Algebraic and topological interplay of algebraic varieties

The discrete charms of Kähler Geometry
(a view of June Huh's Kähler package)

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Prelude

Pictures: Caroline Gutman for Quanta Magazine

Nomination (5 July 2022): For bringing the ideas of Hodge theory to combinatorics, the proof of the Dowling–Wilson conjecture for geometric lattices, the proof of the Heron-Rota-Welsh conjecture for matroids, the development of the theory of Lorentzian polynomials, and the proof of the strong Mason conjecture.

Fields Medal Lecture (6 July 2022): Combinatorics and Hodge Theory, [1]. Paper in Proceedings: [2]. Laudatio: [3].

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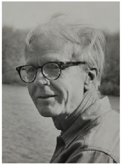
Combinatorics

Background ▪ Results of J. Huh and colleagues

NON-SEPARABLE AND PLANAR GRAPHS[1]



BY HASSLER WHITNEY

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated January 14, 1931

1. *Introduction.*—We shall give here an outline of the main results of a research on non-separable and planar graphs. The methods used are entirely of a combinatorial character; the concepts of rank and nullity play a fundamental rôle. The results will be given in detail in a later paper.

TAMS **34** (1932), 339-362

Hassler Whitney (March 23, 1907 – May 10, 1989)

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ON THE ABSTRACT PROPERTIES OF LINEAR DEPENDENCE.[1]

By Hassler Whitney.

AJM 1935 (509-533)

1. Introduction. Let $C_1, C_2, \cdots, C_n$ be the columns of a matrix $\boldsymbol{M}$. Any subset of these columns is either linearly independent or linearly dependent; the subsets thus fall into two classes. These classes are not arbitrary; for instance, the two following theorems must hold:

(a) Any subset of an independent set is independent.

(b) If $\boldsymbol{N}_p$ and $\boldsymbol{N}_{p+1}$ are independent sets of p and $p+1$ columns respectively, then $\boldsymbol{N}_p$ together with some column of $\boldsymbol{N}_{p+1}$ forms an independent set of $p+1$ columns.

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There are other theorems not deducible from these; for in § 16 we give an example of a system satisfying these two theorems but not representing any matrix. Further theorems seem, however, to be quite difficult to find. Let us call a system obeying (a) and (b) a "matroid." The present paper is devoted to a study of the elementary properties of matroids. The fundamental

General

[4] Graham, Grötschel, Lovasz (1995): *Handbook of Combinatorics* (2 volumes, 44 papers). *Graphs* (5 papers), *Matroids* (3 papers), *Methods* (5 papers), *Applications* (7 papers).

[5] Kung, Rota, Yan (2009): *Combinatorics: the Rota way*.
Exponential identity (§4.1): $exp(\sum_{r=1}^{\infty} \frac{t^r}{r}) = \frac{1}{1-t}$ (**EI**).

[6, 7] Stanley (1999 and 2012): *Enumerative Combinatorics*
(Vol 2 and Vol 1 (2nd ed)).

Matroids

Books: [8] (Welsh 1976) and [9] (Oxley 2011).

Summaries: [10] Welsh (1995), 46 pp, [11] Oxley (2003), 45 pp, [12] Oxley (2021), 13 pp.

A number within double square brackets in the text, say [[36]], refers to the reference item [36] in Huh's ICM paper ([2] Huh (2022) in our reference list). Such labels are linked to an online file, whenever possible, and the correspondence with our reference list is indicated in red if it is included in that list ([13] Grothendieck (1969) in the case of [[36]]).

Let $a_0, \ldots, a_m$ be a sequence of non-negative real numbers. It is

- *Unimodal*: if $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_m$ for some $j \in 0..m$.

- *Staircase*: Unimodal and symmetric. Example: The sequences of Betti numbers $\mathrm{b}_0, \mathrm{b}_2, \ldots, \mathrm{b}_{2n}$ and $\mathrm{b}_1, \mathrm{b}_3, \ldots, \mathrm{b}_{2n-1}$ of a Kähler manifold are staircases. ▷ A 54

- *Log-concave*: if $a_j^2 \geqslant a_{j-1}a_{j+1}$ for all $j \in 1..(m-1)$. A log-concave sequence of *positive* terms is unimodal. The symmetric sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is log-concave, hence also unimodal.

- *Ultra-log-concave*: When $a_j/\binom{m}{j}$, $j \in 0..m$, is log-concave.

- *Top-heavy*: if $a_j \leqslant a_{m-j}$ for $j \in 0..(m /\!\!/ 2)$.

For the ubiquity of these notions in algebra, combinatorics and geometry, see the surveys [14] Stanley (1989) and [15] Brenti (2016).

For specific occurrencies in the theory of projective hypersurface singularities, see [16] Huh (2012).

See also [17] Aluffi (2023) for further examples (and generalizations) in intersection theory.

Theorem (I. Newton). Let $\sum_{j=0}^{n} b_j x^j = \sum_{j=0}^{n} \binom{n}{j} a_j x^j$ be a real polynomial with *real roots*. Then $b_0, b_1, \ldots, b_n$ is ultra-log-concave ($\Leftrightarrow a_0, a_1, \ldots, a_n$ is log-concave). Moreover, if $b_j \geqslant 0$, then $b_0, b_1, \ldots, b_n$ has no internal zeros.

[18] Stanley (2013) (Th. 5.12).

