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Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables

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Abstract

Two area-preserving twist maps are associated to a smooth closed convex table: the (classical) billiard map and the dual billiard map. When the table is circular, these maps are integrable and their phase spaces are foliated by invariant curves. The invariant curves with rational rotation numbers are resonant and do not persist under generic perturbations of the circle. We present a sufficient condition for the break-up of these curves. This condition is expressed directly in terms of the Fourier coefficients of the perturbation. It follows from a standard Melnikov argument.

Keywords: Billiards; Dual billiards; Invariant curves; Melnikov methods

1. Introduction

One of the fundamental questions in dynamical systems is the effect that small perturbations of a dynamical system cause on its unperturbed invariant sets. Probably, the most remarkable examples of persistent invariant sets are the Diophantine tori of completely integrable Hamiltonian flows or symplectic maps [1]. On the other hand, the resonant tori generically break up although, if the perturbation is small enough, some periodic orbits always persist in a small neighborhood of each resonant tori [2]. The standard tool in continuous systems for determining the effect of a concrete perturbation over a resonant tori is the subharmonic Melnikov function [9,24]. (Originally, Melnikov functions were introduced to study the splitting of separatrices, which is just another problem concerning the break-up of delicate invariant structures.) The development of subharmonic Melnikov methods for discrete systems (that is, for maps) is more recent [23,25], but these works are focussed rather on abstract theorems than in physical applications. Our goal is to present a simple and self-contained derivation of the subharmonic Melnikov method for integrable twist maps and, next, to apply it to some billiard problems. The novel result of this paper is to present an application of the subharmonic Melnikov method for maps in which all the computations can be carried out. These computations are interesting because they allow us to estimate the amplitude of the perturbed resonances; see Section 5.

To begin with, we shall establish a sufficient condition for the break-up of a given resonant invariant curve of an integrable area-preserving twist map under a concrete perturbation of the map. Our condition follows from the study of a function defined on the unperturbed resonant curve in terms of the Lagrangian of the perturbed map. Following the literature, this function should be called the *subharmonic Melnikov potential*, but we call it the *radial Melnikov potential* for some reasons that will become clear in Section 2.

The definition of subharmonic Melnikov potentials for perturbed integrable area-preserving maps is not new. For instance, it is used in [23] to estimate the difference of the frequencies on two invariant curves bounding a given resonance, whereas its generalization to $2n$-dimensional perturbed integrable twist maps is contained in [25]. In those papers, the Melnikov potential is written in terms of the second type generating function of the perturbed map. Second type generating functions depend on the old angle and the new action, whereas the first type generating functions (that is, the Lagrangians) used in our work depend on both angles (old and new). This is a minor difference, but first type generating functions are the natural choice for billiard systems.

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We shall prove that a resonant curve breaks up when its Melnikov potential is not constant; see Corollary 8. We stress that this is a first order condition, and so it is sufficient but not necessary for the break-up. Similar conditions for the break-up of resonant invariant surfaces in the frame of volume-preserving maps have been obtained in [15].

Next, we shall study the classical and dual billiard maps associated to small perturbations of circular tables. A particularly nice aspect of both problems is that the break-up condition can be stated *directly* in terms of the Fourier coefficients of the perturbation of the circle; see Theorem 1. We note that, with the definition of integrability used in this paper, circular tables are the only ones giving rise to integrable twist maps [3,26]. The billiard map inside an ellipse is also "integrable", but it has a separatrix so that global action-angle coordinates cannot be introduced over its whole phase space.

Birkhoff [4] introduced the problem of *convex billiard tables* more than 75 years ago as a way to describe the motion of a free particle inside a closed convex curve such that it reflects at the boundary according to the law "angle of incidence equals angle of reflection". He realized that this billiard motion can be modeled by an area-preserving twist diffeomorphism defined on the open cylinder $\mathbb{T} \times (-1, 1)$. Since then, billiards have become a paradigm of conservative dynamical systems, because *in the billiard problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting questions need to be considered* [4, page 170]. We refer to the monographs [13,21] for the latest developments in billiard dynamics. The invariant curves of a classical billiard map are related to its caustics; see [12].

On the other hand, following J.K. Moser, the dual billiard problem *may serve as a crude model for planetary motion* [16, page 66]. The dual billiard map associated to a smooth (strictly) convex closed plane curve $\Gamma \subset \mathbb{R}^2$ is defined on the unbounded component of $\mathbb{R}^2 \setminus \Gamma$ as follows: the image of a point equals its reflection in the tangency point of the oriented tangent line to Γ through the initial point. The dual billiard motion can also be modeled by an area-preserving twist diffeomorphism defined on the half-cylinder $\mathbb{T} \times (0, +\infty)$. Smooth dual billiards have been studied, for instance, in [19,20,10,6,22].

The periodic (classical) billiard trajectories are inscribed polygons whose consecutive sides make equal angles with the curve Γ, whereas the periodic dual billiard trajectories are circumscribed polygons whose sides are bisected by their tangency points about Γ. Given a rational number p/q, where p and q are relatively prime, we say that the trajectory is p/q-periodic when the polygon has q sides and makes p turns inside (resp., about) Γ in the classical (resp., dual) billiard. We shall call these polygons *p/q-gons*. In classical billiards, $q \geq 2$. In dual billiards, $q \geq 3$. Besides, it is well known that the classical (resp., dual) billiard dynamics verifies a variational principle in which it is established that its p/q-periodic trajectories are critical points of the length functional defined on the set of p/q-gons inscribed in Γ (resp., of the extrema of the area functional defined on the set of p/q-gons circumscribed about Γ).

The following existence results are obtained using the above characterization of periodic billiard trajectories as critical points of the functionals length and area. They hold when the billiard curve is smooth and strictly convex, and essentially go back to G.D. Birkhoff:

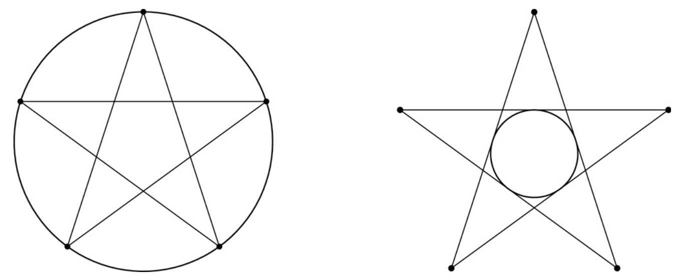

Fig. 1. An element of $\mathcal{I}_0^{2/5}$ (left) and an element of $\mathcal{C}_0^{2/5}$ (right).

- *(Classical version)* For any integers $q \geq 2$ and $1 \leq p \leq q/2$ relatively prime, there exist at least two geometrically distinct p/q-periodic classical billiard trajectories [21, Section 2.5].
- *(Dual version)* For any integers $q \geq 3$ and $1 \leq p \leq q/2$ relatively prime, there exist at least a p/q-periodic dual billiard trajectory [21, Section 4.1].

We ask p and q to be relatively prime because, otherwise, the p/q-periodic trajectories could be a polygon with fewer vertices traversed several times.

Let us now consider the billiard dynamics associated to a circle Γ_0 of radius r_0. It is an exercise in elementary geometry to check that:

- *(Classical version)* For any integers $q \geq 2$ and $1 \leq p \leq q/2$ relatively prime, there exists a continuous family $\mathcal{I}_0^{p/q}$ of regular p/q-gons inscribed in Γ_0 whose consecutive sides make equal angles with Γ_0.
- *(Dual version)* For any integers $q \geq 3$ and $1 \leq p \leq q/2$ relatively prime, there exists a continuous family $\mathcal{C}_0^{p/q}$ of regular p/q-gons circumscribed about Γ_0 whose sides are bisected by their tangency points about Γ_0.

