Waldspurger formula in higher cohomology

S. Molina

May 13, 2021

Index

Automorphic forms

Let F number field, $\mathcal{O}_F$ ring of integers

$$\Sigma_F := \{\sigma : F \hookrightarrow \mathbb{C}\}/\text{conjugation}, \qquad \mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal}.$$

$F_\sigma = \mathbb{R}$, or $\mathbb{C}$, and $F_\mathfrak{p}$ completion of localization at $\mathfrak{p}$.

Automorphic forms

Let F number field, $\mathcal{O}_F$ ring of integers

$$\Sigma_F := \{\sigma : F \hookrightarrow \mathbb{C}\}/\text{conjugation}, \qquad \mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal.}$$

$F_\sigma = \mathbb{R}$, or $\mathbb{C}$, and $F_\mathfrak{p}$ completion of localization at $\mathfrak{p}$.

- Ring of adeles:

$$\mathbb{A}_F := \prod_{\sigma \in \Sigma_F} F_\sigma \times \prod_{\mathfrak{p}}{}' F_\mathfrak{p}, \quad \mathbb{A}_F^\infty := \prod_{\mathfrak{p}}{}' F_\mathfrak{p}$$

Automorphic forms

Let F number field, $\mathcal{O}_F$ ring of integers

$$\Sigma_F := \{\sigma : F \hookrightarrow \mathbb{C}\}/\text{conjugation}, \qquad \mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal.}$$

$F_\sigma = \mathbb{R}$, or $\mathbb{C}$, and $F_\mathfrak{p}$ completion of localization at $\mathfrak{p}$.

- Ring of adeles:

$$\mathbb{A}_F := \prod_{\sigma\in\Sigma_F} F_\sigma \times \prod_{\mathfrak{p}}{}' F_\mathfrak{p}, \quad \mathbb{A}_F^\infty := \prod_{\mathfrak{p}}{}' F_\mathfrak{p} \qquad F \hookrightarrow \mathbb{A}_F, \mathbb{A}_F^\infty.$$

Automorphic forms

Let F number field, $\mathcal{O}_F$ ring of integers

$$\Sigma_F := \{\sigma : F \hookrightarrow \mathbb{C}\}/\text{conjugation}, \qquad \mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal.}$$

$F_\sigma = \mathbb{R}$, or $\mathbb{C}$, and $F_\mathfrak{p}$ completion of localization at $\mathfrak{p}$.

- Ring of adeles:

$$\mathbb{A}_F := \prod_{\sigma \in \Sigma_F} F_\sigma \times \prod_{\mathfrak{p}}{}' F_\mathfrak{p}, \quad \mathbb{A}_F^\infty := \prod_{\mathfrak{p}}{}' F_\mathfrak{p} \qquad F \hookrightarrow \mathbb{A}_F, \mathbb{A}_F^\infty.$$

- Let G an algebraic group (e.g. $G = \mathrm{GL}_n$): automorphic form

$$\phi : G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C},$$

Automorphic forms

Let F number field, $\mathcal{O}_F$ ring of integers

$$\Sigma_F := \{\sigma : F \hookrightarrow \mathbb{C}\}/\text{conjugation}, \qquad \mathfrak{p} \subset \mathcal{O}_F \text{ prime ideal.}$$

$F_\sigma = \mathbb{R}$, or $\mathbb{C}$, and $F_\mathfrak{p}$ completion of localization at $\mathfrak{p}$.

- Ring of adeles:

$$\mathbb{A}_F := \prod_{\sigma\in\Sigma_F} F_\sigma \times \prod_{\mathfrak{p}}{}' F_\mathfrak{p}, \quad \mathbb{A}_F^\infty := \prod_{\mathfrak{p}}{}' F_\mathfrak{p} \qquad F \hookrightarrow \mathbb{A}_F, \mathbb{A}_F^\infty.$$

- Let G an algebraic group (e.g. $G = \mathrm{GL}_n$): automorphic form

$$\phi : G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C},$$

- The space of automorphic forms

$$\mathcal{A} := \{\phi : G(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C}\} \quad \circlearrowleft G(\mathbb{A}_F)$$

$\mathcal{A} = \prod \pi$, π irreducible components automorphic representations

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

well-known kernel.

