UB-UAB-UPC

SEMINARI DE GEOMETRIA ALGEBRAICA

The discrete charm of algebraic geometry (after JUNE HUH)

S. Xambó

IMTech & BSC

17/2/2023

Index

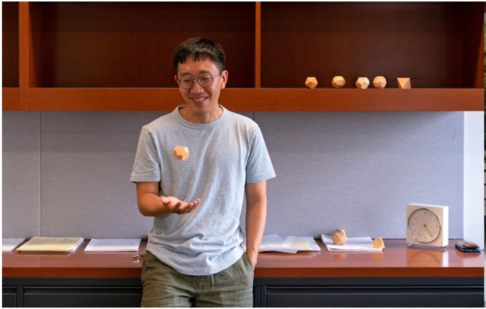
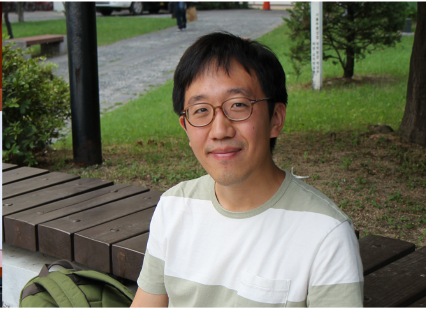
- Prelude
- Manifolds (background)
  - Riemann manifolds
  - Kähler manifolds
- Combinatorics
- Lorentzian polynomials
- The Kähler package
- Applications
- Outlook

Prelude

Thanks to Anna de Mier for insights about combinatorial matters, particularly on graphs and matroids, while writing the joint report "Combinatorics and Hodge theory, after June Huh" NL04 (23-25).

SGA talk 5/5/2017: *Error correcting codes: maths and computations*, that used (page 38) *A bootstrap for the number of $\mathbb{F}_{q^m}$-rational points on a curve over $\mathbb{F}_q$* (S. Molina, N. Sayols, and SX, arXiv:1704.04661, 2017).

Complex algebraic geometry
$\Rightarrow$ Abstract algebraic geometry
$\Rightarrow$ Weil conjectures
$\Rightarrow$ Grothendieck's standard conjectures
$\Rightarrow$ Huh's Kähler package + applications.

Pictures: Caroline Gutman for Quanta Magazine

Nomination (5 July 2022): For bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron-Rota-Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture.

Fields Medal Lecture (6 July 2022): Combinatorics and Hodge Theory, [1]. Paper in Proceedings: [2]. Laudatio: [3].

- Birth: Stanford, 1983. Grew up in South Korea.
- Master's degree: Seoul National University 2002-09 (mentored by HEISUKE HIRONAKA)
- PhD: University of Michigan 2014 (MIRCEA MUSTAȚĂ)
- Institute of Advanced Study, Stanford University, Princeton University

From JORDANA CEPELEWICZ article in the Quantamagazine of July 5th, 2022: "his ability to wander through mathematical landscapes and find just the right objects ... that he then uses *to get the seemingly disparate fields of geometry and combinatorics to talk to each other in new and exciting ways*. Starting in graduate school, *he has solved several major problems in combinatorics, forging a circuitous route by way of other branches of math* to get to the heart of each proof."

Manifolds

[4] Weil58, [5] Hirz66, [6] GH78, [7] Wells80
[8] Warner83, [9] Voisin02, [10] Huy05
[11] Voisin10, [12] Cat10, [13] Lee13

- Manifolds X are assumed to be *compact* and *connected*. $n = \dim(X)$.
- $A^*(X) = \bigoplus_{k=0}^n A^k(X)$: graded algebra of $\mathcal{C}^\infty$ forms.
- $C^*(X) = \bigoplus_{k=0}^n C^k(X)$: graded subalgebra of closed forms.
- $E^*(X) = \bigoplus_{k=0}^n E^k(X)$: graded C^*-ideal of exact forms.
- $H^*_{\rm dR}(X) = C^*(X)/E^*(X)$.
- $H_*(X) = \bigoplus_{k=0}^n H_k(X)$ and $H^*(X) = \bigoplus_{k=0}^n H^k(X)$.
- $H_k(X) \times H^k_{\rm dR}(X) \to \mathbf{R}$, $([z],[\varphi]) \mapsto \int_z \varphi$.
- $H^k_{\rm dR}(X) \simeq H_k(X)^* \simeq H^k(X)$. $\left(H^*_{\rm dR}(X), \wedge\right) \simeq \left(H^*(X), \smile\right)$.
- Betti numbers: $\mathrm{b}_k(X) = \dim H^k(X)$.
- $\chi(X) = \sum_k (-1)^k \mathrm{b}_k(X)$.
- $H^*_{\rm dR}(X, \mathbf{C})$, $H_*(X, \mathbf{C})$, $H^*(X, \mathbf{C})$.

Poincaré duality. If X is an oriented n-manifold, the Poincaré map $P: H_k(X) \to H_{n-k}(X)^* = H^{n-k}(X)$, $(P\alpha)(\beta) = \alpha \cdot \beta$ (intersection product) is an isomorphism.

Via the isomorphism $H^{n-k}_{\text{dR}}(X) \simeq H^{n-k}(X)$, we see that given a cycle $z \in Z_k(X)$ there exists a closed $(n-k)$-form φ such that $[z] \cdot [z'] = \int_{z'} \varphi$ for any $(n-k)$-cycle z'. Abusing notation, let φ_z denote any φ satisfying that integral relation.

Cohomology class of z: $\text{cl}(z) = [\varphi_z] \in H^{n-k}$.

