Cerednik-Drinfeld's Models of Shimura Curves and Reduction of CM Points

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Shimura curves $X_0(D, N)$

Let $D = p_1 \cdot \dots \cdot p_{2r}$ and $N \geq 1$, $(D, N) = 1$, $\Box \nmid DN$.

- $B_D = \mathbb{Q} + \mathbb{Q}i + \mathbb{Q}j + \mathbb{Q}ij, \quad i^2, j^2 \in \mathbb{Q}^*,\ ij = -ji$, quaternion algebra of discriminant D.

- $B_D \otimes \mathbb{R} \xrightarrow{\simeq} \mathrm{M}_2(\mathbb{R})$

- $\mathcal{O}$ = Eichler order of level N in B_D.

- $\Gamma_{D,N} := \{\gamma \in \mathcal{O} \mid \det(\gamma) = 1\} \subseteq SL_2(\mathbb{R})$

- Acts on the Poincaré half-plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ discrete and discontinuously.

Shimura curves $X_0(D,N)$

- If $D=1$, $Y_0(N) := \Gamma_{1,N}\backslash\mathcal{H}$ is not compact.
  We make it compact adding *cuspidal points*: $X_0(N)$.
- If $D>1$, $X_0(D,N) := \Gamma_{D,N}\backslash\mathcal{H}$ is compact: the *Shimura curve of discriminant D and level N*.
- (Shimura) $X_0(D,N)$ admits a canonical model over $\mathbb{Q}$.
- (Morita) $X_0(D,N)$ has good reduction at $p \nmid DN$. At $p \mid DN$ has stable reduction.
- If $D=1$, $z \mapsto z+1 \in \Gamma_{D,N}$: Fourier coefficients of modular forms at cusps give us arithmetic information and allows us to compute equations.
- If $D>1$, there are no cusps, the forms are not periodic. It is harder to find equations.

Shimura curves for genus 0, 1 and 2.

D	N	g	$X_0(D,N)$	
6	1	0	$x^2+y^2+3=0$	Ihara
10	1	0	$x^2+y^2+2=0$	Ihara
22	1	0	$x^2+y^2+11=0$	Kurihara
14	1	1	$(x^2-13)^2+7^3+2y^2=0$	Kurihara
15	1	1	$(x^2+3^5)(x^2+3)+3y^2=0$	Jordan
21	1	1	$x^4-658x^2+7^6+7y^2=0$	Kurihara
33	1	1	$x^4+30x^2+3^8+3y^2=0$	Kurihara
34	1	1	$3x^4-26x^3+53x^2+26x+3+y^2=0$	González-Rotger
46	1	1	$(x^2-45)^2+23+2y^2=0$	Kurihara
6	5	1	$y^2=-x^4+61\,x^2-1024$	González-Rotger
6	7	1	$y^2=-3\,x^4-34\,x^2-2187$	González-Rotger
6	13	1	$y^2=-x^4-115\,x^2-4096$	González-Rotger
10	3	1	$y^2=-2\,x^4-11\,x^2-32$	González-Rotger
10	7	1	$y^2=-27\,x^4-40\,x^3+6\,x^2+40\,x-27$	González-Rotger
26	1	2	$y^2=-2x^6+19x^4-24x^2-169$	González-Rotger
38	1	2	$y^2=-16x^6-59x^4-82x^2-19$	González-Rotger
58	1	2	$2y^2=-x^6-39x^4-431x^2-841$	González-Rotger

Aim

- Compute equations for all hyperelliptic Shimura curves $X_0(D, N)$ and $X_0(D, N)/\langle\omega\rangle$ over $\mathbb{Q}$, where ω is an (Atkin-Lehner) involution.
- Even with $g(X) = 2$, González-Rotger method does not work when $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$