Intersection cohomology staircases. If X is an irreducible complex projective variety of dimension n, Goresky and MacPherson [19, 20] introduced the *intersecction cohomology* of X,

$$IH^*(X) = IH^0(X) \oplus IH^1(X) \oplus \cdots \oplus IH^{2n}(X).$$

Let $\beta_j = \dim IH^j(X)$ ('Betti' numbers). Then the sequences $\beta_0, \beta_2, \ldots, \beta_{2n}$ and $\beta_1, \beta_3, \ldots, \beta_{2n-1}$ are staircases. For an overview of the development of IH, see [21] Kleiman (2007).

Given a graph $G = (V, E)$, and a positive integer q, a *proper coloring* of G with q colors is a map $c : V \to [q]$ such that $c(a) \neq c(b)$ when $ab \in E$.

The number of proper colorings of G with q colors turns out to be a polynomial in q (the *chromatic polynomial* of G) of the form

$$\chi_G(q) = a_n q^n - a_{n-1} q^{n-1} + \cdots + (-1)^{n-1} a_1 q,$$

where $n = |V|$ and $a_j \geqslant 0$ for $j = 1, \ldots, n$.

The Read-Hoggar conjecture (1968, 1974) says that $a_1, \ldots, a_n$ is *log-concave*.

It was proved by Huh in 2009 in his PhD research. The sequence is also *unimodal* (this was Read's conjecture).

This turns out to be a special case of the conjecture considered next.

A *matroid* is a pair $M = (E, \mathcal{I})$, where E is a finite set and $\mathcal{I}$ is a family of subsets of E (called *independent sets*) that satisfy:
(**i0**) the empty subset is independent;
(**i1**) any subset of an independent set is independent; and
(**i2**) if X, X' are independent and $|X| > |X'|$, then there exists $x \in X - X'$ such that $X' \cup \{x\}$ is independent.

Thus a matroid is an abstraction of the notion of linearly independent sets of a finite set of vectors in a K-vector space (such matroids are said to be *representable* over the field K).

It is also important to note that *a graph gives rise to a matroid by declaring a subset of edges independent if it contains no cycles*.

For a matroid $M = (E, \mathcal{I})$, the *rank* $r(X)$ of a subset X of E is defined by $r(X) = \max\{|X'| : X' \subseteq X, X' \in \mathcal{I}\}$.

The *characteristic polynomial* of M, $\chi_M(q)$, is defined as

$$\chi_M(q) = \sum_{X\subseteq E} (-1)^{|X|} q^{r(E)-r(X)} = \sum_{j=0}^{r(E)} (-1)^j w_j q^{d-j},$$

where the coefficients w_j are called *Whitney numbers* (of the first kind).

The characteristic polynomial *generalizes the notion of chromatic polynomial of a graph*.

[9] Oxley (2911), p. 588.

The Heron-Rota-Welsh conjecture asserts that $w_0, w_1, \ldots, w_{r(E)}$ is log-concave.

It was proved in [[1]] [22] Adiprasito, Huh, Katz (2018).

Let $\mathcal{L}$ be a finite lattice, $r : \mathcal{L} \to \mathbb{N}$ its *rank* function, $\mathcal{L}^k = \{x \in \mathcal{L} : r(x) = k\}$, and $d = \operatorname{rank}(\mathcal{L})$ (the rank of its maximum element). $\mathcal{L}$ is said to be *geometric* if it is generated by $\mathcal{L}^1$ (the *atoms* of $\mathcal{L}$) and r satisfies the *submodular* property, namely $r(x) + r(x') \geqslant r(x \vee x') + r(x \wedge x')$ for all $x, x' \in \mathcal{L}$.

The *Dowling-Wilson top-heavy* conjecture (1974) asserts that

$$|\mathcal{L}^k| \leqslant |\mathcal{L}^{d-k}| \text{ for all } k \leqslant d/2. \qquad (*)$$

It was proved in [[41]] [23] Huh,Wang (2017) (see also [[12]] [24] Braden, Huh, Matherne, Proudfoot, Wang (2020) for further enhancements).

Remark. The conjecture was phrased for the lattice $\mathcal{L}(M)$ of *flats* of a matroid $M = (E, \mathcal{I})$.[1] But this is not a more general statement than Eq. $(*)$, as the class of geometric lattices agrees with the class of lattices of flats of matroids.

[1] A *flat* is a subset of E that is maximal for its rank.

Let $i_k = i_k(M)$ be the number of independent sets of cardinal k in a finite matroid $M = (E, \mathcal{I})$.

Mason's ultra-strong conjecture says that the i_k form an ultra log-concave sequence, i.e.

$$i_k^2 \geqslant (1 + \frac{1}{k})(1 + \frac{1}{n-k}) i_{k-1} i_{k+1}, \;\; n = |E|.$$

This conjecture was proved in [[17]] [25] Brändén, Huh (2020).

As explained in the footnote 2 of [2] Huh (2022), it was independently proved in the series [[2]], [[3]], [[4]] [26] Anari, Liu, Gharan, Vinzant (2018).

Weil's conjectures

(When the discrete charmed algebraic geometry)

- $X/\mathbb{F}_q$ an variety over $\mathbb{F}_q$, $n = \dim(X)$, $N_r = \#X(\mathbb{F}_{q^r})$, $Z_X(t) = \exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r})$ (the *Hasse-Weil Zeta function of X*). $N_r = \frac{1}{(r-1)!}\frac{d^r}{dt^r}\log Z(t)\big|_{t=0}$.

Example. $Z_{\mathbb{A}^n}(t) = \frac{1}{1-q^n t}$. Indeed: $N_r = q^{rn}$, $\exp(\sum_{r=1}^{\infty} N_r \frac{t^r}{r}) = \exp(\sum_{r=1}^{\infty} \frac{(q^n t)^r}{r}) = \frac{1}{1-q^n t}$ (**EI**).