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

well-known kernel.
Galois character

$$\chi : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathbb{C}$$

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

well-known kernel.
Galois character

$$\chi : \mathrm{Gal}(F^{\mathrm{ab}}/F) \longrightarrow \mathbb{C}$$

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

well-known kernel.
Galois character

$$\bar{\chi} : \mathbb{A}_F^\times / F^\times \xrightarrow{\rho} \mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\chi} \mathbb{C}$$

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

well-known kernel.
Galois character

$$\bar{\chi} : \mathrm{GL}_1(\mathbb{A}_F)/\mathrm{GL}_1(F) \longrightarrow \mathbb{C}$$

automorphic character.

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

   well-known kernel.
   Galois character

$$\bar{\chi} : \mathrm{GL}_1(\mathbb{A}_F)/\mathrm{GL}_1(F) \longrightarrow \mathbb{C}$$

   automorphic character.

2. Modular form:

$$f : \mathcal{H} \to \mathbb{C}, \qquad f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

   well-known kernel.
   Galois character

$$\bar{\chi} : \mathrm{GL}_1(\mathbb{A}_F)/\mathrm{GL}_1(F) \longrightarrow \mathbb{C}$$

   automorphic character.

2. Modular form:

$$f : \mathcal{H} \to \mathbb{C}, \qquad f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

   the form f gives rise to ϕ_f

$$\phi_f : \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}.$$

Motivation: CFT and modular forms

1. Class Field Theory: There exists surjective *Artin map*

$$\rho : \mathbb{A}_F^\times / F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) = (\text{Galois of max'l abelian ext'n})$$

   well-known kernel.
   Galois character

$$\bar{\chi} : \mathrm{GL}_1(\mathbb{A}_F)/\mathrm{GL}_1(F) \longrightarrow \mathbb{C}$$

   automorphic character.

2. Modular form:

$$f : \mathcal{H} \to \mathbb{C}, \qquad f(\gamma\tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z}).$$

   the form f gives rise to ϕ_f

$$\phi_f : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}.$$

   If f newform $\pi_f := \mathrm{GL}_2(\mathbb{A}_f)\phi_f$ irreducible automorphic rep'n.

Waldspurger formula I

Let G multiplicative group quaternion algebra B over F.

$$G(F_v) = \mathrm{GL}_2(F_v) \text{ for almost all } v = \sigma, \mathfrak{p}.$$

Waldspurger formula I

Let G multiplicative group quaternion algebra B over F.

$$G(F_v) = \mathrm{GL}_2(F_v) \text{ for almost all } v = \sigma, \mathfrak{p}.$$

Assume that $K \hookrightarrow B$, where K/F quadratic ext'n and let $d^\times t = \prod_v d^\times t_v$ be a *Haar measure* of $\mathbb{A}_K^\times$

$$\int_{\mathbb{A}_K^\times} \phi(t) d^\times t = \int_{\mathbb{A}_K^\times} \phi(\gamma t) d^\times t, \quad \forall \gamma \in \mathbb{A}_K^\times.$$

Waldspurger formula I

Let G multiplicative group quaternion algebra B over F.

$$G(F_v) = \mathrm{GL}_2(F_v) \text{ for almost all } v = \sigma, \mathfrak{p}.$$

Assume that $K \hookrightarrow B$, where K/F quadratic ext'n and let $d^\times t = \prod_v d^\times t_v$ be a *Haar measure* of $\mathbb{A}_K^\times$

$$\int_{\mathbb{A}_K^\times} \phi(t) d^\times t = \int_{\mathbb{A}_K^\times} \phi(\gamma t) d^\times t, \quad \forall \gamma \in \mathbb{A}_K^\times.$$

Given π an irred. automorphic rep'n of $G(\mathbb{A}_F)$, and $\chi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}$ character (irred. automorphic rep'n of $\mathrm{GL}_1(\mathbb{A}_K)$)

$$\ell(\phi, \chi) := \int_{\mathbb{A}_K^\times / K^\times} \chi(t)\phi(t) d^\times t, \qquad \phi \in \pi.$$

Waldspurger formula II

Theorem (Waldspurger)

