SPLITTING OF SEPARATRICES IN HAMILTONIAN SYSTEMS AND SYMPLECTIC MAPS

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1. Introduction

A century ago, the phenomenon of the *splitting of separatrices* was discovered by Henri Poincaré in his celebrated memoir on the three-body problem [39]. While trying to integrate the problem of the three bodies, expanding the solutions with respect to a small parameter, Poincaré noticed that the main obstruction was due to the possibility of transversal intersection of invariant manifolds that were coincident (separatrices) for the unperturbed integrable problem. To measure the size of such splitting, he developed a perturbative method in the parameter of perturbation, say ε, and he was confronted with a *singular* separatrix splitting problem, in the sense that the separatrices of the unperturbed problem depended on ε in an essential way. He already noticed that the size of the splitting of the separatrices predicted by his perturbative method was exponentially small with respect to ε [39, page 223], a fact which prevented him to provide rigorous results, since the remainder of his perturbative expansion was, in principle, $O(\varepsilon^2)$.

Seventy years later, the Poincaré perturbative method was rediscovered by Melnikov and Arnold [36, 2], giving rise to the well-known Poincaré-Melnikov-Arnold theory, more shortly addressed as the Melnikov method [27, 48].

The goal of this lecture is to review such theory, for flows as well as for maps. For flows, we will address specifically the singular separatrix splitting in Hamiltonians with one and a half degrees of freedom. For maps, the Melnikov method is just in its first steps, and we will only consider here the *regular* case, where a direct application of a first order theory is enough. It is worth remarking that we will restrict ourselves to the case of separatrices to a periodic orbit for flows, and to a fixed point for a map. This means that the more interesting cases of separatrices to invariant tori [48] will not be dealt here, in spite of their crucial interest for the problem of the Arnold diffusion, for which we refer to the lecture by Pierre Lochak [31].

The method we present here to handle the singular separatrix splitting for Hamiltonian flows was initiated by Lazutkin and co-workers [29, 25], and it is based on the construction of a splitting function which is invariant under the action of the perturbed flow. The analyticity of the problem is pushed forward to compute this splitting function for complex values, and to recover it in the real world in form of an exponentially small in ε measure of the splitting. This method can also handle rapidly quasiperiodic perturbations of one degree of freedom Hamiltonian systems, as is explained in the lecture by Vassili Gelfreich [7]. We are firmly convinced that it can be successfully applied to the singular case for planar area preserving maps. At the present time, the only drawback of this theory, that may prevent its application to more general frameworks, is the existence of a convergent normal form in a neighborhood of the invariant object that possesses the separatrix.

Concerning related work, let us recall that other complete proofs of lower bounds or asymptotic expressions for the rapidly forced pendulum or very similar systems, using different kinds of methods and hypotheses, can be found in [12, 23, 16, 20, 47, 41]. Upper estimates, but valid for more general systems, can be found in [38, 18, 43, 19, 17].

For maps, the Melnikov method is not so well-developed [14, 21, 22, 26], so we expound here the first order theory, giving special emphasis to the case of analytic symplectic maps, where a computable framework is available [9, 10].

In particular, we frequently rely on the *Melnikov potential*, a function defined on the unperturbed separatrix, as a useful tool in the framework of symplectic maps, as well as in Hamiltonian flows. Its importance is even bigger in the high-dimensional case, which will not be considered in this lecture, for the sake of brevity. Thus, we will restrict ourselves to *planar* twist maps, whereas the case of the $2n$-dimensional twist maps will be addressed in the lecture [11].

The main difference between the Melnikov potential (or function) in these two settings, is that for maps, the complex period of the unperturbed homoclinic solution is not lost, and hence the Melnikov potential is a *doubly* periodic function. This extra property makes easier for maps the proof of the splitting of separatrices for a very wide kind of perturbations, since the complex variable methods are readily applicable.

Full details of the ideas presented here are spread out in several papers [13, 9, 10], where the required framework and hypotheses, as well as the results obtained here, are thoroughly detailed, and other more general situations are dealt with. We hope that this survey can provide a good starting point for those who want to know about some of the tools utilized in the search of homoclinic orbits, and may want to try to overcome the difficult points not addressed in this lecture.

2. Hamiltonian systems with one and a half degrees of freedom

Along the first part of this lecture, we will deal with Hamiltonians of the form

$$h(x, t/\varepsilon) = h^0(x) + \varepsilon^p h^1(x, t/\varepsilon),$$

where $h^0(x) = h^0(x_1, x_2)$ is a Hamiltonian with one degree of freedom, $h^1(x, \theta)$ is 2π-periodic in $\theta = t/\varepsilon$, $\varepsilon > 0$ is an small parameter, and $p > 0$.

The Hamiltonian system associated to h^0 is integrable. For simplicity, we shall assume that it is a classical Hamiltonian: $h^0(x) = x_2^2/2 + V(x_1)$. Its associated system of differential equations is given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = f(x_1), \tag{1}$$

where $f(x_1) = -V'(x_1)$, which can be written also as a second order equation $\ddot{x}_1 = f(x_1)$.

The Hamiltonian system associated to the complete Hamiltonian is:

$$\begin{aligned} \dot{x}_1 &= x_2 + \varepsilon^p \partial_2 h^1(x, t/\varepsilon), \\ \dot{x}_2 &= f(x_1) - \varepsilon^p \partial_1 h^1(x, t/\varepsilon). \end{aligned} \tag{2}$$

The complete Hamiltonian $h(x, t/\varepsilon)$ is a perturbation of h^0, which is very rapidly oscillating in time.

We will assume some hypotheses. First, we will require the existence of a *separatrix* for the unperturbed solution to a saddle point, i.e., $f(0) = 0$, $f'(0) > 0$, and $\omega^0 := \sqrt{f'(0)} > 0$:

H1 The unperturbed system (1) has a saddle point at the origin with characteristic exponents $\pm\omega^0$, with $\omega^0 > 0$, and there exists a homoclinic solution $x^0(t) = (x_1^0(t), x_2^0(t))$ to this point: $x^0(t) \longrightarrow 0$ for $t \to \pm\infty$.

To visualize better the dynamical properties of the $2\pi\varepsilon$-periodic in time system (2), we can consider the associated *Poincaré map* defined by:

$$P(x_0) = x(2\pi\varepsilon), \tag{3}$$

where $x(t)$ is the solution of system (2) that begins at x_0 when $t = 0$.

For $h^1 \equiv 0$, system (2) becomes autonomous and therefore the phase portraits of the Poincaré map P and system (1) are identical. In fact, this phase portrait is foliated by the level curves of the Hamiltonian h^0. Assuming, without loss of generality, $V(0) = 0$, the homoclinic orbit x^0 is contained in the level curve $h^0(x) = 0$.

For $h^1 \not\equiv 0$ and $0 < |\varepsilon| \ll 1$, the dynamics of system (2) becomes more intricate and the phase portrait of the Poincaré map P looks different. There exists a hyperbolic fixed point x^∞ close to $(0, 0)$, whose unstable and stable curves C^{u}, C^{s} intersect—P is an area preserving map—but generically they do not coincide. We denote $x^h := (x_1^h, x_2^h)$ the homoclinic point that is closest to the unperturbed one $x^0(0)$. The evolution of x^∞ under the flow of system (2) gives rise to a $2\pi\varepsilon$*-periodic orbit* γ_p, as well as to its associated invariant manifolds $W^{\mathrm{u}}(\gamma_p)$, $W^{\mathrm{s}}(\gamma_p)$.

