One benchmark of our understanding of the geometry of $H'_{3,0,3}$ *[the component of the Hilbert scheme parametrizing the twisted cubics and their degenerations] is being able to solve rigorously a famous enumerative problem: Given 12 quadrics in* $\mathbb{P}^3$*, how many twisted cubics are tangent to all 12?*

(Harris [1980], bottom of p. 38)

Sketch of a Verification of SCHUBERT'S NUMBER 5819539783680 OF TWISTED CUBICS

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ABSTRACT. This sketch is fairly complete. The verification is completely reduced in Sects. 2-3 to proving 4 lemmas. Their proofs are sketched in Sects. 4-7, and the new ideas are emphasized. Also, the enumerative significance of the number is fully treated.

1. INTRODUCTION

On January 29, 1875, the royal Danish Academy awarded a gold medal to Hermann Schubert for his response to its 1873 prize problem, whose statement here is translated from the original Danish (Zeuthen [1875], p. 14):

> "To extend the theory of characteristics to systems of geometric entities formed by the points and the osculating planes of space curves of degree 3, and to determine the characteristics of the systems that must be considered as elementary".

Schubert's work was published solely in his book, Schubert [1879] (see Lit. 35, p.339). On p. 184, the treatment of twisted cubics culminates with the famous example, the determination of the number tangent to 12 quadrics.

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The rigorous verification of Schubert's number has been a particular challenge: twisted cubics are curves of higher degree and are not complete intersections. This paper announces the first successful response ; it sketches a complete proof of the following theorem.

Theorem 1. *Given* 12 *smooth quadrics in general position in* $\mathbb{P}^3$ *over an algebraically closed field of characteristic* 0 , *let* N *be the number of twisted cubics tangent to all* 12 . *Then, just as Schubert found,*

$$N = 5819539783680$$

Moreover, each cubic appears in the count with weight 1 , *and it intersects each quadric in* 5 *points:* 4 *with multiplicity* 1 , *and* 1 *with multiplicity* 2 .

The spirit of the proof of Theorem 1 is essentially that of Schubert's treatment, but there are 2 notable modifications. The first concerns the choice of aspects of the cubics. To solve the prize problem, Schubert found it necessary to incorporate the tangent lines with the points and the osculating planes. (Such liberty to modifly the statement of the problem is granted by its words, "which must be considered as elementary", according to Zeuthen [1875] , middle of p. 155). Now, Theorem 1 is proved by going one step further and employing these two aspects: the points and the tangent planes. Currently, it is an open problem to describe the geometric structure and the intersection ring of Schubert's space. (In fact, there are two distinct spaces: one parametrizes the locus of points, the locus of tangent lines, and the locus of osculating planes as subschemes; the other parametrizes them as cycles. See Piene [1983], Sect. 4, pp. 334-336.)

Secondly, Theorem 1 is proved without a determination of all of the characteristic numbers of the nodal cubics in a variable plane. Only the number, 12960, of such cubics cutting 11 lines is necessary (see Sect. 3) and it may be determinated directy (see Sect. 6). Schubert too could have, in the same way, made do with 12960 to find 5819539783680. However, he had a general interest in finding geometric numbers. Moreover, his work rested logically on his own version of the (independent) work of Zeuthen (1873) and Maillard (1872) on the determination of the characteristic numbers of the nodal cubics in a fixed plane, and this groundwork does not support a separate and direct determination of the number 12960.

The proof of Theorem 1 does not fit the traditional 20th century idea of how to solve an enumerative problem: no determination is made of the intersection ring of a suitable complete, smooth variety of cubics. Rather, in Sect. 2, the Contact Theorem is used much as Schubert used it, and it applies equally to varieties of any dimension and degree. Thus Theorem 1 is reduced to Theorem 2, which gives the characteristic numbers of the twisted cubics. Theorem 2 is then established in Sect. 3 by considering certain intersection numbers of curves and divisors on an appropriate *open* subscheme U of a component, $H'_{3,0,3}$, of the Hilbert scheme.

The curves are defined by the 12 various combinations of 11 of the two elementary conditions - to cut a general line, and to touch a general plane. In Section 5, the curves are shown to be complete - this is a key result - by using the Chow variety of the cubics' conormal cycles. Now the divisors' linear equivalence classes satisfy 2 key relations, found by Schubert [1879]. (Relations 1) and 2) on p. 168 involve additional terms, but they vanish in the case at hand according to the top of page 178.) The relations are established in Sect. 4. They are stated in Lemma 1, Sect. 3 and used in Sect. 3 to reduce the determination of the 13 characteristic numbers of the twisted cubics to the determination of the single characteristic number, 12960, of the nodal cubics and that of the 12 characteristic numbers of the unions of a smooth conic and a unisecant line. These subsidiary characteristic numbers are determined in Sect. 6, resp. Sect. 7, using a suitable compactification of the space of nodal cubics, resp. of unions. The two compactifications are not directly related to any compactification of the space of twisted cubics.