Of course, each family is obtained just by rotating any of its elements with respect to the center of the circle. We show in Fig. 1 an element of $\mathcal{I}_0^{2/5}$ and another of $\mathcal{C}_0^{2/5}$. In particular, all the polygons of $\mathcal{I}_0^{p/q}$ have the same perimeter, $l_0^{p/q} = 2q \sin(\pi p/q) r_0$, and all the polygons of $\mathcal{C}_0^{p/q}$ have the same area, $A_0^{p/q} = q \tan(\pi p/q) r_0^2$. This also follows from the variational principles, since any functional is constant on each continuum set formed by critical points. Thus, if a family $\mathcal{I}_0^{p/q}$ (resp., $\mathcal{C}_0^{p/q}$) persists under some perturbation $\Gamma_\epsilon = \Gamma_0 + \mathrm{O}(\epsilon)$ of the circle, then the polygons of the perturbed family $\mathcal{I}_\epsilon^{p/q}$ (resp., $\mathcal{C}_\epsilon^{p/q}$) do not need to be regular, but they shall still have the same perimeter (resp., area).

A sufficient condition for the break-up of these continuous families of polygons under small perturbations of Γ_0 is given below. Let $\mathbf{n}_\theta = (\cos\theta, \sin\theta)$. We write the perturbation in polar coordinates as follows:

$$\Gamma_\epsilon = \{r_\epsilon(\theta)\mathbf{n}_\theta : \theta \in \mathbb{T}\}, \quad r_\epsilon(\theta) = r_0 + \epsilon r_1(\theta) + \mathrm{O}(\epsilon^2), \quad (1)$$

for some smooth function $r_\epsilon : \mathbb{T} \to \mathbb{R}$.

Theorem 1. *Let $\sum_{j\in\mathbb{Z}} \hat{r}_1^j \mathrm{e}^{\mathrm{i}j\theta}$ be the Fourier expansion of $r_1(\theta)$. Let $q \geq 2$ be any integer. If there exists some $j \in q\mathbb{Z}\backslash\{0\}$ such that $\hat{r}_1^j \neq 0$, then the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ do not persist under the perturbation (1).*

Remark 2. If we write the perturbation (1) in Cartesian coordinates as

$$\Gamma_\epsilon = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + \epsilon P_1(x, y) + \mathrm{O}(\epsilon^2) = r_0^2\right\}, \quad (2)$$

then the $\mathrm{O}(\epsilon)$-terms of (1) and (2) verify the relation $2r_0 r_1(\theta) + P_1(r_0 \mathbf{n}_\theta) = 0$.

In Section 2 we introduce the maps and the variational principles with which we will be working. It also contains the general Melnikov theory for resonant curves of twist maps. The proofs concerning classical billiards and dual billiards are contained in Section 3 and Section 4, respectively. In order to illustrate the strength and the limits of the theory, several examples are studied in Section 5. Finally, some open problems are listed in the last section.

2. Break-up of resonant invariant curves in twist maps

For the sake of simplicity, we will assume that the objects considered here are smooth. For a general background on twist maps we refer, for instance, to the book [11, Section 9.3] or to the review [14].

2.1. Twist maps

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\mathbb{A} = \mathbb{T} \times \mathbb{R}$, and $\pi_1 : \mathbb{A} \to \mathbb{T}$ be the natural projection. Sometimes it is convenient to work in the universal cover $\mathbb{R}$ of $\mathbb{T}$. We will use the coordinates (s, y) for $\mathbb{A}$ and (x, y) for $\mathbb{R}^2$. The horizontal and vertical directions in these coordinates will be called the angular and radial directions, respectively. A tilde will always denote the lift of a point, function or set to the universal cover. If g is a real-valued function, $\partial_i g$ denotes the derivative with respect to the ith variable. We will consider certain diffeomorphisms defined on an open cylinder of the form $C = \mathbb{T} \times Y$, $Y = (y_-, y_+)$, for some $-\infty \leq y_- < y_+ \leq +\infty$. Then $\tilde{C} = \mathbb{R} \times Y$ is an open strip of the plane.

A diffeomorphism $f : C \to C$ is called an *area-preserving twist map* when it preserves area, orientation, and verifies the classical *twist condition*: if $\tilde{f}(x, y) = (x', y')$ is a lift of f then

$$\partial_2 x'(x, y) > 0.$$

In what follows, the lift $\tilde{f}$ remains fixed. We also assume, although it is not essential, that f verifies some *rigid boundary conditions*. To be more precise, if y_- and y_+ are finite, we suppose that the twist map f can be extended continuously to the closed cylinder $\bar{C} = \mathbb{T} \times \bar{Y}$, $\bar{Y} = [y_-, y_+]$, as a rigid rotation on the boundaries. That is, there exist some *boundary frequencies* $\omega_\pm \in \mathbb{R}$, $\omega_- < \omega_+$, such that $\tilde{f}(x, y_\pm) = (x + \omega_\pm, y_\pm)$. When $y_- = -\infty$ (resp., $y_+ = +\infty$), the lower (resp., upper) boundary condition is stated in terms of the corresponding limit. These rigid boundary conditions are verified by the classical and dual billiard maps considered in this paper.

Let $D = \{(x, x') \in \mathbb{R}^2 : \omega_- < x' - x < \omega_+\}$. Then there exists a function $h : D \to \mathbb{R}$ such that $\tilde{f}(x, y) = (x', y')$ if and only if

$$y = -\partial_1 h(x, x'), \quad y' = \partial_2 h(x, x'). \quad (3)$$

The function h is called the *Lagrangian* or *generating function* of f.

Remark 3. In order to give a geometric interpretation of h, we note that if y_- is finite (as for classical and dual billiard maps), then the quantity $h(x, x')$ is equal to the area of the region in the strip $\tilde{C}$ enclosed by the lower boundary $\tilde{C}_- = \mathbb{R} \times \{y_-\}$, the vertical line $\tilde{Y}(x') = \{x'\} \times Y$, and the $\tilde{f}$-image of the vertical line $\tilde{Y}(x) = \{x\} \times Y$; see [11, Section 9.3].

The term Lagrangian is due to the fact that twist maps satisfy a variational principle, since their orbits are in correspondence with the critical points of some real-valued functionals, called *actions*. To clarify this, we describe the periodic version of this principle. A point $(x, y) \in \tilde{C}$ is said to be a *p/q-periodic* point of $\tilde{f}$, for some integers p and q relatively prime, whenever $\tilde{f}^q(x, y) = (x + 2\pi p, y)$. Obviously, $\omega_- < 2\pi p/q < \omega_+$. Then the corresponding point $(s, y) \in C$, $x = \tilde{s}$, is a periodic point of period q by f that is translated by $2\pi p$ in the base by the lift. Analogously, a $\tilde{f}$-orbit $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ is *p/q-periodic* when $(x_{k+q}, y_{k+q}) = (x_k + 2\pi p, y_k)$ for all $k \in \mathbb{Z}$. In the periodic variational principle [14, Section V], it is stated that the p/q-periodic points of the lift are in one-to-one correspondence with the critical points of the *p/q-periodic action* $W^{p/q} : \mathbb{R}^q \to \mathbb{R}$ defined by

$$W^{p/q}(x_0, \dots, x_{q-1}) = h(x_0, x_1) + h(x_1, x_2) + \cdots + h(x_{q-1}, x_0 + 2\pi p).$$

2.2. Integrable twist maps

Let $f_0 : C \to C$ be an integrable twist map, that is, an area-preserving twist map with a lift $\tilde{f}_0 : \tilde{C} \to \tilde{C}$ of the form

$$(x', y') = \tilde{f}_0(x, y) = (x + \omega(y), y)$$

such that $\omega'(y) > 0$ for all $y \in Y$ and the limits $\omega_\pm = \lim_{y\to y_\pm} \omega(y)$ are finite. Its Lagrangian $h_0(x, x')$ depends only on the difference $x' - x$, since $(\partial_1 + \partial_2)h_0(x, x') = y' - y \equiv 0$. In fact, $h_0(x, x') = l_0(x' - x)$ for some function $l_0(\omega)$ such that $l_0'(\omega(r)) = r$.