Example. If $Y \subset X$ is open, $Z_X(t) = Z_Y(t) Z_{X-Y}(t)$. Indeed: $N_r(X) = N_r(Y) + N_r(X - Y)$.

Example. $Z_{\mathbb{P}^n}(t) = \frac{1}{1-q^n t} Z_{\mathbb{P}^{n-1}}(t) = \prod_{j=0}^{n} \frac{1}{1-q^j t}$.

[27] Weil (1949) $a_0x_0^{n_0}+a_1x_1^{n_1}+\cdots+a_rx_r^{n_r}=b$ N

Let $X/\mathbb{F}_q$ be a non-singular projective algebraic variety over $\mathbb{F}_q$, $n=\dim(X)$, and $Z=Z_X(t)$.

W1 (*Rationality*). $Z(t)=\frac{P_1(t)\cdots P_{2n-1}(t)}{P_0(t)\cdots P_{2n}(t)}$, where the P_j are polynomials with integer coefficients with $P_0(t)=1-t$, $P_{2n}(t)=1-q^nt$.

Example. For $X=\mathbb{P}^n$, $P_{2j}=1-q^jt$ $(j\in 0..n)$, $P_{2j-1}=1$ $(j\in 1..n)$.

W2 (*Functional equation*). $Z_X(1/(q^nt))=\pm q^{nE/2}t^EZ_X(t)$, where $E=E(X)$ is the Euler characteristic of X.

Example. For $X=\mathbb{P}^n$, the map $t\mapsto 1/q^nt$ replaces the factor $\frac{1}{1-q^jt}$ $(j\in 0..n)$ of $Z(t)$ by $\frac{1}{1-1/q^{n-j}t}=-q^{n-j}t\frac{1}{1-q^{n-j}t}$. Hence it replaces $Z(t)$ by $(-1)^{n+1}q^{(n+1)n/2}t^{n+1}=(-1)^{n+1}q^{nE/2}t^EZ(t)$, with $E=n+1=E(\mathbb{P}^n)$ (in accord with the Chow ring $A^*(\mathbb{P}^n)=\mathbb{Z}[h]/(h^{n+1})$, h the hyperplane class).

For $j = 1, \ldots, 2n-1$, let $P_j(t) = \prod_k (1 - \alpha_{jk} t)$, $\alpha_{jk} \in \mathbb{C}$.

W3 (*Riemann hypothesis*). $|\alpha_{jk}| = q^{j/2}$ ($j = 1, \ldots, 2n-1$, all k). This means that with the change of variable $t = q^{-s}$, the roots $1/\alpha_{jk}$ of the P_j lie on the line $\mathrm{re}(s) = j/2$. Note that for $\mathbb{P}^n$, we have $1/\alpha_{2j,1} = q^j$.

W4 (*Betti numbers*). If X' is a non-singular projective variety defined over a number field embedded in $\mathbb{C}$ (e.g. $\mathbb{Q}$) and it has good reduction mod $\mathfrak{p}$ to $X/\mathbb{F}_p$, then $\deg P_j = \mathrm{b}_j(\mathbb{C}(X'))$.

[28] Hartshorne (1977), 449-458
[29] Chambert-Loir, Nicaise, Sebag (2018), p17.

[30] Molina, Sayols, Xambó (2017) (algorithm for $N_r(C)$, C curve)

- $X/\mathbb{C}$ smooth irreducible projective variety, $Y \subset X$ a hyperplane section, $f : X \to X$ an endomorphism.

Kählerian analogue of Weil's Riemann hypothesis ▷ A 53

If $f^*(Y) \sim_{\text{alg}} qY$ for some positive integer q, then the modulus of the eigenvalues of $f_j^* : H^j(X, \mathbb{C}) \to H^j(X, \mathbb{C})$ are all equal to $q^{j/2}$.

We see the analogy on replacing $\mathbb{C}$ by $\mathbb{F}_q$ and letting f be the Frobenius endomorphism, which satisfies $f^*(Y) \sim qY$.

[31] Serre (1960), Theorem 1.

[13] Grothendieck (1969)

[32] Kleiman (1968), [33] Kleiman (1994)

- Weil's conjectures are formal consequences of Grothendieck's standard conjectures on algebraic cycles. They are phrased in terms of a "Weil cohomology" $H^*(X)$, for example the ℓ-adic étale cohomology. They are inspired in Lefschetz theory ▷ A 56 and Hodge theory ▷ A 52.

- Actually conjectures **W1** and **W2** could be proved using the Grothendieck's formalism, but not **W3**, which was proved by Pierre Deligne (1973). [28] Hartshorne (1977), 449-458.

- The standard conjectures, which remain conjectures in general, are the main inspiration for the "Kähler package".

- The first conjecture is analogous to "Lefschetz's structure theorem on the cohomology of a smooth projective variety over the complex field" (▷ A 56, ▷ A 57), and the second is a positivity statement "generalizing Weil's positivity theorem for abelian varieties; it is formally analogous to the famous Hodge inequalities, and is in fact a consequence of these in characteristic zero" (▷ A 58).

For a thorough discussion, see [33] Kleiman (1994): "We discuss the context in which the conjectures arose, [...] the way they explain the Weil conjectures, [...] eight important forms of the Lefscbetz standard conjecture, and finally the Hodge standard conjecture and its implications."

Lorentzian polynomials

Let H_n^d be the space of real homogeneous polynomials of degree d in n variables.

