	1	1	0	0	1	1	0	0	1	1	1	−1

10.1.1. Remark. If we take into account only the degeneration D_0, which is enough to compute the characteristic numbers (see Table **1** in Section 9), then the relations above for X_0 and X_1 become the following:

$$5X_0 = 3c + 2D_0, \quad 5X_1 = -3c + 8D_0.$$

These relations were obtained for the first time, using coincidence formulas, by Zeuthen [1872] and were recently verified by Kleiman-Speiser [1986]. Notice that a priori we know, by **1.3**, that $5X_0$ and $5X_1$ are linear combinations of c and the degenerations with integer coefficients.

Proof: The proof of the seven degeneration relations can be done by a judicious choice of equations of the form **8.9**. To write such equations we need to know enough degeneration numbers. Those that will be used are contained in the tables given in the preceeding section. Since the procedure is straightforward, here we will prove only the first two relations. We shall write a_i and a to denote the coefficients of X_0 with respect to D_i and c and b_i and b for the coefficients of X_1.

We want to determine the values of $a, a_0, \ldots, a_{12}, b, b_0, \ldots, b_{12}$. To this end first notice that $X_0^5c^2 = 2$ and $X_0^4X_1c^2 = 8$. From these relations we obtain, taking into account the degeneration numbers given in Table c^2 and using **8.7 (b)**, the equations $5a_0 = 2,\ 5b_0 = 8,\ 8a_0 + 6a_7 = 8$. Hence

$$a_0 = 2/5, b_0 = 8/5, a_7 = 4/5.$$

In what follows we briefly point out what relation we take, the equations it leads to and the value of the coefficients they determine.

From $X_0(X_0^2X_1^2c^2) = X_1(X_0^3X_1c^2)$ we get the relation $8a_0 + 21a_7 = 8b_0 + 6b_7$. So

$$b_7 = 6/5.$$

From $X_0(X_0X_1^3c^2) = X_1(X_0^2X_1^2c^2)$ we get the relation $5a_0+15a_7+12a_{12} = 8b_0+21b_7$, and so

$$a_{12} = 9/5.$$

From $X_0(X_1^4c^2) = X_1(X_0X_1^3c^2)$ we get the relation $2a_0 + 24a_{12} = 5b_0 + 18b_7 + 12b_{12}$, and so

$$b_{12} = 6/5.$$

As a corollary we get, using **8.7 (b)**, the following numbers:

$$c^2 = 2,\ 8,\ 20,\ 38,\ 44,\ 32.$$

[By this we mean the numbers $X_0^{5-i}X_1^ic^2$, $i = 0, \dots, 5$].

Using table c and and the numbers for c^2 just obtained we can determine the coefficients a and b. In fact, from the relation $X_0(X_0^4X_1c) = X_1(X_0^5c)$ we get the equation $8a + 45a_0 + 24a_7 = 2b + 27b_0$. Similarly, from the relation $X_0(X_0^3X_1^2c) = X_1(X_0^4X_1c)$ we get the equation $20a + 54a_0 + 78a_7 = 8b + 45b_0 + 24b_7$. Solving for a and b we obtain

$$a = -b = 3/5.$$

From $X_0(X_1^3c^2v) = X_1(X_0X_1^2c^2v)$ we obtain $15a_{12} = 9b_5 + 9b_7 + 9b_{12}$ and so

$$b_5 = 3/5.$$

From $X_0(X_0X_1^2c^2v) = X_1(X_0^2X_1c^2v)$ we obtain $9a_5 + 9a_7 + 9a_{12} = 15b_5 + 15b_7$ which implies that

$$a_5 = 2/5.$$

From $X_0(X_0^2X_1c^2v) = X_1(X_0^3c^2v)$ we obtain $15a_5 + 15a_7 = 3b_2 + 6b_5 + 6b_7$ which implies that

$$b_2 = 12/5.$$

As a corollary we obtain the following numbers:

$$c^2v = 9,\ 18,\ 27,\ 27,\ 18.$$

Using table cv and the numbers for c^2v just obtained we can determine a_2. From the relation $X_0(X_0^3X_1cv) = X_1(X_0^4cv)$ we obtain $60a_7 + 9a_2 + 54a_5 + 8a_0 + 18a = 24b_7 + 18b_2 + 24b_5 + 5b_0 + 9b$ and so

