HIGH-ORDER PERSISTENCE OF RESONANT CAUSTICS IN PERTURBED CIRCULAR BILLIARDS

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ABSTRACT. We find necessary and sufficient conditions for high-order persistence of resonant caustics in perturbed circular billiards. The main tool is a perturbation theory based on the Bialy-Mironov generating function for convex billiards. All resonant caustics with period q persist up to order $\lceil q/n \rceil - 1$ under any polynomial deformation of the circle of degree n.

1. INTRODUCTION

The goal of this work is two-fold. First, to extend the first-order perturbation theory for exact twist maps developed in [41, 39, 17] to a higher-order theory. Second, to apply that theory to the study of high-order persistence of resonant caustics in perturbed circular billiards. The second goal is strongly motivated by some of the numerical experiments discussed in [37].

The computational aspects of our analysis are greatly simplified when working with the Bialy-Mironov generating function for convex billiards discovered in [7, 4]. We were also inspired by [10]. We know just a few practical high-order Melnikov theories for time-periodic perturbations of integrable continuous systems (ODEs) —see, for instance, [14, 15]—, but none for perturbations of integrable discrete systems (maps). In that sense, our theory is novel.

The fragility of resonant caustics is a key idea behind recent proofs of local versions of the Birkhoff conjecture (see below) and related results about the rigidity of the length spectrum of strictly convex domains [1, 29, 25, 24, 30, 23]. See also the surveys [28, 19]. Almost all these works describe the first-order persistence condition of resonant caustics contained in [41]. We hope that our new high-order persistence conditions will be equally useful.

A *caustic* is a curve such that any billiard trajectory, once tangent to the curve, stays tangent after every reflection. The robustness of a convex caustic is closely related to the arithmetic properties of its *rotation number* $\rho \in (0,1)$, a number that measures the number of turns around the caustic per impact. Tangent lines to the caustic can be counterclockwise or clockwise oriented. We fix the counterclockwise orientation, so $\rho \in (0,1/2]$. Lazutkin [33] showed that for any smooth strictly convex domain there is a positive measure Cantor set $\mathcal{R} \subset (0,1/2)$ of Diophantine rotation numbers that accumulates to 0 such that there is a caustic for any rotation number $\rho \in \mathcal{R}$. These caustics persist under smooth deformations of the domain [40].

Let $\rho = p/q \in (0,1/2]$ be a rational rotation number such that $\gcd(p,q) = 1$. A convex caustic is called *p/q-resonant* (or *p/q-rational*) when all its tangent trajectories form closed polygons with q sides that make p turns around the caustic. We say that q is the *period* of the caustic. Resonant caustics generically break up under perturbation. Recent results in [20] confirm their fragility. Once fixed $q \geq 2$, the space of convex domains with a resonant caustic

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of period q has infinite dimension and codimension [2]. The space of convex domains with at least one resonant caustic is dense in the space of all convex domains [31].

We shall not deal with the case $\rho = 1/2$, since convex domains with $1/2$-resonant caustics are easily characterized as the *constant width* domains [32, 22]. Centrally symmetric convex domains with a $1/4$-resonant caustic have also been completely characterized in terms of the Fourier coefficients of the square of the support function of the convex domain in [11]. Some non-circular convex domains with a $1/3$-resonant caustic were constructed in [26].

Circles and ellipses are the only known strictly convex smooth domains almost completely foliated by convex caustics. The centenary *Birkhoff conjecture* claims that they are the only ones [42]. Bialy [3] proved the following weak version of this conjecture. If almost every billiard trajectory in a convex domain is tangent to a convex caustic, then the domain is a disk. A much stronger version of the Birkhoff conjecture for centrally symmetric C^2-domains, based on the structure of the $1/4$-resonant caustic, was recently established by Bialy and Mironov [9]. See also [11, 6] for effective (that is, quantitative) versions on these two results. Near centrally symmetric domains were considered in [27].

We are interested in two practical problems. First, to characterize the deformations of the circle that preserve a given resonant caustic. Second, to determine all resonant caustics that are preserved under a given deformation of the circle. In that regard, we recall that any $\mathbb{Z}_2$-symmetric analytical deformation of a circle (with certain Fourier decaying rate) preserving both its $1/2$-resonant and $1/3$-resonant caustics has to be an isometric transformation [44].

In what follows we introduce some notations and state our two main results.

Let Γ_ϵ be a deformation of the unit circle with smooth support function

$$h(\psi;\epsilon) = h_\epsilon(\psi) \asymp 1 + \sum_{k\geq 1} \epsilon^k h_k(\psi) \quad \text{as } \epsilon \to 0, \tag{1}$$

where $\psi \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the normal angle and $\epsilon \in [-\epsilon_0, \epsilon_0]$ is the perturbative parameter. We say that a resonant caustic of the unit circle $\mathrm{O}(\epsilon^m)$-*persists* under Γ_ϵ when the billiard in Γ_ϵ is $\mathrm{O}(\epsilon^{m+1})$-close to having that resonant caustic. See Definition 2 for more details.

Let $\nu_l : (0,1/2) \to \mathbb{R} \cup \{\infty\}$, with $l \in \mathbb{Z}$ and $|l| \geq 2$, be the sequence of functions given by

$$\nu_l(\rho) = \nu_{-l}(\rho) = \begin{cases} \dfrac{\tan(l\pi\rho) - l\tan(\pi\rho)}{\tan(\pi\rho)\tan(l\pi\rho)}, & \text{if } 2l\rho \notin \mathbb{Z}, \\ 1/\tan(\pi\rho), & \text{if } 2l\rho \in \mathbb{Z} \text{ but } l\rho \notin \mathbb{Z}, \\ \infty, & \text{if } l\rho \in \mathbb{Z}. \end{cases} \tag{2}$$

Once again, we realize that Gutkin's equation $\tan(l\pi\rho) = l\tan(\pi\rho)$ is ubiquitous in billiard problems. See [22, 4, 8, 12] for other examples. Cyr [16] proved that $\nu_l(\rho)$ has no rational roots $\rho = p/q \in (0,1/2)$ when $|l| \geq 2$. The case $l\rho \in \mathbb{Z}$ never takes place in our computations. The singular value $\nu_l(\rho) = \infty$ has been written just for definitness. It is irrelevant.