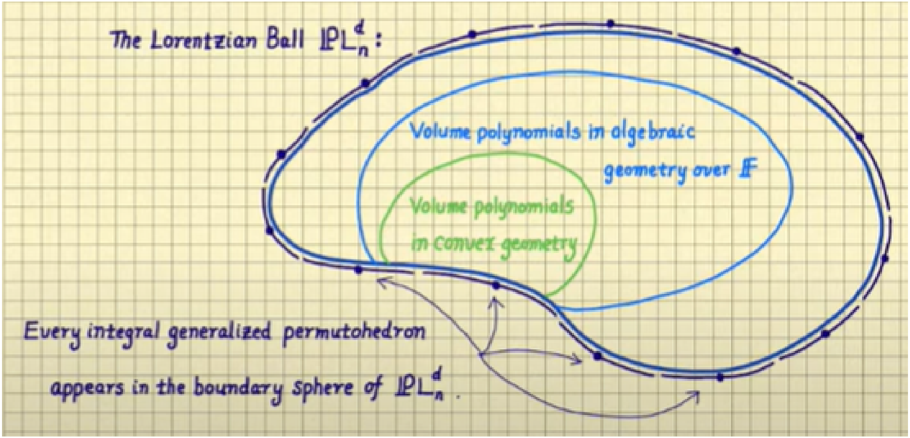
The set of *Lorentzian polynomials* L_n^d is defined as follows.

The elements of L_n^2 are specified by two conditions:
(a_2) their coefficients are non-negative, and
(b_2) their signature has at most one positive sign.

For degrees $d > 2$ the set L_n^d is defined recursively by the following conditions:
(a_d) $\partial_j f \in L_n^{d-1}$ for all $j \in [n]$, and
(b_d) the set of (exponents of) monomials of f is the set of lattice points of an *integral generalized permutohedron* (that is, a polytope whose edges' directions have the form $e_j - e_k$, with $e_1, \ldots, e_n$ the standard basis of $\mathbf{R}^n$; for a reference on these objects, see [34] Doker (2011)).

One of the crucial results in [[17]] [25] Brändén, Huh (2020) is that L_n^d is the closure of $\mathring{L}_n^d$, a set defined by the conditions:
($\mathring{a}_2$) their coefficients are *positive* real numbers,
($\mathring{b}_2$) their signature has *exactly* one positive sign, and, for $d > 2$,
($\mathring{a}_d$) $\partial_j f \in \mathring{L}_n^{d-1}$ for all $j \in [n]$.

Theorem 2.28 of the same paper proves that the compact set $\mathbb{P}L_n^d \subset \mathbb{P}H_n^d$ is contractible, with contractible interior $\mathbb{P}\mathring{L}_n^d$, and conjectured that it is homeomorphic to a closed Euclidean ball (proved by Brändén [[16]] [35] Brändén (2021)).



Detail of slide number 13 of HUH's lecture at the ICM-22. Note the statement on the boundary sphere.

Example. If $C = C_1, \dots, C_n$ are convex bodies in $\mathbb{R}^d$, $\mathrm{vol}_C : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$, $w \mapsto \frac{1}{d!}\mathrm{vol}(w_1 C_1 + \cdot + w_n C_n)$ is a Lorentzian polynomial [2] Huh (2022), Example 6.

Example. Let $D = D_1, \dots, D_n$ be nef Cartier divisors on d-dimensional irreducible projective variety X over an algebraically closed field. Consider the polynomial function

$$\mathrm{vol}_D : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}, \quad w \mapsto \frac{1}{d!}\deg(w_1 D_1 + \cdot + w_n D_n)^d.$$

If X admits a resolution of singularties Y and the Hodge-Riemann relations hold in degree $\leqslant 1$ for the ring of algebraic cycles $A(Y)$, then $\mathrm{vol}_D(w)$ is Lorentzian [2] Huh (2022), Example 7.

The Kähler package

As presented by HUH, the scheme has three ingredients and three postulates (dubbed the *Kähler package* by HUH, for "KÄHLER first emphasized the importance of the respective objects in topology and geometry").

For the Kähler geometry background that inspires these definitions, see see the Appendix on Manifolds ...

Ingredients

(1) A graded real vector space $A = \bigoplus_{j=0}^{d} A^j$;
(2) A convex cone K of graded linear maps $L : A^\star \to A^{\star+1}$; and
(3) A symmetric bilinear pairing $P : A^\star \times A^{d-\star} \to \mathbf{R}$.

Postulates

For any $j \leqslant d/2$,

- *Poincaré Duality*: $P : A^j \to (A^{d-j})^*$ is an isomorphism;
- *Hard Lefschetz Propery*: For any $L \in K$, $L^{d-2j} : A^j \to A^{d-j}$ is an isomorphism;
- *Hodge-Riemann Relations*: The pairing

$$A^j \times A^j \to \mathbf{R}, \quad (x, y) \mapsto (-1)^j P(x, L^{d-2j} y),$$

is positive definite on the kernel of L^{d-2j+1}
(*primitive part* of A^j, to borrow the name from Lefschetz theory).

In the examples known so far, $A = A(X)$ depends on the objects X of some species.

- X a smooth projective variety, $A(X)$ a cohomology ring (ℓ-adic, for instance). The package agrees essentially with GROTHENDIECK's standard conjectures.

- X is a convex polytope and $A(X)$ its combinatorial cohomology [[45]] [36] Karu (2004).

- X a matroid and $A(X)$ can be its:
(i) Chow ring [[1]] [22] Adiprasito, Huh, Katz (2018);
(ii) Conormal Chow ring [[6]] [37] Ardilla, Denham, Huh (2002); or
(iii) Intersection cohomology [[12]], [24] Braden, Huh, Matherne, Proudfoot, Wang (2020).