Theorem 1 is probably still valid in characteristic $p > 3$. At any rate, the proof works if p divides neither 5819539783680 nor any of the 13 characteristic numbers of the twisted cubics; see Sect. 2. Moreover, a variation of the proof, described at the end of Sect. 3, shows that if $p > 3$, then: (i) 5819539783680 is the weighted number of cubics, (ii) the weights are all equal to the same power q of p (so $q = 1$ if $p \nmid 5819539783680$), (iii) each cubic touches each quadric only once, and (iv) the intersection multiplicity at a point of contact is equal to 2 if $q = 1$. This variation involves intersecting on U the 12 divisors of cubics touching the quadrics, and it may be possible using a little deformation theory to determine the tangent spaces of the divisors and to find conditions to guarantee that they are independent. The conditions might be of this sort (cf. Fulton [1984], Ex. 9.1.9, p. 158): no 2 quadrics are tangent, no 3 have more than a finite number of common points or common tangent planes, no 4 have any common point or tangent plane, etc.

Theorem 1 is, of course, only an example. It is a trivial matter now to rigorously enumerate the twisted cubics that cut any c curves and touch any s surfaces, $c + s = 12$, in general position, given the degree of each curve and the degree and rank of each surface. (The rank of a smooth surface of degree n, or one with only finitely many singular points, is $n(n-1)$; see, for example, Kleiman [1984], II-(9), II-(5). A little additional arithmetic yields the number, and the present theory guarantees its significance.

A variation of the proof of Theorem 2 verifies Schubert's values for the number of twisted cubics that pass through i general points, cut j general lines, and touch $12-2i-j$ general planes for $0 \leq i \leq 6$ and $0 \leq j \leq 12-2i$. The modification is similar in spirit to that in Kleiman-Speiser [1984], Sect. 8. Vainsencher [1985] verified the cases $i = 5,6$ differently; he parametrized the cubics via the pencils of quadrics through a variable line containing a fixed point.

The case $i = 6$ stands apart and may be handled by direct elementary means. For $i < 6$, consider the curve on U defined by the i points and by all but one of the lines and planes. It is complete, because the closure in the Hilbert scheme of the variety of twisted cubics through a general point cuts each orbit in a set of codimension 2 or more. Applying the two key divisorial relations in Lemma 1 then reduces the problem to determining the corresponding numbers for the nodal cubics and for the conic-line cubics. Finally, Schubert's values for the latter numbers may be verified by proceeding as in Sections 6 and 7 and carrying the work further.

Some of Schubert's numbers involving the conditions to osculate a given plane, to send an osculating plane through a given line, and to send a tangent line through a given point now follow on considering the "strict" dual curve, the curve in the dual $\mathbb{P}^3$ parametrizing the osculating planes. However, it is still an open problem to verify the remainder of Schubert's numbers such as these two: the number 1146960 of cubics cutting 6 general lines and sending 6 osculating planes through 6 other general lines, and the number 120 of twisted cubics that touch 3 general lines and cut 3 others. And, of course, it is an open problem to treat most other curves. (Coray and Vainsencher [1985] have proved that there are 105 rational quintics through 10 general points by parametrizing the quintics via pencils of cubic surfaces doubled along a common line.) So, with an eye toward the future, an attempt has been made here to develop and use as many general arguments as possible.

Assumption. From now on, the ground field will be algebraically closed of arbitrary characteristic p, but $p \neq 2,3$.

2. Reduction to theorem 2

Theorem 1 will now be reduced to Theorem 2 below. Then, at the end of this section, the proof of Theorem 2 will be begun.

Theorem 2. *For* $1 \leq m \leq 13$, *let* N_m *denote the* mth *characteristic number of the twisted cubics, that is, the weighted number that cut* 13-m *lines and touch* m-1 *planes in general position. Then the* N_m *are just the numbers tabulated by* Schubert ([1879], *middle of* p. 178): 80160, 134400, 209760, etc.

Moreover, (i) *each cubic counts in* N_m *with the same weight* q_m, *and* $q_m = 1$ *if* $p = 0$ *or if* $p \nmid N_m$, *and* $q_m = p^e$ *for some* $e \geq 0$ *if* $p > 3$; (ii) *each cubic cuts each of the* 13-m *lines at only one point, and it touches each of the* m-1 *planes at only one point; and* (iii) *it is osculated by none of the planes if* $q_m = 1$.

Indeed, Theorem 2 and the Contact Theorem (Fulton-Kleiman-MacPherson [1982], pp. 161-2, or Kleiman [1984], III-(9)) yield

$$N = 2^{12}[80160 + \binom{12}{1}134400 + \binom{12}{2}209760 + \ldots] ,$$

and a little arithmetic yields 5819539783680. (The arithmetic here and elsewhere has all been double checked, so it may be said that Schubert made no arithmetic mistake in determining N.)

The Contact Theorem asserts a priori that the number N is finite and given by the formula,

$$N = (r_o L_o + r_1 L_1 + r_2 L_2)^{12} .$$

The r_i are the ranks of a general quadric, and it is well-known that $r_i = 2$ for all i . (In fact, these 2's characterize the smooth quadric; see for example, Kleiman [1984], II-(10)). The expression on the right is evaluated by expanding it formally and replacing each monomial $L_o^{j_o} L_1^{j_1} L_2^{j_2}$ by the weighted number $N(j_o, j_1, j_2)$ of cubics passing through j_o points, cutting j_1 lines, and touching j_2 planes. Also, $N(j_o, j_1, j_2) = 0$ if $j_o > 0$; this vanishing is part of the Contact Theorem too.