Fourier coefficients of 2π-periodic functions are denoted with a hat: $a(t) = \sum_{l\in\mathbb{Z}} \hat{a}_l \mathrm{e}^{\mathrm{i}lt}$. Given a 2π-periodic smooth function $a(t)$ and a subset $R \subset \mathbb{Z}$, let $\mu_R\{a(t)\} = \sum_{l\in R} \hat{a}_l \mathrm{e}^{\mathrm{i}lt}$ be the projection of $a(t)$ onto its R-harmonics. We only consider the cases $R = q\mathbb{Z}$ or $R = q\mathbb{Z}^\star$ with $\mathbb{Z}^\star = \mathbb{Z} \setminus \{0\}$.

Theorem 1. *Let $\rho = p/q \in (0, 1/2)$ be any rational rotation number such that* $\gcd(p, q) = 1$. *The high-order persistence of the p/q-resonant caustic of the unit circle under the deformation with support function* (1) *can be determined as follows.*

a) It $\mathrm{O}(\epsilon)$-persists if and only if $\mu_{q\mathbb{Z}^\star}\{h_1\} = 0$.

b) It $\mathrm{O}(\epsilon^2)$-persists if and only if it $\mathrm{O}(\epsilon)$-persists and $\mu_{q\mathbb{Z}^\star}\{h_2 + \theta_1^2/2\} = 0$, where

$$\theta_1(t) = \sum_{l \notin q\mathbb{Z} \cup \{-1,1\}} \nu_l(\rho) \hat{h}_{1,l}\, \mathrm{e}^{\mathrm{i}lt} \quad \text{if} \quad h_1(\psi) = \sum_{l \in \mathbb{Z}} \hat{h}_{1,l}\, \mathrm{e}^{\mathrm{i}l\psi}.$$

c) It $\mathrm{O}(\epsilon^m)$-persists for some $m \geq 3$ if and only if it $\mathrm{O}(\epsilon^{m-1})$-persists and $\mu_{q\mathbb{Z}^\star}\{h_m + \zeta_m\} = 0$, where ζ_m is a smooth 2π-periodic function, only depending on $h_1, \ldots, h_{m-1}$, that can be explicitly computed from recurrences given along the paper.

The $\mathrm{O}(\epsilon)$-persistence result in Theorem 1 is just a reformulation of the main theorem in [41]. Condition $\mu_{q\mathbb{Z}^\star}\{h_m + \zeta_m\} = 0$ is equivalent $\mu_{q\mathbb{Z}}\{h'_m + \zeta'_m\} = 0$. In particular, condition $\mu_{q\mathbb{Z}^\star}\{h_2 + \theta_1^2/2\}$ is equivalent to $\mu_{q\mathbb{Z}}\{h'_2 + \theta_1\theta'_1\} = 0$.

Let $T_n[\psi]$ be the space of 2π-periodic real trigonometric polynomials of degree $\leq n$ in ψ.

Definition 1. *We say that a deformation Γ_ϵ of the unit circle with support function* (1) *is* polynomial *of* degree $\leq n$ *when*

$$h_k(\psi) \in T_{nk}[\psi], \quad \forall k \geq 1, \tag{3}$$

and is centrally *or* anti-centrally symmetric *when $h_\epsilon(\psi + \pi) = h_\epsilon(\psi)$ or $h_\epsilon(\psi + \pi) = h_{-\epsilon}(\psi)$.*

Being centrally symmetric is a property of single curves. Being anti-centrally symmetric is a property of deformations.

Theorem 2. *Let $\lceil x \rceil = \min\{k \in \mathbb{Z} : k \geq x\}$ be the ceil function. If $\rho = p/q \in (0, 1/2)$ is a rational rotation number such that* $\gcd(p, q) = 1$ *and Γ_ϵ is a polynomial deformation of the unit circle of degree $\leq n$, then the p/q-resonant caustic $\mathrm{O}(\epsilon^{\chi-1})$-persists under Γ_ϵ, where*

$$\chi = \chi(\Gamma_\epsilon, q) = \begin{cases} 1 + 2\left\lceil (q-n)/2n \right\rceil, & \text{for anti-centrally symmetric } \Gamma_\epsilon \text{ and odd } q, \\ 2\lceil q/2n \rceil, & \text{for anti-centrally symmetric } \Gamma_\epsilon \text{ and even } q, \\ \lceil 2q/n \rceil, & \text{for centrally symmetric } \Gamma_\epsilon \text{ and odd } q, \\ \lceil q/n \rceil, & \text{otherwise.} \end{cases}$$

The idea behind this theorem is quite simple. For non-symmetric deformations, it suffices to check that $\zeta_m(t) \in T_{nm}[t]$ for $m = 1, \ldots, \chi - 1$, where $\zeta_m(t)$ are the functions introduced in Theorem 1. Symmetric deformations require to check that, in addition, those polynomials $\zeta_m(t)$ are π-periodic or π-antiperiodic.

Polynomial deformations of the unit circle of degree $\leq n$ can be defined without mentioning support functions. For instance, we can define them in Cartesian coordinates (x, y) as

$$\Gamma_\epsilon = \big\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = P(x, y; \epsilon)\big\}, \tag{4}$$

for some smooth function $P(x, y; \epsilon)$ of the form $P(x, y; \epsilon) \asymp 1 + \sum_{k \geq 1} \epsilon^k P_k(x, y)$ as $\epsilon \to 0$ with $P_k(x, y) \in \mathbb{R}_{kn}[x, y]$ for all $k \geq 1$. Alternatively, we can also define them in polar coordinates (r, ϕ) as

$$\Gamma_\epsilon = \big\{r(\phi; \epsilon) \cdot (\cos\phi, \sin\phi) : \phi \in \mathbb{T}\big\},$$