$$a_2 = 3/5.$$

From $X_1(X_0^3c^2y) = X_0(X_0^2X_1c^2y)$ and the table of c^2y we get $6b_7 + 2b_3 = 15a_7$ and hence

$$b_3 = 12/5.$$

From $X_1(X_0^2c^2y^2) = X_0(X_0X_1c^2y^2)$ we obtain $b_7 + 2b_3 = 3a_3 + 6a_8$ and hence

$$a_8 = 3/5.$$

From $X_1(X_0X_1c^2y^2) = X_0(X_1^2c^2y^2)$ we obtain $3b_7 + 6b_8 = 2a_{12} + 4a_8$, and so

$$b_8 = 2/5.$$

Now we have $X_1X_0^2c^2y^2 = 6$ and $X_0^3c^2y^2 = a_7 + 2a_3$.

From the relation $X_0(X_0^2X_1cy^2) = X_1(X_0^3cy^2)$ we obtain $15a_7 + 6a_3 + 18a_8 + 6a = 6b_7 + 10b_3 + b(a_7 + 2a_3)$, so

$$a_3 = 3/5.$$

From $X_0(X_0^2X_1c^2z) = X_1(X_0^3c^2z)$ and $X_0(X_0^2X_1c^2q) = X_1(X_0^3c^2q)$ we obtain

$$\left.\begin{aligned} 4a_6 + a_7 + 15a_5 + 2a_0 &= 2b_1 + 6b_5 + b_0 \\ 4a_6 + a_7 + 2a_0 &= b_1 + b_0 \end{aligned}\right\}$$

which yields

$$b_1 = 12/5, \quad a_6 = 3/5.$$

Now we have $X_0(X_0^3c^2q) = a_0 + a_1$, $X_0^3X_1c^2q = 4$, $X_0^2X_1^2c^2q = 10$.

From $X_1(X_0^4cq) = X_0(X_0^3X_1cq)$ we obtain $7b_0 + 6b_1 + b(a_0 + a_1) = 18a_6 + 6a_7 + 3a_1 + 13a_0 + 4a$, and so

$$a_1 = 3/5.$$

From $X_1(X_0^3X_1cq) = X_0(X_0^2X_1^2cq)$, $X_1(X_0^2X_1c^2q) = X_0(X_0X_1^2c^2q)$, and $X_1(X_0X_1^2c^2q) = X_0(X_1^3c^2q)$, we obtain

$$\left.\begin{aligned} 18b_6 + 6b_7 + 3b_1 + 13b_0 + 4b &= 21a_6 + 21a_7 + 9a_{11} + 16a_0 + 10a \\ 4b_6 + b_7 + 2b_0 &= a_6 + 4a_7 + 3a_{11} + 2a_0 \\ b_6 + 4b_7 + 3b_{11} + 2b_0 &= 3a_{11} + 4a_{12} + a_0 \end{aligned}\right\}$$

Solving for b_6, a_{11} and b_{11} we obtain

$$b_6 = 7/5, \quad a_{11} = 9/5, \quad b_{11} = 6/5.$$

From $X_1(X_0^2X_1c^2z) = X_0(X_0X_1^2c^2z)$ we obtain $4b_6 + b_7 + 15b_5 + 2b_0 = a_6 + 4a_7 + 6a_{10} + 9a_5 + 2a_0$, and so

$$a_{10} = 9/5.$$

From $X_1(X_0X_1^2c^2z) = X_0(X_1^3c^2z)$ we obtain $b_6 + 4b_7 + 6b_{10} + 9b_5 + 2b_0 = 4a_{12} + 8a_{10} + a_0$, and so

$$b_{10} = 6/5.$$

From $X_1(X_0^3c^2w) = X_0(X_0^2X_1c^2z)$ and $X_1(X_0^2c^2qw) = X_0(X_0X_1c^2qw)$ we obtain

$$\left.\begin{aligned} 4b_1 + 6b_4 + b_0 &= 4a_6 + a_7 + 15a_4 + 2a_0 \\ 3b_1 + b_4 &= 3a_6 + 3a_4 + 6a_8 \end{aligned}\right\}$$