The splitting of separatrices can be measured by different quantities, like the distance d between the two invariant curves near $x^0(0)$, the angle α between the invariant curves at the homoclinic point x^h, or the area A of the lobe that remains between the two invariant curves from x^h to their next intersection. Among them, the area A has the property of being *invariant*, that is, it does not depend on the homoclinic point x^h. Even more, it is invariant under canonical changes of variables.

The standard tool to measure such splitting quantities is the *Melnikov function*

$$M(s, \varepsilon) = \int_{-\infty}^{+\infty} \{h^0, h^1\}(x^0(t + s), t/\varepsilon)\, \mathrm{d}t = \sum_{k \neq 0} M_k(\varepsilon)\, \mathrm{e}^{\mathrm{i}ks/\varepsilon}, \tag{4}$$

called also the Melnikov integral. A direct approach of the Melnikov theory [27] gives, for instance for the distance d, the following asymptotic expression:

$$d = d(s) = \varepsilon^p \frac{M(s, \varepsilon)}{\|x^0(s)\|} + \mathrm{O}(\varepsilon^{2p}). \tag{5}$$

The variable s simply parameterizes the separatrix $x^0(s)$, so that $d(s)$ is the distance between invariant curves in the normal direction to $x^0(s)$.

We make now some comments on the features of the Melnikov method that will hold also in other frameworks. The Melnikov function is $2\pi\varepsilon$-periodic with respect to s, in spite of the fact that $d(s)$ is not. In other words, the Melnikov function is invariant under the action of the *unperturbed Poincaré map* P_0, i.e., the Poincaré map (3) for $h^1 \equiv 0$. It is also worth remarking that $M(s, \varepsilon)$ has zero mean, and in fact it can be written as

$$M(s, \varepsilon) = \frac{\partial L}{\partial s}(s, \varepsilon), \quad L(s, \varepsilon) = -\int_{-\infty}^{+\infty} h^1(x^0(t + s), t/\varepsilon)\, \mathrm{d}t,$$

where $L(s,\varepsilon)$ is called the *Melnikov potential.* Using a Lagrangian formalism for our model, $h(x_1,x_2,t/\varepsilon)-x_2\cdot\dot{x}_1=\mathcal{L}(x_1,\dot{x}_1,t/\varepsilon)=\mathcal{L}_0+\varepsilon^p\mathcal{L}_1+\mathrm{O}\left(\varepsilon^{2p}\right)$, the Melnikov potential takes the form $L(s,\varepsilon)=\int_{-\infty}^{+\infty}\mathcal{L}_1(x_1^0(t+s),\dot{x}_1^0(t+s),t/\varepsilon)\,\mathrm{d}t$, and can also be obtained via a variational approach [1]. Moreover, L is as smooth as the function $t\mapsto h^1(x^0(t),\theta)$.

We are going to consider the analytic case. (In particular, we will able to compute the Melnikov function using residue theory.) This means that we will assume some analyticity properties on the unperturbed system, as well as on the separatrix:

H2 The function $f(x_1)$ is real entire, and $x_2^0(u)=\dot{x}_1^0(u)$ is analytic on a strip $|\Im u|<a$, with a pole of order r at $u=\pm a\mathrm{i}$ as its only singularity on each line $\Im u=\pm a\mathrm{i}$.

For an entire function $f(x_1)$, it is not difficult to check that the homoclinic solution $x^0(u)$ behaves very well for large $|\Re u|$. In particular, $x^0(u)$ is $T\mathrm{i}$-periodic, for $T=2\pi/\omega_0$, and the analyticity of $x_2^0(u)$ on a complex strip $|\Im u|<a$, for some $a<T$, follows from the analyticity of the unperturbed system. The main *restriction* of this hypothesis is the assumption that the *only* singularity of $x_2^0(u)$ on each component of the boundary of this strip is a *pole* of some order $r\geq 1$, which implies a severe restriction on the behavior of $f(x_1)$ for x_1 big enough. More precisely, if $r\geq 2$, it is easy to check that $f(x_1)$ has to be a polynomial of degree 2 or 3, for $r=3$ or $r=2$, respectively. Analogously, $r=1$ can only take place if $f(x_1)$ is a trigonometric polynomial of degree 1 or 2, and then $x_1^0(u)\sim \mathrm{i}k\log(u\mp a\mathrm{i})$ for $u\to\pm a\mathrm{i}$, with k equal to 2 or 1, respectively. All the other values of r and of the degree of the (trigonometric) polynomial $f(x_1)$ give rise to branching points ("poles of fractional order") as singularities of $x_2^0(u)$.

A direct application of the Melnikov theory is useful as long as the Melnikov term dominates the reminder in (5). This is the typical case for the regular cases, where the Melnikov function does not depend on the parameter of perturbation. Unfortunately, in our model the Melnikov function not only depends on ε, but it is also exponentially small in ε (in fact $\mathrm{o}(\,\mathrm{e}^{-a/\varepsilon})$, as will be shown in Corollary 2), and a direct application of equation (5) only gives that the $\mathrm{O}(\varepsilon^{2p})$ term is the one that dominates.

To be able to validate the role of the Melnikov term in equation (5), we consider s complex. A crucial point is to control the perturbative function h^1, as well as its derivatives, over the separatrix $x^0(u)$ near the singularity $u=a\mathrm{i}$. For the clearness of the exposition, we simply will assume that h^1 is of polynomial type in x:

H3 The function $h^1(x,\theta)$ is 2π-periodic and $\mathcal{C}^1$ in θ, with zero mean: $\int_0^{2\pi}h^1(x,\theta)\,\mathrm{d}\theta=0$. With respect to x, it can take either of the following forms:

(a) if f is 2π-periodic, h^1 is a trigonometric polynomial in x_1 and a polynomial in x_2; $h^1(x,\theta)=x_1 g(\theta)$ is also allowed,

(b) h^1 is a polynomial in x, in the case that f is not 2π-periodic.

As a consequence of Hypothesis *H3*, $h^1(x,\theta)$ can be written as a sum of monomials in the variable x, each of which has a pole at $u=\pm a\mathrm{i}$, when $x=x^0(u)$, for every θ. We will denote by ℓ the greatest order of this pole among these monomials, and we will call it the *order of the perturbation on the separatrix* or, even more precisely, the *order of the perturbation on the singularity of the homoclinic solution.*

By its definition, it is not difficult to observe that ℓ satisfies $\ell\geq r-1$. In general, ℓ will be the order of the pole of $h^1(x^0(u),\theta)$ at $u=\pm a\mathrm{i}$, if there is no cancellation between the different monomials of h^1, when evaluated on $x^0(u)$. An example where these cancellations take place is provided by $h^1=h^0(x)$. In such case, $h^0(x^0(u))$ is constant (and hence with no pole at all), but for instance the monomial $x_2^2/2$ has a pole of order $2r$. The same happens if h^1 is functionally dependent on h^0.

Let us note that in the case $h^1(x,\theta) = x_1 g(\theta)$, system (2) is equivalent to the scalar equation $\ddot{x}_1 = f(x_1) + \varepsilon^p g(t/\varepsilon)$, i.e., the perturbation only depends on time. In the trigonometric case, $x_1^0(u)$ has logarithmic singularities, but we take, by convention, $\ell = 0$.