for some smooth *polar function* $r(\phi;\epsilon)$ of the form $r(\phi;\epsilon) \asymp 1 + \sum_{k\geq 1} \epsilon^k r_k(\phi)$ as $\epsilon \to 0$ with $r_k(\phi) \in T_{kn}[\phi]$ for all $k \geq 1$. The Cartesian setting was considered in [37] with $P(x,y;\epsilon) = 1 - \epsilon y^n$. The polar setting was considered in [41] with $r(\phi;\epsilon) = 1 + \epsilon r_1(\phi) + \mathrm{O}(\epsilon^2)$ and in [44] with $r(\phi;\epsilon) = 1 + \epsilon r_1(\phi) + \epsilon^2 r_2(\phi) + \mathrm{O}(\epsilon^3)$. Any deformation of the unit circle expressed in Cartesian coordinates as (4) for some $P(x,y;\epsilon) = 1 + \epsilon P_1(x,y)$ with $P_1(x,y) \in \mathbb{R}_n[x,y]$ is a polynomial deformation of degree $\leq n$ in the sense of Definition 1 with $h_1(\psi) = \frac{1}{2} P_1(\cos\psi, \sin\psi)$. See Lemma 10. We are interested in such deformations because we want to understand the numerical experiments discussed in [37]. However, for brevity, we omit the corresponding proofs for deformations written in polar coordinates or in Cartesian coordinates with more than the first order term $\epsilon P_1(x,y)$. Such proofs are just a slew of boring computations based in the Taylor, multinomial and Lagrange inversion theorems. We only stress that $h_1(\psi) = r_1(\psi)$, which justifies that the $\mathrm{O}(\epsilon)$-persistence result in Theorem 1 is just a reformulation of the main theorem in [41].

The map $q \mapsto \chi(\Gamma_\epsilon, q)$ is unbounded for any polynomial deformation Γ_ϵ of degree $\leq n$, since $\chi(\Gamma_\epsilon, q) \asymp 2q/n$ as odd $q \to +\infty$ for centrally symmetric deformations, and $\chi(\Gamma_\epsilon, q) \asymp q/n$ as $q \to +\infty$ otherwise. The experiments described in [37, Numerical Result 5], which cover degrees $3 \leq n \leq 8$ and periods $3 \leq q \leq 100$, suggest that none of the p/q-resonant caustics $\mathrm{O}(\epsilon^\chi)$-persists under monomial deformations (4) with $P(x,y;\epsilon) = 1 - \epsilon y^n$ and $n \geq 3$. Its proof requires to check that $\mu_{q\mathbb{Z}^*}\{h_\chi + \zeta_\chi\} \neq 0$, which is a challenge. If it were true, that monomial deformations *would break all resonant caustics* in such a way that *there would be breakups of any order*, because the map $q \mapsto \chi(\Gamma_\epsilon, q) \in \mathbb{N}$ is exhaustive.

The paper is organized as follows. Section 2 begins with a description of the Bialy-Mironov generating function and ends with a list of necessary conditions for the existence of smooth convex resonant caustics in smooth strictly convex domains. The general notion of high-order persistence of convex resonant caustics in deformed smooth convex domains is presented in Section 3 and applied to deformed circles in Section 4, where Theorem 1 is proved. The results about polynomial deformations of circles, including Theorem 2, are presented in Section 5. Finally, we discuss three open problems: the co-preservation of resonant caustics with different rotation numbers, the convergence of a procedure to correct the original deformation in order to preserve a chosen resonant caustic and the asymptotic measure of some exponentially small phenomena as the period q grows. See Section 6. Several technical proofs have been relegated to the appendices.

2. Existence of smooth convex resonant caustics

To begin with, we introduce coordinates in the space of oriented lines, define the support function and the billiard map of a convex domain, and describe the Bialy-Mironov generating function following [10]. We also recall the periodic version of the variational principle for twist maps following [38]. Next, we combine all those elements to find necessary conditions for the existence of smooth convex resonant caustics in Theorem 4. This part is inspired by the computations in [10, Theorem 2.2] and the Lagrangian approach to the existence of rotational invariant curves (RICs) of twist maps described in [34, 31]. Finally, we discuss five simple examples: circles, ellipses, constant width curves, Gutkin billiard tables (also called constant angle curves) and centrally symmetric curves with $1/4$-resonant caustics.

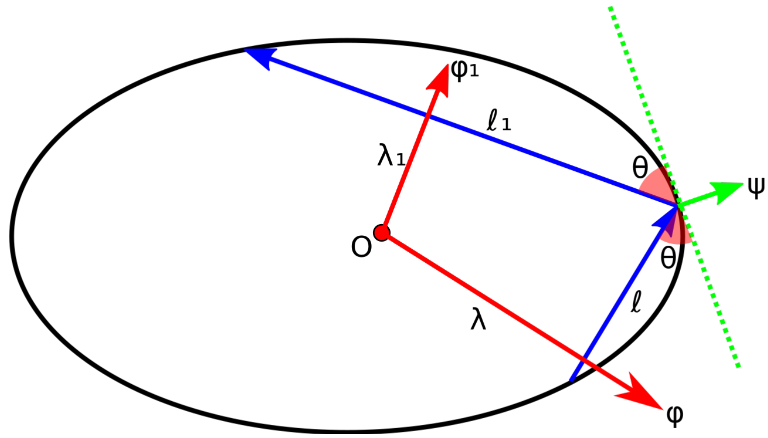

FIGURE 1. The billiard map $f(\varphi,\lambda) = (\varphi_1,\lambda_1)$, the normal angle $\psi = (\varphi_1+\varphi)/2$ and the incidence-reflection angle $\theta = (\varphi_1-\varphi)/2$.

The billiard dynamics acts on the subset of oriented lines (rays) that intersect the boundary of the convex domain. An oriented line ℓ can be written as

$$\cos\varphi \cdot x + \sin\varphi \cdot y = \lambda,$$

where $\varphi \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the direction of the right normal to the oriented line and $\lambda \in \mathbb{R}$ is the signed distance to the origin. Thus, $(\varphi,\lambda) \in \mathbb{A} := \mathbb{T}\times\mathbb{R}$ are coordinates in the space of oriented lines, which is topologically a cylinder.