Moduli interpretation of Shimura curves

- An abelian surface A/K has QM by $\mathcal{O}$ if
  - i) $\exists i : \mathcal{O} \hookrightarrow \mathrm{End}_K(A)$
  - ii) $\Lambda := H_1(A, \mathbb{Z}) \cong \mathcal{O}$ as a $\mathcal{O}$-module
- $X_0(D, N)(\mathbb{C})$ parametrizes such pairs (A, i) over $\mathbb{C}$.
- $\Gamma_{D,N} \backslash \mathcal{H} \ni \tau \longmapsto \Lambda_\tau = \mathcal{O} \begin{pmatrix} \tau \\ 1 \end{pmatrix} \longmapsto \mathbb{C}^2/\Lambda_\tau$
- $X_0(D, N)/\mathbb{Z}$ is the coarse moduli space that parametrizes *special* pairs (A, i), where A/T is a abelian 2-dimensional scheme,

$$i : \mathcal{O} \hookrightarrow \mathrm{End}_T(A).$$

CM points of a Shimura curve

Let R be an order in an imaginary quadratic field K.

- $(A, i) \in CM(R)$ when $A \cong E_1 \times E_2$, E_i CM elliptic curve by R.
- The points $P \in CM(R)$ correspond to optimal embeddings $\varphi : R \hookrightarrow \mathcal{O}$, up to conjugation by $\mathcal{O}^*$.
- To such φ we assign the single $\tau \in \Gamma_{D,N} \backslash \mathcal{H}$ fixed by $\varphi(R) \subset M_2(\mathbb{R})$.
- $\mathrm{End}(A, i) = \{\alpha \in \mathrm{End}(A) \ : \ \alpha i(o) = i(o)\alpha \ \forall o \in \mathcal{O}\} \cong R$.

Cerednik-Drinfeld's Model of $X_0(D, N)$ at $p \mid D$

Fix $p \mid D$, $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$.

- $\widetilde{X_0(D,N)}(\mathbb{F})$ classifies *mixed* pairs $(\widetilde{A}, \widetilde{i})$; $\widetilde{A}/\mathbb{F}$, $\widetilde{i} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{F}}(\widetilde{A})$.
- $P = (\widetilde{A}, \widetilde{i})$ is non-singular on $\widetilde{X_0(D,N)} \Rightarrow \exists! H \subset \widetilde{A}$ $\mathcal{O}$-stable torsion subgroup isomorphic to α_p.

$\longrightarrow$ $\widetilde{i}$ induces $j : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathbb{F}}(\widetilde{A}/H)$.

$\longrightarrow$ $\mathrm{End}(\widetilde{A}/H, j)$ is a level-N-Eichler order in the definite quaternion algebra $B_{D/p}$.

- $P = (\widetilde{A}, \widetilde{i})$ singular $\Rightarrow \mathrm{End}(\widetilde{A}, \widetilde{i})$ level-Np-Eichler order $\subset B_{D/p}$.

Cerednik-Drinfeld's Model

An orientation of $\mathcal{O}$ at $\ell \mid DN$ is

$$\pi_\ell : \mathcal{O}_\ell \longrightarrow \overline{\mathbb{F}_\ell}$$

An oriented Eichler order is an Eichler order $\mathcal{O}$ plus orientations

$$\pi_\ell \text{ for each } \ell \mid DN$$

For d, n, $\square \nmid dn$, $(d, n) = 1$, define
$\mathrm{Pic}(d, n) = \{$Oriented Eichler orders of level n in B_d $\}$.

$$\left\{ \begin{array}{c} \text{Singular points} \\ \text{of } \widetilde{X_0(D, N)}(\mathbb{F}) \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \mathrm{Pic}(\frac{D}{p}, Np)$$

$$\left\{ \begin{array}{c} \text{Connected} \\ \text{components of} \\ \widetilde{X_0(D, N)}(\mathbb{F}) \end{array} \right\} \stackrel{1:1}{\longleftrightarrow} \mathrm{Pic}(\frac{D}{p}, N) \sqcup \mathrm{Pic}(\frac{D}{p}, N)$$

Optimal embeddings

Let $R \subset K$ of conductor $c \geq 1$.

$$CM_{d,n}(R) := \bigsqcup_{\mathcal{O} \in \mathrm{Pic}(d,n)} \{\varphi : R \hookrightarrow \mathcal{O} \text{ optimal up to conj. by } \mathcal{O}^*\}.$$

$\#CM_{d,n}(R) = h(R) \prod_{p|d}(1 - (\frac{R}{p})) \prod_{p|n}(1 + (\frac{R}{p}))$.