The main point in measuring an exponentially small splitting of separatrices consists of defining a $2\pi\varepsilon$-periodic distance $d(s)$ in (5), which means that it is invariant under the action of the Poincaré map (3). In this way, we will introduce in (20) the so-called *splitting function* ψ, after introducing some suitable "flow-box" canonical coordinates (S,E). In these coordinates, S is a common parameter for both the stable and the unstable manifolds, $E = 0$ is the equation of the stable manifold, and $E = \psi(S)$ is the equation for the unstable one. It is important to notice that the splitting function is $2\pi\varepsilon$-periodic and independent of time, and hence it gives an *invariant* measure of the distance between the invariant manifolds. In particular, its zeros give rise to homoclinic orbits, and all the splitting quantities are obtained from it. Thus, the area A and the angle α given in Theorem 1 are expressed in terms of the integral and the derivative of the splitting function ψ.

The next theorem states a better approximation than (5) for the area A and the splitting angle α, for $p :=$ power of $\varepsilon > \ell =$ order of the perturbation on the separatrix.

Theorem 1 (Upper estimate) *Under hypotheses H1–H3, assume that $\gamma := p - \ell > 0$. Then, for $\varepsilon \to 0^+$, the following formulae hold:*

$$\begin{aligned} A &= \varepsilon^p \left[\int_{s_0}^{\bar{s}_0} M(\sigma,\varepsilon)\,\mathrm{d}\sigma \right] + \mathrm{O}(\varepsilon^{2\gamma+r}, \varepsilon^{p+2})\,\mathrm{e}^{-a/\varepsilon}, \\ \sin\alpha &= \varepsilon^p \frac{M'(s_0,\varepsilon)}{\|x^0(s_0)\|^2} + \mathrm{O}(\varepsilon^{2\gamma+r-2}, \varepsilon^p)\,\mathrm{e}^{-a/\varepsilon}, \end{aligned}$$

where $s_0 < \bar{s}_0$ are the two zeros of the Melnikov function (4) closest to zero.

This theorem gives upper sharp estimates of exponentially small order for the area A and the angle α. We now introduce an additional hypothesis on the Poisson bracket of h^0 and h^1 over $x^0(u)$:

H4 $J_{\pm 1}(x^0(u))$ has a pole of order exactly $\ell + 1$ at $u = \pm a\mathrm{i}$, where

$$J(x,\theta) := \{h^0, h^1\}(x,\theta) = \sum_{k \neq 0} J_k(x)\,\mathrm{e}^{\mathrm{i}k\theta}.$$

If one writes the Laurent expansion $J_{\pm 1}(x^0(u)) = \sum_{k \le \ell+1} J_{\pm 1,k}(u \mp a\mathrm{i})^{-k}$ of $J_{\pm 1}(x^0(u))$ at $u = \pm a\mathrm{i}$, hypothesis *H4* is equivalent to assume that the coefficient $J_{1,\ell+1} = \overline{J_{-1,\ell+1}}$ is not zero. Under this generic additional hypothesis, a direct computation of the Melnikov function shows that Theorem 1 provides asymptotic expressions:

Corollary 2 (Asymptotic expression) *If moreover hypothesis H4 holds, the first terms of A and $\sin\alpha$ in Theorem 1—those containing the Melnikov function (4)—are not zero and are dominant with respect to the second ones, for $\varepsilon \to 0^+$:*

$$\begin{aligned} A &= 4\,|J_{1,\ell+1}|\,\varepsilon^{\gamma+1}\,\mathrm{e}^{-a/\varepsilon}\left[1 + \mathrm{O}(\varepsilon^{\gamma+r-1}, \varepsilon^{\ell+1})\right], \\ \sin\alpha &= -\frac{2\,|J_{1,\ell+1}|}{\|x^0(0)\|^2}\varepsilon^{\gamma-1}\,\mathrm{e}^{-a/\varepsilon}\left[1 + \mathrm{O}(\varepsilon^{\gamma+r-1}, \varepsilon)\right]. \end{aligned}$$

Concerning optimality of p, our estimates are valid for $p > \ell$, which is the condition required for the Extension Theorem 5 in some complex strip (15). We believe that this Extension Theorem is not true if $p < \ell$ (this has to do with the fact that the term $\varepsilon^p M(s, \varepsilon)$ of the Melnikov method is not small in the complex strip $|\Im s| \leq a - \varepsilon$ for $p \leq \ell$). Of course, we do not claim that $p > \ell$ is the optimal lower bound, but it is clear that new methods are needed for lower ranges of p. For instance, D. Treschev [47], using a continuous averaging method, proves for an specific trigonometric example with $\ell = 2$, that the splitting is given by the Melnikov method for $p > 0 = \ell - 2$. Also in the trigonometric case, G. Gallavotti [20] gives $p > \ell - 1$ as a probably optimal lower bound, and recent papers by C. Simó [43] and V. Gelfreich [24], as well as numerical experiments, seem to indicate that the lower bound can be $p > \ell - 2$.

Let us now discuss some examples satisfying hypotheses *H1–H4*:

1. A *forced pendulum* equation, with Hamiltonian

$$h = \frac{x_2^2}{2} - (\cos x_1 + 1) + \varepsilon^p x_2 \cos x_1 \sin \frac{t}{\varepsilon} \,. \tag{6}$$

   The pendulum equation has homoclinic orbits $\Gamma_\pm = \{(x_1^0(\pm t), \pm x_2^0(t))\}$, where $x_1^0(t) = 2\arctan(\sinh t)$, $x_2^0(u) = \dot{x}_1^0(u)$ has poles of order $r = 1$ at $u = \pm\pi\,\mathrm{i}/2$, and $\ell = 3$.
2. A perturbed *Duffing* equation, with Hamiltonian

$$h = \frac{x_2^2}{2} - \frac{x_1^2}{2} + \frac{x_1^4}{4} + \varepsilon^p \left(\frac{x_2^2}{2} \cos \frac{t}{\varepsilon} + x_1 x_2 \sin \frac{2t}{\varepsilon} \right) . \tag{7}$$

   For the Duffing equation, the homoclinic orbits are $\Gamma_\pm = \{(\pm x_1^0(t), x_2^0(\pm t))\}$, where $x_1^0(t) = \sqrt{2}/\cosh t$, $x_2^0(u) = \dot{x}_1^0(u)$ has poles of order $r = 2$ at $u = \pm\pi\,\mathrm{i}/2$, and $\ell = 4$.
3. A perturbed *cubic potential* equation, with Hamiltonian

$$h = \frac{x_2^2}{2} - \frac{x_1^2}{2} + \frac{x_1^3}{3} + \varepsilon^p x_1 \cos \frac{t}{\varepsilon} \,. \tag{8}$$

   The unperturbed cubic potential has the homoclinic orbit $\Gamma = \{(x_1^0(t), x_2^0(t))\}$, where $x_1^0(t) = (\sqrt{3}/2)\,(\cosh(t/2))^{-2}$, $x_2^0(u) = \dot{x}_1^0(u)$ has poles of order $r = 3$ at $u = \pm\pi\,\mathrm{i}$, and $\ell = 2$.

Applying Theorem 1 and Corollary 2 to the examples (6), (7) and (8), we get the following corollary.