Let Γ be a smooth strictly convex closed curve of $\mathbb{R}^2$. We fix its counterclockwise orientation and assume its interior contains the origin O, so there is a positive smooth 2π-periodic function $h(\varphi)$, called the *support function* of Γ, such that $\{\lambda = h(\varphi)\}$ and $\{\lambda = -h(\varphi+\pi)\}$ are the 1-parameter families of oriented lines positively and negatively tangent to Γ. Then

$$z : \mathbb{T} \to \Gamma \subset \mathbb{R}^2 \simeq \mathbb{C}, \qquad z(\varphi) = \big(x(\varphi), y(\varphi)\big) = h(\varphi)\mathrm{e}^{\mathrm{i}\varphi} + h'(\varphi)\mathrm{e}^{\mathrm{i}(\varphi+\pi/2)},$$

is a parametrization of Γ, where $\varphi \in \mathbb{T}$ is the counterclockwise angle between the positive x-axis and the outer normal to Γ at the point $z(\varphi)$.

The space of the oriented lines that intersect the interior of Γ is the open cylinder

$$\mathbb{A}_\Gamma = \{(\varphi,\lambda) \in \mathbb{A} : -h(\varphi+\pi) < \lambda < h(\varphi)\}$$

and the *billiard map* $f : \mathbb{A}_\Gamma \to \mathbb{A}_\Gamma$ acts by the reflection law in Γ. That is, $f(\varphi,\lambda) = (\varphi_1,\lambda_1)$ means that the oriented line ℓ_1 with coordinates (φ_1,λ_1) is the reflection of the oriented line ℓ with coordinates (φ,λ) with respect to the tangent to Γ at the second intersection of ℓ with Γ. See Figure 1. The shocking discovery by Bialy and Mironov was that the billiard map f is an exact twist map with generating function

$$S(\varphi,\varphi_1) = 2h(\psi)\sin\theta, \quad \psi = \frac{\varphi_1+\varphi}{2}, \quad \theta = \frac{\varphi_1-\varphi}{2}.$$

To be precise, $\lambda_1\,\mathrm{d}\varphi_1 - \lambda\,\mathrm{d}\varphi = f^*(\lambda\,\mathrm{d}\varphi) - \lambda\,\mathrm{d}\varphi = \mathrm{d}S$, so

$$f(\varphi,\lambda) = (\varphi_1,\lambda_1) \Leftrightarrow \begin{cases} \lambda = -\partial_1 S(\varphi,\varphi_1) = h(\psi)\cos\theta - h'(\psi)\sin\theta, \\ \lambda_1 = \partial_2 S(\varphi,\varphi_1) = h(\psi)\cos\theta + h'(\psi)\sin\theta, \end{cases} \tag{5}$$

and f preserves the standard area form $\mathrm{d}\varphi \wedge \mathrm{d}\lambda$. See, for instance, [10, Proposition 2.1]. Here, $\partial_i S$ denotes the derivative with respect to the i-th variable. The strict convexity of Γ implies the twist condition: $\partial_{12} S(\varphi, \varphi_1) = \frac{1}{2}\rho(\psi)\sin\theta > 0$, where $\rho(\psi) = h''(\psi) + h(\psi)$ is the *radius of curvature* of Γ at the point $z(\psi)$.

We say that $\psi \in \mathbb{T}$ and $\theta \in (0, \pi)$ are the *normal angle* and the *angle of incidence/reflection* at each impact point, whereas $\varphi \in \mathbb{T}$ is the *side angle*. We consider $S(\varphi, \varphi_1)$ defined on the universal cover $\{(\varphi, \varphi_1) \in \mathbb{R}^2 : \varphi < \varphi_1 < \varphi + 2\pi\}$ because $S(\varphi + 2\pi, \varphi_1 + 2\pi) = S(\varphi, \varphi_1)$.

Let us briefly recall the classical variational principle for exact twists maps. The interested reader can find more details in [38, §V]. We will introduce several operators which are not standard in the variational approach, but they simplify computations and shorten formulas.

Given any sequence $\{a_j\}$, we consider the *shift*, *sum*, *difference* and q*-average* operators

$$\tau\{a_j\} = a_{j+1}, \quad \sigma\{a_j\} = a_{j+1} + a_j, \quad \delta\{a_j\} = a_{j+1} - a_j, \quad \mu\{a_j\} = \frac{a_j + \cdots + a_{j+q-1}}{q}.$$

For simplicity, we omit the dependence of μ on q and we just say that μ is the *average* operator.

Let $p/q \in (0, 1)$ be a rational number with $\gcd(p, q) = 1$. We call the elements of the space

$$X = \big\{\{\varphi_j\} \in \mathbb{R}^{\mathbb{Z}} : \varphi_j < \varphi_{j+1} < \varphi_j + 2\pi,\ \varphi_{j+q} = \varphi_j + 2\pi p,\ \forall j \in \mathbb{Z}\big\}$$

p/q*-periodic sequences*. We define the p/q*-periodic action* $A : X \to \mathbb{R}$ as

$$A\{\varphi_j\} = \sum_{j=0}^{q-1} S(\varphi_j, \varphi_{j+1}) = 2\sum_{j=0}^{q-1} h(\psi_j)\sin\theta_j,$$

where $\psi_j = \sigma\{\varphi_j\}/2$ and $\theta_j = \delta\{\varphi_j\}/2$. Periodicities $\varphi_{j+q} = \varphi_j + 2\pi p$, $\psi_{j+q} = \psi_j + 2\pi p$ and $\theta_{j+q} = \theta_j$ imply that $\mu\{S(\varphi_{j_0+j}, \varphi_{j_0+j+1})\} = 2\mu\{h(\psi_{j_0+j})\sin\theta_{j_0+j}\} = A\{\varphi_j\}/q$ for all $j_0 \in \mathbb{Z}$. Critical points of the action, which we call p/q*-periodic configurations*, can be lifted to full p/q*-periodic orbits* of the billiard map f by taking $\lambda_j = -\partial_1 S(\varphi_j, \varphi_{j+1}) = \partial_2 S(\varphi_{j-1}, \varphi_j)$. Thus, any p/q-periodic configuration defines a p/q-periodic billiard trajectory inside Γ with side angles φ_j, normal angles $\psi_j = \sigma\{\varphi_j\}/2$ and incidence-reflection angles $\theta_j = \delta\{\varphi_j\}/2$. We characterize such configurations in the next proposition, where we also recall a formula for the length of their corresponding periodic billiard trajectories.