- $\longrightarrow$ If some $p \mid gcd(d, c)$, $CM_{d,n}(R) = \emptyset$.
- $\longrightarrow$ If some $p \mid d$ splits in R, $CM_{d,n}(R) = \emptyset$.
- $\longrightarrow$ If some $p \mid n$ is inert in R, $CM_{d,n}(R) = \emptyset$.

Reduction of CM points

Theorem. (M.)

Any CM point $P \in CM(R)$ of $X_0(D,N)$ reduces to a singular point of $\widetilde{X_0(D,N)}$ modulo $p \mid D$ if and only if p ramifies in K.

Idea of the proof: $\Rightarrow$) Suppose $\widetilde{P} = (\widetilde{A}, \widetilde{i})$ is singular, then

$$R \cong \mathrm{End}(A,i) \hookrightarrow \mathrm{End}(\widetilde{A}, \widetilde{i}) \in \mathrm{Pic}(D/p, Np),$$

is optimal. Hence

$$\left.\begin{array}{l} CM_{D/p,Np}(R) \neq \emptyset \Rightarrow p \text{ is not inert in } K. \\ P \in CM_{D,N}(R) \neq \emptyset \Rightarrow p \text{ does not split in } K \end{array}\right\} \Rightarrow p \text{ ramifies in } K.$$

$\Leftarrow$) Harder part. Exploits Ribet's theory of bimodules.

Compairing CM sets

Let $p \mid D$, $p \mid \mathrm{disc}(K)$ so that $CM(R) \overset{\text{red}}{\to} \mathrm{Sing}(\widetilde{X_0(D,N)})$.

- $P \in CM(R)$ can be thought as $\varphi \in CM_{D,N}(R)$.
- The reduction $\mod p$ gives an optimal embedding

$$(R \cong \mathrm{End}(A,i) \hookrightarrow \mathrm{End}(\widetilde{A},\widetilde{i})) \in CM_{\frac{D}{p},Np}(R)$$

- We have constructed an application

$$\phi : CM_{D,N}(R) \longrightarrow CM_{\frac{D}{p},Np}(R)$$

Compatibilities on the CM sets

- For any pair (d, n), $\mathrm{Pic}(R)$ acts faithfully on $CM_{d,n}(R)$.
- *Shimura reciprocity law*:

$$\begin{array}{ccccc} CM_{D,N}(R) & \longleftrightarrow & CM(R) & \subseteq X_0(D,N)(H_R) \\ \circlearrowleft & & \circlearrowleft & \\ \mathrm{Pic}(R) & \stackrel{rec(\cdot^{-1})}{\longrightarrow} & \mathrm{Gal}(H_R/K) & \end{array}$$

- On $CM_{d,n}(R)$ also acts the Atkin-Lehner group $W \cong (\mathbb{Z}/2\mathbb{Z})^{\#\{p|dn\}}$.

Theorem. (M.)

$\phi : CM_{D,N}(R) \longrightarrow CM_{\frac{D}{p},Np}(R)$ *is a bijection compatible under the actions of* $\mathrm{Pic}(R)$ *and* W.

How can we use it?

- A Shimura curve X has a stable model $\mathcal{X}$ over $\mathbb{Z}$.
- Any singular point P of $\mathcal{X} \times \mathbb{F}_p$ has attached an integer $e_P \geq 1$, called the *thickness* of P.
- Assume X hyperelliptic over $\mathbb{Q}$. Then $\mathcal{X}$ admits a hyperelliptic model.
- Let $\mathcal{X} : y^2 = P(x)$ over $\mathbb{Z}[1/2]$. Then e_P's provide information about the valuation of the difference of the roots of $P(x)$ at $\mathfrak{P} \mid p \neq 2$.

Application: Equations of hyperelliptic Shimura curves

Let $X = X_0(D, N)$ hyperelliptic over $\mathbb{Q}$ and fix $p \mid D$.