Integrable twist maps leave invariant all the horizontal lines $\mathbb{R} \times \{y\}$. We restrict our study to the resonant ones, that is, those whose rotation number is rational. If $p/q \in \mathbb{Q}$ and $\omega_- < 2\pi p/q < \omega_+$, let $y_0^{p/q} = \omega^{-1}(2\pi p/q)$ and $T_0^{p/q} = \mathbb{R} \times \{y_0^{p/q}\}$. Any orbit

$$\left(x_k^{p/q}, y_k^{p/q}\right) = \tilde{f}_0^k\left(x_0^{p/q}, y_0^{p/q}\right), \quad x_k^{p/q} = x + 2\pi kp/q, \quad y_k^{p/q} = y_0^{p/q},$$

contained in $T_0^{p/q}$ is p/q-periodic. Finally, we stress that all these p/q-periodic orbits have the same p/q-periodic action, namely

$$\sum_{k=1}^{q} h_0\left(x_{k-1}^{p/q}, x_k^{p/q}\right) = \sum_{k=1}^{q} l_0\left(x_k^{p/q} - x_{k-1}^{p/q}\right) \equiv q l_0(2\pi p/q). \tag{4}$$

2.3. Radial curves and radial potentials

Let f_ϵ be a perturbation of an integrable twist map and let $\tilde{f}_\epsilon$ be a fixed lift. Moreover, let $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$ be the perturbed Lagrangian.

As is well known, the resonant invariant curves $T_0^{p/q}$ do not persist under generic perturbations but, following Birkhoff [4, Section VI] and Arnold [1, Section 20], there exist a couple of *radial curves*

$$T_\epsilon^{p/q} = \left\{\left(x, y_\epsilon^{p/q}(x)\right) : x \in \mathbb{R}\right\},$$
$$\widehat{T}_\epsilon^{p/q} = \left\{\left(x, \hat{y}_\epsilon^{p/q}(x)\right) : x \in \mathbb{R}\right\}$$

close to $T_0^{p/q}$ such that $\tilde{f}_\epsilon^q$ projects $T_\epsilon^{p/q}$ onto $\widehat{T}_\epsilon^{p/q}$ along the radial direction. This follows directly from the next lemma.

Lemma 4. *For any small enough ϵ, there exists a couple of 2π-periodic functions $y_\epsilon^{p/q}, \hat{y}_\epsilon^{p/q} : \mathbb{R} \to Y$ such that:*

(i) $y_\epsilon^{p/q}(x) = y_0^{p/q} + \mathrm{O}(\epsilon)$ *and* $\hat{y}_\epsilon^{p/q}(x) = y_0^{p/q} + \mathrm{O}(\epsilon)$, *uniformly in $x \in \mathbb{R}$;*
(ii) $f_\epsilon^q\left(x, y_\epsilon^{p/q}(x)\right) = \left(x, \hat{y}_\epsilon^{p/q}(x)\right)$, *for all $x \in \mathbb{R}$.*

Proof. To obtain the 2π-periodic function $y_\epsilon^{p/q}(x)$, it suffices to realize that

$$G^{p/q}(y, \epsilon; x) := \pi_1\left(\tilde{f}_\epsilon^q(x, y)\right) - x - 2\pi p$$

is 2π-periodic in x and verifies the hypotheses

$$G\left(y_0^{p/q}, 0; x\right) = 0, \quad \partial_1 G\left(y_0^{p/q}, 0; x\right) = q\omega'\left(y_0^{p/q}\right) \neq 0$$

of the Implicit Function Theorem at the point $(y, \epsilon) = (y_0^{p/q}, 0)$. Here, we have used the twist condition $\omega'(y) > 0$. Besides, y and ϵ are the variables, whereas x is considered just a parameter. Once we have proved the existence and uniqueness of $y_\epsilon^{p/q} : \mathbb{R} \to Y$, we determine $\hat{y}_\epsilon^{p/q} : \mathbb{R} \to Y$ by means of property (ii). Finally, the uniformity in $x \in \mathbb{R}$ follows from the periodicity of the functions. □

Corollary 5. *The radial curves $T_\epsilon^{p/q}$ and $\widehat{T}_\epsilon^{p/q}$ have the following properties:*

(i) *they have a non-empty intersection;*
(ii) *their intersection points are p/q-periodic points of $\tilde{f}_\epsilon$;*
(iii) *they coincide identically if and only if $T_0^{p/q}$ persists.*

Therefore, it is rather useful to quantify the separation between the radial curves $T_\epsilon^{p/q}$ and $\widehat{T}_\epsilon^{p/q}$, which is done in the next lemma.

Lemma 6. *$\hat{y}_\epsilon^{p/q}(x) - y_\epsilon^{p/q}(x) = (L_\epsilon^{p/q})'(x)$, where $L_\epsilon^{p/q} : \mathbb{R} \to \mathbb{R}$ is the function*

$$L_\epsilon^{p/q}(x) = \sum_{k=1}^{q} h_\epsilon\left(\bar{x}_{k-1}^{p/q}(x; \epsilon), \bar{x}_k^{p/q}(x; \epsilon)\right),$$
$$\bar{x}_k^{p/q}(x; \epsilon) = \pi_1\left(\tilde{f}_\epsilon^k\left(x, y_\epsilon^{p/q}(x)\right)\right).$$

Proof. We shall not write the dependence on ϵ and p/q. Given any $x \in \mathbb{R}$, we introduce the notations

$$(\bar{x}_k, y_k) = \tilde{f}^k(x, y(x)), \quad d_k = d_k(x) := \partial_1 \bar{x}_k(x; \epsilon), \quad k = 0, \ldots, q.$$

Then $\bar{x}_0 = x$ and $\bar{x}_q = x + 2\pi p$, so $d_0 = d_q = 1$. Besides, $y_0 = y(x)$ and $y_q = \hat{y}(x)$. From the implicit equation (3), we get that $\partial_1 h(\bar{x}_0, \bar{x}_1) = -y_0$, $\partial_2 h(\bar{x}_{q-1}, \bar{x}_q) = y_q$, and $\partial_2 h(\bar{x}_{k-1}, \bar{x}_k) + \partial_1 h(\bar{x}_k, \bar{x}_{k+1}) = 0$ for $k = 1, \ldots, q-1$. Therefore, $L'(x) = \partial_1 h(\bar{x}_0, \bar{x}_1) d_0 + \sum_{k=1}^{q-1}(\partial_2 h(\bar{x}_{k-1}, \bar{x}_k) + \partial_1 h(\bar{x}_k, \bar{x}_{k+1})) d_k + \partial_2 h(\bar{x}_{q-1}, \bar{x}_q) d_q = \hat{y}(x) - y(x)$. □

Corollary 7. *The resonant curve $T_0^{p/q}$ persists if and only if $(L_\epsilon^{p/q})'(x) \equiv 0$.*

Since the function $L_\epsilon^{p/q}(x)$ is 2π-periodic, it can be considered as a function defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. We shall say that $L_\epsilon^{p/q} : \mathbb{T} \to \mathbb{R}$ is the *radial potential* of the resonant radial invariant curve $T_0^{p/q}$ under the perturbation f_ϵ.