Proposition 3. *A p/q-periodic sequence $\{\varphi_j\}$ is a p/q-periodic configuration if and only if*

$$\sigma\{h'(\psi_j)\sin\theta_j\} - \delta\{h(\psi_j)\cos\theta_j\} = 0, \quad \forall j \in \mathbb{Z},$$

where $\psi_j = \sigma\{\varphi_j\}/2$ and $\theta_j = \delta\{\varphi_j\}/2$, in which case

$$2q\mu\big\{h(\psi_j)\sin\theta_j\big\} = A\{\varphi_j\} = L, \quad \mu\{h'(\psi_j)\sin\theta_j\} = 0, \quad \mu\{\theta_j\} = \pi p/q,$$

where L is the length of the p/q-periodic configuration $\{\varphi_j\}$.

Proof. A p/q-periodic sequence $\{\varphi_j\}$ is a critical point of the action if and only if

$$\sigma\{h'(\psi_j)\sin\theta_j\} - \delta\{h(\psi_j)\cos\theta_j\} = \partial_2 S(\varphi_j, \varphi_{j+1}) + \partial_1 S(\varphi_{j+1}, \varphi_{j+2}) = 0$$

for all $j \in \mathbb{Z}$. The first equality above follows from $S(\varphi_j, \varphi_{j+1}) = 2h(\psi_j)\sin\theta_j$ and relations $\psi_j = \sigma\{\varphi_j\}/2 = (\varphi_{j+1} + \varphi_j)/2$ and $\theta_j = \delta\{\varphi_j\}/2 = (\varphi_{j+1} - \varphi_j)/2$.

Identities $2q\mu\big\{h(\psi_j)\sin\theta_j\big\} = L$ and $\mu\{h'(\psi_j)\sin\theta_j\} = 0$ are proved in [10, Theorem 2.2]. By q-periodicity of $\{\theta_j\}$, we get

$$\mu\{\theta_j\} = \frac{1}{q}\sum_{i=0}^{q-1}\theta_{j+i} = \frac{1}{q}\sum_{i=0}^{q-1}\frac{\theta_{j+i}+\theta_{j+i+1}}{2} = \frac{1}{q}\sum_{i=0}^{q-1}\frac{\psi_{j+i+1}-\psi_{j+i}}{2} = \frac{\psi_{j+q}-\psi_j}{2q} = \frac{\pi p}{q}. \quad \square$$

We look for necessary conditions for the existence of convex resonant caustics inside Γ. Tangent lines to such a caustic can be counterclockwise or clockwise oriented. We fix the counterclockwise orientation, so we assume that $p/q < 1/2$ from now on. Any p/q-resonant convex caustic gives rise to a 1-parameter family of p/q-periodic orbits of the billiard map. We want to parametrize this family using a *dynamical parameter* $t \in \mathbb{R}$ in which the billiard map acts as the constant shift $t \mapsto t + \omega$ with angular frequency $\omega = 2\pi p/q$. The dynamical parameter is not unique. If $a(t)$ is any smooth ω-periodic function such that $1 + a'(t) > 0$, then $s = t + a(t)$ is another dynamical parameter.

Let us stress the three main differences between this setting, where we deal with functions of a continuous variable $t \in \mathbb{R}$, and the previous setting, where we had sequences whose elements are labeled by a discrete index $k \in \mathbb{Z}$.

Firstly, we define the *shift*, *sum*, *difference* and *average* operators as

$$\tau\{a(t)\} = a(t+\omega), \quad \sigma\{a(t)\} = a(t+\omega) + a(t), \quad \delta\{a(t)\} = a(t+\omega) - a(t)$$

and $\mu\{a(t)\} = \frac{1}{q}\sum_{j=0}^{q-1} a(t + j\omega)$. These operators diagonalize in the Fourier basis. Operator μ is the projection onto the resonant $q\mathbb{Z}$-harmonics: $\mu = \mu_{q\mathbb{Z}}$, but we omit the $q\mathbb{Z}$ subscript for simplicity. Both claims are proved in Appendix A.

Secondly, we define the *p/q-periodic action* of a *side function* $\varphi(t)$ as

$$A\{\varphi(t)\} = q\mu\left\{S(\varphi(t), \varphi(t+\omega))\right\} = 2q\mu\{h(\psi(t)) \cdot \sin\theta(t)\},$$

where $\varphi(t)$, the *normal function* $\psi(t)$ and the *incidence-reflection function* $\theta(t)$ are related by

$$\text{(6)} \qquad \psi = \varphi + \theta = \sigma\{\varphi\}/2, \quad 2\theta = \delta\{\varphi\}, \quad \mu\{\psi - t\} = 0, \quad \mu\left\{t - \varphi\right\} = \omega/2 = \mu\{\theta\}.$$

We ask functions $\varphi(t) - t$, $\psi(t) - t$ and $\theta(t)$ to be 2π-periodic, as a continuous analogue of the discrete periodicity conditions $\varphi_{j+q} = \varphi_j + 2\pi p$, $\psi_{j+q} = \psi_j + 2\pi p$ and $\theta_{j+q} = \theta_j$. Hence, they are lifts of some functions $\varphi, \psi : \mathbb{T} \to \mathbb{T}$ and $\theta : \mathbb{T} \to \mathbb{R}$. To simplify the exposition, we sometimes abuse the notation and use the same symbol for an object and its lift. Other times we denote lifts with a tilde. We also ask that $\varphi'(t) > 0$, so that $\varphi : \mathbb{R} \to \mathbb{R}$ can be inverted.