In terms of the de Rham cohomology, the pairing $C^k(X) \times C^{n-k}(X) \to C^n(X)$, $(\alpha, \alpha') \mapsto \alpha \wedge \alpha'$, induces a pairing $H^k_{\text{dR}}(X) \times H^{n-k}_{\text{dR}}(X) \to H^n_{\text{dR}}(X) \simeq \mathbf{R}$ which is a duality.

Theorem. If $z \in Z_k(X)$ and $z' \in Z_{n-k}(X)$, then $[z] \cdot [z'] = \int_X \varphi_z \wedge \varphi_{z'}$, or $\varphi_z \wedge \varphi_{z'} = \varphi_{z \cdot z'}$, or $\text{cl}(z \cdot z') = \text{cl}(z) \wedge \text{cl}(z')$.

Let (X,g) be an oriented riemannian manifold. Then g can be extended to a symmetric bilinear map $g : A^k(X)\times A^k(X)\to A^0(X)$ and $A^k(X)$ inherits the symmetric bilinear form $(\alpha,\beta)=\int_X g(\alpha,\beta)\mathfrak{v}$ ($\mathfrak{v}$ the volume form). From linear algebra we know that there is a unique linear isomorphism $*: A^k(X)\to A^{n-k}$ (the Hodge $*$-operator) such that $\alpha\wedge *\beta = g(\alpha,\beta)\mathfrak{v}$, hence $(\alpha,\beta)=\int_X \alpha\wedge *\beta$. It satisfies (1) $** = (-1)^{k(n-k)}$; and (2) $\alpha\wedge *\alpha = 0 \Leftrightarrow g(\alpha,\alpha)=0 \Leftrightarrow \alpha = 0$.

Then the Laplacian is defined by $\Delta=\Delta_d = d\delta+\delta d$, where $\delta : A^k(X)\to A^{k-1}(X)$ is the adjoint of $d: A^{k-1}(X)\to A^k(X)$. Set $\mathcal{H}^k_\Delta(X)=\{\alpha\in A^k(X)\,|\,\Delta(\alpha)=0\}$ (*harmonic* k-forms).

Theorem (Hodge). The natural map $\mathcal{H}^k_\Delta(X)\to H^k_{\mathrm{dR}}(X)$ is an isomorphism. Moreover, these isomorphisms provide a graded algebra isomorphim $\left(\mathcal{H}^*_\Delta(X),\wedge\right)\simeq\left(H^*_{\mathrm{dR}}(X),\wedge\right)$, hence also a graded algebra isomorphism $\left(\mathcal{H}^*_\Delta(X),\wedge\right)\simeq\left(H^*(X),\smile\right)$.

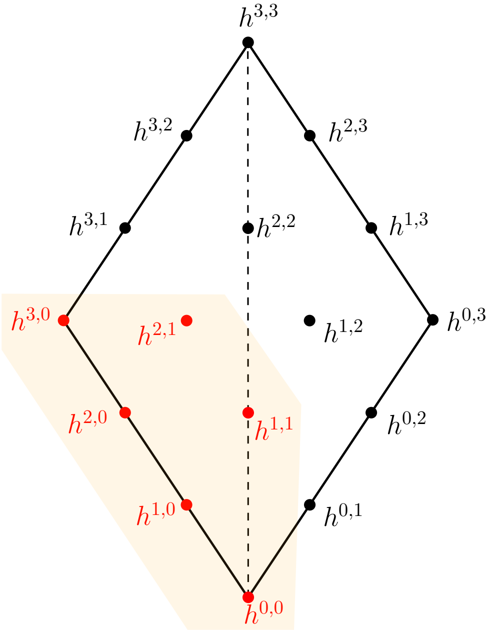
On a complex manifold X of (complex) dimension n, we have a decomposition $A^k(X,\mathbf{C}) = \bigoplus_{p+q=k} A^{p,q}(X,\mathbf{C})$. The forms in $A^{p,q}(X,\mathbf{C})$ are said to be of type (p,q).

A *Kähler manifold* is a complex manifold equipped with a Hermitian metric (*Kähler metric*) whose imaginary part ω, which is a 2-form of type (1, 1), is *closed*. This 2-form is called the *Kähler form* of the Kähler metric.

Submanifolds of a Kähler manifold are Kähler.

A Kähler manifold is in particular a riemannian manifold of dimension $2n$ and it turns out that the (p,q) components of an harmonic k-form are harmonic. This and the Hodge theorem imply a *Hodge decomposition* of cohomology: $H^k(X,\mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbf{C})$ $(k = 0, 1, \ldots, 2n)$. Thus $\left(H^*(X,\mathbf{C}), \wedge\right)$ is a bigraded algebra.

Note $\overline{H}^{p,q}(X) = H^{q,p}$. *Hodge numbers*: $h^{p,q} = \dim_{\mathbf{C}} H^{p,q}(X,\mathbf{C})$.