We have that

$$\ell(\phi, \chi)^2 = C \cdot L(\pi, \chi, 1/2) \cdot \prod_v \alpha_v(\phi_v, \chi_v).$$

Waldspurger formula II

Theorem (Waldspurger)

We have that

$$\ell(\phi,\chi)^2 = C \cdot L(\pi,\chi,1/2) \cdot \prod_v \alpha_v(\phi_v,\chi_v).$$

- Let $\omega : \mathbb{A}_F^\times / F^\times \to \mathbb{C}$ be such that

$$\phi(ag) = \omega(a)\phi(g); \quad a \in \mathbb{A}_F^\times,$$

  central character. Then, if $\chi\,|_{\mathbb{A}_F^\times} \neq \omega$, the equality is $0 = 0$.

Waldspurger formula II

Theorem (Waldspurger)

We have that

$$\ell(\phi,\chi)^2 = C\cdot L(\pi,\chi,1/2)\cdot\prod_v \alpha_v(\phi_v,\chi_v).$$

- Let $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}$ be such that

  $$\phi(ag)=\omega(a)\phi(g);\quad a\in\mathbb{A}_F^\times,$$

  central character. Then, if $\chi\mid_{\mathbb{A}_F^\times}\neq\omega$, the equality is $0=0$.
- The term C is explicit.

Waldspurger formula II

Theorem (Waldspurger)

We have that

$$\ell(\phi,\chi)^2 = C \cdot L(\pi,\chi,1/2) \cdot \prod_v \alpha_v(\phi_v,\chi_v).$$

- Let $\omega : \mathbb{A}_F^\times / F^\times \to \mathbb{C}$ be such that

$$\phi(ag) = \omega(a)\phi(g); \quad a \in \mathbb{A}_F^\times,$$

  central character. Then, if $\chi \mid_{\mathbb{A}_F^\times} \neq \omega$, the equality is $0 = 0$.
- The term C is explicit.
- The terms $\alpha_v(\phi_v, \chi_v)$ are computable and

$$\alpha_v(\phi_v,\chi_v) = 1, \text{ for almost all } v.$$

Waldspurger formula II

Theorem (Waldspurger)

We have that

$$\ell(\phi,\chi)^2 = C \cdot L(\pi,\chi,1/2) \cdot \prod_v \alpha_v(\phi_v,\chi_v).$$

- Let $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}$ be such that

$$\phi(ag) = \omega(a)\phi(g); \quad a \in \mathbb{A}_F^\times,$$

  central character. Then, if $\chi \mid_{\mathbb{A}_F^\times} \neq \omega$, the equality is $0 = 0$.
- The term C is explicit.
- The terms $\alpha_v(\phi_v,\chi_v)$ are computable and

$$\alpha_v(\phi_v,\chi_v) = 1, \text{ for almost all } v.$$

- Other formulas for $|\ell(\phi,\chi)|^2$.

Eichler-Shimura

Let $d = [F : \mathbb{Q}]$ and

$$\Sigma_G := \{\sigma \in \Sigma_F;\ G(F_\sigma) = \mathrm{GL}_2(F_\sigma)\}, \qquad r_G := \#\Sigma_G$$

Eichler-Shimura

Let $d = [F : \mathbb{Q}]$ and

$$\Sigma_G := \{\sigma \in \Sigma_F;\ G(F_\sigma) = \mathrm{GL}_2(F_\sigma)\}, \qquad r_G := \#\Sigma_G$$

Eichler-Shimura map:

$$\left\{\begin{array}{l}\text{Automorphic} \\ \text{forms of weight} \\ (k_1+2, \cdots, k_d+2)\end{array}\right\} \longrightarrow \left\{\begin{array}{l} r_G\text{-cohomology} \\ \text{classes with coeff.} \\ \bigotimes_{i=1}^d \mathrm{Sym}^{k_i}\mathbb{C}^2 \end{array}\right\}$$

Eichler-Shimura

Let $d = [F : \mathbb{Q}]$ and

$$\Sigma_G := \{\sigma \in \Sigma_F;\ G(F_\sigma) = \mathrm{GL}_2(F_\sigma)\}, \qquad r_G := \#\Sigma_G$$