Solving for a_4 and b_4 we obtain

$$a_4 = 6/5, b_4 = 9/5.$$

From $X_1(X_0^2X_1c^2w) = X_0(X_0X_1^2c^2w)$ we obtain $4b_6 + b_7 + 15b_4 + 2b_0 = a_6 + 4a_7 + 9a_4 + 12a_9 + 2a_0$, and so

$$a_9 = 9/5.$$

From $(X_0X_1^2c^2w) = X_0(X_1^3c^2w)$ we obtain $b_6 + 4b_7 + 9b_4 + 12b_9 + 2b_0 = 4a_{12} + 18a_9 + a_0$, and so

$$b_9 = 6/5.$$

11. Fundamental numbers

Once we know degeneration relations for the first order conditions and the degeneration numbers, the computation of fundamental numbers is reduced to arithmetic operations (see **8.7 (b)**). This has been applied in the proof of **10.1** to find several fundamental numbers that were needed along the way. Here we include a couple of examples that will further illustrate the use of **8.7**.

11.1. $N' = X_0^3 c^2 v^2$

Since $X_0^2 c^2 v^2$ only contains degenerations of type D_2, D_5 and D_7 (see Table c^2v^2 in section 9), with degeneration numbers 1, 4 and 1, respectively, we have, by **10.1 (1)**, that

$$N' = a_2 + 4a_5 + a_7 = (3 + 8 + 4)/5 = 3.$$

Notice that the term $\frac{3}{5}c$ in the expression of X_0 does not give any contribution to N', because numbers with c^3 are 0 (see **8.8**).

11.2. $N = X_0^4 cv^2$

Since $X_0^3 cv^2$ only contains degenerations of type D_2, D_5 and D_7 (see Table cv^2 in section 9), with degeneration numbers 6, 12 and 6, respectively, we have, by **10.1 (1)**, that

$$N = 6a_2 + 12a_5 + 6a_7 + aN' = (18 + 24 + 24 + 9)/5 = 15.$$

The value of this number that we find in Schubert [1879] (p. 141, line 4) is 17. This looks like a misprint, rather than a mistake, for on p. 138, line −11, we find that the value given to the dual number is 15.

11.3. $M'' = X_0^2 c^2 vyz$

Here it is not hard to see that $X_0 c^2 vyz = X_0 c^2 v^2 y$ and hence this only contains degenerations of type D_5 and D_7 (see Table c^2v^2y in section 9), with degeneration numbers 3 and 1, respectively. Therefore we have, by **10.1 (1)**, that

$$M'' = 3a_5 + a_7 = (6 + 4)/5 = 2.$$

11.4. $M' = X_0^3 cvyz$

Since $X_0^2 cvyz$ only contains degenerations of type D_2, D_3, D_5 and D_7 (see Table $cvyz$ in section 9), with degeneration numbers 1, 2, 13 and 6, respectively, we have, by **10.1 (1)**, that

$$M' = a_2 + 2a_3 + 13a_5 + 6a_7 + aM'' = (3 + 6 + 26 + 24 + 6)/5 = 13.$$

11.5. $M = X_0^3 X_1 vyz$

Here $X_0^3 vyz$ contains degenerations of type D_0, D_1, D_2, D_3, D_5 and D_7 (see Table vyz in section 9), with degeneration numbers 1, 2, 6, 10, 18 and 6, respectively, we have, by **10.1 (2)**, that

$$M = b_0 + 2b_1 + 6b_2 + 10b_3 + 18b_5 + 6b_7 + bM' = (8 + 24 + 72 + 120 + 54 + 36 - 39)/5 = 55.$$

This is one of the numbers that we can not find in Schubert's book.