Hodge diamond

Betti numbers

$$\mathsf{b}_k = \sum_{p+q=k} h^{p,q}$$

Symmetry about vertical bisector

$H^{q,p} = \overline{H^{p,q}}$

$\Rightarrow \quad h^{q,p} = h^{p,q}$

$\Rightarrow$ odd betti numbers are even

Symmetry about center of diamond

$H^{n-p,n-q} = *H^{p,q}$

$\Rightarrow \quad h^{n-p,n-q} = h^{p,q}$

Symmetry about horizontal bisector

$$h^{0,0} = h^{n,n} = 1$$

The restriction to S^{2n+1} of the Fubini-Study hermitian metric $ds^2 = \sum_{j=0}^{n} dz_j \otimes d\bar{z}_j$ on $\mathbf{C}^{n+1}$ is invariant by the action of S^1 and hence it induces a hermitian metric on $S^{2n+1}/S^1 = \mathbf{P}^n(\mathbf{C})$. Setting $z_j = x_j + iy_j$, the imaginary part of ds^2 is $\omega = \sum_j dx_j \wedge dy_j$. This form has type $(1,1)$ and is closed. Therefore it induces a Kähler structure ω on $\mathbf{P}^n(\mathbf{C})$. The class $[\omega] \in H^{1,1}(X,\mathbf{C}) \subset H^2(X,\mathbf{C})$ coincides with the cohomology class $\mathrm{cl}(Y)$ of a hyperplane section Y of X.

Complex submanifolds of the complex projective space are Kähler, and they are projective subvarieties by Chow's theorem.

Kodaira's theorem. A compact complex manifold admits a holomorphic embedding into complex projective space [*and hence is a smooth algebraic variety*] if and only if it admits a Kähler metric whose Kähler form is a rational class (i.e, belongs to the image of $H^2(X,\mathbf{Q}) \to H^2(X,\mathbf{C})$).

Let $h=[\omega]\in H^2(X,\mathbf{C})$ (the cohomology class of the Kähler form).

$L: H^k(X,\mathbf{C})\to H^{k+2}(X,\mathbf{C})$, $\alpha\mapsto h\wedge\alpha$. In the Hodge diamond, L moves each node one vertical step up.

Hard Lefschetz Theorem

(1) L is injective, and hence $b_k\leqslant b_{k+2}$ and $h^{k-i,i}\leqslant h^{k-i+1,i+1}$, for $k<n$. By Poincaré duality, $b_{n-k}\leqslant b_{n-k-2}$ and $h^{n-i,n-k+i}\leqslant h^{n-i-1,n-k+i-1}$ for $n>k$.

These properties are dubbed *Hodge staircases*: the Hodge numbers on a vertical line of the Hodge diamond are non-decreasing in the bottom half and non-increasing in the top half; and the even or odd Betti numbers have the same property.

Note that $L: H^{n-1}(X,\mathbf{C})\to H^{n+1}(X,\mathbf{C})$ is a isomorphism, as it is injective and both spaces have the same dimension. This is a special case of next statement.

(2) $L^j : H^{n-j}(X) \to H^{n+j}(X)$ is an isomorphism for all $j \geqslant 0$.

If $H^{p,q}$ is a Hodge component of $H^{n-j}(X)$, so $p+q=n-j$, then L^j maps it isomorphically to $H^{p+j,q+j} = H^{n-q,n-p}$. We get again that the Hodge diamond is symmetric about the horizontal diagonal, which can also be accounted for by composing the symmetry about the center of the diamond (induced by $*$) and the symmetry about the vertical line (induced by the conjugation).

For $k \leqslant n$, the *primitive subspace* of $H^k(X)$ is defined as the kernel of $L^{n-k+1} : H^k(X) \to H^{2n-k+2}$, and is denoted by $H_0^k(X)$.

Lefschetz Decomposition Theorem (Let $q_k = \lfloor k/2 \rfloor = k /\!\!/ 2$)

$H^k(X,\mathbf{C}) = \bigoplus_{j \geqslant (k-n)^+} L^j H_0^{k-2j}(X)$. $H_0^k(X) = H^k(X)$, $k = 0,1$.

For $k \leqslant n$, $H^k = H_0^k \oplus LH_0^{k-2} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k} = H_0^k \oplus LH^{k-2}$.

For $k = n+k'$, $1 \leqslant k' \leqslant n$, $H^k = L^{k'} H_0^{k-2k'} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k}$.

Hodge-Riemann pairing

$Q : H^k(X, \mathbf{C}) \times H^k(X, \mathbf{C}) \to \mathbf{C}$,
$Q(\alpha, \alpha') = (-1)^{k/\!\!/2} \int_X \alpha \wedge \alpha' \wedge \omega^{n-k}$.

Theorem

The Hodge decomposition $H^k(X, \mathbf{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ satisfies:

(1) $Q(H^{p,q}, H^{p',q'}) = 0$ if $(p', q') \neq (q, p)$, and
(2) $i^{p-q} Q(\alpha, \bar{\alpha}) > 0$ for $0 \neq \alpha \in H_0^{p,q}(X)$.

Let X is a Kähler manifold and Z a submanifold of codimention k.

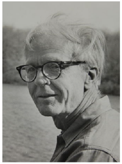
Then $\text{cl}(Z) \in H^{2k}(X, \mathbf{Q}) \cap H^{k,k}(X) = H^{k,k}(X, \mathbf{Q})$.

The same is true if $Z \in \mathcal{Z}^k_{\mathbf{Q}}$, the group of rational linear combinations of submanifolds of codimension k (rational cycles of codimension k).

The Hodge conjecture states that if X is a smooth projective variety (or a Kähler manifold of integral type), then $\text{cl} : \mathcal{Z}^k_{\mathbf{Q}} \to H^{k,k}(X, \mathbf{Q})$ is surjective.

Combinatorics

NON-SEPARABLE AND PLANAR GRAPHS[1]



By Hassler Whitney

Department of Mathematics, Harvard University

Communicated January 14, 1931

1. *Introduction.*—We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a combinatorial character; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper.