Corollary 3 *For $\varepsilon \to 0^+$, the following estimates hold:*

1. $A = \frac{16}{3}\pi\varepsilon^{p-2}\,\mathrm{e}^{-\pi/2\varepsilon}[1 + \mathrm{O}(\varepsilon^{p-3}, \varepsilon^4)]$*, for the pendulum equation (6), if* $p > 3$*;*
2. $A = \frac{4}{3}\pi\varepsilon^{p-3}\,\mathrm{e}^{-\pi/2\varepsilon}[1 + \mathrm{O}(\varepsilon^{p-3}, \varepsilon^5)]$*, for the Duffing equation (7), if* $p > 4$*;*
3. $A = 24\pi\varepsilon^{p-1}\,\mathrm{e}^{-\pi/\varepsilon}[1 + \mathrm{O}(\varepsilon^{p}, \varepsilon^3)]$*, for the cubic potential equation (8), if* $p > 2$*.*

2.1. SKETCH OF THE PROOF

First of all, to deal with the local behavior of system (2), we use a Normal Form Theorem, which asserts that the Birkhoff normal form is convergent in a neighborhood of the origin, whose size is independent of ε. Besides, the normal form and the change of variables to normal form are, respectively, $\mathrm{O}(\varepsilon^{p+2})$ and $\mathrm{O}(\varepsilon^{p+1})$-close to the unperturbed ones, that is for $h^1 \equiv 0$. The proof of this fact is based on a parameterized version of a well known theorem due to Moser [37]. More recent proofs, valid for more degrees of freedom, can be found in [4, 6].

The Normal Form Theorem provides "natural" parameterizations $x^{\mathrm{u}}(t,s)$ and $x^{\mathrm{s}}(t,s)$ for the local invariant manifolds $W^{\mathrm{u}}_{\mathrm{loc}}(\gamma_p)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(\gamma_p)$, respectively for $\Re(t+s) < -T$ and $\Re(t+s) > T$. These parameterizations $x^{\mathrm{u}}(t,s)$, $x^{\mathrm{s}}(t,s)$ are called "natural" [9], since they are formed by solutions of system (2) in the real variable t, and the action of the Poincaré map is simply a shift of amount $2\pi\varepsilon$ in the complex variable s. It is worth mentioning that they are uniquely determined except for a change of parameter $s = S + \phi(S))$, for a $2\pi\varepsilon$-periodic function ϕ of size $\mathrm{O}(\varepsilon^{p+1})$.

From the explicit solution of the Normal Form system, one obtains the existence of local flow box coordinates outside of the local unstable invariant manifold $W^{\mathrm{u}}_{\mathrm{loc}}(\gamma_p)$.

Theorem 4 (Flow Box Theorem) *There exists a canonical change of variables*

$$(x, \theta = t/\varepsilon) \in \mathcal{U} \longmapsto (S, E, \theta) = (\mathcal{S}(x,\theta), \mathcal{E}(x,\theta), \theta) \in \mathcal{V}, \tag{9}$$

analytic in x, 2π-periodic and $\mathcal{C}^2$ in θ on $\mathcal{U} = \{(x,\theta) \in \mathbb{C}^2 \times \mathbb{R} : \|x - \gamma_p(\theta)\| < r_0^2\} \setminus W^{\mathrm{u}}_{\mathrm{loc}}(\gamma_p)$, with r_0 independent of ε, such that transforms system (2) in a flow box system

$$\dot S = 1, \qquad \dot E = 0, \tag{10}$$

and satisfies:

1. $$\mathcal{S}(x,\theta) = \mathcal{S}^0(x) + \mathrm{O}(\varepsilon^{p+1}), \qquad \mathcal{E}(x,\theta) = \mathcal{E}^0(x) + \mathrm{O}(\varepsilon^{p+1}), \tag{11}$$
   where $x \mapsto \big(\mathcal{S}^0(x), \mathcal{E}^0(x) = h^0(x)\big)$ is the corresponding change for system (1).
2. *Denoting $(S,E,\theta) \in \mathcal{V} \longmapsto (\mathcal{X}(S,E,\theta),\theta) \in \mathcal{U}$ the inverse change to (9), the following estimate holds*
   $$\mathcal{X}(S,E,\theta) = \mathcal{X}^0(S,E) + \mathrm{O}(\varepsilon^{p+1}), \tag{12}$$
   where $x = \mathcal{X}^0(S,E)$ is the inverse change to $x \mapsto (S,E) = (\mathcal{S}^0(x), \mathcal{E}^0(x))$.
3. *Along the local stable manifold $x^{\mathrm{s}}(t,s)$, the flow-box functions (9) satisfy*
   $$\mathcal{S}(x^{\mathrm{s}}(t,s), t/\varepsilon) = t + s, \qquad \mathcal{E}(x^{\mathrm{s}}(t,s), t/\varepsilon) = 0. \tag{13}$$

This is a local result. Now, to extend the parameterization $x^{\mathrm{u}}(t,s)$ of the unstable manifold for other values of (t,s), we compare $x^{\mathrm{u}}(t,s)$ with the unperturbed separatrix $x^0(t+s)$. For $s \in \mathbb{R}$, a standard *real* comparison of solutions gives

$$x^{\mathrm{u}}(t,s) - x^0(t+s) = \mathrm{O}(\varepsilon^{p+1}), \tag{14}$$

for $-T \le t+s \le T$, and $t,s \in \mathbb{R}$. We need an analogous version for complex s.

Since $x^0(u)$ has a singularity in the complex field at $u = \pm a\mathrm{i}$, we will restrict ourselves, as in [12], to a complex strip $\mathcal{D}^{\mathrm{u}}_\varepsilon$ of imaginary width equal to $a - \varepsilon$:

$$\mathcal{D}^{\mathrm{u}}_\varepsilon := \{(t,s) \in \mathbb{R} \times \mathbb{C} : \quad |\Im s| \le a - \varepsilon, \quad |t + \Re s| \le T\}. \tag{15}$$

The following Extension Theorem ensures us that the parameterization $x^{\mathrm{u}}(t,s)$ of the unstable invariant manifold $W^{\mathrm{u}}(\gamma_p)$ is still defined and close enough to the unperturbed separatrix.

Theorem 5 (Extension Theorem) *Let $x^0(t+s)$ be the unperturbed separatrix of system (1), and $x^{\mathrm{u}}(t,s)$ the local parameterization of the unstable invariant manifold, where $s \in \mathbb{C}$, $|\Im s| \leq a - \varepsilon$, and $t + \Re s = -T$.*

Then, if $\gamma = p - \ell > 0$, $x^{\mathrm{u}}(t,s)$ is defined on $\mathcal{D}^{\mathrm{u}}_{\varepsilon}$ given in (15) and satisfies there:

$$x^{\mathrm{u}}(t,s) - x^0(t+s) = \mathrm{O}(\varepsilon^{\gamma}).$$

The proof [13] of this theorem is based on a good choice of the solutions of the variational equations associated to the separatrix and the partition of the strip $\mathcal{D}^{\mathrm{u}}_{\varepsilon}$ in different regions.

By hypothesis *H1*, for $t + \Re s \geq T/2$, $x^0(t+s)$ arrives and remains at the open set $\mathcal{U}$ where the flow-box functions (9) are defined. By the Extension Theorem, the same happens to $x^{\mathrm{u}}(t,s)$ for $T/2 \leq t + \Re s \leq T$, and $|\Im s| \leq a - \varepsilon$, and we can therefore define, for $|\Im s| \leq a - \varepsilon$, the functions:

$$\mathcal{S}^{\mathrm{u}}(s) := \mathcal{S}(x^{\mathrm{u}}(t,s), t/\varepsilon) - t, \qquad \mathcal{E}^{\mathrm{u}}(s) := \mathcal{E}(x^{\mathrm{u}}(t,s), t/\varepsilon), \tag{16}$$

which do not depend on t, by the Flow Box Theorem 4. Moreover, by the natural parameterization of the manifolds, it turns out that $\mathcal{S}^{\mathrm{u}}(s) - s$ and $\mathcal{E}^{\mathrm{u}}(s)$ are analytic for $|\Im s| \leq a - \varepsilon$ and $2\pi\varepsilon$-periodic in s.