Thirdly, condition $\sigma\{h'(\psi_j)\sin\theta_j\} - \delta\{h(\psi_j)\cos\theta_j\} = 0$ becomes the difference equation

$$\text{(7)} \qquad \sigma\{h' \circ \psi \cdot \sin\theta\} - \delta\{h \circ \psi \cdot \cos\theta\} = 0.$$

Remark 1. Relations (6) are redundant, but we have listed all of them for future references. They can be used to determine all three functions $\varphi(t)$, $\psi(t)$ and $\theta(t)$ from any one of them. Usually, we will determine $\varphi(t)$ and $\psi(t)$ from $\theta(t)$. If $\theta(t)$ is a smooth 2π-periodic function such that $\mu\{\theta\} = \omega/2$, then there is a unique smooth 2π-periodic function $\varphi(t) - t$ such that $\delta\{\varphi\} = 2\theta$ and $\mu\{t - \varphi\} = \omega/2$. See Lemma 12 in Appendix A for a proof. Then $\psi = \varphi + \theta$ implies that $\psi = \sigma\{\varphi\}/2$ and $\mu\{\psi - t\} = 0$.

Theorem 4. *Let $p/q \in (0, 1/2)$ be any rational rotational number with* $\gcd(p,q) = 1$. *If there is a smooth convex p/q-resonant caustic inside Γ, the following necessary conditions hold.*

(f) *There are three smooth* 2π*-periodic* **f***unctions* $\varphi(t) - t$*,* $\psi(t) - t$ *and* $\theta(t)$ *related by* (6) *that satisfy the difference equation* (7)*. Besides,* $\varphi'(t) > 0$.

(p) *There is a smooth* **p***arametrization* $c : \mathbb{T} \to \mathbb{A}_\Gamma$ *such that* $G = c(\mathbb{T})$ *is a graph and* $f(c(t)) = c(t + \omega)$*.*

(a) *The* p/q*-periodic* **a***ction is constant on the side function:* $A\{\varphi(t)\} = L$*, where* L *is the length of all* p/q*-periodic billiard trajectories in* Γ*.*

Proof. **(f)** Let $g(\varphi)$ be the smooth support function of the p/q-resonant caustic. Set

$$G = \operatorname{graph}(g) := \big\{(\varphi, \lambda) \in \mathbb{A} : \lambda = g(\varphi)\big\}.$$

The caustic is inside Γ, so $G \subset \mathbb{A}_\Gamma$. Clearly, G is f-invariant, so $f|_G$ defines a smooth preserving orientation circle diffeomorphism $r : \mathbb{T} \to \mathbb{T}$ such that

$$\tilde{f}\big(\varphi, \tilde{g}(\varphi)\big) = \big(\tilde{r}(\varphi), \tilde{g}(\tilde{r}(\varphi))\big), \qquad \tilde{r}^q(\varphi) = \varphi + 2\pi p = \varphi + q\omega,$$

where $\tilde{f} : \tilde{\mathbb{A}}_\Gamma \to \tilde{\mathbb{A}}_\Gamma$, $\tilde{g} : \mathbb{R} \to \mathbb{R}$ and $\tilde{r} : \mathbb{R} \to \mathbb{R}$ are the corresponding lifts. Identity $\tilde{r}^q(\varphi) = \varphi + q\omega$ is the key. It follows from the definition of p/q-resonant caustic. It implies that $\varphi \mapsto \tilde{r}(\varphi)$ becomes the constant shift $t \mapsto t + \omega$ in the smooth parameter

$$t = \tilde{s}(\varphi) := \frac{1}{q} \sum_{j=0}^{q-1} \big(\tilde{r}^j(\varphi) - j\omega\big).$$

Inversion of $\tilde{s}(\varphi(t)) = t$ defines a smooth function such that $\varphi'(t) > 0$, $\varphi(t) - t$ is 2π-periodic, and any sequence $\{\varphi(t + j\omega)\}$, $t \in \mathbb{R}$, is a p/q-periodic configuration of f. Then Proposition 3 implies that the normal function $\psi(t)$ and the incidence-reflection function $\theta(t)$ obtained from $\varphi(t)$ by relations (6) satisfy the difference equation (7).

(p) We define

$$\lambda(t) = -\partial_1 S(\varphi(t), \varphi(t + \omega)) = \partial_2 S(\varphi(t - \omega), \varphi(t)).$$

Since $\varphi(t) - t$ and $\lambda(t)$ are 2π-periodic, the map $c = (\varphi, \lambda) : \mathbb{R} \to \mathbb{R}^2$ can be projected to a map from $\mathbb{T}$ to $\mathbb{A}$. Since $g(\varphi)$ is the support function of the caustic, we get that $\lambda(t) = g(\varphi(t))$ and $c(\mathbb{T}) = G \subset \mathbb{A}_\Gamma$. Condition $\varphi'(t) > 0$ implies that $c'(t) \neq (0, 0)$, so $c : \mathbb{T} \to G$ is a parametrization. Implicit equations (5) imply $f(c(t)) = c(t + \omega)$.

(a) The p/q-periodic sequences $\{\varphi(t+j\omega)\}_{j\in\mathbb{Z}}$ form a 1-parameter family of critical points of the p/q-periodic action, being $t \in \mathbb{R}$ a smooth parameter. Therefore, the action is constant on this 1-parameter family. That is, all periodic billiard trajectories tangent to the p/q-resonant caustic have the same length. □

Remark 2. If these three necessary conditions hold, then $\mu\{h' \circ \psi \cdot \sin\theta\} = 0$. This relation follows by applying operator μ to (7), since $\mu \circ \sigma = 2\mu$ and $\mu \circ \delta = 0$ on the space of smooth 2π-periodic functions.

To provide a first insight into the usefulness of Condition (**f**) in Theorem 4, let us give some information about functions $\varphi(t)$, $\psi(t)$ and $\theta(t)$ in five examples.