- X is an element of a Coxeter group and $A(X)$ its Soergel bimodule [[26]], [38] Elias, Williamson (2014). Other references: [39], [40].

The general strategy was summarized in slide number 14 of [1] Huh (2022), while pointing out [[40]] [41] Huh, Matherne, Meszaros, Stdizier (2022) and [[27]] [42] Eur, Huh (2020) for examples and conjectures for various X:

(1) Given X, search for interesting multivariate generating functions from it;

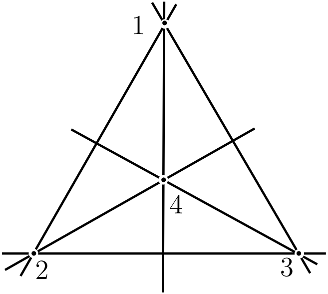
(2) Do we see any generalized permutohedra?

(3) Do we see any Lorentzian polynomials?

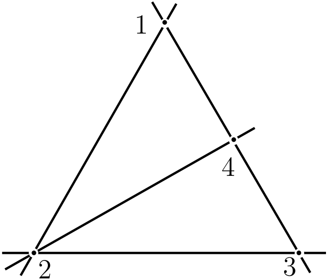
(4) Can we guess $A(X)$, $K(X)$, $P(X)$?

Let us end by describing how the Dowling–Wilson conjecture was solved, after [1] Huh22-lecture.

Given a geometric lattice $\mathcal{L}$ of rank d, consider the set $\mathbb{B}$ of its *bases*, that is, subsets of size d of $E = \mathcal{L}^1$ (the set of atoms) whose join has rank d. Then $\mathbb{B}$ is the set of *lattice points of an integral generalized permutohedron*, and the basis generating function $g = \sum_{\nu\in\mathbb{B}} w^\nu$ is a Lorentzian polynomial.



$g = w_1w_2w_3 + w_1w_2w_4 + w_1w_3w_4 + w_2w_3w_4$



$g = w_1w_2w_3 + w_1w_2w_4 + w_2w_3w_4$

Now define $\mathbf{H}(\mathcal{L}) = \{f : \mathcal{L} \to \mathbb{Q}\} = \bigoplus_{F\in\mathcal{L}} \mathbb{Q}\delta_F$ and make it a graded $\mathbb{Q}$-algebra (the *Möbius algebra* of $\mathcal{L}$) with the multiplication determined by

$$\delta_F \cdot \delta_{F'} = \begin{cases} \delta_{F\vee F'} & \text{if } r(F \vee F') = r(F) + r(F') \\ 0 & \text{otherwise.} \end{cases}$$

The *basis generating function* of $\mathcal{L}$ is $\frac{1}{d!}(\sum_{j\in E} w_j\delta_j)^d$. This suggests taking:

- $A(\mathcal{L}) = \mathbf{H}(\mathcal{L})$;
- $K(\mathcal{L})$, the set of multiplications by positive linear combinations of the δ_j; and
- $P(\mathcal{L})$, multiplication in $\mathbf{H}(\mathcal{L})$ composed with $\mathbf{H}^d(\mathcal{L}) \simeq \mathbb{Q}$.

But $\mathbf{H}(\mathcal{L})$ already fails to satisfy Poincaré duality, for $\dim \mathbf{H}^j(\mathcal{L}) = |\mathcal{L}^j|$ and in general $|\mathcal{L}^j| \neq |\mathcal{L}^{d-j}|$.

As shown in [[12]] [24] Braden, Huh, Matherne, Proudfoot, Wang (2020), the rescue from this failure came from the *intersection cohomology* of $\mathcal{L}$, $\mathbf{IH}(\mathcal{L})$, which is an indecomposable graded $\mathbf{H}(\mathcal{L})$-module endowed with a map $P : \mathbf{IH}(\mathcal{L}) \to \mathbf{IH}(\mathcal{L})^*[-d]$ that satisfies the following properties for every $j \leqslant d/2$ and every $L \in K(\mathcal{L})$:

Poincaré duality $P : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})^*$ is an isomorphism;
Hard Lefschetz $L^{d-2j} : \mathbf{IH}^j(\mathcal{L}) \to \mathbf{IH}^{d-j}(\mathcal{L})$ is an isomorphism; and
Hodge-Riemann relations: The pairing $\mathbf{IH}^j(\mathcal{L}) \times \mathbf{IH}^j(\mathcal{L}) \to \mathbb{Q}$, $(x, y) \mapsto (-1)^j P(x, L^{d-2j} y)$ is positive definite on the kernel of L^{d-2j+1}. In addition, $\mathbf{IH}^0(\mathcal{L})$ generates a submodule isomorphic to $\mathbf{H}(\mathcal{L})$.

The construction relies on the resolution of singularities of algebraic varieties, and in particular on [43] Concini, Procesi (1995) 'wonderful models' (see [44] Concini, Procesi (2010), a wonderful book).