Eichler-Shimura map:

$$\left\{\begin{array}{l}\text{Automorphic} \\ \text{forms of weight} \\ (k_1+2, \cdots, k_d+2)\end{array}\right\} \longrightarrow \left\{\begin{array}{l} r_G\text{-cohomology} \\ \text{classes with coeff.} \\ \bigotimes_{i=1}^{d} \mathrm{Sym}^{k_i}\mathbb{C}^2 \end{array}\right\}$$

1. $f : \mathcal{H} \to \mathbb{C}$ modular form weight 2 for $\Gamma_0(N)$

$$c_f(\gamma) = \int_z^{\gamma z} f(\tau)d\tau \in H^1(\Gamma_0(N), \mathbb{C}).$$

Eichler-Shimura

Let $d = [F : \mathbb{Q}]$ and

$$\Sigma_G := \{\sigma \in \Sigma_F;\ G(F_\sigma) = \mathrm{GL}_2(F_\sigma)\}, \qquad r_G := \#\Sigma_G$$

Eichler-Shimura map:

$$\left\{\begin{array}{l}\text{Automorphic} \\ \text{forms of weight} \\ (k_1+2, \cdots, k_d+2)\end{array}\right\} \longrightarrow \left\{\begin{array}{l} r_G\text{-cohomology} \\ \text{classes with coeff.} \\ \bigotimes_{i=1}^d \mathrm{Sym}^{k_i}\mathbb{C}^2 \end{array}\right\}$$

1. $f : \mathcal{H} \to \mathbb{C}$ modular form weight 2 for $\Gamma_0(N)$

$$c_f(\gamma) = \int_z^{\gamma z} f(\tau) d\tau \in H^1(\Gamma_0(N), \mathbb{C}).$$

2. in general; given $\phi \in \pi$ (weight $(2, \cdots, 2)$)

$$c_\phi \in H^{r_G}(G(F), \mathcal{A}^\infty(\mathbb{C})), \qquad \mathcal{A}^\infty(\mathbb{C}) := \{\phi : G(\mathbb{A}_F^\infty) \to \mathbb{C}\}$$

Eichler-Shimura

Let $d = [F : \mathbb{Q}]$ and

$$\Sigma_G := \{\sigma \in \Sigma_F;\ G(F_\sigma) = \mathrm{GL}_2(F_\sigma)\}, \qquad r_G := \#\Sigma_G$$

Eichler-Shimura map:

$$\left\{ \begin{array}{l} \text{Automorphic} \\ \text{forms of weight} \\ (k_1+2, \cdots, k_d+2) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} r_G\text{-cohomology} \\ \text{classes with coeff.} \\ \bigotimes_{i=1}^d \mathrm{Sym}^{k_i}\mathbb{C}^2 =: V_{\underline{k}} \end{array} \right\}$$

1. $f : \mathcal{H} \to \mathbb{C}$ modular form weight 2 for $\Gamma_0(N)$

$$c_f(\gamma) = \int_z^{\gamma z} f(\tau)d\tau \in H^1(\Gamma_0(N), \mathbb{C}).$$

2. in general; given $\phi \in \pi$ (weight $(k_1 + 2, \cdots, k_d + 2)$)

$$c_\phi \in H^{r_G}(G(F), \mathcal{A}^\infty(V_{\underline{k}})), \qquad \mathcal{A}^\infty(V_{\underline{k}}) := \{\phi : G(\mathbb{A}_F^\infty) \to V_{\underline{k}}\}$$

Fundamental class I

K/F quadratic ext'n. Let T alg. group associated with $K^\times/F^\times$.

Fundamental class I

K/F quadratic ext'n. Let T alg. group associated with $K^\times/F^\times$.

$$\Sigma_T := \{\sigma \in \Sigma_F;\ T(F_\sigma) = F_\sigma^\times\}, \qquad r_T := \#\Sigma_T.$$

Fundamental class I

K/F quadratic ext'n. Let T alg. group associated with $K^\times/F^\times$.

$$\Sigma_T := \{\sigma \in \Sigma_F;\ T(F_\sigma) = F_\sigma^\times\}, \qquad r_T := \#\Sigma_T.$$