Example 1. The simplest example is the completely integrable circular billiard. If Γ is a circle of radius one centered at the origin, then $h(\psi) \equiv 1$, so $\mathbb{A}_\Gamma = \mathbb{T} \times (-1, 1)$ and the billiard map $f : \mathbb{A}_\Gamma \to \mathbb{A}_\Gamma$ is given by $f(\varphi, \lambda) = (\varphi + \varpi(\lambda), \lambda)$ with $\varpi(\lambda) = 2 \arccos \lambda$. In particular, we can take $\varphi(t) = t - \omega/2$, $\psi(t) = t$, $\theta(t) \equiv \omega/2$, $\omega = 2\pi p/q$ and $L = 2q\sin(\omega/2)$ for any $p/q \in (0, 1/2)$. Straightforward computations show that these functions satisfy Condition (**f**).

Example 2. Elliptic billiards are integrable too, but their computations are harder. If Γ is the ellipse $\{x^2/a^2 + y^2/b^2 = 1\}$, then $h(\psi) = \sqrt{a^2\cos^2\psi + b^2\sin^2\psi}$; see [5, Lemma 1]. The explicit expression of $\varphi(t)$, $\psi(t)$ and $\theta(t)$ as functions of a dynamical parameter t requires the use of elliptic functions whose modulus depends on the eccentricity of the ellipse and the rotation number p/q of each resonant caustic. See [39, 17, 5] for similar computations. We omit the details, since we only deal with deformations of circles in this work, but we stress that $\varphi'(t), \psi'(t) \not\equiv 1$ in elliptic billiards, unlike circular billiards.

Example 3. Constant width curves are a classic example. Any such curve, other than a circle, has a nonsmooth (with cusps) and nonconvex $1/2$-resonant caustic [32]. The curve Γ has constant width $w > 0$ when $h(\psi)+h(\psi+\pi) \equiv w$, which implies that all $(2\mathbb{Z}\setminus\{0\})$-harmonics of $h(\psi)$ vanish, In that case, we take $\varphi(t) = t-\omega/2$, $\psi(t) = t$, $\theta(t) \equiv \omega/2$, $\omega = \pi$ and $L = 2w$ for $p/q = 1/2$. These functions satisfy Condition (**f**) because $\sin\theta(t) \equiv 1$, $\cos\theta(t) \equiv 0$ and $h'(\psi) + h'(\psi+\pi) \equiv 0$.

Example 4. *Gutkin billiard tables* [22, 4], also called *constant angle curves*, are another classic example. We claim that circles are the only convex billiard tables with a p/q-resonant caustic, with $p/q \in (0, 1/2)$, whose incidence-reflection function $\theta(t)$ is constant. In that case, $\theta(t) \equiv \omega/2$ with $\omega = 2\pi p/q$, so $\varphi(t) = t - \omega/2$ and $\psi(t) = t$ are the functions determined from $\theta(t)$ by relations (6). Therefore, the difference equation (7) becomes

$$\tan(\omega/2)\big(h'(t+\omega) + h'(t)\big) = h(t+\omega) - h(t).$$

If $h(t) = \sum_{l\in\mathbb{Z}} \hat{h}_l \mathrm{e}^{\mathrm{i}lt}$ satisfies this equation, then $\hat{h}_l = 0$ for any index $l \in \mathbb{Z}$ such that

$$\tan(l\pi p/q) \neq l\tan(\pi p/q).$$

Cyr [16] proved that given any integer $l \notin \{-1, 0, 1\}$, equation $\tan(l\pi p/q) = l\tan(\pi p/q)$ has no rational solution $p/q \in (0, 1/2)$. This proves the claim, because circles are the only convex curves whose support function is a trigonometric polynomial of degree one.

Example 5. A smooth centrally symmetric convex curve Γ with support function $h(\psi)$ has a convex $1/4$-resonant caustic if and only if

$$h^2(\psi) = \hat{c}_0 + \sum_{l\in 2+4\mathbb{Z}} \hat{c}_l \mathrm{e}^{\mathrm{i}l\psi} \qquad \text{and} \qquad h + h'' > 0.$$

This claim is proved in [11, Proposition 3.1]. Along the proof, the authors check three facts. First, all such curves are centrally symmetric: $h(\psi+\pi) = h(\psi)$. Second, once fixed one of such curves, there is a constant $R > 0$ such that $h^2(\psi) + h^2(\psi+\pi/2) = R^2$. Third, then we can take $\psi(t) = t$ and determine the incidence-reflection function $\theta(t)$ by means of

$$h(t) = R\sin\theta(t), \quad h(t+\pi/2) = R\cos\theta(t).$$

Note that $\theta(t+\pi/2) = \pi/2 - \theta(t)$ and $\psi(t+\pi/2) = \psi(t) + \pi/2$. The last relation means that the tangent lines to Γ at the impacts of any $1/4$-periodic trajectory form a rectangle. This fact plays a key role in the Bialy-Mironov proof of a strong version of the Birkhoff conjecture [9]. If we set $\varphi = \psi - \theta$ and $\omega = \pi/2$, then functions $\varphi(t)$, $\psi(t)$ and $\theta(t)$ satisfy Condition (**f**).

3. Persistence of smooth convex resonant caustics

Necessary conditions in Theorem 4 are not sufficient for the existence of *smooth convex* resonant caustics, see Example 3. However, if the envelope of the 1-parameter family of lines from $z(\psi(t))$ to $z(\psi(t+\omega))$, where t is the dynamical parameter, is a smooth convex curve, then those conditions are sufficient too. That is the case when we consider small enough smooth deformations $\Gamma_\epsilon = \Gamma_0 + \mathrm{O}(\epsilon)$ of a smooth strictly convex curve Γ_0, not necessarily a circle, with a smooth strictly convex resonant caustic.

We consider that setting. To be precise, we assume the following hypothesis from here on.