Since the composition of $\mathbf{H}^j(\mathcal{L}) \hookrightarrow \mathbf{IH}^j(\mathcal{L})$ with the Hard-Lefschetz isomorphism $\mathbf{IH}^j(\mathcal{L}) \simeq \mathbf{IH}^{d-j}(\mathcal{L})$ is injective, it follows that $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ composed with $\mathbf{H}^{d-j} \to \mathbf{IH}^{d-j}(\mathcal{L})$ is injective (see diagram below) and consequently $L^{d-2j} : \mathbf{H}^j(\mathcal{L}) \to \mathbf{H}^{d-j}(\mathcal{L})$ is injective, which proves that $|\mathcal{L}^j| \leqslant |\mathcal{L}^{d-j}|$.

$$\begin{array}{ccc} \mathbf{H}^j(\mathcal{L}) & \hookrightarrow & \mathbf{IH}^j(\mathcal{L}) \\ L^{n-2j} \downarrow & & \downarrow L^{n-2j} \\ \mathbf{H}^{d-j}(\mathcal{L}) & \to & \mathbf{IH}^{d-j}(\mathcal{L}) \end{array}$$

Outlook

- July 1, 2022 to June 30, 2023, The Fields Institute

  Matroids - Combinatorics, Algebra and Geometry Seminar

  http://www.fields.utoronto.ca/activities/22-23/matroids-seminar

"Matroids are abstractions of (in)dependence structures in mathematics. There were several open conjectures concerning sequences of combinatorial invariants of matroids. Recently, June Huh along with his collaborators resolved these conjectures ... This spurred a lot of activity in the area. In this seminar series, we will exhibit these developments. We aim at mainstreaming the algebraic geometry of matroids into a mathematical research landscape."

- Connections with mirror symmetry? ([45] (cox-katz-1999))
- Connections with Enumerative geometry? ([46] (katz-2006), [47] (okounkov-2018))
- Standard conjectures!!

Thanks to Anna de Mier for insights about combinatorial matters, particularly on graphs and matroids, while writing the joint report "Combinatorics and Hodge theory, after June Huh", NL04 (23-25).

Thank you!!

Notes

- Birth: Stanford, 1983. Grew up in South Korea.
- Master's degree: Seoul National University 2002-09 (mentored by HEISUKE HIRONAKA)
- PhD: University of Michigan 2014 (MIRCEA MUSTAȚĂ)
- Institute of Advanced Study, Stanford University, Princeton University

From JORDANA CEPELEWICZ article in the Quantamagazine of July 5th, 2022: "his ability to wander through mathematical landscapes and find just the right objects ... that he then uses *to get the seemingly disparate fields of geometry and combinatorics to talk to each other in new and exciting ways*. Starting in graduate school, *he has solved several major problems in combinatorics, forging a circuitous route by way of other branches of math* to get to the heart of each proof." P

HASSLER WHITNEY was an American mathematician.

He was one of the founders of *singularity theory*.

He did foundational work in *manifolds*, *embeddings*, *immersions*, *characteristic classes*, and *geometric integration theory*.

P

[10] (Welsh 1995: *Matroids: fundamental concepts*), in Vol. 1 of the *Handbook of Combinatorics*:

"As the word suggests, Whitney conceived a matroid as an abstract generalisation of a matrix, and much of the language of the theory is based on that of linear algebra. However, Whitney's approach was also motivated by his work in graph theory and as a result some of the matroid terminology has a distinct graphical flavour.

Apart from [several] isolated papers [up to 1949] ... the subject lay virtually dormant until the late fifties when William Thomas Tutte (1958,1959), published his fundamental papers on matroids and graphs and Rado (1957) studied the representability problem for matroids. Since then interest in matroids and their application in combinatorial theory has accelerated rapidly. Indeed it was realized that matroids have important applications in the field of *combinatorial optimization* and also that they *unify and simplify* apparently diverse areas of pure combinatorics"

From WP: “During the Second World War, he made a brilliant and fundamental advance in cryptanalysis of the Lorenz cipher, a major Nazi German cipher system which was used for top-secret communications within the Wehrmacht High Command.” P

"This will contain nothing new, except perhaps in the mode of presentation of the final results, which will lead to the *statement of some conjectures concerning the numbers of solutions of equations over finite fields*, and their relation to the topological properties of the varieties defined by the corresponding equations over the field of complex numbers" (page 498).

"This, and other examples which we cannot discuss here, seem to lend some support to the following conjectural statements, which are known to be true for curves, but which I have not so far been able to prove for varieties of higher dimension" (page 507). P

Appendix on manifolds

[48] Weil (1958)

[49] Hirzebruch (1966)

[50] Griffiths, Harris (1978)

[51] Wells (1980)

[52] Warner (1983)

[53] Voisin (2002)

[54] Huybrechts (2005)

[55] Voisin (2010)

[56] Cattani (2010)

[57] Lee (2013)

- Manifolds X are assumed to be *compact* and *connected*. $n = \dim(X)$.
- $A^*(X) = \bigoplus_{k=0}^n A^k(X)$: graded algebra of $\mathcal{C}^\infty$ forms.
- $C^*(X) = \bigoplus_{k=0}^n C^k(X)$: graded subalgebra of closed forms.
- $E^*(X) = \bigoplus_{k=0}^n E^k(X)$: graded C^*-ideal of exact forms.
- $H^*_{\mathrm{dR}}(X) = C^*(X)/E^*(X)$ (*de Rham cohomology*).
- $H_*(X) = \bigoplus_{k=0}^n H_k(X)$ and $H^*(X) = \bigoplus_{k=0}^n H^k(X)$.
- $H_k(X) \times H^k_{\mathrm{dR}}(X) \to \mathbb{R}$, $([z],[\varphi]) \mapsto \int_z \varphi$.
- $H^k_{\mathrm{dR}}(X) \simeq H_k(X)^* \simeq H^k(X)$. $(H^*_{\mathrm{dR}}(X), \wedge) \simeq (H^*(X), \smile)$.
- Betti numbers: $\mathrm{b}_k(X) = \dim H^k(X)$.
- $\chi(X) = \sum_k (-1)^k \mathrm{b}_k(X)$.
- $H^*_{\mathrm{dR}}(X,\mathbb{C})$, $H_*(X,\mathbb{C})$, $H^*(X,\mathbb{C})$.