A twisted cubic is reflexive, because its dual variety is a surface of degree 4 and because $p \neq 2$ (apply, for example, Kleiman [1984], II-(16), II-(2)(iv), I-(4)). So, the Contact Theorem yields this too: (1) each cubic counts in N , resp. in $N(j_o, j_1, j_2)$, with the same weight q ; (2) $q = 1$ if $p = 0$, and $q = p^e$ for some $e \geq 0$ if $p \geq 3$; (3) each cubic touches each quadrics at only one point, resp. each cubic cuts each of the j_1 lines in only one point, and it touches each of the j_2 planes at only one point; (4) if $q = 1$, then each cubic intersects each quadric, resp. each plane, with multiplicity 2 at the point of contact ((4) is a refinement added to the Contact

Theorem in Kleiman [1984], III-4, on the basis of Goldstein's theory (Goldstein [1984], Sect.5; Kleiman [1984], III-1) of a generalized second fundamental form).

Assume $p = 0$. Then, by (1), (2) and Theorem 2, all the q and q_m are equal to 1 . Hence, for all m ,

$$N(0,13-m,m-1) = N_m .$$

Thus, Schubert's value of N is verified on the basis of Theorem 2. Moreover, (2), (3) and (4) yield the rest of Theorem 1.

Assume $p > 3$. Then $N(0,13-m,m-1)$ and N_m still count the same cubics, but, a priori, possibly with different weights, q and q_m say, because the technical setups of the counts are different. By Theorem 2, the value of N_m is the same as it is when $p = 0$. The value of $N(0,13-m,m-1)$ may, on the other hand, be less: it is less iff some of the cubics degenerate under the reduction to characteristic p (see Kleiman [1984], III-(5), paragraph before (13)). Hence $q \leq q_m$ (in fact, q divides q_m), and $q = q_m$ iff no cubic in characteristic 0 degenerates under reduction.

Suppose p does not divide N_m . Then, by Theorem 2, $q_m = 1$. So, by the above, $q = q_m$, and no cubic degenerates under reduction. Suppose instead that no cubic degenerates under reduction. Then $q = q_m$ by the above. In any event, as before, whenever $q = q_m$ for all m , Schubert's value of N is correct; moreover, the remaining assertions of Theorem 1 are valid if $p \nmid N$.

Finally, if $q_m = 1$, then $q = 1$ because $q \leq q_m$. Hence, (3) and (4) yield the assertions (ii) and (iii) of Theorem 2.

The first two assertions of Theorem 2 are reformulated as Theorem 2* in the next section, and they are derived there from 4 lemmas. The proofs of the lemmas are sketched in the subsequent sections. The general formal setup is introduced in the next section. However, additional notation and hypotheses will vary from one later section to the next.

3. Reduction to 4 lemmas

Consider the irreducible component $H'_{3,0,3}$ of the Hilbert scheme of $\mathbb{P}^3$ formed by closing the open subscheme representing the twisted cubics. It is 12-dimensional and smooth by Piene-Schlessinger [1982], Theorem, p. 761. The group of linear transformations of $\mathbb{P}^3$ acts naturally on it, and there are finitely many orbits. (The orbits are enumerated in Piene [1981] and in Harris [1982], pp. 39-41.) The following three orbits and their union U play a major role in what follows:

S := the orbit of a twisted cubic;

A := the orbit of an irreducible cubic with an ordinary node plus an embedded point situated at the node and not contained in the plane of the cubic;

B := the orbit of an union of a smooth conic and a unisecant line;

U := the union of S , A and B .

The remaining orbits fill out the boundaries of A and of B . So, A and B are the only orbits of codimension 1, and U is open.

Fix, once and for all, a line L and a plane H . Form the following 2 closed subsets of U :

ZL := the closure of the set of twisted cubics cutting L ;

ZH := the closure of the set of twisted cubics touching H .

Then ZL and ZH are irreducible; in fact, ZL (resp. ZH) is the closure of an orbit under the subgroup fixing L (resp. H).

Lemma 1. *The following two relations are valid in* A^1U :

(1) $2[ZL] = 3[A]+[B]$; (2) $3[ZH] = 2[ZL]+2[B]$.

A proof of Lemma 1 will be sketched in the next section. (Of course, analogous relations hold on $H'_{3,0,3}$ because the complement of U is of codimension 2. However, the relations are proved and used on U .)

Fix 12 general linear transformations: $g_1,\ldots,g_{12}$. The term "general" means that in each of the following 4 results, the 12-tuple of g_i may be chosen arbitrarily from a certain dense (=nonempty) open set, whose existence is implicitly being asserted.