By the Extension Theorem, it turns out that $\mathcal{E}^{\mathrm{u}}(s)$ is well-approximated by the Melnikov function on the complex strip $|\Im s| \leq a - \varepsilon$:

$$\mathcal{E}^{\mathrm{u}}(s) = \varepsilon^p M(s,\varepsilon) + \mathrm{O}(\varepsilon^{2\gamma + r - 1}, \varepsilon^{p+1}). \tag{17}$$

As a consequence, the difference between $\mathcal{E}^{\mathrm{u}}(s) - \mathcal{E}^{\mathrm{u}}_0$ and $\varepsilon^p M(s,\varepsilon)$ is $\mathrm{O}(\mathrm{e}^{-a/\varepsilon})$ for real s, where $\mathcal{E}^{\mathrm{u}}_0$ is the zero order Fourier coefficient of $\mathcal{E}^{\mathrm{u}}$:

$$\mathcal{E}^{\mathrm{u}}(s) - \mathcal{E}^{\mathrm{u}}_0 = \varepsilon^p M(s,\varepsilon) + \mathrm{O}(\varepsilon^{2\gamma + r - 1}, \varepsilon^{p+1})\, \mathrm{e}^{-a/\varepsilon}. \tag{18}$$

From the Flow Box Theorem 4, the local stable manifold $x^{\mathrm{s}}(t,s)$ has a very simple expression in the (S,E) coordinates:

$$(S,E) = (\mathcal{S}(x^{\mathrm{s}}(t,s), t/\varepsilon), \mathcal{E}(x^{\mathrm{s}}(t,s), t/\varepsilon)) = (t+s, 0),$$

i.e., $E = 0$. By (16), the arriving unstable manifold $x^{\mathrm{u}}(t,s)$ has in these coordinates the expression

$$(S,E) = (\mathcal{S}(x^{\mathrm{u}}(t,s), t/\varepsilon), \mathcal{E}(x^{\mathrm{u}}(t,s), t/\varepsilon)) = (t + \mathcal{S}^{\mathrm{u}}(s), \mathcal{E}^{\mathrm{u}}(s))$$

and, in particular, the unstable curve C^{u} of the Poincaré map P defined in (3), is given by $(S,E) = (2\pi n\varepsilon + \mathcal{S}^{\mathrm{u}}(s), \mathcal{E}^{\mathrm{u}}(s))$.

Therefore, it is very natural to introduce the *splitting function* ψ given implicitly by $\psi(2\pi n\varepsilon + \mathcal{S}^{\mathrm{u}}(s)) = \mathcal{E}^{\mathrm{u}}(s)$, or simply by $\psi(\mathcal{S}^{\mathrm{u}}(s)) = \mathcal{E}^{\mathrm{u}}(s)$, using that $\mathcal{S}^{\mathrm{u}}(s) - s$ and $\mathcal{E}^{\mathrm{u}}(s)$ are $2\pi\varepsilon$-periodic in s.

Now, one checks that for $s \in \mathbb{R}$, $S = \mathcal{S}^{\mathrm{u}}(s)$ is real analytic and invertible, and its inverse $s = s^{\mathrm{u}}(S)$ satisfies that $s^{\mathrm{u}}(S) - S$ is $\mathrm{O}(\varepsilon^{p+1})$ and $2\pi\varepsilon$-periodic in S.

Therefore, $\psi(S)$ is explicitly defined, for real values of S, as

$$\psi(S) = \mathcal{E}^{\mathrm{u}}(s^{\mathrm{u}}(S)). \tag{19}$$

Since $s^{\mathrm{u}}(S) - S$ is $\mathrm{O}(\varepsilon^{p+1})$ and $2\pi\varepsilon$-periodic in s, we can introduce a new natural parameterization for the unstable invariant manifold

$$\tilde{x}^{\mathrm{u}}(t,S) = x^{\mathrm{u}}(t, s^{\mathrm{u}}(S)),$$

in such a way that $\psi(S)$ can be also written as

$$\psi(S) = \mathcal{E}(x^{\mathrm{u}}(t, s^{\mathrm{u}}(S)), t/\varepsilon) - \mathcal{E}(x^{\mathrm{s}}(t,S), t/\varepsilon) = \mathcal{E}(\tilde{x}^{\mathrm{u}}(t,S), t/\varepsilon). \tag{20}$$

The next Proposition contains the properties of the splitting function ψ. From this Proposition, and mainly from the estimate (21), Theorem 1 and Corollary 2 follow.

Proposition 6 *The function ψ is a $2\pi\varepsilon$-periodic, real analytic function that satisfies the following properties.*

1. *There exists $h^{\mathrm{u}} \in \mathbb{R}$ such that $x^{\mathrm{u}}(t,h^{\mathrm{u}}) = x^{\mathrm{s}}(t,h^{\mathrm{s}})$ (giving an homoclinic connection), with $h^{\mathrm{s}} = \mathcal{S}^{\mathrm{u}}(h^{\mathrm{u}})$. Consequently, $\psi(h_n) = 0$, for $h_n = h^{\mathrm{s}} + 2\pi\varepsilon n$, $n \in \mathbb{N}$. Moreover, $\psi'(h_n)$ is independent of n, and*

$$\psi'(h_n) = \frac{\partial x^{\mathrm{s}}}{\partial S}(t,h_n) \wedge \frac{\partial \tilde{x}^{\mathrm{u}}}{\partial S}(t,h_n) = \left\| \frac{\partial x^{\mathrm{s}}}{\partial S}(t,h_n) \right\| \cdot \left\| \frac{\partial \tilde{x}^{\mathrm{u}}}{\partial S}(t,h_n) \right\| \sin\alpha(t,h_n),$$

   where $\wedge$ denotes the exterior product on $\mathbb{R}^2$, and $\alpha(t,h_n)$ denotes the angle between $x^{\mathrm{u}}(t,h^{\mathrm{u}} + 2\pi\varepsilon n) = \tilde{x}^{\mathrm{u}}(t,h_n)$ and $x^{\mathrm{s}}(t,h_n)$.
2. *The area of the lobe between the invariant curves is given by $A = \left| \int_{h_n}^{\bar{h}_n} \psi(S)\, \mathrm{d}S \right|$, where h_n and $\bar{h}_n$ are the two consecutive zeros of $\psi(S)$ closest to zero.*
3. *$\psi_0 = \int_{h_n}^{h_n + 2\pi\varepsilon} \psi(S)\, \mathrm{d}S = 0$.*
4. *$\psi(S)$ satisfies for $S \in \mathbb{R}$ the estimate*

$$\psi(S) \;=\; \varepsilon^p M(S,\varepsilon) + \mathrm{O}(\varepsilon^{2\gamma+r-1}, \varepsilon^{p+1})\, \mathrm{e}^{-a/\varepsilon}. \tag{21}$$

3. Twist maps

Assume that $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth diffeomorphism with a *separatrix* Γ to a *saddle point* z_0^∞ and a non-trivial first integral H_0. We can assume that F_0 preserves the orientation, taking the square of the map if necessary. Thus, $\mathrm{Spec}[DF_0(z_0^\infty)] = \{\lambda, \lambda^{-1}\}$, where $\lambda > 1$. Let $h = \ln\lambda$ be the associated *characteristic exponent.*