Poincaré duality. If X is an oriented n-manifold, the Poincaré map $P: H_k(X) \to H_{n-k}(X)^* = H^{n-k}(X)$, $(P\alpha)(\beta) = \alpha \cdot \beta$ (intersection product) is an isomorphism.

Via the isomorphism $H^{n-k}_{\mathrm{dR}}(X) \simeq H^{n-k}(X)$, we see that given a cycle $z \in Z_k(X)$ there exists a closed $(n-k)$-form φ such that $[z] \cdot [z'] = \int_{z'} \varphi$ for any $(n-k)$-cycle z'. Abusing notation, let φ_z denote any φ satisfying that integral relation.

Cohomology class of z: $\mathrm{cl}(z) = [\varphi_z] \in H^{n-k}$.

In terms of the de Rham cohomology, the pairing $C^k(X) \times C^{n-k}(X) \to C^n(X)$, $(\alpha, \alpha') \mapsto \alpha \wedge \alpha'$, induces a pairing $H^k_{\mathrm{dR}}(X) \times H^{n-k}_{\mathrm{dR}}(X) \to H^n_{\mathrm{dR}}(X) \simeq \mathbb{R}$ which is a duality.

Theorem. If $z \in Z_k(X)$ and $z' \in Z_{n-k}(X)$, then $[z] \cdot [z'] = \int_X \varphi_z \wedge \varphi_{z'}$, or $\varphi_z \wedge \varphi_{z'} = \varphi_{z \cdot z'}$, or $\mathrm{cl}(z \cdot z') = \mathrm{cl}(z) \wedge \mathrm{cl}(z')$.

Let (X, g) be an oriented riemannian manifold. Then g can be extended to a symmetric bilinear map $g : A^k(X) \times A^k(X) \to A^0(X)$ and $A^k(X)$ inherits the symmetric bilinear form $(\alpha, \beta) = \int_X g(\alpha, \beta)\mathfrak{v}$ ($\mathfrak{v}$ the volume form). From linear algebra we know that there is a unique linear isomorphism $* : A^k(X) \to A^{n-k}(X)$ (*Hodge $*$-operator*) such that $\alpha \wedge *\beta = g(\alpha, \beta)\mathfrak{v}$, hence $(\alpha, \beta) = \int_X \alpha \wedge *\beta$. It satisfies (1) $** = (-1)^{k(n-k)}$; and (2) $\alpha \wedge *\alpha = 0 \Leftrightarrow g(\alpha, \alpha) = 0 \Leftrightarrow \alpha = 0$.

Then the Laplacian is defined by $\Delta = \Delta_d = d\delta + \delta d$, where $\delta : A^k(X) \to A^{k-1}(X)$ is the adjoint of $d : A^{k-1}(X) \to A^k(X)$. Set $\mathcal{H}^k_\Delta(X) = \{\alpha \in A^k(X) \,|\, \Delta(\alpha) = 0\}$ (*harmonic* k-forms).

Theorem (Hodge). The natural map $\mathcal{H}^k_\Delta(X) \to H^k_{\mathrm{dR}}(X)$ is an isomorphism. Moreover, these isomorphisms provide a graded algebra isomorphim $\big(\mathcal{H}^*_\Delta(X), \wedge\big) \simeq \big(H^*_{\mathrm{dR}}(X), \wedge\big)$, hence also a graded algebra isomorphism $\big(\mathcal{H}^*_\Delta(X), \wedge\big) \simeq \big(H^*(X), \smile\big)$.

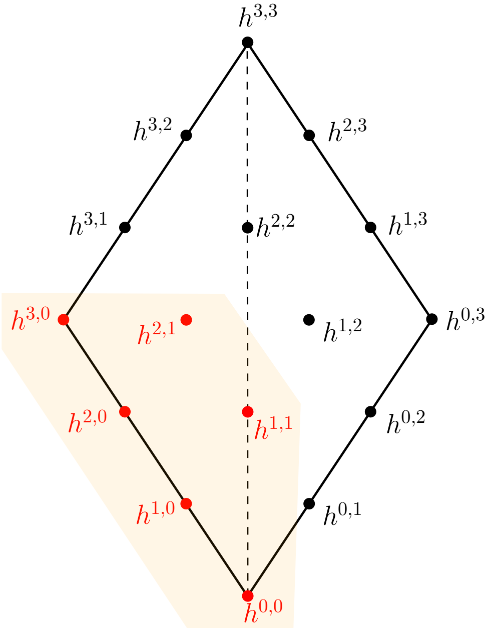
On a complex manifold X of (complex) dimension n, we have a decomposition $A^k(X,\mathbb{C}) = \bigoplus_{p+q=k} A^{p,q}(X,\mathbb{C})$. The forms in $A^{p,q}(X,\mathbb{C})$ are said to be of type (p,q).

A *Kähler manifold* is a complex manifold equipped with a Hermitian metric (*Kähler metric*) whose imaginary part ω, which is a 2-form of type (1, 1), is *closed*. This 2-form is called the *Kähler form* of the Kähler metric.

Submanifolds of a Kähler manifold are Kähler.

A Kähler manifold is in particular a riemannian manifold of dimension $2n$ and it turns out that the (p,q) components of an harmonic k-form are harmonic. This and the Hodge theorem imply a *Hodge decomposition* of cohomology: $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X,\mathbb{C})$ $(k = 0, 1, \ldots, 2n)$. Thus $\big(H^*(X,\mathbb{C}),\wedge\big)$ is a bigraded algebra.

Note $\overline{H}^{p,q}(X) = H^{q,p}$. *Hodge numbers*: $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X,\mathbb{C})$.