For $1 \leq m \leq 12$, form in U the following scheme intersection:

$$T_m := (g_1 ZH) \ldots (g_{m-1} ZH) \ldots (g_{m+1} ZL) \ldots (g_{12} ZL) \quad .$$

Lemma 2. T_m *is a complete curve.*

Lemma 3. $\int [T_1].[A] = 12960$, *the number given by* Schubert [1879], *top* *p.* 178, *for the number of nodal cubics meeting* 11 *lines.*

Lemma 4. $\int [T_m].[B] = 121440, 180240, 236160$, etc., *the numbers given by Schubert* [1879], *top p.* 178, *for the numbers of unions of a conic and a unisecant line cutting* 12-m *lines and touching* m-1 *planes.*

Theorem 2*. *For* $1 \leq m \leq 13$, *form the sum of local intersection numbers,*

$$N_m := \int [g_1 ZH] \ldots [g_{m-1} ZH].[g_m ZL] \ldots [g_{12} ZL] \ .$$

Then the N_m *are:* 80160, 134400, 209760, etc. *Moreover, the points of intersection all lie in* S , *and each local intersection number is equal to* 1 *if* $p = 0$ *and, for fixed* m , *to the same power* q_m *of* p *if* $p > 0$.

Proofs of the Lemmas 2-4 will be sketched in Sects. 5-7 respectively.

Theorem 2* may now be derived from the lemma as follows. Observe that

$$N_m = \int [T_m].[g_m ZL] \qquad \text{and} \qquad N_{m+1} = \int [T_m].[g_m ZH]$$

for $1 \leq m \leq 12$. Now, T_m is complete by Lemma 2. Hence, Lemma 1 yields this:

$$2N_m = 3 \int [T_m].[A] + \int [T_m].[B] \quad \text{and}$$

$$3N_{m+1} = 2N_m + 2 \int [T_m].[B] \quad .$$

Therefore, Lemmas 3 and 4 yield this:

$$N_1 = (3 \times 12960 + 121440)/2 = 80160 \; ,$$

$$N_2 = 2(80160 + 180240)/3 = 134400 \; ,$$

$$N_3 = 2(134400 + 236160)/3 = 209760 \; ,$$

etc.

The second assertion of Theorem 2* is a standard consequence of the general theory of transversality of a general translate; see Kleiman [1974] and Vainsencher [1978], (7.2).

A variation of the proof of Theorem 1 runs as follows. First, if the proof of Lemma 1 is modified slightly about (4.5) and (4.6), then it yields this: Given a surface G of degree n with only finitely many singularities, then

$$[ZG] = n(n-1)[ZL] + n[ZH] \quad .$$

Secondly, the proof of Lemma 2 yields this: The intersection of 11 general translates of ZG , ZL and ZH is a complete curve. Finally, proceeding as in the proof of Theorem 2* yields (without an appeal to the Contact Theorem) statements (i) and (ii), which were asserted in the fifth paragraph after Theorem 1. Statements (iii) and (iv) there follow now from the reasoning in Sect. 2.

4. Proof of lemma 1

(In an earlier proof, the relations were derived from corresponding relations on a lovely compactification of the scheme of maps from $\mathbb{P}^1$ to $\mathbb{P}^3 =: \mathbb{P}(E)$. The compactification is the Quot scheme parametrizing the quotients of rank 1 and degree 3 of $E_{\mathbb{P}^1}$. The corresponding relations are derived by identifying the quotients corresponding to the nodal cubics and to the unions of a conic and a unisecant line, and then computing some Chern classes. The basic setup is discussed in Strømme [1985].)

Consider the total space of the universal (flat) family.

$$C \subset \mathbb{P}^3 \times U .$$

Let A' (resp. B') denote the set of points in the preimage of A (resp. of B) that are singular in their fibers. Clearly, C is nonsingular off the union of A' and B'. By Serre's criterion, C is reduced, and normal off A' . Obviously, C is irreducible.

Consider the (birational) normalization map,

$$n : X \to C .$$

By the above, n is an isomorphism off $n^{-1}A'$. Let

$$f : X \to U$$

be the canonical map. Then f is flat; indeed, it is homogeneous, and flat over the generic points of A and B , because X is integral and U is nonsingular.

Let u be a geometric point of A , and consider the fiber of n ,

$$n(u) : X(u) \to C(u) .$$

It is finite, and an isomorphism over the nonsingular locus of $C(u)$. Moreover, $X(u)$ has no embedded points; indeed, this claim holds if u is generic in A

because X is normal and A is a divisor, so it holds if u is arbitrary by homogeneity. Hence $X(u)$ is reduced and irreducible. Now, $X(u)$ is of arithmetic genus 0, because f is flat. Therefore, $X(u)$ is isomorphic to $\mathbb{P}^1$, and it is equal to the normalization of $C(u)_{red}$.

Consider the following projection and invertible sheaf on X :

$$p_1 : C \to \mathbb{P}^3 \qquad \text{and} \qquad L := n^*p_1^* \mathcal{O}(1) \quad .$$

For each geometric point u of U, the fiber $L(u)$ is very ample on $X(u)$, and $H^1(L(u))$ vanishes. Hence, by standard base-change theory, the direct image

$$D := f_*L$$

is locally free of rank 4 on U, its formation commutes with base-change, the natural map $f^*D \to L$ is surjective, and the corresponding morphism

$$i : X \to P := \mathbb{P}(D)$$

is a closed embedding. In fact, i is a regular embedding of codimension 2, because its fibers are so and because X and P are flat over U.