12. Old and new tables of fundamental numbers of cuspidal cubics

Here we collect the values of all non-zero fundamental numbers (see the Remarks at the end). They have been calculated, as illustrated in the preceeding section, by means of formula **8.7 (b)**, using the degeneration formulas **10.1** (basically **(1)** and **(2)**). Most have been calculated in more than one way. Those not listed in Schubert [1879] (nor anywhere else, as far as we know) are distinguished with a **. A few numbers are marked with *; this means that their value can be deduced from some table of Schubert corresponding to space cuspidal cubics. The arrangement of the tables is as follows. A number like $M = X_0^3X_1vyz$ is located at the second place of the row that begins with $vyz =$. The row ends with $= yzq$ because by duality M is equal to $X_0X_1^3qzy$. The rows are ordered lexicographically by the leading monomials. To the monomial **1** there corresponds the list of *characteristic numbers*:

$$24, 60, 114, 168, 168, 114, 60, 24.$$

Order 1

$c =$	12	42	96	168	186	132	$72 = w$
$v =$	66	123	177	168	105	51	$18 = q$
$y =$	48	96	150	168	132	78	$36 = z$

Order 2

$c^2 =$	2	8	20	38	44	$32 = w^2$
$cv =$	47	89	128	119	71	$32 = qw$
$cy =$	32	62	92	92	62	$32 = zw$
$^{**}cz =$	22	52	94	112	88	$52 = yw$
$^{*}cq =$	7	25	58	85	79	$52 = vw$
$cw =$	52	106	166	166	106	$52 = cw$
$v^2 =$	20	35	47	38	17	$5 = q^2$
$vy =$	59	89	92	65	35	$14 = zq$
$^{**}vz =$	40	79	121	112	61	$25 = yq$
$vq =$	34	79	139	139	79	$34 = vq$
$y^2 =$	20	44	74	74	44	$20 = z^2$
$yz =$	34	70	112	112	70	$34 = yz$

Order 3

$c^2v =$	9	18	27	27	$18 = qw^2$
$c^2y =$	6	12	18	18	$12 = zw^2$
$c^2z =$	4	10	19	22	$16 = yw^2$
$c^2q =$	1	4	10	13	$10 = vw^2$
$c^2w =$	10	22	37	40	$28 = cw^2$
$cv^2 =$	$\|15\|$	27	36	27	$9 = q^2w$
$cvy =$	33	48	45	27	$12 = zqw$
$^{**}cvz =$	19	37	55	49	$25 = yqw$
$^{**}cvq =$	19	49	64	49	$28 = vqw$
$^{*}cvw =$	43	67	73	49	$19 = cqw$
$cy^2 =$	12	30	36	24	$12 = z^2w$
$^{**}cyz =$	22	46	55	40	$22 = yzw$
$^{**}cyq =$	13	34	46	37	$22 = vzw$
$^{**}cyw =$	40	70	73	46	$22 = czw$
$cz^2 = c^2z =$	4	10	19	22	$16 = y^2w$
$^{**}czq =$	7	19	37	43	$31 = vyw$
$cq^2 = c^2q =$	1	4	10	13	$10 = v^2w$
$v^2y =$	15	21	18	9	$3 = zq^2$
$v^2z =$	10	19	28	22	$7 = yq^2$
$v^2q =$	10	22	37	31	$10 = vq^2$
$vy^2 =$	21	30	27	15	$6 = z^2q$
$^{**}vyz =$	31	55	55	31	$13 = yzq$
$^{**}vyq =$	31	61	64	37	$16 = vzq$
$vz^2 = v^2z =$	10	19	28	22	$7 = y^2q$
$y^2z =$	10	22	37	34	$16 = yz^2$

Order 4

$c^2v^2 =$	3	6	9	9	$= q^2w^2$
$c^2vy =$	6	9	9	6	$= zqw^2$
$c^2vz = c^2v^2 =$	3	6	9	9	$= yqw^2$
$c^2vq =$	3	9	9	6	$= vqw^2$
$^{*}c^2vw =$	9	15	18	15	$= cqw^2$
$c^2y^2 =$	2	6	6	4	$= z^2w^2$
$c^2yz =$	4	9	9	6	$= yzw^2$
$c^2yq = c^2y^2 =$	2	6	6	4	$= vzw^2$
$^{**}c^2yw =$	8	15	15	10	$= czw^2$
$c^2zq =$	1	3	6	5	$= vyw^2$
$c^2zw =$	4	9	15	14	$= cyw^2$
$c^2qw =$	3	9	12	9	$= cvw^2$
$c^2w^2 =$	6	9	9	6	$= c^2w^2$
$cv^2y =$	9	12	9	3	$= zq^2w$
$cv^2z = c^2v^2 =$	3	6	9	9	$= yq^2w$
$^{**}cv^2q =$	6	15	18	12	$= vq^2w$
$cv^2w =$	9	12	9	3	$= cq^2w$