We can construct

$$\xi \in H_{r_T}(T(F), \mathcal{C}_c^\infty(\mathbb{Z})), \qquad \mathcal{C}_c^\infty(\mathbb{Z}) := \{f : T(\mathbb{A}_F^\infty) \to \mathbb{Z},\ \text{comp. supp}\}.$$

Fundamental class I

K/F quadratic ext'n. Let T alg. group associated with $K^\times/F^\times$.

$$\Sigma_T := \{\sigma \in \Sigma_F;\ T(F_\sigma) = F_\sigma^\times\}, \qquad r_T := \#\Sigma_T.$$

We can construct

$$\xi \in H_{r_T}(T(F), \mathcal{C}_c^\infty(\mathbb{Z})), \qquad \mathcal{C}_c^\infty(\mathbb{Z}) := \{f : T(\mathbb{A}_F^\infty) \to \mathbb{Z},\ \text{comp. supp}\}.$$

- It is constructed by means of

$$\Lambda = \mathcal{O}_K^\times/\mathcal{O}_F^\times, \quad \mathbb{Z} - \mathrm{rank} = r_T.$$

Fundamental class I

K/F quadratic ext'n. Let T alg. group associated with $K^\times/F^\times$.

$$\Sigma_T := \{\sigma \in \Sigma_F;\ T(F_\sigma) = F_\sigma^\times\}, \qquad r_T := \#\Sigma_T.$$

We can construct

$$\xi \in H_{r_T}(T(F), \mathcal{C}_c^\infty(\mathbb{Z})), \qquad \mathcal{C}_c^\infty(\mathbb{Z}) := \{f : T(\mathbb{A}_F^\infty) \to \mathbb{Z},\ \text{comp. supp}\}.$$

- It is constructed by means of

$$\Lambda = \mathcal{O}_K^\times/\mathcal{O}_F^\times, \quad \mathbb{Z} - \mathrm{rank} = r_T.$$

- We have r_T copies of $\mathbb{R}_+$ in $T(\mathbb{A}_F)$, variables $x_1, \cdots, x_{r_T}$

$$0 \longrightarrow C^{\sigma_i}(T(\mathbb{A}_F), \mathbb{C}) \longrightarrow C(T(\mathbb{A}_F), \mathbb{C}) \xrightarrow{\frac{\partial}{\partial x_i}} C(T(\mathbb{A}_F), \mathbb{C}) \longrightarrow 0$$

Fundamental class II

The above short exact seq. provides in long exact seq.

$$\cdots \longrightarrow H^n(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \xrightarrow{d^i} H^{n+1}(T(F), C^{\sigma_i}(T(\mathbb{A}_F), \mathbb{C})) \longrightarrow \cdots$$

Fundamental class II

The above short exact seq. provides in long exact seq.

$$\cdots \longrightarrow H^n(T(F), C^{S}(T(\mathbb{A}_F), \mathbb{C})) \xrightarrow{d^i} H^{n+1}(T(F), C^{S\cup\sigma_i}(T(\mathbb{A}_F), \mathbb{C})) \longrightarrow \cdots$$

Fundamental class II

The above short exact seq. provides in long exact seq.

$$\cdots \longrightarrow H^n(T(F), C^S(T(\mathbb{A}_F), \mathbb{C})) \xrightarrow{d^i} H^{n+1}(T(F), C^{S\cup\sigma_i}(T(\mathbb{A}_F), \mathbb{C})) \longrightarrow \cdots$$

Giving rise

$$\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \stackrel{d_1\circ\cdots\circ d_{r_T}}{\longrightarrow} H^{r_T}(T(F), C^{\Sigma_T}(T(\mathbb{A}_F), \mathbb{C}))$$

Fundamental class II

The above short exact seq. provides in long exact seq.

$$\cdots \longrightarrow H^n(T(F), C^S(T(\mathbb{A}_F), \mathbb{C})) \xrightarrow{d^i} H^{n+1}(T(F), C^{S\cup\sigma_i}(T(\mathbb{A}_F), \mathbb{C})) \longrightarrow \cdots$$

Giving rise

$$\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \stackrel{d_1\circ\cdots\circ d_{r_T}}{\longrightarrow} H^{r_T}(T(F), C^\infty(\mathbb{C}))$$

Fundamental class II

The above short exact seq. provides in long exact seq.