Consider the base-change-like map on U associated to $p_1 n : X \to \mathbb{P}^3$,

$$v : H^0(\mathbb{P}^3, \mathcal{O}(1)) \otimes \mathcal{O}_U \to D \ .$$

Its formation commutes with passage to the fibers. So, v is an isomorphism off A, because if u lies off A, then $X(u)$ is linearly normal in $\mathbb{P}^3$. In A^1U, therefore, $c_1(D)$ is a multiple of $[A]$. In fact,

$$(4.1) \qquad [A] = c_1 D$$

because this equation holds after being pulled back to the parameter space, A^1, of the following family of cubics X_t in $\mathbb{P}^3$:

$$X_t : x_0 = tu^3 \ , \ x_1 = u(u^2-v^2) \ , \ x_2 = v(u^2-v^2) \ , \ x_3 = v^3 \ , \ (u,v) \in \mathbb{P}^1 .$$

Consider the singular locus, $\mathrm{Sing}(f)$, the subscheme of X defined by the Fitting ideal $F^1(\Omega_{X/U})$. Its formation commutes with base-change. Now,

$$f_*[\mathrm{Sing}(f)] = [B] \quad . \tag{4.2}$$

Indeed, it suffices to show that (4.2) holds after pullback to a suitable family of cubics X_t , $t \in A^1$; the formation of the cycle $[\mathrm{Sing}(f)]$ commutes with pullback because the higher tor's vanish. A suitable family is this:

$$X_t : x_0x_2-t(x_1)^2 = 0 \ , \ x_0x_3-tx_1x_2 = 0 \ , \ x_1x_3-(x_2)^2 = 0 \quad .$$

(In fact, $\mathrm{Sing}(f)$ is smooth, because its pullback is. Hence f induces a map,

$$\mathrm{Sing}(f) \to B \quad ,$$

and it is an isomorphism because its fibers are reduced points.)

Let J denote the ideal of the embedding $i : X \to P$. Consider the standard sequence,

$$0 \to i^*J \to i^*\Omega_{P/U} \to \Omega_{X/U} \to 0 \ . \tag{4.3}$$

It is exact on the left, because it is so generically, i^*J is locally free, and X is reduced. Apply Porteous's formula. Then (4.2) yields this:

$$[B] = f_*c_2\Omega_{X/U} \ . \tag{4.4}$$

Consider the closed subscheme $Y := (p_1n)^{-1}H$ of X . Since X is integral, Y is a divisor. Since X is regularly embedded in P , so is Y . Obviously, Y does not contain either $f^{-1}A$ or $f^{-1}B$. So Y is the set closure of its trace on $f^{-1}S$. This trace is clearly the set closure of an orbit under the subgroup fixing H . Thus Y is irreducible and Cohen-Macaulay.

Let $g : Y \to U$ be the restriction of f. Obviously, g is smooth on an open subset of Y meeting both $g^{-1}A$ and $g^{-1}B$.

Consider the ramification locus R of g ; it is the closed subscheme defined by the Fitting ideal $F^{o}\Omega_{Y/U}$. Because of the above, R is a divisor with no component contained in $g^{-1}A$ or in $g^{-1}B$. So R is the set closure of its trace on $g^{-1}S$. Obviously, this trace dominates ZH, in fact,

$$(4.5) \qquad [ZH] = g_*[R] = g_*c_1\Omega_{Y/U} .$$

Indeed, as with (4.2), it suffices to check the first equation after pullback to A^1 for a suitable family of cubics X_t, $t \in A^1$. A suitable family is this:

$$X_t : (x_1+tx_3)x_3-(x_2)^2 = 0 , \; x_0x_3-x_2(x_1+tx_3) = 0 , \; x_0x_2-(x_1+tx_3)^2 = 0 ,$$

provided the plane H is defined by $x_1 = 0$.

To compute the c_1, consider the inclusion $j : Y \to X$ and the sequence

$$0 \to \mathcal{O}_Y(-1) \to j^*\Omega_{X/U} \to \Omega_{Y/u} \to 0 .$$

This sequence, (4.5) and the projection formula yield this:

$$(4.6) \qquad [ZH]= f_*j_*j^*(c_1\Omega_{X/U}+c_1\mathcal{O}_X(1)) = f_*(c_1\Omega_{X/U} \;\; c_1\mathcal{O}_X(1)+c_1\mathcal{O}_X(1)^2) .$$

Almost by definition,

$$(4.7) \qquad [ZL] = f_*(p_1n)^*[L] .$$

Indeed, the two sides obviously have the same support. So it remains to check that the cycle on the right is reduced. To check it, fix a twisted cubic V that cuts L once and is not tangent to it, fix a smooth quadric that contains V but not L, and on Q consider the complete linear system of V. Then, on the projective space parametrizing the system, the trace of the cycle in question is equal to the trace of the cycle of the union of 2 distinct hyperplanes; whence, it is reduced.