2.4. The radial Melnikov potential

Once the radial potential $L_\epsilon^{p/q}(x) = L_0^{p/q}(x) + \epsilon L_1^{p/q}(x) + \mathrm{O}(\epsilon^2)$ is introduced, it is rather natural to extract information from its low-order terms. This is the main idea behind any Melnikov approach to a perturbative problem; see [9,24]. The zero-order term $L_0^{p/q}(x)$ is constant (and so useless), since

$$L_0^{p/q}(x) = \sum_{k=1}^{q} h_0\left(x_{k-1}^{p/q}, x_k^{p/q}\right) = \sum_{k=1}^{q} l_0\left(x_k^{p/q} - x_{k-1}^{p/q}\right) \equiv q l_0(2\pi p/q),$$

where $x_k^{p/q} = x_k^{p/q}(x) := \bar{x}_k^{p/q}(x; 0) = \pi_1(\tilde{f}_0^k(x, y_0^{p/q})) = x + 2\pi k p/q$; compare with (4). We shall say that the first-order term $L_1^{p/q} : \mathbb{R} \to \mathbb{R}$ is the *radial Melnikov potential* of the resonant invariant curve $T_0^{p/q}$ under the perturbation f_ϵ. The following corollary displays the most important property of the radial Melnikov potential in relation to the goals of this paper; moreover, the proposition below provides a closed formula for its computation.

Corollary 8. *If $L_1^{p/q}(x)$ is not constant, the curve $T_0^{p/q}$ does not persist.*

Proposition 9. $L_1^{p/q}(x) = \sum_{k=1}^{q} h_1(x_{k-1}^{p/q}, x_k^{p/q})$, *where $x_k^{p/q} = x + 2\pi k p/q$. In particular, $L_1^{p/q}(x)$ is a $2\pi/q$-periodic function.*

Proof. Given any $x \in \mathbb{R}$, we introduce the notations

$x_k^{p/q} = x_k^{p/q}(x) := \bar{x}_k^{p/q}(x; 0)$,

$z_k^{p/q} = z_k^{p/q}(x) := \partial_2 \bar{x}_k^{p/q}(x; 0)$,

for $k = 0, \ldots, q$. The perturbed Lagrangian is $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$. Then the first-order term of the radial potential is

$$L_1^{p/q}(x) = \sum_{k=1}^{q} h_1\left(x_{k-1}^{p/q}, x_k^{p/q}\right) + \partial_1 h_0\left(x_0^{p/q}, x_1^{p/q}\right) z_0^{p/q} + \partial_2 h_0\left(x_{q-1}^{p/q}, x_q^{p/q}\right) z_q^{p/q} + \sum_{k=2}^{q-1}\left(\partial_1 h_0\left(x_k^{p/q}, x_{k+1}^{p/q}\right) + \partial_2 h_0\left(x_{k-1}^{p/q}, x_k^{p/q}\right)\right) z_k^{p/q}.$$

Using the implicit Eq. (3) for the integrable twist map, the third line vanishes. The second line of the above equation also vanishes, due to the fact that $\bar{x}_0^{p/q}(x; \epsilon) = x$ and $\bar{x}_q^{p/q}(x; \epsilon) = x + 2\pi p$ for small values of ϵ. Besides, $x_k^{p/q} = x_k^{p/q}(x) = \bar{x}_k^{p/q}(x; 0) = \pi_1(\tilde{f}_0^k(x, y_0^{p/q})) = x + 2\pi k p/q$.

Finally, it is clear that $L_1^{p/q}(x)$ is a periodic function with periods 2π and $2\pi p/q$. But p and q are relatively prime. Thus, $2\pi/q$ is also a period. □

3. Classical billiards

Let Γ be a smooth closed strictly convex curve in the plane $\mathbb{R}^2$. Without loss of generality, we can assume that Γ has length 2π; it is just a technical normalization condition. Let $\gamma : \mathbb{T} \to \Gamma$ be an arc-parameterization of this curve and $\tilde{\gamma} : \mathbb{R} \to \Gamma$ be a fixed lift. Finally, let us consider the open cylinder $C = \mathbb{T} \times (y_-, y_+)$, with $y_\pm = \pm 1$. Then we can model the classical billiard dynamics inside Γ by means of a map $f : C \to C$, $f(s, y) = (s', y')$, defined as follows; see Fig. 2. If the particle hits Γ at a point $\gamma(s)$ under an angle of incidence $\phi \in (0, \pi)$, then the next impact point is $\gamma(s')$ and the next angle of incidence is $\phi' \in (0, \pi)$. Here, $y = -\cos\phi$ and $y' = -\cos\phi'$. It is well known [11, Section 9.2] that, in these coordinates, the billiard map f is an area-preserving twist map whose Lagrangian is

$$h(x, x') = \left|\tilde{\gamma}(x) - \tilde{\gamma}(x')\right|.$$

Besides, f verifies the rigid boundary conditions with $\omega_- = 0$ and $\omega_+ = 2\pi$.

Let us now consider the classical billiard dynamics inside a circle Γ_0 of radius $r_0 = 1$, since Γ_0 must have length 2π. Then $\gamma_0(s) = \mathbf{n}_s = (\cos s, \sin s)$ is an arc-parameterization. Let $f_0 : C \to C$ be the area-preserving twist map that models the classical billiard dynamics inside Γ_0 in the coordinates (s, y). It is integrable. In fact, it is easy to check that

$$\tilde{f}_0(x, y) = (x + \omega(y), y), \quad \omega(y) = 2\cos^{-1}(-y),$$

where $\cos^{-1} : (-1, 1) \to (0, \pi)$ stands for the inverse of the cos function, and so $\omega : (-1, 1) \to (0, 2\pi)$ is a diffeomorphism.

Thus, given any integers $q \geq 2$ and $1 \leq p < q$ relatively prime, there exists only one radial coordinate $y_0^{p/q} \in (-1, 1)$ such that $\omega(y_0^{p/q}) = 2\pi p/q$, and so there exists just one resonant invariant curve $T_0^{p/q} = \mathbb{R} \times \{y_0^{p/q}\}$ with rotation number p/q. The curves $T_0^{p/q}$, $1 \leq p < q$, are in two-to-one correspondence with the continuous families $\mathcal{I}_0^{p/q}$, $1 \leq p \leq q/2$, of regular p/q-gons inscribed in Γ_0 mentioned in the Introduction (but in the case $q = 2$, of course). This has to do with the fact that a p/q-gon can be traversed clockwise or counterclockwise. So we can restrict our attention to the case $1 \leq p \leq q/2$.

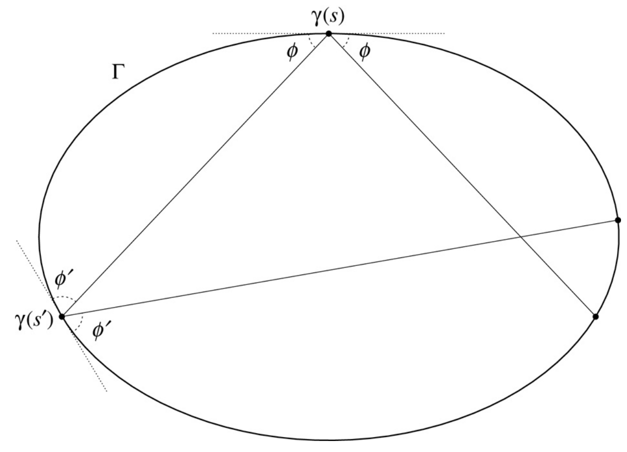

Fig. 2. The classical billiard.

This correspondence between resonant invariant curves of a (classical) billiard map and continuous families of inscribed polygons whose consecutive sides make equal angles with the billiard curve holds for any smooth strictly convex curve. Therefore, in order to prove the claim contained in Theorem 1 about the families $\mathcal{I}_0^{p/q}$, it suffices to prove the same claim about the invariant curves $T_0^{p/q}$, which follows directly from Corollary 8 and the next proposition.