Hodge diamond

Betti numbers

$$\mathsf{b}_k = \sum_{p+q=k} h^{p,q}$$

Symmetry about vertical bisector

$$H^{q,p} = \overline{H^{p,q}}$$

$\Rightarrow \quad h^{q,p} = h^{p,q}$

$\Rightarrow$ odd Betti numbers are even

Symmetry about center of diamond

$$H^{n-p,n-q} = *H^{p,q}$$

$\Rightarrow \; h^{n-p,n-q} = h^{p,q}$

Symmetry about horizontal bisector

$$h^{0,0} = h^{n,n} = 1$$

The restriction to S^{2n+1} of the Fubini-Study hermitian metric $ds^2 = \sum_{j=0}^{n} dz_j \otimes d\bar{z}_j$ on $\mathbb{C}^{n+1}$ is invariant by the action of S^1 and hence it induces a hermitian metric on $S^{2n+1}/S^1 = \mathbb{P}^n(\mathbb{C})$. Setting $z_j = x_j + iy_j$, the imaginary part of ds^2 is $\omega = \sum_j dx_j \wedge dy_j$. This form has type $(1,1)$ and is closed. Therefore it induces a Kähler structure ω on $\mathbb{P}^n(\mathbb{C})$. The class $[\omega] \in H^{1,1}(X,\mathbb{C}) \subset H^2(X,\mathbb{C})$ coincides with the cohomology class $\mathrm{cl}(Y)$ of a hyperplane section Y of X.

Complex submanifolds of the complex projective space are Kähler, and they are projective subvarieties by Chow's theorem.

Kodaira's theorem. A compact complex manifold admits a holomorphic embedding into complex projective space [*and hence is a smooth algebraic variety*] if and only if it admits a Kähler metric whose Kähler form is a rational class (i.e, belongs to the image of $H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{C})$).

Let $h = [\omega] \in H^2(X, \mathbb{C})$ (the cohomology class of the Kähler form).

$L : H^k(X, \mathbb{C}) \to H^{k+2}(X, \mathbb{C})$, $\alpha \mapsto h \wedge \alpha$. In the Hodge diamond, L moves each node one vertical step up.

Hard Lefschetz Theorem

(1) L is injective, and hence $b_k \leqslant b_{k+2}$ and $h^{k-i,i} \leqslant h^{k-i+1,i+1}$, for $k < n$. By Poincaré duality, $b_{n-k} \leqslant b_{n-k-2}$ and $h^{n-i,n-k+i} \leqslant h^{n-i-1,n-k+i-1}$ for $n > k$.

The Hodge numbers on a vertical line of the Hodge diamond are non-decreasing in the bottom half and non-increasing in the top half; and the even or odd Betti numbers have the same property. All these sequences are symmetrical, and hence are *staircases*.

Note that $L : H^{n-1}(X, \mathbb{C}) \to H^{n+1}(X, \mathbb{C})$ is a isomorphism, as it is injective and both spaces have the same dimension. This is a special case of next statement.

(2) $L^j : H^{n-j}(X) \to H^{n+j}(X)$ is an isomorphism for all $j \geqslant 0$.

If $H^{p,q}$ is a Hodge component of $H^{n-j}(X)$, so $p+q=n-j$, then L^j maps it isomorphically to $H^{p+j,q+j} = H^{n-q,n-p}$. We get again that the Hodge diamond is symmetric about the horizontal diagonal, which can also be accounted for by composing the symmetry about the center of the diamond (induced by $*$) and the symmetry about the vertical line (induced by the conjugation).

For $k \leqslant n$, the *primitive subspace* of $H^k(X)$ is defined as the kernel of $L^{n-k+1} : H^k(X) \to H^{2n-k+2}$, and is denoted by $H_0^k(X)$.

Lefschetz Decomposition Theorem (Let $q_k = \lfloor k/2 \rfloor = k/\!\!/2$)

$H^k(X,\mathbb{C}) = \bigoplus_{j \geqslant (k-n)^+} L^j H_0^{k-2j}(X)$. $H_0^k(X) = H^k(X)$, $k=0,1$.

For $k \leqslant n$, $H^k = H_0^k \oplus LH_0^{k-2} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k} = H_0^k \oplus LH^{k-2}$.

For $k = n+k'$, $1 \leqslant k' \leqslant n$, $H^k = L^{k'} H_0^{k-2k'} \oplus \cdots \oplus L^{q_k} H_0^{k-2q_k}$.

Hodge-Riemann pairing

$Q : H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) \to \mathbb{C}$,
$Q(\alpha,\alpha') = (-1)^{k/\!\!/2} \int_X \alpha \wedge \alpha' \wedge \omega^{n-k}$.

Theorem

The Hodge decomposition $H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ satisfies:

(1) $Q(H^{p,q}, H^{p',q'}) = 0$ if $(p',q') \neq (q,p)$, and
(2) $i^{p-q} Q(\alpha,\bar{\alpha}) > 0$ for $0 \neq \alpha \in H_0^{p,q}(X)$.

Let X is a Kähler manifold and Z a submanifold of codimention k.

Then $\mathrm{cl}(Z) \in H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X) = H^{k,k}(X,\mathbb{Q})$.

The same is true if $Z \in \mathcal{Z}^k_{\mathbb{Q}}$, the group of rational linear combinations of submanifolds of codimension k (rational cycles of codimension k).

The Hodge conjecture states that if X is a smooth projective variety (or a Kähler manifold of integral type), then $\mathrm{cl} : \mathcal{Z}^k_{\mathbb{Q}} \to H^{k,k}(X,\mathbb{Q})$ is surjective.

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