Now, consider a family of smooth diffeomorphisms $F = F_0 + \varepsilon F_1 + \mathrm{O}(\varepsilon^2) : \mathbb{R}^2 \to \mathbb{R}^2$. For $0 < |\varepsilon| \ll 1$, F is a general perturbation of the integrable map F_0, and it has a saddle point z^∞ "close" to z_0^∞ and the local stable and unstable manifolds W^{u}, W^{s} of z^∞ are "close" to Γ, but, in general, they no longer coincide. To compute their distance, for every point z in the separatrix Γ, we denote by z^{u} (respectively z^{s}), the "first" intersection of W^{u} (respectively W^{s}) with the normal to Γ at z. Following Poincaré and Arnold [40, 2], we introduce the difference of first integrals ("energies") as the distance between the invariant curves:

$$\Delta(z) := H_0(z^{\mathrm{u}}) - H_0(z^{\mathrm{s}}) =: \varepsilon M(z) + \mathrm{O}(\varepsilon^2), \qquad z \in \Gamma, \tag{22}$$

and we introduce the *Melnikov function* M as the first order term in ε of the difference of energies. To obtain the expression of M, we first observe that for every fixed z and any $m > 0$, we can write:

$$\Delta(z) \quad = \quad H_0(F^{-m}(z^{\mathrm{u}})) - H_0(F^m(z^{\mathrm{s}})) + \sum_{k=1-m}^{m} H_0(F^k(z^\alpha)) - H_0(F^{k-1}(z^\alpha)),$$

where $\alpha = \alpha(k)$ is given by $\alpha = \mathrm{u}$ if $k \leq 0$, and $\alpha = \mathrm{s}$ if $k > 0$. Passing to the limit $m \to +\infty$, we obtain

$$\Delta(z) = \sum_{k\in\mathbb{Z}} (H_0 \circ F - H_0) \left(F^k(z^\alpha)\right) . \tag{23}$$

Now, since $z^\alpha \in W^\alpha$ is $\mathrm{O}(\varepsilon)$-close to z, it turns out that $F^k(z^\alpha) = F_0^k(z) + \mathrm{O}(\varepsilon)$, uniformly in k. Moreover, $H_0 \circ F - H_0 = \varepsilon\langle\nabla H_0 \circ F_0, F_1\rangle + \mathrm{O}(\varepsilon^2)$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^2$. Putting all this together in (23), we obtain the following expression for the Melnikov function in (22):

$$M(z) = \sum_{k\in\mathbb{Z}} \langle\nabla H_0(z_{k+1}), F_1(z_k)\rangle, \qquad z_k = F_0^k(z),\ z \in \Gamma. \tag{24}$$

Since the present framework is regular, i.e., the characteristic exponent h (or equivalently, the eigenvalue $\lambda = \mathrm{e}^h$) does *not* depend on the parameter of perturbation ε, the Melnikov theory simply says that if M has a simple zero at $z = z_0$, then for $0 < |\varepsilon| \ll 1$, the perturbed invariant manifolds W^{u}, W^{s}, intersect transversally on a homoclinic point near z_0. In particular, one has the following formula for the *angle* of intersection α:

$$|\tan\alpha| = \frac{|\,\mathrm{d}M(z_0)\varepsilon|}{\|\nabla H(z_0)\|^2} + \mathrm{O}(\varepsilon^2).$$

From (24) we see that the Melnikov function is invariant under the action of the unperturbed map: $M \circ F_0 = M$, but $\Delta(z)$ is not (we found the same situation in the case of flows). Consequently M can be defined on the *reduced separatrix* $\Gamma^* = \Gamma/F_0$ which is the quotient of the separatrix by the unperturbed map, and is homeomorphic to the one-dimensional torus $\mathbb{T}$, at least if we only take into account one branch of the separatrix.

A very important case takes place when F is a *twist map*, that is, when there exists a smooth function $\mathcal{L}(x, X)$ such that

$$(X, Y) = F(x, y) \Longleftrightarrow y = -\partial_1\mathcal{L}(x, X), \quad Y = \partial_2\mathcal{L}(x, X). \tag{25}$$

We fix $\mathcal{L}$ by imposing $\mathcal{L}(x^\infty, x^\infty) = 0$, where $z^\infty = (x^\infty, y^\infty)$. Planar twist maps are the simplest case of exact symplectic maps. See the lecture [11] (resp., the paper [10]) for the generalization of these results to the context of twist maps (resp., exact sympectic maps).

By straightforward expansion in ε, setting $\mathcal{L} = \mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \mathrm{O}(\varepsilon^2)$, it follows that

$$M(z) = \mathrm{d}L(z), \qquad L(z) = \sum_{k\in\mathbb{Z}} \mathcal{L}_1(x_k, x_{k+1}), \qquad z_k = (x_k, y_k) = F_0^k(z),\ z \in \Gamma.$$

We note that, since $\mathcal{L}(x^\infty, x^\infty) = 0$, $\mathcal{L}_1(x_0^\infty, x_0^\infty) = 0$, if $z_0^\infty = (x_0^\infty, y_0^\infty)$. Hence, the previous series is uniformly and absolutely convergent over $z \in \Gamma$, giving rise to a smooth function L called the *Melnikov potential*. The critical points of the Melnikov potential are the zeros of the Melnikov function. Again, it is worth remarking that the Melnikov potential is invariant under the action of unperturbed map: $L \circ F_0 = L$ and, consequently, L can be defined on the *reduced separatrix* $\Gamma^* = \Gamma/F_0$.

For twist maps, the Melnikov theory asserts that if there exist non-degenerate critical points of the Melnikov potential L, then F has transverse homoclinic orbits for $0 < |\varepsilon| \ll 1$. Moreover, if z_0, z_0' denote consecutive non-degenerate critical points of L, the *area* A of the lobe between the invariant manifolds has a nice expression:

$$A = \varepsilon \left(L(z_0') - L(z_0) \right) + \mathrm{O}(\varepsilon^2).$$

Notice that the above formula does not change when z_0, z_0' are replaced by $F_0^k(z_0)$, $F_0^m(z_0')$, as it should be.

This formula can also be obtained from a variational principle, due to MacKay, Meiss and Percival [33, 15], which establishes that the homoclinic orbits of the twist map (25) are the extremals of the action

$$W[\mathcal{O}] := \sum_{k\in\mathbb{Z}} \mathcal{L}(x_k, x_{k+1}), \qquad \mathcal{O} = (x_k)_{k\in\mathbb{Z}},$$

and that the area between consecutive homoclinic orbits $\mathcal{O} = (x_k)_{k\in\mathbb{Z}}$, $\mathcal{O}' = (x_k')_{k\in\mathbb{Z}}$ is given by its difference of actions $A = W[\mathcal{O}'] - W[\mathcal{O}]$.

From now on, we will restrict ourselves to the analytic twist case, i.e., we will assume that F in (25) is a real analytic twist map, as well as the twist generating function $\mathcal{L}$ and the first integral H_0. As a first result, we note that if the Melnikov potential L is not constant, it has a maximum and a minimum on the reduced separatrix (i.e., on $\mathbb{T}$), which are of finite order, and as a consequence the Melnikov function has zeros of odd finite order, which implies that F is non-integrable [5].