$cvy^2 =$	14	15	9	$4 = z^2qw$
$^{**}cvyz =$	13	21	18	$9 = yzqw$
$^{**}cvyq =$	17	24	18	$10 = vzqw$
$^{**}cvyw =$	23	27	18	$7 = czqw$
$cvz^2 = c^2v^2 =$	3	6	9	$9 = y^2qw$
$^{**}cvzq =$	7	18	24	$17 = vyqw$
$^{**}cvzw =$	13	21	24	$17 = cyqw$
$cvq^2 = c^2vq =$	3	9	9	$6 = v^2qw$
$^{**}cvqw =$	21	33	33	$21 = cvqw$
$^{**}cy^2z =$	6	15	15	$8 = yz^2w$
$cy^2q = c^2y^2 =$	2	6	6	$4 = vz^2w$
$cy^2w =$	14	15	9	$4 = cz^2w$
$cyz^2 = c^2yz =$	4	9	9	$6 = y^2zw$
$^{**}cyzq =$	7	18	21	$13 = vyzw$
$^{**}cyzw =$	16	30	30	$16 = cyzw$
$cyq^2 = c^2y^2 =$	2	6	6	$4 = v^2zw$
$cz^2q = c^2qz =$	1	3	6	$5 = vy^2w$
$czq^2 = c^2qz =$	1	3	6	$5 = v^2yw$
$v^2y^2 =$	5	6	3	$1 = z^2q^2$
$v^2yz =$	7	12	9	$3 = yzq^2$
$^{**}v^2yq =$	8	15	12	$4 = vzq^2$
$v^2zq =$	4	9	15	$11 = vyq^2$
$v^2q^2 =$	3	9	9	$3 = v^2q^2$
$^{**}vy^2z =$	9	15	12	$5 = yz^2q$
$vy^2q =$	11	15	9	$4 = vz^2q$
$vyz^2 = v^2yz =$	7	12	9	$3 = y^2zq$
$^{**}vyzq =$	13	27	27	$13 = vyzq$
$y^2z^2 =$	4	9	9	$4 = y^2z^2$

Order 5

$c^2v^2y =$	2	3	3	$= zq^2w^2$
$c^2v^2q =$	1	3	3	$= vq^2w^2$
$c^2v^2w =$	2	3	3	$= cq^2w^2$
$c^2vy^2 =$	3	3	2	$= z^2qw^2$
$c^2vyz = c^2v^2y =$	2	3	3	$= yzqw^2$
$c^2vyq = c^2vy^2 =$	3	3	2	$= vzqw^2$
$^{**}c^2vyw =$	5	6	5	$= czqw^2$
$c^2vzq = c^2v^2q =$	1	3	3	$= vyqw^2$
$c^2vzw = c^2v^2w =$	2	3	3	$= cyqw^2$
$^{*}c^2vqw =$	4	6	5	$= cvqw^2$
$c^2vw^2 = c^2v^2w =$	2	3	3	$= c^2qw^2$
$c^2y^2z =$	1	3	2	$= yz^2w^2$
$c^2y^2w =$	3	3	2	$= cz^2w^2$
$c^2yzq = c^2y^2z =$	1	3	2	$= vyzw^2$
$^{**}c^2yzw =$	3	6	5	$= cyzw^2$