Proposition 10. *The radial Melnikov potential of the resonant invariant curve $T_0^{p/q}$ of the classical billiard map inside the perturbed circle (1) is*

$$L_1^{p/q}(x) = 2\sin(\pi p/q)\sum_{k=1}^{q} r_1(x + 2k\pi p/q) = 2q\sin(\pi p/q)\sum_{j\in q\mathbb{Z}} \hat{r}_1^j \mathrm{e}^{\mathrm{i}jx}.$$

Proof. The second equality is trivial: the sum $\sum_{k=1}^{q} \mathrm{e}^{\mathrm{i}j(x+2k\pi p/q)}$ is equals to $q\mathrm{e}^{\mathrm{i}jx}$ when $j \in q\mathbb{Z}$, and vanishes otherwise. It remains to prove the first equality.

Let $\gamma_\epsilon(s)$ be an arc-parameterization of Γ_ϵ. Then $h_\epsilon(x, x') = |\tilde{\gamma}_\epsilon(x) - \tilde{\gamma}_\epsilon(x')|$ is the Lagrangian of the classical billiard map inside Γ_ϵ. We have shown in Proposition 9 that, if $h_\epsilon = h_0 + \epsilon h_1 + \mathrm{O}(\epsilon^2)$, the radial Melnikov potential is

$$L_1^{p/q}(x) = \sum_{k=1}^{q} h_1\left(x_{k-1}^{p/q}, x_k^{p/q}\right),$$

$$x_k^{p/q} = \pi_1\left(\tilde{f}_0^k\left(x, y_0^{p/q}\right)\right) = x + 2\pi k p/q.$$

Next, we compute the term $h_1(x_{k-1}^{p/q}, x_k^{p/q})$ that appears in this formula.

From Eq. (1), we get that the arc-parameterization has the form

$$\gamma_\epsilon(s) = r_\epsilon(\theta_\epsilon(s))\mathbf{n}_{\theta_\epsilon(s)}$$

for some 2π-periodic function $\theta_\epsilon(s)$. We know that $\gamma_0(s) = r_0\mathbf{n}_s$, so $\theta_0(s) = s$. Let $\theta_\epsilon(s) = s + \epsilon\theta_1(s) + \mathrm{O}(\epsilon^2)$ and $\gamma_\epsilon(s) = r_0\mathbf{n}_s + \epsilon\gamma_1(s) + \mathrm{O}(\epsilon^2)$. Then it turns out that $\gamma_1(s) = r_0\theta_1(s)\mathbf{t}_s + r_1(s)\mathbf{n}_s$, where $\mathbf{t}_s = (-\sin s, \cos s)$ and $\mathbf{n}_s = (\cos s, \sin s)$. In particular,

$$\tilde{\gamma}_1\left(x_k^{p/q}\right) = r_0\theta_1\left(x_k^{p/q}\right)\mathbf{t}_k^{p/q} + r_1\left(x_k^{p/q}\right)\mathbf{n}_k^{p/q},$$
$$\mathbf{n}_k^{p/q} = \mathbf{n}_{x_k^{p/q}}, \quad \mathbf{t}_k^{p/q} = \mathbf{t}_{x_k^{p/q}}.$$

On the other hand, $h_1(x_{k-1}^{p/q}, x_k^{p/q}) = \langle u_k^{p/q}, \tilde{\gamma}_1(x_{k-1}^{p/q}) - \tilde{\gamma}_1(x_k^{p/q})\rangle$, where

$$u_k^{p/q} = \frac{\mathbf{n}_{k-1}^{p/q} - \mathbf{n}_k^{p/q}}{|\mathbf{n}_{k-1}^{p/q} - \mathbf{n}_k^{p/q}|}.$$

Finally, we obtain the radial Melnikov potential in the desired form:

$$\begin{aligned} L_1^{p/q}(x) &= \sum_{k=1}^{q} h_1\left(x_{k-1}^{p/q}, x_k^{p/q}\right) \\ &= \sum_{k=1}^{q}\left\langle u_k^{p/q}, \tilde{\gamma}_1\left(x_k^{p/q}\right) - \tilde{\gamma}_1\left(x_{k-1}^{p/q}\right)\right\rangle \\ &= \sum_{k=1}^{q}\left\langle u_k^{p/q} - u_{k+1}^{p/q}, \tilde{\gamma}_1\left(x_k^{p/q}\right)\right\rangle \\ &= 2\sin(\pi p/q)\sum_{k=1}^{q} r_1\left(x_k^{p/q}\right), \end{aligned}$$

since $\langle u_k^{p/q} - u_{k+1}^{p/q}, \mathbf{t}_k^{p/q}\rangle = 0$, $\langle u_k^{p/q} - u_{k+1}^{p/q}, \mathbf{n}_k^{p/q}\rangle = 2\sin(\pi p/q)$, $x_{k+q}^{p/q} = x_k^{p/q} + 2\pi p$, and $u_{k+q}^{p/q} = u_k^{p/q}$. □

4. Dual billiards

Let Γ be a smooth closed strictly convex curve in the plane $\mathbb{R}^2$. Let U_Γ be the unbounded component of $\mathbb{R}^2 \setminus \Gamma$, that is, the "exterior" of Γ. The dual billiard map $f : U_\Gamma \to U_\Gamma$ is defined as follows (see Fig. 3): $f(z)$ is the reflection of z in the tangency point of the oriented tangent line to Γ through z. This map is area-preserving [19]. Next we shall describe some useful coordinates on U_Γ. We skip many details, which can be found in the nice paper of Boyland [6].

To begin with, let $\gamma : \mathbb{T} \to \Gamma$ be an *envelope parameterization* of Γ, that is, a parameterization of the curve Γ such that

$$\gamma'(s) = \rho(s)\mathbf{t}_s, \quad \mathbf{t}_s = (-\sin s, \cos s),$$

for some function $\rho(s) > 0$, which is the *radius of curvature* of Γ. Given any $s \in \mathbb{T}$, let $\mathcal{L}_s$ be the tangent line to Γ at the point $\gamma(s)$. As is shown in Fig. 3, we consider the decomposition $\mathcal{L}_s = \mathcal{L}_s^- \cup \{\gamma(s)\} \cup \mathcal{L}_s^+$, where

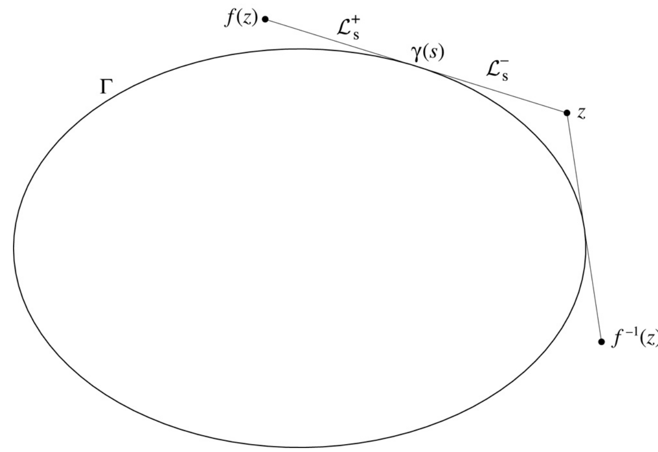

Fig. 3. The dual billiard. The segment from z to $f(z)$ is tangent to Γ at $\frac{1}{2}(z + f(z))$. The change $U_\Gamma \ni z \mapsto (s, y) \in \mathbb{T} \times (0, +\infty)$ is defined by means of the relations $\gamma(s) = \frac{1}{2}(z + f(z))$ and $y = \frac{1}{2}|z - \gamma(s)|^2$.

$$\mathcal{L}_s^- = \{z \in \mathcal{L}_s : \langle z - \gamma(s), \mathbf{t}_s\rangle < 0\},$$
$$\mathcal{L}_s^+ = \{z \in \mathcal{L}_s : \langle z - \gamma(s), \mathbf{t}_s\rangle > 0\}.$$

The *height function* $p(s)$ of Γ is the distance from the origin to the line $\mathcal{L}_s$. Then $\beta(s) = p(s)\mathbf{n}_s$ is the nearest point to the origin on the line $\mathcal{L}_s$. Boyland proves that the height function verifies the second-order differential equation

$$p'' + p = \rho, \tag{5}$$

and that $p'(s)$ equals the signed distance from $\gamma(s)$ to $\beta(s)$.