Moreover, it is worth introducing a *natural parameterization* of Γ (with regard to F_0), i.e., a bijective analytic map $z_0 : \mathbb{R} \longrightarrow \Gamma$ such that:

$$F_0(z_0(t)) = z_0(t+h), \quad \forall\, t \in \mathbb{R},$$

where we recall that $h = \ln \lambda$. One way of obtaining such parameterization, consists of looking for the standard parameterization $\varphi : \mathbb{R} \longrightarrow \Gamma$ that conjugates the action of F_0 to a multiplication by the eigenvalue $\lambda = \mathrm{e}^h$: $F_0(\varphi(r)) = \varphi(\lambda r)$, and making the change of variable $t = \log r$, i.e., $r = \mathrm{e}^t$, obtaining $z_0(t) = \varphi(\mathrm{e}^t)$. However, since F_0 is integrable, there is an easier way of finding $z_0(t)$, based on the fact that, maybe multiplying the first integral by a suitable constant, the above natural parameterization is a *solution* of the Hamiltonian vector field associated to H_0, i.e., $\dot{z}_0 = J \cdot \nabla H_0 \circ z_0$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

With the change of variable $z = z_0(t)$, the Melnikov function is given now by

$$M = \dot{L}\,, \qquad L(t) = \sum_{k\in\mathbb{Z}} f(t+hk), \qquad f(t) = \mathcal{L}_1(x_0(t), x_0(t+h))\,, \tag{26}$$

where $x_0(t)$ is the first component of $z_0(t)$.

From the change $r = \mathrm{e}^t$ used to find the natural parameterization, it turns out that $z_0(t)$ can be extended for *complex* t, and it is $2\pi\mathrm{i}$-periodic. Hence, the Melnikov function M and the Melnikov potential L given in (26) are *doubly periodic* functions with periods h, $T\mathrm{i} = 2\pi\mathrm{i}$. When extra symmetries are present, T can be a divisor of 2π (for instance, $T = \pi$). An important case takes place when the function f given in (26) has only isolated singularities in the complex field. Then, the same happens to M and L, and a powerful *criterion of non-integrability* holds:

$$L \text{ has a singularity } \Longrightarrow L \not\equiv \text{constant} \Longrightarrow F \text{ is non-integrable.}$$

Before passing to some applications, let us mention that the explicit computation of the Melnikov potential can be carried out with the help of a *Summation Formula*, which has not been developed here but can be found in [9].

3.1. APPLICATION: NON INTEGRABILITY OF BILLIARDS CLOSE TO ELLIPSES

Let us consider the problem of the "convex billiard table": let C be an (analytic) closed convex curve of the plane $\mathbb{R}^2$, parameterized by $\gamma : \mathbb{T} \longrightarrow C$, in such a way that C is traveled counterclockwise. A material point moves inside C and collides with C according to the law "the angle of incidence is equal to the angle of reflection". Following Birkhoff [3], we consider the annulus $\mathbb{A} = \{z = (\varphi, v) \in \mathbb{T} \times \mathbb{R} : |v| < |\dot\gamma(\varphi)|\}$, where the coordinate φ is the parameter on C and $v = |\dot\gamma(\varphi)| \cos\vartheta$, with $\vartheta \in (0, \pi)$ the angle of incidence-reflection of the material point. In this way, we obtain a twist map $T : \mathbb{A} \longrightarrow \mathbb{A}$ given by $(\varphi, v) \longmapsto (\Phi, V)$ that models the billiard. Its twist generating function is $S(\varphi, \Phi) = |\gamma(\varphi) - \gamma(\Phi)|$. It is geometrically clear that if C' is another closed convex curve obtained from C by a similarity, then its associated map T' has an equivalent dynamics to T.

The map T has no fixed points but it has periodic orbits of period 2, corresponding to opposite points with the "maximum" and "minimum" distance between them. Instead of studying them as fixed points of T^2, we introduce a simplification, as is usual in the literature [30, 46, 32].

We will assume that C is *symmetric* with regard to a point. Modulo a similarity, we can assume that this point is the origin: $C = -C$. Consequently, we choose a parameterization γ of C such that $\gamma(\varphi + \pi) = -\gamma(\varphi)$ in such a way that the 2-periodic orbits are of the form $(\varphi_0, 0)$, $(\varphi_0 + \pi, 0)$, that is, two opposite points over C.

Introducing the *involution* $R : \mathbb{A} \to \mathbb{A}$ given by $R(\varphi, v) = (\varphi + \pi, v)$, we now define a new map $F : \mathbb{A} \to \mathbb{A}$ by $F = R \circ T$. Since $F^2 = T^2$, the dynamics of F and T are equivalent. Moreover, since $\gamma(\Phi + \pi) = -\gamma(\Phi)$, it is easy to check that

$$\mathcal{L}(\varphi, \Phi) = |\gamma(\varphi) + \gamma(\Phi)| \tag{27}$$

is a twist generating function for F, and consequently F is a twist map. We note that the variable φ can be defined modulo π in the symmetric case.

As a first example, consider now a *non-circular ellipse*: $C_0 = \{\gamma_0(\varphi) : \varphi \in \mathbb{T}\}$, where $\gamma_0(\varphi) = (a\cos\varphi, b\sin\varphi)$, with $a^2 \neq b^2$. Modulo a similarity, we can assume that $a^2 - b^2 = 1$. Thus $a > 1$, $b > 0$ and the foci of the ellipse are $(\pm 1, 0)$. Let us denote $T_0 : \mathbb{A} \longrightarrow \mathbb{A}$ the analytic twist map associated to the ellipse C_0, and $F_0 = R \circ T_0$.

The points $(0,0)$ and $(\pi, 0)$ form a 2-periodic orbit for T_0 that corresponds to the vertexes $(\pm a, 0)$ of the ellipse, and hence $z_0^\infty := (0,0) = (\pi, 0)$ is a fixed point for F_0. (Remember that φ is defined modulo π in the symmetric case.) The main properties of F_0 are listed in the following Lemma [9].

Lemma 7 *a)* $z_0^\infty = (0,0) = (\pi, 0)$ *is a saddle fixed point of* F_0 *and* $Spec[DF_0(z_\infty)] = \{\lambda, \lambda^{-1}\}$, *with* $\lambda = (a+1)(a-1)^{-1} > 1$. *Moreover, if* $h := \ln\lambda$ *the following expressions hold*

$$a = \coth(h/2), \quad b = \operatorname{cosech}(h/2).$$

b) $H_0(\varphi, v) = \sin^2\varphi - v^2$ *is a first integral of* F_0*, and* $\Gamma^\pm = \{(\varphi, \pm\sin\varphi) : 0 < \varphi < \pi\}$ *are the* separatrices *of* F_0*.*

c) *Let* $\varphi(t) = \arcsin(\operatorname{sech} t)$ *and* $v(t) = \operatorname{sech} t$*. Then,* $z_0^\pm(t) := (\varphi(\pm t), \pm v(t))$ *are natural parameterizations of* $\Gamma^\pm$ *with regard to* F_0*.*

d) *Let* $\Phi(t) = \varphi(t+h)$*. Then*

$$b\frac{\sin\varphi(t) + \sin\Phi(t)}{|\gamma_0(\varphi(t)) + \gamma_0(\Phi(t))|} = \operatorname{sech}(t + h/2). \tag{28}$$

Birkhoff conjectured that the elliptic billiard is the only integrable convex billiard. Our goal is to see that this is locally true for the symmetric billiards, i.e., any non-trivial symmetric entire perturbation is non-integrable. (Non-trivial perturbation means not reducible to an ellipse.) Thus, we now consider an arbitrary symmetric perturbation $C_\varepsilon = -C_\varepsilon$ of the ellipse C_0. Modulo $\mathrm{O}(\varepsilon^2)$ terms (which do not play any rôle in our first order analysis) and a similarity, C_ε can be put in the following "normal" form

$$C_\varepsilon = \{\gamma_\varepsilon(\varphi) = (a\cos\varphi, \sin\varphi[b + \varepsilon\eta(\varphi)]) : \varphi \in \mathbb{T}\}, \quad \begin{array}{l}\text{i) } \eta \text{ analytic,}\\ \text{ii) } \eta\ \pi\text{-periodic.}\end{array} \tag{29}$$

From the expression above, it is clear that C_ε is an ellipse (up to first order in ε) if and only if $\eta(\varphi)$ is a constant function. As a consequence, we will say that C_ε is a *non-trivial symmetric entire perturbation* of the ellipse C_0 when it can be put in the normal form (29) and moreover, $\eta(\varphi)$ is a non-constant entire function.