To proceed, use (4.3) and the projection formula to obtain this in A^*P :

$$i_*c\Omega_{X/U} = i_*((ci^*\Omega_{P/U})(ci^*J)^{-1}) = c\Omega_{P/U} \cdot i_*(ci^*J)^{-1} .$$

A formula of Mumford and Fulton (see Fulton [1984], 15.3.5, p.298, and 18.21, p.353) now yields this:

(4.8) $$i_*c\Omega_{X/U} = c\Omega_{P/U} \cdot (1-ci_*\mathcal{O}_X) .$$

To find $ci_*\mathcal{O}_X$, use the following resolution on P :

(4.9) $$0 \to (h^*F)(-3) \to (h^*E)(-2) \to \mathcal{O}_P \to i_*\mathcal{O}_X \to 0$$

where E , resp. F , is locally free on U of rank 3, resp. 2, and

$$h : P \to U$$

is the structure map. Such a resolution exists on each fiber of P/U , and it may be globalized as follows. Set

$$E := h_*J(2) .$$

Then E is locally free of rank 3, its formation commutes with base-change, and

$$a : h^*E \to J(2)$$

is surjective by standard base-change theory. Set

$$F := h_*\ker(a(1)) .$$

Then F is locally free of rank 2, its formation commutes with base-change, and

$$h^*F \to \ker(a(1))$$

is an isomorphism by base-change theory.

Work modulo elements in A^*U of degree ≥ 2 for convenience. Set

$$d := c_1 D \ , \qquad e := c_1 E (= c_1 F) \ , \qquad \text{and} \qquad x := c_1 \mathcal{O}_P(1) \ .$$

Then (4.8) and (4.9) and a little "algebra" yields this:

$$i_* c\Omega_{X/U} = (-ex+3x^2) + (3dx^2-ex^2-2x^3) + (dx^3-2ex^3+2x^4) \ .$$

Since $f = hi$ and $h_* x^4 = d$, therefore (4.4), (4.6) and (4.7) yield this:

$$[B] = h_* i_* c_2 \Omega_{X/U} = 3d-2e \ ;$$

$$[ZH] = h_*(x \cdot i_* c_1 \Omega_{X/U} + x^2 \cdot i_* c_0 \Omega_{X/U}) = 4d-2e \ ;$$

$$[ZL] = f_*(c_1 \mathcal{O}_X(1))^2 = h_*(x^2 \cdot i_* c_0 \Omega_{X/U}) = 3d-e \ .$$

Finally, these formulas plus (4.1) readily yield the relations of Lemma 1.

5. Proof of lemma 2

Let I denote the graph of the point-plane incidence correspondence. Given a subscheme V of $\mathbb{P}^3$, let CV' denote its conormal variety; that is, the closure in I of the point-plane pairs (Q,M) such that Q is a simple point of V and such that M contains the tangent space $T_Q V$. Denote by V' the dual variety of V ; that is, V' is the image of CV in the dual projective space $\mathbb{P}^{3*}$.

It is basic general fact in characteristic $p = 0$ (Kleiman [1983], Sect. 3) that, given a specialization $V \Rightarrow V^*$ (that is, V is the generic fiber and V^* is the special fiber of a flat family over a discrete valuation ring), then the induced specialization $CV \Rightarrow C^*$, is such that

$$[C^*] = \sum_{W \leq V^*} m_W [CW]$$

where (A) the W include every component of V^* , (B) if W is a component that is not multiple, then $m_W = 1$, and (C) if W is not a component, then it lies in the singular locus of V^* .

If $p > 0$, the proceding result is not always valid. However, it is valid if V and V^* are reduced curves; a simple direct argument shows this. Moreover, in the case at hand where V is a twisted cubic and $p > 3$, the result is also valid. Indeed, the proof of the general result works in this case; the separability that is required holds, because, on the one hand, the degree of inseparability is a power of p and, on the other hand, in the present case this degree can be at most 3 .

Consider the subset of the Chow variety of I representing the "conormal" cycles $[CV]$ of the twisted cubics V . Denote the normalization of the subset's closure by T. Normalize in the function field of U (the open subscheme of the Hilbert scheme), because, if $p > 0$, then the natural map possibly is not birational. Since U is smooth, there is an induced map, $U \to T$, and it is clearly an open embedding.

In the product $I^{\times 11} \times S$, form the following set:

$$X := \{x:=((Q_1,M_1),\ldots,(Q_{11},M_{11}),V) \mid (Q_i,M_i) \text{ in } CV \text{ , and } V \text{ in } S\} \quad .$$

Then X is a closed subvariety, because S parametrizes the flat family of all twisted cubics. Denote the closure of X in $I^{\times 11} \times T$ by X^* . Consider an arbitrary point x^* of X^* ; say,

$$x^* = ((Q_1^*,M_1^*),\ldots,(Q_{11}^*,M_{11}^*),c^*) \ .$$

Then x^* is the specialization of a point x of X ; say

$$x = ((Q_1,M_1),\ldots,(Q_{11},M_{11}),V) .$$

Let Z^* denote the cycle corresponding to c^* , and C^* its support. Then (Q_i,M_i) is in C^* for all i . Consider the image Z' of Z^* in $\mathbb{P}^3$. Then Z' is the specialization of the cycle $[V']$ of the dual variety V' of V . Hence, C' is of pure dimension 2 and of degree 4, because V' is. Let V^* be a corresponding specialization of V as a subscheme. In view of the facts discussed above, it is clear that the only probabilities for the cycle Z^* are these:

(1) $[CV^*]$ if V^* is twisted or nodal.