TAMS **34** (1932), 339-362

Hassler Whitney (March 23, 1907 – May 10, 1989) was an American mathematician. He was one of the founders of singularity theory, and did foundational work in manifolds, embeddings, immersions, characteristic classes, and geometric integration theory.

ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

AJM 1935 (509-533)

1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental

"As the word suggests, Whitney conceived a matroid as an abstract generalisation of a matrix, and much of the language of the theory is based on that of linear algebra. However, Whitney's approach was also motivated by his work in graph theory and as a result some of the matroid terminology has a distinct graphical flavour.

Apart from [several] isolated papers [up to 1949] ... the subject lay virtually dormant until me late fifties when Tutte (1958,1959), published his fundamental papers on matroids and graphs and Rado (1957) studied the representability problem for matroids. Since then interest in matroids and their application in combinatorial theory has accelerated rapidly. Indeed it was realized that matroids have important applications in the field of *combinatorial optimization* and also that they *unify and simplify* apparently diverse areas of pure combinatorics." D. J. A. Welsh, *Matroids: Fundamental concepts*, 481-526 in Handbook of Combinatorics, Volume I, Elsevier-MIT Press, 1995. For a comprehensive presentation, [14] Ox11, or the earlier (1976) and slimmer *Matroid theory* by Welsh (republished by Dover in 2010).

A number within double square brackets in the text, say [[36]], refers to the reference item [36] in Huh's ICM paper ([2] in our reference list). Such labels are linked to an online file, whenever possible, and the correspondence with our reference list is indicated in red if it is included in that list ([15] in the case of [[36]]).

In what follows, we first state the *main combinatorial results* and summarize the theory of *Lorentzian polynomials*. Then we focus on the *Kähler package*. Finally we look into *examples* of how this machinery works for solving conjectures in combinatorics that hitherto had been unreachable by other means.

As we will see, the Hodge section is connected to the work of a number of Fields medalists: Michael Atiyah [[7]], Pierre Deligne [[10]], Alexander Grothendieck [[36]] [15], Jean-Pierre Serre [[67]] and Shing-Tung Yau [[74]].

Let $a_0, \ldots, a_m$ be a sequence of non-negative real numbers. It is

- *Unimodal*: if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$ for some $j \in 0..m$. The sequences of Betti numbers $\mathrm{b}_0, \mathrm{b}_2, \ldots, \mathrm{b}_{2n}$ and $\mathrm{b}_1, \mathrm{b}_3, \ldots, \mathrm{b}_{2n-1}$ of a Kähler manifold and unimodal and *symmetric*.
- *Log-concave*: if $a_j^2 \geqslant a_{j-1}a_{j+1}$ for all $j \in 1..(m-1)$. A log-concave sequence of *positive* terms is unimodal. The symmetric sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is log-concave, hence also unimodal.
- *Ultra-log-concave*: If $a_j/\binom{m}{j}$, $j \in 0..m$, is log-concave.
- *Top-heavy*: if $a_j \leqslant a_{m-j}$ for $j \in 0..(m/\!\!/2)$.

For the ubiquity of these notions in algebra, combinatorics and geometry, see the surveys [16] Stan89 and [17] Bren16. For specific occurrencies in the theory of projective hypersurface singularities, see [18] Huh12.

Theorem (I. Newton). Let $\sum_{j=0}^{n} b_j x^j = \sum_{j=0}^{n} \binom{n}{j} a_j x^j$ be a real polynomial with *real roots*. Then $b_0, b_1, \ldots, b_n$ is ultra-log-concave ($\Leftrightarrow a_0, a_1, \ldots, a_n$ is log-concave). Moreover, if $b_j \geqslant 0$, then $b_0, b_1, \ldots, b_n$ has no internal zeros. [19] Stan13 (Theorem 5.12).

Intersection cohomology staircases. If X is an irreducible complex projective variety of dimension n, Goresky and MacPherson [20, 21] introduced the *intersecction cohomology* of X,

$$IH^*(X) = IH^0(X) \oplus IH^1(X) \oplus \cdots \oplus IH^{2n}(X).$$

Let $\beta_j = \dim IH^j(X)$ ('Betti' numbers). Then the sequences $\beta_0, \beta_2, \ldots, \beta_{2n}$ and $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ are symmetric and unimodal. For an detailed overview of its development of IH, see [22] Klei07.

Given a graph $G = (V, E)$, and a positive integer q, a *proper coloring* of G with q colors is a map $c : V \to [q]$ such that $c(a) \neq c(b)$ when $ab \in E$.

The number of proper colorings of G with q colors turns out to be a polynomial in q (the *chromatic polynomial* of G) of the form

$$\chi_G(q) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^{n-1} a_1 q,$$

where $n = |V|$ and $a_j \geqslant 0$ for $j = 1, \ldots, n$.

The Read-Hoggar conjecture (1968, 1974) says that $a_1, \ldots, a_n$ is *log-concave*.

It was proved by HUH in 2009 in his PhD research. The sequence is also *unimodal* (this was Read's conjecture).

This turns out to be a special case of the conjecture considered next.

A *matroid* is a pair $M = (E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a family of subsets of E (called *independent sets*) that satisfy:
(**i0**) the empty subset is independent;
(**i1**) any subset of an independent set is independent; and
(**i2**) if X, X' are independent and $|X| > |X'|$, then there exists $x \in X - X'$ such that $X' \cup \{x\}$ is independent.

Thus a matroid is an abstraction of the notion of linearly independent sets of a finite set of vectors in a K-vector space (such matroids are said to be *representable* over the field K).