Let T_ε be the map in the annulus associated to the billiard in C_ε, and $F_\varepsilon = R \circ T_\varepsilon$. For $0 < |\varepsilon| \ll 1$, C_ε is an analytic convex closed curve, and thus F_ε is an analytic twist map, being $\mathcal{L}_\varepsilon(\varphi, \Phi) = |\gamma_\varepsilon(\varphi) + \gamma_\varepsilon(\Phi)| = \mathcal{L}_0(\varphi, \Phi) + \varepsilon\mathcal{L}_1(\varphi, \Phi) + \mathrm{O}(\varepsilon^2)$ its twist generating function, where $\mathcal{L}_0(\varphi, \Phi) = |\gamma_0(\varphi) + \gamma_0(\Phi)|$ and

$$\mathcal{L}_1(\varphi, \Phi) = b\frac{\sin\varphi + \sin\Phi}{|\gamma_0(\varphi) + \gamma_0(\Phi)|}[\sin\varphi\,\eta(\varphi) + \sin\Phi\,\eta(\Phi)]. \tag{30}$$

From now on, we consider only Γ^+. Using the natural parameterization provided by Lemma 7, the formula of $\mathcal{L}_1$ given in equation (30), and the formula (28), the function $f(t)$ in (26) takes the form

$$f(t) = \mathcal{L}_1(\varphi(t), \varphi(t+h)) = \operatorname{sech}(t + h/2)[\operatorname{sech}(t)\eta(\varphi(t)) + \operatorname{sech}(t+h)\eta(\varphi(t+h))]. \tag{31}$$

Now, assume we are given a non-trivial symmetric entire perturbation C_ε of the ellipse. Our aim is to prove the non-integrability of T_ε, and for this purpose we only have to find a singularity of $L(t) = \sum_{k\in\mathbb{Z}} f(t + hk)$.

By hypothesis, $\eta(\varphi)$ is a non-constant entire function, and by ii), it is π-periodic. By Lemma 7, $\sin\varphi(t) = \operatorname{sech}(t)$ and $\cos\varphi(t) = \tanh(t)$ have simple poles at $t = \pi\mathrm{i}/2$ and no more singularities on $\Im t = \pi/2$. Since $\eta(\varphi)$ is non-constant, $t = \pi\mathrm{i}/2$ is a singularity of $\eta(\varphi(t))$. So, we concentrate on $t = \pi\mathrm{i}/2$. It is not difficult to check [8] that

$$t \mapsto \sum_{k\in\mathbb{Z}} f(t + hk) - \frac{2a}{b}\operatorname{sech}(t + h/2)\operatorname{sech}(t - h/2)\eta(\varphi(t))$$

is an analytic function on $t = \pi\mathrm{i}/2$. Since $\operatorname{sech}(t + h/2)\operatorname{sech}(t - h/2)$ is also analytic and non-zero on $t = \pi\mathrm{i}/2$, $t = \pi\mathrm{i}/2$ is a singularity of $L(t) = \sum_{k\in\mathbb{Z}} f(t + hk)$. We have proved:

Theorem 8 *Let C_ε be any non-trivial symmetric entire perturbation of an ellipse. Then the billiard in C_ε is non-integrable for* $0 < |\varepsilon| \ll 1$.

3.2. APPLICATION: PLANAR STANDARD-LIKE MAPS

We consider the following planar standard-like map

$$F_0(x,y) = \left(y, -x + 2y \frac{\mu + \beta y}{1 - 2\beta y + y^2} \right), \qquad -1 < \beta < 1 < \mu. \tag{32}$$

These maps were introduced by Suris [45]. (Standard-like maps are twist maps: the twist generating function of $(x,y) \mapsto (X,Y) = (y, -x + U'(y))$ is $\mathcal{L}(x,X) = -xX + U(X)$.)

The origin is a hyperbolic fixed point of the Suris map F_0; its characteristic multipliers are $\mathrm{e}^{\pm h}$, where the *characteristic exponent* $h > 0$ is given by $\cosh h = \mu$. Moreover, the polynomial $H_0(x,y) = [x^2 - 2\mu xy + y^2 - 2\beta xy(x+y) + x^2y^2]/2$ is a first integral of F_0, and the energy level $\{H_0 = 0\}$ is a necklace containing two separatrices $\Gamma^\pm = \Gamma^\pm_{\mu,\beta}$ to the origin. Their natural parameterizations are given by [9]

$$\Gamma^\pm = \left\{ z_0^\pm(t) = (x_0^\pm(t), x_0^\pm(t+h)) : t \in \mathbb{R} \right\}, \qquad x_0^\pm(t) = \frac{\pm c}{\Delta \cosh t \mp b}, \tag{33}$$

where $c = \sqrt{\mu^2 - 1} = \sinh h$, $\Delta = \sqrt{1 - 2\beta^2/(\mu - 1)}$, and $b = \beta\sqrt{(\mu+1)/(\mu-1)}$. We note that $\Gamma^-_{\mu,\beta} = -\Gamma^+_{\mu,-\beta}$, so we study only $\Gamma = \Gamma^+$, $z_0 = z_0^+$ and $x_0 = x_0^+$. In the particular case $\beta = 0$, $\Gamma^- = -\Gamma^+$, and the map (32) is an odd map, called the *McMillan map* [35].

We consider now a perturbation formed by standard-like maps

$$F(x,y) = \left(y, -x + 2y \frac{\mu + \beta y}{1 - 2\beta y + y^2} + \varepsilon V'(y) \right), \quad -1 < \beta < 1 < \mu\,, \quad \varepsilon \in \mathbb{R}, \tag{34}$$

where $V(y)$ is determined by imposing $V(0) = 0$. The twist generating function has the form $\mathcal{L}(x,X) = \mathcal{L}_0(x,X) + \varepsilon V(X)$, where $\mathcal{L}_0$ comes from the unperturbed map. Therefore, the Melnikov potential is simply $L(t) = \sum_{k \in \mathbb{Z}} V(x_0(t + hk))$. If, for instance, V is a non-constant real entire function, then $V(x_0(t))$ has the same isolated singularities in the complex variable t as $x_0(t)$, and it is not difficult to check that they remain as singularities for the Melnikov potential $L(t)$. In this way we have established the following result.

Theorem 9 *If V is a non-constant real entire function, the map (34) is non-integrable for* $0 < |\varepsilon| \ll 1$.

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1 SPLITTING OF SEPARATRICES IN HAMILTONIAN SYSTEMS AND SYMPLECTIC MAPS **1**