$c^2yqw = c^2y^2q =$	3	3	2	$= cvzw^2$
$c^2yw^2 = c^2y^2w =$	3	3	2	$= c^2zw^2$
$c^2zqw =$	1	3	4	$= cvyw^2$
$cv^2y^2 =$	4	3	1	$= z^2q^2w$
$cv^2yz = c^2v^2y =$	2	3	3	$= yzq^2w$
$^{**}cv^2yq =$	5	6	4	$= vzq^2w$
$cv^2yw = cv^2y^2 =$	4	3	1	$= czq^2w$
$cv^2zq = c^2v^2q =$	1	3	3	$= vyq^2w$
$cv^2zw = c^2v^2w =$	2	3	3	$= cyq^2w$
$cv^2q^2 = c^2v^2q =$	1	3	3	$= v^2q^2w$
$^{*}cv^2qw = cvqw^2 =$	5	6	4	$= cvq^2w$
$^{**}cvy^2z =$	5	6	3	$= yz^2qw$
$cvy^2q = c^2vy^2 =$	3	3	2	$= vz^2qw$
$cvy^2w = cv^2y^2 =$	4	3	1	$= cz^2qw$
$cvyz^2 = c^2v^2y =$	2	3	3	$= y^2zqw$
$^{**}cvyzq =$	6	9	6	$= vyzqw$
$^{**}cvyzw =$	7	9	6	$= cyzqw$
$cvyq^2 = c^2vy^2 =$	3	3	2	$= v^2zqw$
$^{**}cvyqw =$	8	9	6	$= cvzqw$
$cvz^2q = c^2v^2q =$	1	3	3	$= vy^2qw$
$cvz^2w = c^2v^2w =$	2	3	3	$= cy^2qw$
$cvzq^2 = c^2v^2q =$	1	3	3	$= v^2yqw$
$cy^2z^2 = c^2y^2z =$	1	3	2	$= y^2z^2w$
$cy^2zq = c^2y^2z =$	1	3	2	$= vyz^2w$
$^{**}cy^2zw = cyzw^2 =$	5	6	3	$= cyz^2w$
$cyz^2q = c^2y^2z =$	1	3	2	$= vy^2zw$
$cyzq^2 = c^2y^2z =$	1	3	2	$= v^2yzw$
$v^2y^2z =$	2	3	1	$= yz^2q^2$
$v^2y^2q =$	3	3	1	$= vz^2q^2$
$^{**}v^2yzq =$	3	6	4	$= vyzq^2$
$v^2yq^2 = v^2y^2q =$	3	3	1	$= v^2zq^2$
$vy^2z^2 = v^2y^2z =$	2	3	1	$= y^2z^2q$
$^{**}vy^2zq = vyzq^2$	4	6	3	$= vyz^2q$

12.1. Remark. For any condition α in the list $\{c, v, y, z, q, w\}$, it is clear that if a fundamental number N contains α^3, then $N = 0$. We may conveniently phrase this by writing $\alpha^3 = 0$. Similarly, if (α, β) is any pair on the list

$$\{(c,q),(c,z),(v,z),(v,w),(y,w),(y,q)\},$$

then $\alpha^2\beta^2 = 0$, for whenever α and β refer to incident elements of the singular triangle we cannot fix both independently. Finally it is also clear that if (α, β) is a pair of distinct vertices or sides of the singular triangle and γ is the side or vertex defined by the pair, then $\alpha^2\beta^2\gamma = 0$.

12.2. Remark. In the tables above we have used identities of the form $\alpha^2\beta = \alpha\beta^2$, which is valid for any pair (α, β) on the list

$$\{(c,q),(c,z),(v,z),(v,w),(y,w),(y,q)\},$$

inasmuch as they are valid for triangles.

12.3. Remark. We have not listed the table corresponding to order 6. In this case, if the order six monomial involves at least one square and it is not in one of the cases in 12.1, or amenable to such a case by 12.2, then the row corresponding to it is (1,1), for it is not hard to see that such a monomial fixes the singular triangle. On the other hand, the list corresponding to the unique square free monomial $cvyzqw$ is (2,2), for there are 2 triangles satisfying this condition. In any case, the cuspidal cubics having a given triangle as a singular triangle form a pencil and so there is a unique cubic in it going through a point or (by duality) tangent to a line (cf. Schubert [1879], Remark on top of p. 143).

12.4. Remark. For reasons of dimensions, it is clear that all monomials of degree 7 not involving X_0 and X_1 are 0.

12.5. Remark. It turns out that the fundamental numbers which do not satisfy one of the vanishing conditions given in the preceeding remarks are automatically non-zero.