Finally, let $\ell(s,t)$ be the distance from $\gamma(s)$ to the intersection $\mathcal{L}_s \cap \mathcal{L}_t$. We note that $\ell(s,t) \neq \ell(t,s)$. This function can be expressed in terms of the radius of curvature or the height function. Concretely,

$$\begin{aligned} \ell(s,t) &= \frac{\int_s^t \sin(v-s)\rho(v)\mathrm{d}v}{\sin(t-s)} \\ &= \frac{p(s)}{\sin(t-s)} - \frac{p(t)}{\tan(t-s)} + p'(t). \end{aligned}$$

The first equality is contained in [6]. The second one follows from Eq. (5); it suffices to integrate by parts twice.

Given any point $z \in U_\Gamma$, let $s = s(z)$ be the unique angle such that $z \in \mathcal{L}_s^-$, and let $y = y(z) = \ell(z)^2/2$ with $\ell(z) = |z - \gamma(s)|$. The map $U_\Gamma \ni z \mapsto (s, y) \in C$ is an area-preserving diffeomorphism, where $C = \mathbb{T} \times (y_-, y_+)$ with $y_- = 0$ and $y_+ = +\infty$. The coordinates (s, y) will be called *envelope coordinates*. The dual billiard map $f : C \to C$, $f(s,y) = (s', y')$, is an area-preserving twist map in these envelope coordinates [10,6]. Besides, it verifies the rigid boundary conditions with $\omega_- = 0$ and $\omega_+ = \pi$.

From the the area-preserving character of the change $z \mapsto (s, y)$ and the geometric interpretation given in Remark 3, the Lagrangian of f is

$$h(x, x') = \int_x^{x'} y(x,u)\mathrm{d}u, \quad y(x,u) = \frac{1}{2}\tilde{\ell}(x,u)^2.$$

Hence, the quantity $h(x, x')$ is the area bounded by the curve Γ and the tangent lines $\mathcal{L}_x^+$ and $\mathcal{L}_{x'}^-$.

Let us now consider the dual billiard dynamics outside a circle Γ_0 of radius r_0. Its radius of curvature and its height function are $\rho_0(s) = p_0(s) = r_0$. Then $\gamma_0(s) = r_0\mathbf{n}_s = (r_0 \cos s, r_0 \sin s)$ is an envelope parameterization of Γ_0, and $\ell_0(s,t) = r_0(1-\cos(t-s))/\sin(t-s) = r_0 \tan\frac{t-s}{2}$.

Let $f_0 : C \to C$ be the area-preserving twist map that models the dual billiard dynamics outside Γ_0 in the coordinates (s, y). The map f_0 is integrable. In fact, it is easy to check that

$$\tilde{f}_0(x,y) = (x + \omega(y), y), \quad \omega(y) = 2\tan^{-1}\left(\sqrt{2y}/r_0\right),$$

where $\tan^{-1} : (0,+\infty) \to (0, \pi/2)$ stands for the inverse of the tan function, and so $\omega : (0,+\infty) \to (0,\pi)$ is a diffeomorphism.

Thus, given any integers $q \geq 3$ and $1 \leq p < q/2$ relatively prime, there exists only one radial coordinate $y_0^{p/q} \in (0,+\infty)$ such that $\omega(y_0^{p/q}) = 2\pi p/q$, and so there exists just one resonant invariant curve $T_0^{p/q} = \mathbb{R} \times \{y_0^{p/q}\}$ with rotation number p/q. It is clear that the curves $T_0^{p/q}$, $1 \leq p < q/2$, are in one-to-one correspondence with the continuous families $\mathcal{C}_0^{p/q}$, $1 \leq p < q/2$, of regular p/q-gons circumscribed about Γ_0 mentioned in the Introduction. Therefore, to prove the claim contained in Theorem 1 about the families $\mathcal{C}_0^{p/q}$, it suffices to prove the same claim about the invariant curves $T_0^{p/q}$, which follows directly from Corollary 8 and the next proposition.

Proposition 11. *The radial Melnikov potential of the resonant invariant curve $T_0^{p/q}$ of the dual billiard map outside the perturbed circle (1) is*

$$\begin{aligned} L_1^{p/q}(x) &= 2r_0 \tan(\pi p/q) \sum_{k=1}^{q} r_1(x + 2k\pi p/q) \\ &\quad - pr_0 \int_0^{2\pi} r_1(x)\mathrm{d}x \\ &= 2r_0 q \tan(\pi p/q) \sum_{j\in q\mathbb{Z}} \hat{r}_1^j \mathrm{e}^{\mathrm{i}jx} - pr_0 \hat{r}_1^0. \end{aligned}$$

Proof. The second equality follows from a trivial computation with Fourier coefficients, as in the previous section. It remains to prove the first one.

Let $p_\epsilon(s) = r_0 + \epsilon p_1(s) + \mathrm{O}(\epsilon^2)$ and $\gamma_\epsilon(s) = r_0\mathbf{n}_s + \mathrm{O}(\epsilon)$ be the height function and an envelope parameterization of the perturbed circle (1). Then the perturbed Lagrangian is

$$\begin{aligned} h_\epsilon(x,x') &= h_0(x,x') + \epsilon h_1(x,x') + \mathrm{O}(\epsilon^2) \\ &= \frac{1}{2}\int_x^{x'} \tilde{\ell}_\epsilon(x,u)^2 \mathrm{d}u, \end{aligned}$$

where $\ell_\epsilon(s,t) = \ell_0(s,t) + \epsilon\ell_1(s,t) + \mathrm{O}(\epsilon^2) = \frac{p_\epsilon(s)}{\sin(t-s)} - \frac{p_\epsilon(t)}{\tan(t-s)} + p'_\epsilon(t)$.

First, we are going to show that $p_1(s) = r_1(s)$, and consequently

$$\ell_1(s,t) = \frac{r_1(s)}{\sin(t-s)} - \frac{r_1(t)}{\tan(t-s)} + r_1'(t).$$

To prove it, let us consider the triangle formed by $A = \alpha_\epsilon(s) = r_\epsilon(s)\mathbf{n}_s \in \Gamma_\epsilon$, $B = \beta_\epsilon(s) = p_\epsilon(s)\mathbf{n}_s \in \mathcal{L}_s$, and $C = \gamma_\epsilon(s) \in \Gamma_\epsilon \cap \mathcal{L}_s$. We recall that $\mathcal{L}_s$ is the tangent line to Γ_ϵ at the point C. We also know that $\mathrm{dist}(C, B) = |p'_\epsilon(s)| = \mathrm{O}(\epsilon)$. If $\hat{A}$, $\hat{B}$, and $\hat{C}$ are the angles of the triangle, then $\hat{B} = \pi/2$ and $\hat{C} = \mathrm{O}(\epsilon)$. The first claim follows from the orthogonality between $\mathcal{L}_s$ and $\mathbf{n}_s$. The second one follows from the fact that the line through the points $A, C \in \Gamma_\epsilon$ tends to the tangent line to the curve Γ_ϵ at the point C (that is, to $\mathcal{L}_s$) as A tends to C, because the curve is smooth. Hence

$$\begin{aligned} |r_\epsilon(s) - p_\epsilon(s)| &= \mathrm{dist}(A,B) = \mathrm{dist}(C,B)\cdot \tan\hat{C} \\ &= |p'_\epsilon(s)| \tan\hat{C} = \mathrm{O}(\epsilon^2), \end{aligned}$$

and so $r_1(s) = p_1(s)$.