It is also important to note that *a graph gives rise to a matroid by declaring a subset of edges independent if it contains no cycles*.

For a matroid $M = (E, \mathcal{I})$, the *rank* $r(X)$ of a subset X of E is defined by $r(X) = \max\{|X'| : X' \subseteq X, X' \in \mathcal{I}\}$.

The *characteristic polynomial* of M, $\chi_M(q)$, is defined as

$$\chi_M(q) = \sum_{X\subseteq E} (-1)^{|X|} q^{r(E)-r(X)} = \sum_{j=0}^{r(E)} (-1)^j w_j q^{d-j},$$

where the coefficients w_j are called *Whitney numbers* (of the first kind).

The characteristic polynomial *generalizes the notion of chromatic polynomial of a graph* (see [14, p. 588]).

The Heron-Rota-Welsh conjecture asserts that $w_0, w_1, \ldots, w_{r(E)}$ is log-concave.

It was proved in [[1]] [23].

Let $\mathcal{L}$ be a finite lattice, $r : \mathcal{L} \to \mathbb{N}$ its *rank* function, $\mathcal{L}^k = \{x \in \mathcal{L} : r(x) = k\}$, and $d = \operatorname{rank}(\mathcal{L})$ (the rank of its maximum element). $\mathcal{L}$ is said to be *geometric* if it is generated by $\mathcal{L}^1$ (the *atoms* of $\mathcal{L}$) and r satisfies the *submodular* property, namely $r(x) + r(x') \geqslant r(x \vee x') + r(x \wedge x')$ for all $x, x' \in \mathcal{L}$.

The Dowling-Wilson *top-heavy* conjecture (1974) asserts that

$$|\mathcal{L}^k| \leqslant |\mathcal{L}^{d-k}| \text{ for all } k \leqslant d/2. \qquad (*)$$

Actually the conjecture was phrased for the lattice $\mathcal{L}(M)$ of flats of a matroid $M = (E, \mathcal{I})$ (a *flat* is a subset of E that is maximal for its rank) and it was proved in [[41]] [24] (see also [[12]] [25] for further enhancements). But this is not a more general statement than Eq. $(*)$, as the class of geometric lattices agrees with the class of lattices of flats of matroids.

Let $i_k = i_k(M)$ be the number of independent sets of cardinal k in a finite matroid $M = (E, \mathcal{I})$.

Mason's ultra-strong conjecture says that the i_k form an ultra log-concave sequence, i.e.

$$i_k^2 \geqslant (1 + \frac{1}{k})(1 + \frac{1}{n-k}) i_{k-1} i_{k+1}, \; n = |E|.$$

This conjecture was proved in [[17]] [26].

As explained in the footnote 2 of [2] Huh22-ICM, it was independently proved in the series [[2]], [[3]], [[4]] [27].

Lorentzian polynomials

Let H_n^d be the space of real homogeneous polynomials of degree d in n variables.

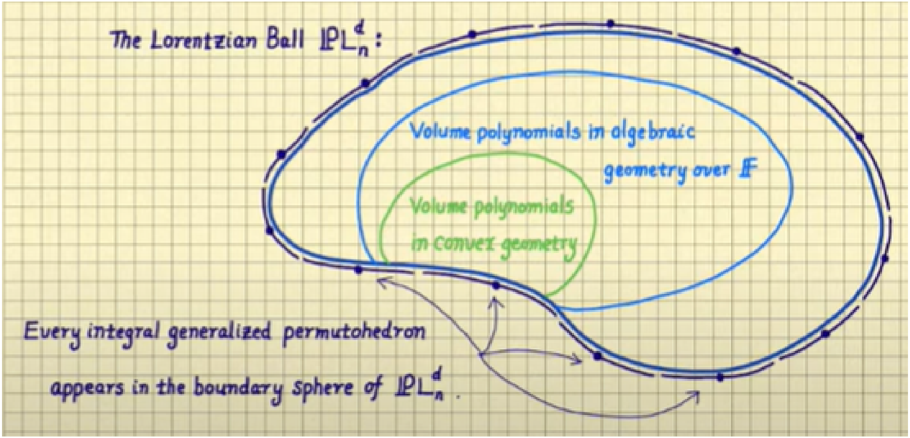
The set of *Lorentzian polynomials* L_n^d is defined as follows.

The elements of L_n^2 are specified by two conditions:
(a_2) their coefficients are non-negative, and
(b_2) their signature has at most one positive sign.

For degrees $d > 2$ the set L_n^d is defined recursively by the following conditions:
(a_d) $\partial_j f \in L_n^{d-1}$ for all $j \in [n]$, and
(b_d) the set of (exponents of) monomials of f is the set of lattice points of an *integral generalized permutohedron* (that is, a polytope whose edges' directions have the form $e_j - e_k$, with $e_1, \ldots, e_n$ the standard basis of $\mathbf{R}^n$; for a reference on these objects, see [28] Doker11).

One of the crucial results in [[17]] [26] is that L_n^d is the closure of $\mathring{L}_n^d$, a set defined by the conditions:
($\mathring{a}_2$) their coefficients are *positive* real numbers,
($\mathring{b}_2$) their signature has *exactly* one positive sign, and, for $d > 2$,
($\mathring{a}_d$) $\partial_j f \in \mathring{L}_n^{d-1}$ for all $j \in [n]$.

Theorem 2.28 of the same paper proves that the compact set $\mathbb{P}L_n^d \subset \mathbb{P}H_n^d$ is contractible, with contractible interior $\mathbb{P}\mathring{L}_n^d$, and conjectured that it is homeomorphic to a closed Euclidean ball (proved by Brändén [[16]] [29]).