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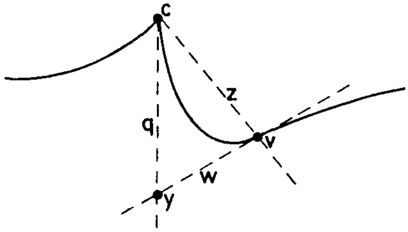
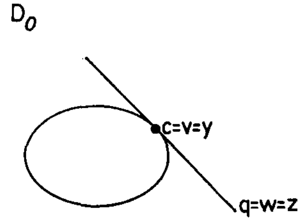
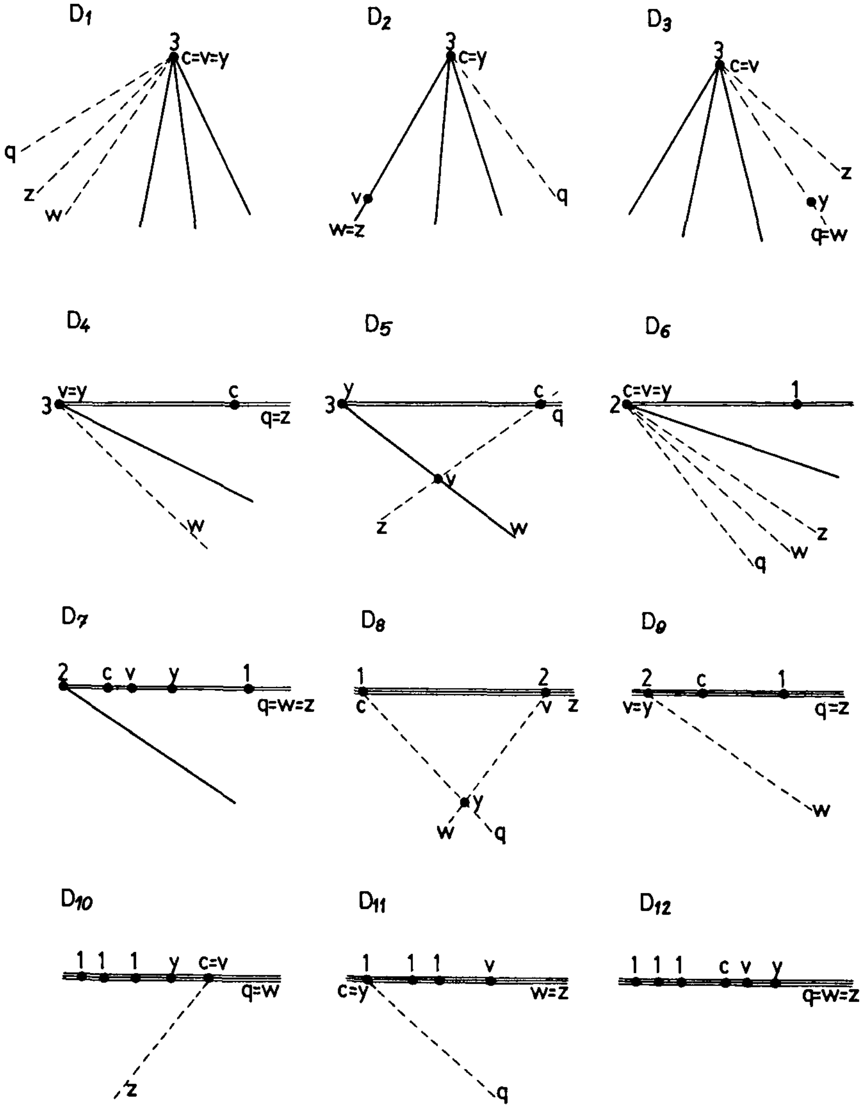

c
z
q
v
w
y

Fig.1


D0
c=v=y
q=w=z


D1
3
c=v=y
q
z
w
D2
3
c=y
v
w=z
q
D3
3
c=v
z
y
q=w
D4
v=y
c
3
q=z
w
D5
y
c
3
q
y
z
w
D6
c=v=y
1
2
z
w
q
D7
2
c
v
y
1
q=w=z
D8
1
2
c
v
z
y
w
q
D9
2
c
1
v=y
q=z
w
D10
1
1
1
y
c=v
q=w
z
D11
1
1
1
v
c=y
w=z
q
D12
1
1
1
c
v
y
q=w=z