(2) $[CV^*] + [CQ]$ if V^* is cuspidal and Q is the cusp.

(3) $[CW] + 2[CQ]$ if V^* is the union of a smooth conic W and a unisecant line L and if Q is the common point.

(4) $[CV^*] + [CQ_1] + [CQ_2] + [CQ_3] + [CQ_4]$ if V^* is a connected union of 3 distinct lines and if each Q_i is the common point of 2 lines.

(5) $[CW_1] + 2[CW_2] + [CQ_1] + [CQ_2] + [CQ_3] + [CQ_4]$ if $[V^*] = [W_1] + 2[W_2]$ where the W_i are distinct lines with a common point Q , say, and if the Q_i are suitable points on W_2 , one which is equal to Q . (One must be equal to Q because C' is a pure surface.)

(6) $3[CW] + [CQ_1] + [CQ_2] + [CQ_3] + [CQ_4]$ if $[V^*] = 3[W]$ and the Q_i are points on W .

In view of the above list, it is clear that, if C is any orbit on T whose preimage in X^* is of codimension 1, then C is equal to A or B . Now, for $1 \leq i \leq 11$, let $p_i : I^{x11}xT \to I$ denote the projection onto the ith factor, and consider the scheme intersection,

$$T^*_m := (p_1^{-1}g_1ZCH)\ldots(p_{m-1}^{-1}g_{m-1}ZCH) . (p_m^{-1}g_{m+1}ZCL)\ldots(p_{11}^{-1}g_{12}ZCL) . X^* .$$

Then, since $g_1,\ldots,g_{12}$ are general, therefore T^*_m is a complete curve, which does not meet the preimage of the complement of U. Consequently, T^*_m projects onto T_m, and T_m is a complete curve. Thus, Lemma 2 holds.

Remark. Let T' be any variety such that there exists a homogeneous birational map $T' \to T$. For example, one obvious choice for T' would be Schubert's model: it is normal and parametrizes the complete cubics with 3 aspects --- a cycle of points, a cycle of tangent lines, and a cycle of osculating planes.

The closed preimages $Z'H$ and $Z'L$ of ZH and ZL contain no orbit on T' as their images in T contain no orbit. So, as $g_1,\ldots,g_{12}$ are general, the intersection,

$$T'_m := (g_1 Z'H)\ldots(g_{m-1}Z'H).(g_{m+1}Z'L)\ldots(g_{12}Z'L) ,$$

is a complete curve in T' that meets no orbit of codimension at least 2; also, the trace of T'_m on the open orbit is dense in T'_m. Clearly, the preimage U' of U maps isomorphically onto U, and the trace of T'_m on U' maps isomorphically onto T'. Hence, Lemma 2 is equivalent to the statement that T'_m meets no orbit on T' of codimension 1 other than the preimages of A and B.

Conceivably, the above statement could be checked directly on Schubert's model. (Schubert [1879], top of p. 178, gives the impression that it can be). However, to check it will require a good description of each of the codimension 1 orbits. Schubert described 11 of them, but possibly there are more; see Piene [1983], Sect. 4.

6. Proof of lemma 3

The proof is, in part, similar to the verification of the number, 92, of conics in $\mathbb{P}^3$ meeting 8 lines that is given in Harris [1980], p. 26, and in Fulton [1983], Ex. 14.7.12, p. 275, and Ex. 3.2.22, p. 63. It is also, in part, similar to the verification of the number, 12, of nodal cubics in a fixed plane meeting 8 points that is given in Sacchiero [1984].

Say $\mathbb{P}^3 :\approx \mathbb{P}(E)$, where E is a 3-dimensional vector space. Then $\mathbb{P}^{3*} = \mathbb{P}(E^*)$. Denote the tautological sequence by

$$0 \to R \to E^*_{\mathbb{P}(E^*)} \to Q \to 0 ,$$

where $rk(R) = 3$, $rk(Q) = 1$. Then $\mathbb{P}(R^*)$ is the total space of the universal family of planes of $\mathbb{P}(E)$. So $\mathbb{P}(\mathrm{Sym}^3 R)$ parametrizes the planar cubics in $\mathbb{P}(E)$.

Let K denote the 2-dimensional subspace of E such that

$$L = \mathbb{P}(E/K) .$$

Consider the following composition of natural maps:

$$c : K_{\mathbb{P}(E^*)} \to V_{\mathbb{P}(E^*)} \to R^* .$$

There exists a maximal open subset W of $\mathbb{P}(E^*)$ on which c is injective and $cok(c)$ is invertible. In geometric terms, W is the subset of planes not containing L . So the complement of W is of dimension 1. On W , moreover,

$$cok(c) = \det(R^*) \otimes \det(K)^{-1} = Q$$

because K and E are vector spaces.