Detail of slide number 13 of HUH's lecture at the ICM-22. Note the statement on the boundary sphere.

Example. If $C = C_1, \ldots, C_n$ are convex bodies in $\mathbf{R}^d$, $\mathrm{vol}_C : \mathbf{R}^n_{\geqslant 0} \to \mathbf{R}$, $w \mapsto \frac{1}{d!}\mathrm{vol}(w_1 C_1 + \cdot + w_n C_n)$ is a Lorentzian polynomial [2, Example 6]

Example. Let $D = D_1, \ldots, D_n$ be nef Cartier divisors on d-dimensional irreducible projective variety X over an algebraically closed field. Consider the polynomial function

$$\mathrm{vol}_D : \mathbf{R}^n_{\geqslant 0} \to \mathbf{R}, \quad w \mapsto \frac{1}{d!}\deg(w_1 D_1 + \cdot + w_n D_n)^d.$$

If X admits a resolution of singularties Y and the Hodge-Riemann relations hold in degree $\leqslant 1$ for the ring of algebraic cycles $A(Y)$, then $\mathrm{vol}_D(w)$ is Lorentzian [2, Example 7]

The Kähler package

As presented by HUH, the scheme has three ingredients and three postulates (dubbed the *Kähler package* by HUH, for KÄHLER "first emphasized the importance of the respective objects in topology and geometry").

Ingredients

(1) A graded real vector space $A = \bigoplus_{j=0}^{d} A^j$;
(2) A convex cone K of graded linear maps $L : A^\star \to A^{\star+1}$; and
(3) A symmetric bilinear pairing $P : A^\star \times A^{d-\star} \to \mathbf{R}$.

Postulates

For any $j \leqslant d/2$,

- *Poincaré Duality*: $P : A^j \to (A^{d-j})^*$ is an isomorphism;
- *Hard Lefschetz Propery* For any $L \in K$, $L^{d-2j} : A^j \to A^{d-j}$ is an isomorphism;
- *Hodge-Riemann Relations*: The pairing

$$A^j \times A^j \to \mathbf{R}, \quad (x, y) \mapsto (-1)^j P(x, L^{d-2j} y),$$

is positive definite on the kernel of L^{d-2j+1}
(*primitive part* of A^j, to borrow the name from Lefschetz theory).

In the examples known so far, $A = A(X)$ depends on the objects X of some species (functorially).

- X a smooth projective variety, $A(X)$ a cohomology ring (ℓ-adic, for instance). The package agrees essentially with GROTHENDIECK's standard conjectures.

- X is a convex polytope and $A(X)$ its combinatorial cohomology [[45]] [30].

- X a matroid and $A(X)$ can be its:
(i) Chow ring [[1]] [23];
(ii) Conormal Chow ring [[6]] [31]; or
(iii) Intersection cohomology [[12]], [25].

- X is an element of a Coxeter group and $A(X)$ its Soergel bimodule [[26]], [32]. Other references: [33], [34].

The general strategy was summarized in slide number 14 of [1], while pointing out [[40]] [35] and [[27]] [36] for examples and conjectures for various X:

(1) Given X, search for interesting multivariate generating functions from it;

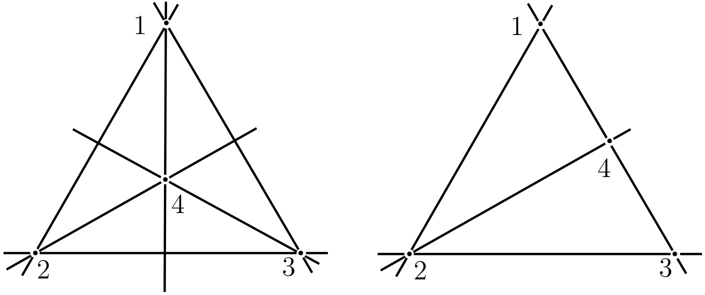
(2) Do we see any generalized permutohedra?

(3) Do we see any Lorentzian polynomials?

(4) Can we guess $A(X)$, $K(X)$, $P(X)$?

Let us end by describing how the Dowling–Wilson conjecture was solved, after [1] Huh22-lecture.

Given a geometric lattice $\mathcal{L}$ of rank d, consider the set $\mathbb{B}$ of its *bases*, that is, subsets of size d of $E = \mathcal{L}^1$ (the set of atoms) whose join has rank d. Then $\mathbb{B}$ is the set of *lattice points of an integral generalized permutohedron*, and the basis generating function $g = \sum_{\nu \in \mathbb{B}} w^\nu$ is a Lorentzian polynomial.



$g = w_1w_2w_3 + w_1w_2w_4 + w_1w_3w_4 + w_2w_3w_4$ $\qquad$ $g = w_1w_2w_3 + w_1w_2w_4 + w_2w_3w_4$

Now define $\mathbf{H}(\mathcal{L}) = \{f : \mathcal{L} \to \mathbb{Q}\} = \bigoplus_{F\in\mathcal{L}} \mathbb{Q}\delta_F$ and make it a graded $\mathbb{Q}$-algebra (the *Möbius algebra* of $\mathcal{L}$) with the multiplication determined by

$$\delta_F \cdot \delta_{F'} = \begin{cases} \delta_{F\vee F'} & \text{if } r(F \vee F') = r(F) + r(F') \\ 0 & \text{otherwise.} \end{cases}$$

The *basis generating function* of $\mathcal{L}$ is $\frac{1}{d!}(\sum_{j\in E} w_j\delta_j)^d$. This suggests taking $A(\mathcal{L}) = \mathbf{H}(\mathcal{L})$; $K(\mathcal{L})$, the set of multiplications by positive linear combinations of the δ_j; and $P(\mathcal{L})$, multiplication in $\mathbf{H}(\mathcal{L})$ composed with $\mathbf{H}^d(\mathcal{L}) \simeq \mathbb{Q}$. But $\mathbf{H}(\mathcal{L})$ already fails to satisfy Poincaré duality, for $\dim \mathbf{H}^j(\mathcal{L}) = |\mathcal{L}^j|$ and in general $|\mathcal{L}^j| \neq |\mathcal{L}^{d-j}|$.