- $\{\text{Weierstrass points}\} = \bigsqcup_i CM_{D,N}(R_i)$.

$$P \in CM(R) \quad \overset{\phi}{\longmapsto} \quad \varphi \in CM_{\frac{D}{p},Np}(R) \quad \longmapsto \quad \mathcal{O}' \in \mathrm{Pic}(\tfrac{D}{p}, Np)$$

$$\| $$

$$\mathrm{Sing}(\widetilde{X})$$

- $\mathrm{Pic}(R) \times W$ acts transitively on $CM_{d,n}(R)$.
- If P corresponds to $\mathcal{O}' \in \mathrm{Pic}(D/p, Np)$, $e_P = \#\mathcal{O}'^*/2$.

Conclusions

- X=$X_0(D,N)$: $y^2 = \prod_j P_j(x)$, P_j irreducible.
- $P_j(x)$ has roots α_j^i, we can compute $K_j = \mathbb{Q}[x]/P_j(x)$.
- We know $\nu_{\mathfrak{P}}(\alpha_j^i - \alpha_l^k)$ for all $\mathfrak{P} \nmid 2$. Thus also $\mathrm{disc}(P_j)$ and $\mathrm{Res}(P_i, P_j)$ modulo $\mathbb{Z}[1/2]^*$.
- *Index form equations* $(\mathcal{O}_{K_j} : \mathbb{Z}[\alpha]) = k$, could help us to compute $P_j(x + t_j)$ with $t_j \in \mathbb{Z}$ indeterminate. The values $\mathrm{Res}(P_i, P_k)$ would allow us to find such indeterminations.
- In case we find more than a candidate curve, we use $\mathrm{Jac}(X_0(D,N)) \overset{\mathbb{Q}}{\simeq} J_0^{\text{new}}(D,N)$ and the Eichler-Shimura congruence.

Hyperelliptic Shimura curves over $\mathbb{Q}$ and Conjectural equations.

D	$g(X_D)$	w	Conjectural equation
$2 \cdot 31$	3	ω_D	$y^2 = -(64x^8 + 99x^6 + 90x^4 + 43x^2 + 8)$
$2 \cdot 47$	3	ω_D	$y^2 = -(8x^8 - 69x^6 + 234x^4 - 381x^2 + 256)$
$3 \cdot 13$	3	ω_D	$y^2 = -(19x^4 + 16x^3 - 88x^2 - 40x + 112)(x^4 + 8x^2 - 24x + 16)$
$3 \cdot 17$	3	ω_D	$y^2 = (x^2 + 3)(-243x^6 - 235x^4 + 31x^2 - 1)$
$3 \cdot 23$	3	ω_D	$y^2 = -(3x^8 + 28x^6 + 74x^4 - 1268x^2 + 2187)$
$5 \cdot 7$	3	ω_D	$y^2 = -(7X^2 + 1)(x^6 + 197x^4 + 51x^2 + 7)$
$5 \cdot 11$	3	ω_D	$y^2 = -(x^4 + 4x^3 + 46x^2 - 4x + 1)(3x^4 - 4x^3 + 10x^2 + 4x + 3)$
$2 \cdot 37$	4	ω_D	$y^2 = -(2x^{10} - 47x^8 + 328x^6 - 946x^4 + 4158x^2 + 1369)$
$2 \cdot 43$	4	ω_D	$y^2 = (16x^{10} - 245x^8 + 756x^6 + 1506x^4 + 740x^2 + 43)$
$3 \cdot 29$	5	ω_D	$y^2 = -(81x^6 - 63x^4 + 43x^2 + 3)(3^7x^6 + 523x^4 + 41x^2 + 1)$
$2 \cdot 67$	6	ω_D	
$2 \cdot 73$	7	ω_D	
$3 \cdot 37$	7	ω_D	
$5 \cdot 19$	7	ω_D	
$2 \cdot 97$	9	ω_D	
$2 \cdot 103$	9	ω_D	
$3 \cdot 53$	9	ω_D	
$7 \cdot 17$	9	ω_D	