$$\cdots \longrightarrow H^n(T(F), C^S(T(\mathbb{A}_F), \mathbb{C})) \xrightarrow{d^i} H^{n+1}(T(F), C^{S\cup\sigma_i}(T(\mathbb{A}_F), \mathbb{C})) \longrightarrow \cdots$$

Giving rise

$$\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \overset{d_1\circ\cdots\circ d_{r_T}}{\longrightarrow} H^{r_T}(T(F), \mathcal{C}^\infty(\mathbb{C}))$$

$\xi \in H_{r_T}(T(F), \mathcal{C}_c^\infty(\mathbb{Z}))$ is characterized by

$$\boxed{\partial f \cap \xi = \int_{T(\mathbb{A}_F)/T(F)} f(t) d^\times t,}$$

for all $f \in H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C}))$.

Waldspurger in higher cohomology I

1. Assume that $\Sigma_G = \Sigma_T$, $\qquad r_G = r_T =: r$.

Waldspurger in higher cohomology I

1. Assume that $\Sigma_G = \Sigma_T, \qquad r_G = r_T =: r$.
2. Notice
$$H^r(G(F), \mathcal{A}^\infty(\mathbb{C})) \circlearrowright G(\mathbb{A}_F^\infty).$$
Given π aut. rep. weight $(2, \cdots, 2)$, by Eichler-Shimura
$$\pi^\infty \subseteq H^r(G(F), \mathcal{A}^\infty(\mathbb{C})).$$

Waldspurger in higher cohomology I

1. Assume that $\Sigma_G = \Sigma_T, \qquad r_G = r_T =: r$.
2. Notice

$$H^r(G(F), \mathcal{A}^\infty(V_{\underline{k}})) \circlearrowright G(\mathbb{A}_F^\infty).$$

Given π aut. rep. weight $(k_1 + 2, \cdots, k_d + 2)$, by Eichler-Shimura

$$\pi^\infty \subseteq H^r(G(F), \mathcal{A}^\infty(V_{\underline{k}})).$$

Waldspurger in higher cohomology I

1. Assume that $\Sigma_G = \Sigma_T, \qquad r_G = r_T =: r$.
2. Notice
$$H^r(G(F), \mathcal{A}^\infty(\mathbb{C})) \circlearrowleft G(\mathbb{A}_F^\infty).$$
Given π aut. rep. weight $(2, \cdots, 2)$, by Eichler-Shimura
$$\pi^\infty \subseteq H^r(G(F), \mathcal{A}^\infty(\mathbb{C})).$$
3. Assume that $\omega = 1$: $\mathbb{A}_F^\times \to \mathbb{C}$, then
$$\mathrm{res} : H^r(G(F), \mathcal{A}^\infty(\mathbb{C})) \longrightarrow H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$$
For any loc. constant character $\chi : T(\mathbb{A}_F^\infty)/T(F) \longrightarrow \mathbb{C}$
$$\boxed{\ell^r(c, \chi) := (\mathrm{res}(c) \cup \chi) \cap \xi \in \mathbb{C}} \qquad c \in \pi^\infty \subseteq H^r(G(F), \mathcal{A}^\infty(\mathbb{C}))$$

Waldspurger in higher cohomology II

Theorem (M.)

Let π *weight* $(2, \cdots, 2)$*; and let* χ *loc. constant*

$$\ell^r(c,\chi)^2 = C \cdot L(\pi,\chi,1/2) \cdot \prod_{\mathfrak{p}} \alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c \in \pi^\infty \subseteq H^r(G(F), \mathcal{A}^\infty(\mathbb{C}))$.

Waldspurger in higher cohomology II

Theorem (M.)

Let π *weight* $(2,\cdots,2)$*; and let* χ *loc. constant*

$$\ell^r(c,\chi)^2 = C\cdot L(\pi,\chi,1/2)\cdot\prod_{\mathfrak{p}}\alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c\in\pi^\infty\subseteq H^r(G(F),\mathcal{A}^\infty(\mathbb{C}))$*.*

- C is totally explicit (non-zero).

Waldspurger in higher cohomology II

Theorem (M.)