As shown in [[12]] [25], the rescue from this failure came from the *intersection cohomology* of $\mathcal{L}$, $\mathbf{IH}(\mathcal{L})$, which is an indecomposable graded $\mathbf{H}(\mathcal{L})$-module endowed with a map $P : \mathbf{IH}(\mathcal{L}) \to \mathbf{IH}(\mathcal{L})^*[-d]$ that satisfies the following properties for every $j \leqslant d/2$ and every $L \in K(\mathcal{L})$:

Poincaré duality $P : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})^*$ is an isomorphism;
Hard Lefschetz $L^{d-2j} : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})$ is an isomorphism; and
Hodge-Riemann relations: The pairing $\mathbf{IH}^j(\mathcal{L}) \times \mathbf{IH}^j(\mathcal{L}) \to \mathbb{Q}$, $(x, y) \mapsto (-1)^j P(x, L^{d-2j} y)$ is positive definite on the kernel of L^{d-2j+1}. In addition, $\mathbf{IH}^0(\mathcal{L})$ generates a submodule isomorphic to $\mathbf{H}(\mathcal{L})$.

The construction relies on the resolution of singularities of algebraic varieties, and in particular on [37] CP95 'wonderful models' (see [38] CP10, a wonderful book).

Since the composition of $\mathbf{H}^j(\mathcal{L}) \hookrightarrow \mathbf{IH}^j(\mathcal{L})$ with the Hard-Lefschetz isomorphism $\mathbf{IH}^j(\mathcal{L}) \simeq \mathbf{IH}^{d-j}(\mathcal{L})$ is injective, it follows that $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ composed with $\mathbf{H}^{d-j} \to \mathbf{IH}^{d-j}(\mathcal{L})$ is injective (see diagram below) and consequently $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ is injective, which proves that $|\mathcal{L}^j| \leqslant |\mathcal{L}^{d-j}|$.

$$\begin{array}{ccc} \mathbf{H}^j(\mathcal{L}) & \hookrightarrow & \mathbf{IH}^j(\mathcal{L}) \\ L^{n-2j} \downarrow & & \downarrow L^{n-2j} \\ \mathbf{H}^{d-j}(\mathcal{L}) & \to & \mathbf{IH}^{d-j}(\mathcal{L}) \end{array}$$

Outlook

- July 1, 2022 to June 30, 2023, The Fields Institute

  Matroids - Combinatorics, Algebra and Geometry Seminar

  http://www.fields.utoronto.ca/activities/22-23/matroids-seminar

"Matroids are abstractions of (in)dependence structures in mathematics. There were several open conjectures concerning sequences of combinatorial invariants of matroids. Recently, June Huh along with his collaborators resolved these conjectures ... This spurred a lot of activity in the area. In this seminar series, we will exhibit these developments. We aim at mainstreaming the algebraic geometry of matroids into a mathematical research landscape."

- Connections with mirror symmetry? ([39] (cox-katz-1999))
- Connections with Enumerative geometry? ([40] (katz-2006), [41] (okounkov-2018))
- Standard conjectures!!

References I

[1] J. Huh, "Combinatorics and Hodge theory," 2022.
ICM Fields Lecture, 6 July 2022 Video (2:34–3:37).

[2] J. Huh, "Combinatorics and Hodge theory," in *Proceedings of the ICM 2022*, IMU, 2022.
pdf.

[3] G. Kalai, "The work of June Huh," in *Proceedings of the ICM 2022*, IMU, 2022.
pdf.

[4] A. Weil, *Variétés Kähleriennes*.
Hermann, Paris, 1958.

References II

[5] F. Hirzebruch, A. Borel, and R. Schwarzenberger, *Topological methods in algebraic geometry*, vol. 131 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*.

Springer Verlag New York, 1966.

[6] P. Griffiths and J. Harris, *Principles of algebraic geometry*.

John Wiley & Sons, 1978.

[7] R. O. Wells, *Differential analysis on complex manifolds (2nd edition)*, vol. 65 of *Graduate Texts in Mathematics*.

Springer, 1980.

The first edition was published by Prentice-Hall in 1972.

References III

[8] F. W. Warner, *Foundations of differentiable manifolds and Lie groups (2nd edition)*, vol. 94 of *Graduate Texts in Mathematics*.

Springer, 1983.

The first edition was published by Scott, Foresman and Co. in 1971.

[9] C. Voisin, *Hodge Theory and Complex Algebraic Geometry I*, vol. 76 of *Cambridge Studies in Advanced Mathematics*.

Cambridge University Press, 2002.

Translated by Leila Schneps.

[10] D. Huybrechts, *Complex geometry: an introduction*.

Universitext, Springer, 2005.

References IV

[11] C. Voisin, "Algebraic geometry versus Kähler geometry," *Milan Journal of Mathematics*, vol. 78, pp. 85–116, 2010.