Now we are ready to compute the first-order term $h_1(x, x')$ of the Lagrangian. To begin with, we recall that $\ell_0(s,t) = r_0 \tan\frac{t-s}{2}$ and so

$$\ell_0(s,t)\ell_1(s,t) = r_0 \frac{\mathrm{d}}{\mathrm{d}t}\left\{\tan\frac{t-s}{2}(r_1(s) + r_1(t))\right\} - r_0 r_1(t).$$

Then, using that $h_1(x,x') = \int_x^{x'} \tilde{\ell}_0(x,u)\tilde{\ell}_1(x,u)\mathrm{d}u$, we get that

$$h_1(x,x') = r_0 \tan\frac{x'-x}{2}(r_1(x) + r_1(x')) - r_0\int_x^{x'} r_1(u)\mathrm{d}u.$$

Finally, the radial Melnikov potential is

$$\begin{aligned} L_1^{p/q}(x) &= \sum_{k=1}^{q} h_1\left(x_{k-1}^{p/q}, x_k^{p/q}\right) \\ &= r_0 \tan(\pi p/q) \sum_{k=1}^{q}\left(r_1\left(x_{k-1}^{p/q}\right) + r_1\left(x_k^{p/q}\right)\right) \\ &\quad - r_0 \int_{x_0^{p/q}}^{x_q^{p/q}} r_1(u)\mathrm{d}u \\ &= 2r_0\tan(\pi p/q)\sum_{k=1}^{q} r_1\left(x_k^{p/q}\right) - pr_0\int_0^{2\pi} r_1(u)\mathrm{d}u, \end{aligned}$$

since $x_k^{p/q} = \pi_1(\tilde{f}_0^k(x, y_0^{p/q})) = x + 2\pi kp/q$ and $r_1(u)$ is 2π-periodic. □

5. Examples

To begin with, we consider classical and dual billiard dynamics associated to the perturbations (1) such that

$$r_1(\theta) = \alpha\frac{\cos\theta - \alpha}{1 - 2\alpha\cos\theta + \alpha^2} = \frac{1}{2}\sum_{0\neq j\in\mathbb{Z}} \alpha^{|j|}\mathrm{e}^{\mathrm{i}j\theta}$$

for some $\alpha \in (0,1)$. We note that no Fourier coefficient of $r_1(\theta)$ vanishes. Therefore, we deduce from Theorem 1 that these perturbations destroy the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ for *any* rational p/q. Hence, no resonant invariant curve of the classical and dual billiard maps persists under these perturbations.

Next, we consider the following monomial perturbations of the unit circle:

$$\Gamma_\epsilon^n = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + \epsilon y^n = 1\right\}, \quad n \geq 0. \tag{6}$$

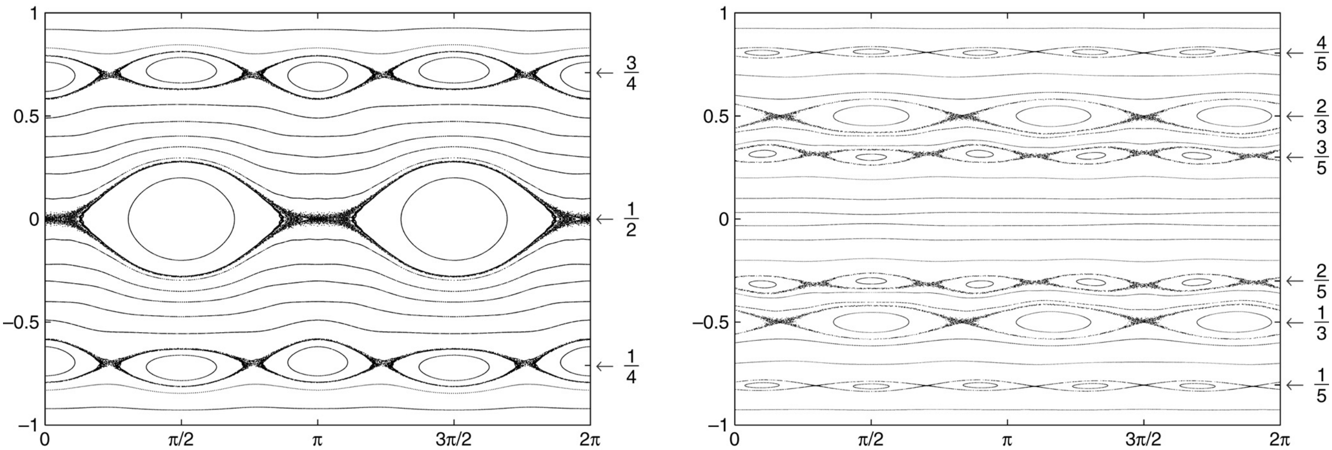

Fig. 4. Some orbits of the billiard map $f : C \to C$ inside the perturbed circle Γ_ϵ^n. Left: $n = 4$ and $\epsilon = 0.1$ displaying the p/q-resonances such that $q \in \mathcal{Q}_4 = \{2, 4\}$. Right: $n = 5$ and $\epsilon = 0.015$ displaying the p/q-resonances such that $q \in \mathcal{Q}_5 = \{3, 5\}$.

The cases $n = 0$ and $n = 1$ have no interest, because Γ_ϵ^0 and Γ_ϵ^1 are again circles. In these cases, the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ are preserved.

The results for $n = 2$ are optimal in the following sense: no break-up is skipped by our method. Let us explain this. We write Γ_ϵ^2 in polar coordinates; see Remark 2. Then

$$r_1(\theta) = -\frac{1}{2}\sin^2\theta = -\frac{1}{4} + \frac{1}{8}\mathrm{e}^{\mathrm{i}2\theta} + \frac{1}{8}\mathrm{e}^{-\mathrm{i}2\theta}.$$

The Fourier expansion of $r_1(\theta)$ contains just three harmonics: $-\frac{1}{4}$ and $\frac{1}{8}\mathrm{e}^{\pm\mathrm{i}2\theta}$. Hence, we deduce from Theorem 1 that the family $\mathcal{I}_0^{1/2}$ is destroyed under the perturbation Γ_ϵ^2. A priori, no information about the other families can be deduced. But Γ_ϵ^2 is an ellipse and (classical and dual) elliptic billiards are integrable,[1] because of the celebrated Poncelet Porism [21, Section 4.3]. This Porism also implies the preservation of the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ for all $q \geq 3$. Therefore, in this case the sufficient condition stated in Theorem 1 is also necessary, and so it is optimal.