Let π *weight* $(2,\cdots,2)$*; and let* χ *loc. constant*

$$\ell^r(c,\chi)^2 = C \cdot L(\pi,\chi,1/2) \cdot \prod_{\mathfrak{p}} \alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c \in \pi^\infty \subseteq H^r(G(F),\mathcal{A}^\infty(\mathbb{C}))$*.*

- C is totally explicit (non-zero).
- F any number field.

Waldspurger in higher cohomology II

Theorem (M.)

Let π *weight* $(k_1+2,\cdots,k_n+2)$ *even; and let* χ *loc. polynomial*

$$\ell^r(c,\chi)^2 = C\cdot L(\pi,\chi,1/2)\cdot\prod_{\mathfrak{p}}\alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c\in\pi^\infty\subseteq H^r(G(F),\mathcal{A}^\infty(V_{\underline{k}}))$.

- C is totally explicit (non-zero).
- F any number field.
- Also works weight $(k_1+2,\cdots,k_n+2)$ even.

Waldspurger in higher cohomology II

Theorem (M.)

Let π weight $(k_1+2,\cdots,k_n+2)$ even; and let χ loc. polynomial

$$\ell^r(c,\chi)^2 = C\cdot L(\pi,\chi,1/2)\cdot\prod_{\mathfrak{p}}\alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c\in\pi^\infty\subseteq H^r(G(F),\mathcal{A}^\infty(V_{\underline{k}}))$.

- C is totally explicit (non-zero).
- F any number field.
- Also works weight $(k_1+2,\cdots,k_n+2)$ even.
  Only proven F tot. real!

Waldspurger in higher cohomology II

Theorem (M.)

Let π *weight* $(2,\cdots,2)$*; and let* χ *loc. constant*

$$\ell^r(c,\chi)^2 = C \cdot L(\pi,\chi,1/2)\cdot\prod_{\mathfrak{p}}\alpha_{\mathfrak{p}}(c_{\mathfrak{p}},\chi_{\mathfrak{p}}).$$

for all $c\in\pi^\infty\subseteq H^r(G(F),\mathcal{A}^\infty(\mathbb{C}))$*.*

- C is totally explicit (non-zero).
- F any number field.
- Also works weight $(k_1+2,\cdots,k_n+2)$ even.
  Only proven F tot. real!
- Interpolation properties (anticyclotomic) p-adic L-functions.

Idea of the proof

- Assume $\chi = 1$ for simplicity.

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\{\phi : G(\mathbb{A}_F) \to \mathbb{C}\} \qquad G(F)$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Then,

$$\ell^r(\mathrm{ES}(\phi), 1) = \mathrm{res}(\mathrm{ES}(\phi)) \cap \xi.$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Then,

$$\ell^r(\mathrm{ES}(\phi), 1) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)}) \cap \xi,$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Then,

$$\ell^r(\mathrm{ES}(\phi), 1) = \int_{T(\mathbb{A}_F)/T(F)} \phi(v)(t) d^\times t,$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Then,

$$\ell^r(\mathrm{ES}(\phi), 1) = \ell(\phi(v), 1),$$

Idea of the proof

- Assume $\chi = 1$ for simplicity.
- Automorphic form weight $(2, \cdots, 2)$:

$$\phi \in \mathrm{Hom}_{(\mathcal{G},K)}\left(\bigotimes_{\sigma\in\Sigma_G} D(2), \{\phi : G(\mathbb{A}_F) \to \mathbb{C}\}\right)^{G(F)}$$

- Recall $\partial : H^0(T(F), C(T(\mathbb{A}_F), \mathbb{C})) \to H^r(T(F), \mathcal{C}^\infty(\mathbb{C}))$

$$\boxed{\mathrm{res}(\mathrm{ES}(\phi)) = \partial(\phi(v) \mid_{T(\mathbb{A}_F)})} \in H^r(T(F), \mathcal{C}^\infty(\mathbb{C})),$$

  for some (explicit) $v \in \bigotimes_{\sigma\in\Sigma_G} D(2)$.

Then,

$$\ell^r(\mathrm{ES}(\phi), 1) = \ell(\phi(v), 1),$$

This it follows from (classical) Waldspurger formula.
HARD PART! compute $\alpha_\sigma(v_\sigma, \chi_\sigma)$

Waldspurger formula in higher cohomology

S. Molina

May 13, 2021