[12] E. Cattani, "Introduction to Kähler manifolds," 2010.

The Abdus Salam International Centre for Theoretical Physics, Summer School and Conference on Hodge Theory and Related Topics, 14 June - 2 July, 2010.

[13] J. M. Lee, *Smooth manifolds (2nd edition)*, vol. 218 of *Graduate Texts in Mathematics*.

Springer, 2013.

References V

[14] J. G. Oxley, *Matroid theory (2nd edition)*, vol. 21 of *Oxford Graduate Texts in Mathematics*.

Oxford University Press, 2011.

The 1st edition was published in 1992.

[15] A. Grothendieck, "Standard conjectures on algebraic cycles," 1969.

In [42], 193-199; in the pdf, 203-211.

[16] R. P. Stanley, "Log-concave and unimodal sequences in algebra, combinatorics, and geometry," *Ann. New York Acad. Sci*, vol. 576, no. 1, pp. 500–535, 1989.

[17] F. Brenti *et al.*, "Stanley's work on unimodality," *The Mathematical Legacy of Richard P. Stanley*, pp. 119–130, 2016.

References VI

[18] J. Huh, "Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs," *Journal of the American Mathematical Society*, vol. 25, no. 3, pp. 907–927, 2012.

[19] R. P. Stanley, *Algebraic combinatorics–walks, trees, tableaux, and more*.

Undergraduate Texts in Mathematics, Springer, 2013.

[20] M. Goresky and R. MacPherson, "Intersection homology theory," *Topology*, vol. 19, no. 2, pp. 135–162, 1980.

[21] M. Goresky and R. MacPherson, "Intersection homology ii," *Inventiones Mathematicae*, vol. 71, pp. 77–129, 1983.

References VII

[22] S. L. Kleiman, "The development of intersection homology theory," *Pure Appl. Math. Q*, vol. 3, no. 1, pp. 225–282, 2007.

[23] K. Adiprasito, J. Huh, and E. Katz, "Hodge theory for combinatorial geometries," *Annals of Mathematics*, vol. 188, no. 2, pp. 381–452, 2018.

[24] J. Huh and B. Wang, "Enumeration of points, lines, planes, etc.," *Acta Mathematica*, vol. 218, 2017.

[25] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang, "Singular hodge theory for combinatorial geometries," 2020. https://arxiv.org/pdf/2010.06088.pdf.

References VIII

[26] P. Brändén and J. Huh, "Lorentzian polynomials," *Annals of Mathematics*, vol. 192, no. 3, pp. 821–891, 2020.

[27] N. Anari, K. Liu, S. O. Gharan, and C. Vinzant, "Log-concave polynomials III: Mason's ultra-log-concavity conjecture for independent sets of matroids," 2018.

https://arxiv.org/pdf/1811.01600.pdf.

[28] J. S. Doker, *Geometry of generalized permutohedra*.

PhD thesis, UC Berkeley, 2011.

pdf.

References IX

[29] P. Brändén, "Spaces of Lorentzian and real stable polynomials are Euclidean balls," in *Forum of Mathematics, Sigma*, vol. 9, p. e73, Cambridge University Press, 2021.

[30] K. Karu, "Hard Lefschetz theorem for nonrational polytopes," *Inventiones Mathematicae*, vol. 157, no. 2, 2004.

`https://arxiv.org/pdf/math/0112087.pdf`.

[31] F. Ardila, G. Denham, and J. Huh, "Lagrangian geometry of matroids," *Journal of the American Mathematical Society*, 2022.

[32] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, "The Hodge theory of Soergel bimodules," *Introduction to Soergel Bimodules*, pp. 347–367, 2014.

References X

[33] N. Libedinsky, "Gentle introduction to soergel bimodules i: the basics," *Sao Paulo Journal of Mathematical Sciences*, vol. 13, pp. 499–538, 2019.

[34] B. Elias, S. Makisumi, U. Thiel, and G. Williamson, *Introduction to Soergel bimodules*, vol. 5.

Springer Nature, 2020.

[35] J. Huh, J. Matherne, K. Mészáros, and A. St Dizier, "Logarithmic concavity of Schur and related polynomials," *Transactions of the American Mathematical Society*, vol. 375, no. 06, pp. 4411–4427, 2022.

.

References XI

[36] C. Eur and J. Huh, "Logarithmic concavity for morphisms of matroids," *Advances in Mathematics*, vol. 367, p. 107094, 2020.

https://arxiv.org/pdf/1906.00481.pdf.

[37] C. De Concini and C. Procesi, "Wonderful models of subspace arrangements," *Selecta Mathematica*, vol. 1, pp. 459–494, 1995.

[38] C. De Concini and C. Procesi, *Topics in hyperplane arrangements, polytopes and box-splines*.

Universitext, Springer Science & Business Media, 2010.

[39] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, vol. 68 of *Mathematical Surveys and Monographs*.

American Mathematical Society, 1999.

References XII

[40] S. Katz, *Enumerative geometry and string theory*, vol. 32 of *Student Mathematical Library/ΩIAS/Park City Mathematical subseries*.

American Mathematical Society, 2006.

[41] A. Okounkov, "On the crossroads of enumerative geometry and geometric representation theory," in *Proceedings of the International Congress of Mathematicians: Rio de Janeiro 2018*, pp. 839–867, World Scientific, 2018.

[42] M. S. Narasimhan, ed., *Algebraic Geometry*, Tata Institute Studies in Mathematics, Oxford University Press & Tata Institute of Fundamental Research, 1969.