The results for $n \geq 3$ are far from optimal. Nevertheless, they are useful to estimate the amplitude of the resonances in the perturbed maps. In order to describe the results, we write Γ_ϵ^n in polar coordinates as before. Then $r_1(\theta) = -\frac{1}{2}\sin^n\theta$. Let $\mathcal{Q}_n$ be the set of integers $q \geq 2$ such that the j-th Fourier coefficient of the function $\sin^n\theta$ is non-zero for some $j \in q\mathbb{Z} \setminus \{0\}$. A straightforward computation shows that

$$\mathcal{Q}_{2l} = \{2, 4, \ldots, 2l\} \cup \{2, 3, \ldots, l\},$$
$$\mathcal{Q}_{2l+1} = \{3, 5, \ldots, 2l + 1\}$$

for any $l \geq 0$. For instance, $\mathcal{Q}_0 = \mathcal{Q}_1 = \emptyset$, $\mathcal{Q}_2 = \{2\}$, $\mathcal{Q}_3 = \{3\}$, $\mathcal{Q}_4 = \{2, 4\}$, $\mathcal{Q}_5 = \{3, 5\}$, and $\mathcal{Q}_6 = \{2, 3, 4, 6\}$. Then, we deduce from Theorem 1 that the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ are destroyed under the perturbation Γ_ϵ^n for any $q \in \mathcal{Q}_n$. A priori, no information about the other families can be deduced. Thus, some natural questions arise. What dynamical information can be deduced from the sets $\mathcal{Q}_n$? Does Γ_ϵ^n destroy the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ for some $q \notin \mathcal{Q}_n$?

The first question has a simple answer. If $q \notin \mathcal{Q}_n$, the break-up is not detected at first order in ϵ, and so the order of the amplitude of the corresponding resonances is smaller than the generic one. (Generically, the amplitude of the resonances in area-preserving maps is $\mathrm{O}(\epsilon^{1/2})$; see [23,17].) For instance, when $n = 4$ (resp., $n = 5$) the biggest resonances are the ones with $q \in \mathcal{Q}_4 = \{2, 4\}$ (resp., $q \in \mathcal{Q}_5 = \{3, 5\}$). These "big" resonances are displayed in Fig. 4.

The second question remains open. We conjecture that, for any monomial perturbation (6) with $n \geq 3$, the classical and dual billiard dynamics is generic in the sense that all the unperturbed resonant curves break up, no matter whether q belongs to $\mathcal{Q}_n$ or not. The proof of this conjecture is a work in progress. Some experimental support on this conjecture is given in Fig. 5, where we see that the 2/3-resonant (resp., 3/4-resonant) invariant curve breaks up under the monomial perturbation of degree four (resp., five). We have checked that other resonant curves also break up.

We also stress that, in accordance with the answer of the first question, the amplitude of the 2/3-resonance is smaller than the one of the $p/4$-resonances when $n = 4$, whereas the amplitude of the 3/4-resonance is smaller than the one of the $p/5$-resonances when $n = 5$; compare Fig. 4 with Fig. 5.

To end this section, we note that the polynomial perturbations

$$\Gamma_\epsilon^{n,\alpha} = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + \epsilon(x\sin\alpha + y\cos\alpha)^n = 1\right\}, \quad \alpha \in \mathbb{T} \tag{7}$$

are equivalent to the monomial perturbations (6). This has to do with the fact that $\Gamma_\epsilon^{\alpha,n}$ is obtained from $\Gamma_\epsilon^n = \Gamma_\epsilon^{n,0}$ by means of a rotation of angle α, and rotations have no effect on the (classical or dual) billiard dynamics. Hence, we know that under the perturbation (7) the p/q-resonant curves with $q \in \mathcal{Q}_n$ break up, and we conjecture that the others also break up for any $n \geq 3$ and $\alpha \in \mathbb{T}$.

[1] Here, integrable only means that there exists a first integral, because elliptic billiard have separatrices, so they are not integrable in the sense of Section 2.2.

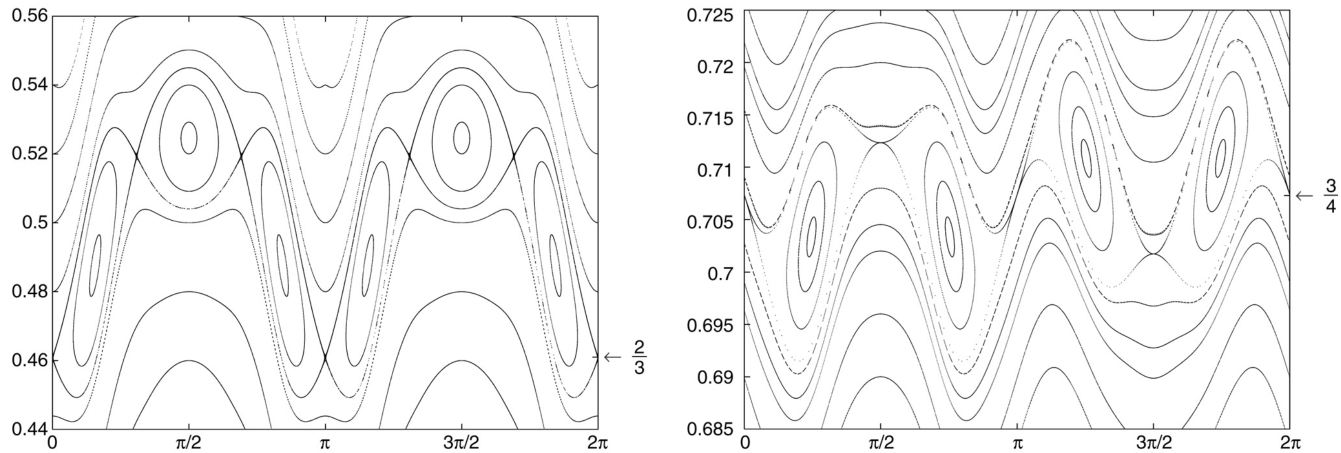

Fig. 5. Some orbits in a narrow strip of the phase space of the billiard map inside Γ_ϵ^n. Left: $n = 4$ and $\epsilon = 0.1$ displaying the 2/3-resonance. Right: $n = 5$ and $\epsilon = 0.015$ displaying the 3/4-resonance.

6. Conclusion and open problems

In this paper we introduced the radial Melnikov potential to study the break-up of resonant invariant curves for perturbations of integrable twist maps. Once we have fixed some resonant curve of an integrable twist map and some perturbation, the curve breaks up when the radial Melnikov potential is not constant. This is a first-order condition, and so it is sufficient but not necessary for the break-up. Then we applied the theory to the classical and dual billiard maps associated to small perturbations of circular tables. These problems are very nice, since the previous first-order condition can be stated directly in terms of the Fourier coefficients of the perturbation of the circle, giving rise to a simple sufficient condition for the break-up of resonant curves. Finally, we studied some perturbations with the above condition and we formulated a conjecture.

This research can be continued in several ways. For instance, one can study perturbations of elliptic billiards which are a bit harder because they have no global "action-angle" variables. In this case, from our experience with related separatrix splitting problems [7], we believe that, to derive the non-constant character of the Melnikov potential, we must study its complex singularities instead of its Fourier coefficients.

Another problem is the extension of the current theory to perturbations of $2n$-dimensional completely integrable twist maps, which is more or less done in [25]. Next, one could try to apply the theory to study the break-up of the resonant invariant tori that appear in billiards inside spheres or ellipsoids. To be more precise, we hope to obtain results about the break-up of those resonant tori very similar to the results about the splitting of separatrices for billiards inside perturbed ellipsoids obtained in [8,5]. The bifurcations of periodic orbits can also be studied, following the methods contained in [18] for the study of homoclinic bifurcations. The fact that the billiard dynamics inside symmetric curves is reversible could be useful, because it reduces the problem of finding periodic orbits to a one-dimensional search.

The last question we want to mention is to find some higher-order conditions to determine whether a break-up takes place if the Melnikov potential is constant. This is a first step towards the conjecture about the break-up of the families $\mathcal{I}_0^{p/q}$ and $\mathcal{C}_0^{p/q}$ under the perturbation (7) for any $n \geq 3$ and $\alpha \in \mathbb{T}$. Besides, once some higher-order condition ensures the break-up of a resonant curve, the amplitude of the corresponding resonance is very small, since the perturbed map is very close to an integrable one in a neighborhood of the resonance. We plan to estimate the amplitude of these small resonances using the techniques explained in [17].

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