The canonical map, $R^* \to cok(c)$, defines a section over W of $\mathbb{P}(R^*)$; the section assigns to a hyperplane its point of intersection with L . Hence, on the preimage of W in $\mathbb{P}(\mathrm{Sym}^3 R)$, the composition

$$\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^3 R)}(-1) \to \mathrm{Sym}^3 R^*_{\mathbb{P}(\mathrm{Sym}^3 R)} \to cok(c)^{\otimes 3}$$

vanishes precisely at the points representing cubics meeting L . The zero scheme of the composition is obviously a reduced divisor (it is a hyperplane in each fibre of $\mathbb{P}(\mathrm{Sym}^3 R)$ over W). Now, the complement in $\mathbb{P}(\mathrm{Sym}^3 R)$ of the preimage of W is of codimension 2, because the preimage of W is. Hence, if the closure of this zero scheme is denoted by DL, then

$$[DL] = c_1(Q^{\otimes 3} \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^3 R)}(1)) .$$

Note that DL is the divisor of all planar cubics meeting L.

Clearly DL contains no orbit on $\mathbb{P}(\mathrm{Sym}^3 R^*)$. So, transversality theory applies. Let A' denote the orbit of nodal cubics. Then, therefore, the intersection of the eleven translates $g_2 DL, \ldots, g_{12} DL$ and the closure of A' is a finite set of points entirely contained in A'.

Consider the third Veronese embedding,

$$\mathbb{P}(R^*) \to \mathbb{P}(\mathrm{Sym}^3 R^*) ,$$

form its normal sheaf, N say, and its conormal variety,

$$C\mathbb{P}(R^*) = \mathbb{P}(N \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^3 R)}(-1)) = \mathbb{P}(N \otimes \mathcal{O}_{\mathbb{P}(R^*)}(-3)) .$$

Then the natural map,

$$f : C\mathbb{P}(R^*) \to \mathbb{P}(\mathrm{Sym}^3 R) ,$$

is birational onto its image. In fact, it is an isomorphism over A'.

There is a canonical isomorphism between A' and the orbit A on U. Indeed, A' parametrizes a natural flat family of nodal cubics, and it is possible to add an embedded point with 3-dimensional Zariski tangent space at each node so that the resulting family is still flat. Therefore, there is a natural map, $A' \to U$, and it obviously factors through A. (The induced map, $A' \to A$, is obviously bijective; hence, in characteristic $p = 0$, it is an isomorphism, alternatively because both source and target are smooth). Now, A parametrizes a flat family of nodal cubics with an embedded point at the node. Reducing the total space yields, by homogeneity, a flat family of nodal cubics. So, there exists a natural map $A \to \mathbb{P}(\mathrm{Sym}^3 R)$, and it obviously factors through A' .

The isomorphism from A' to A identifies the pullback cycle $[DL]\,.\,A'$ as $[ZL].A$. Indeed, it is evident from the definition of DL that $[DL]$ may be described by a formula like (4.7); whence, the assertion.

Thus, to prove Lemma 3, it remains to compute this intersection number:

$$\int_{\mathbb{P}(\mathrm{Sym}^3 R)} f_*[C\mathbb{P}(R^*)].c_1(Q^{\otimes 3} \otimes \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^3 R)}(1))^{11} .$$

It is not hard to compute it using standard methods - the projection formula, the theory of Segre classes, and standard exact sequences - to lift the intersection up to $C\mathbb{P}(R^*)$, then push it down to $\mathbb{P}(R^*)$, then down to $\mathbb{P}^{3*}$, and finally to work it out. Thus Lemma 3 is proved.

7. Proof of lemma 4

The proof follows the lines indicated by Schubert [1879], pp. 99-100, and it is similar to the verification of the characteristic numbers of the cubics of type σ in Sect. 6 of Kleiman-Speiser [1984].

Denote by B the (smooth) variety of complete conics in $\mathbb{P}^3$, and by Π the graph of the point-line incidence correspondence. Denote by F_o the subvariety of $B \times \Pi$ of elements $(X,(P,M))$ such that X is a nondegenerate conic, M is a unisecant line, and P is the common point. Denote by F the closure of F_o . On F , consider the following divisors:

n := the pullback from B of the divisor of complete conics cutting L :

r := the pullback from B of the divisor of complete conics touching H ;

N := the pullback from Π of the divisor of pairs (P,M) such that M cuts L ;

R := the pullback from Π of the divisor of pairs (P,M) such that P lies in H .

The proof has 2 mains steps. The first is to prove that F_o and B are canonically isomorphic and that the following identifications obtain:

$$[ZL].B = ([n]+[N]).F_o \qquad \text{and} \qquad [ZH].B = ([r]+2[R]).F_o \ .$$

The idea is to proceed basically as in the corresponding part of the proof of Lemma 3. Of course, when treating $[ZH].B$, it is necessary to consider the family of the dual surfaces of the cubics and to use (3) of Sect. 5.

The second step is to determine the following intersection numbers:

$$\int_F [n]^i [r]^j [N]^k [R]^l \qquad \text{for} \quad i+j+k+l = 11 \ .$$

The idea is to use the knowledge of the intersection rings of Π and of B (for that of B , see Casas-Xambó [1985]). The determination is reduced via the projection formula to finding the direct images on B of the $N^k R^l$ and to doing dome arithmetic. An example is worked out on p. 100 in Schubert [1879].

With these two steps carried out, the proof of Lemma 4 is easily completed with another application of transversality theory and a little more arithmetic. Thus, the verification of Schubert's number is accomplished.

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