(H) Let $p/q \in (0,1/2)$ be a rational rotational number such that $\gcd(p,q)=1$. Let Γ_0 be a smooth strictly convex curve with support function $h_0(\varphi)$. We assume that there is a smooth convex p/q-resonant caustic with support function $g_0(\varphi)$ inside Γ_0. We also assume that the origin is in the interior of the caustic, so $0 < g_0(\varphi) < h_0(\varphi)$. Let $\Gamma_\epsilon = \Gamma_0 + \mathrm{O}(\epsilon)$, with $\epsilon \in [-\epsilon_0, \epsilon_0]$, be a deformation of the unperturbed curve with smooth support function $h(\varphi;\epsilon)$. Let $(\varphi_1,\lambda_1) = f_\epsilon(\varphi,\lambda)$, $S_\epsilon(\varphi,\varphi_1) = 2h(\psi;\epsilon)\sin\theta$ and $A_\epsilon\{\varphi\} = q\mu\{S_\epsilon(\varphi,\tau\{\varphi\})\} = 2q\mu\{h\circ\psi\cdot\sin\theta\}$ be the perturbed billiard map in Γ_ϵ, the perturbed generating function and the perturbed action, respectively.

We need two parameters in that perturbed setting. The *dynamical parameter* t parametrizes the invariant objects. The *perturbative parameter* $\epsilon \in [-\epsilon_0,\epsilon_0]$ labels the ovals. Then the shift, sum, difference and average operators are applied to functions that depend on t and ϵ, although they only act on t. For instance, $\tau\{a(t;\epsilon)\} = a(t+\omega;\epsilon)$. We will denote the derivatives of the support function $h(\psi;\epsilon) = h_\epsilon(\psi)$ as $h' = \frac{\mathrm{d}h}{\mathrm{d}\psi}$ and $\dot h = \frac{\mathrm{d}h}{\mathrm{d}\epsilon}$. Analogously, we will denote the derivatives of any function $a(t;\epsilon) = a_\epsilon(t)$ as $a' = \frac{\mathrm{d}a}{\mathrm{d}t}$ and $\dot a = \frac{\mathrm{d}a}{\mathrm{d}\epsilon}$.

Next, we state an immediate extension of Theorem 4.

Corollary 5. *If $0 < \epsilon_0 \ll 1$, the unperturbed smooth convex resonant p/q-caustic persists under deformation Γ_ϵ, $\epsilon \in [-\epsilon_0,\epsilon_0]$, if and only if the following three conditions hold.*

(F) *There are three smooth 2π-periodic functions $\varphi(t;\epsilon)-t$, $\psi(t;\epsilon)-t$ and $\theta(t;\epsilon)$ related by* (6) *that satisfy the difference equation* (7)*. Besides, $\varphi'(t;\epsilon) > 0$.*

(P) *There are smooth parametrizations $c_\epsilon : \mathbb{T} \to \mathbb{A}_{\Gamma_\epsilon}$ such that $G_\epsilon = c_\epsilon(\mathbb{T})$ are graphs and*

$$f_\epsilon(c_\epsilon(t)) = c_\epsilon(t+\omega).$$

(A) *The p/q-periodic action is constant on the side function: $A_\epsilon\{\varphi(t;\epsilon)\} = L(\epsilon)$, where $L(\epsilon)$ is the length of all p/q-periodic billiard trajectories in Γ_ϵ for $\epsilon \in [-\epsilon_0,\epsilon_0]$.*

Remark 3. Similarly to Remark 2, if these necessary conditions hold, then

$$2q\mu\{h\circ\psi\cdot\sin\theta\} = L, \quad \mu\{h'\circ\psi\cdot\sin\theta\} = 0, \quad 2q\mu\{\dot h\circ\psi\cdot\sin\theta\} = \dot L.$$

Only the last formula is new. Let us prove it. If we derive the first relation with respect to ϵ, use the summation by parts formula and take advantage of (7), we get

$$\begin{aligned}
\dot L &= 2q\mu\big\{\dot h\circ\psi\cdot\sin\theta + h'\circ\psi\cdot\sin\theta\cdot\dot\psi + h\circ\psi\cdot\cos\theta\cdot\dot\theta\big\} \\
&= 2q\mu\{\dot h\circ\psi\cdot\sin\theta\} + q\mu\big\{h'\circ\psi\cdot\sin\theta\cdot\sigma\{\dot\varphi\} + h\circ\psi\cdot\cos\theta\cdot\delta\{\dot\varphi\}\big\} \\
&= 2q\mu\{\dot h\circ\psi\cdot\sin\theta\} + q\mu\big\{\big[\sigma\{h'\circ\psi\cdot\sin\theta\} - \delta\{h\circ\psi\cdot\cos\theta\}\big]\cdot\tau\{\dot\varphi\}\big\} \\
&= 2q\mu\{\dot h\circ\psi\cdot\sin\theta\}.
\end{aligned}$$

We can normalize Γ_ϵ by a scaling in such a way that $\dot{L} = 0$, but we do not need it.

Full persistence of resonant caustics is a extremely rare phenomenon, so we introduce the more common concept of $\mathrm{O}(\epsilon^m)$-*persistence* for some *order* $m \in \mathbb{N} \cup \{0\}$.

Definition 2. *Let $m \in \mathbb{N} \cup \{0\}$. The unperturbed resonant p/q-caustic* $\mathrm{O}(\epsilon^m)$-persists *under deformation Γ_ϵ if and only if the following three conditions hold.*

(F)$_m$ *There are smooth 2π-periodic functions $\varphi(t;\epsilon) - t$, $\psi(t;\epsilon) - t$ and $\theta(t;\epsilon)$ related by* (6) *such that $\varphi'(t;\epsilon) > 0$ and*

$$\sigma\{h' \circ \psi \cdot \sin\theta\} - \delta\{h \circ \psi \cdot \cos\theta\} = \mathrm{O}(\epsilon^{m+1}) \quad \textit{as } \epsilon \to 0. \tag{8}$$

(P)$_m$ *There are smooth parametrizations $c_\epsilon : \mathbb{T} \to \mathbb{A}_{\Gamma_\epsilon}$ such that $G_\epsilon = c_\epsilon(\mathbb{T})$ are graphs and*

$$f_\epsilon \circ c_\epsilon - \tau\{c_\epsilon\} = \big(0, \mathrm{O}(\epsilon^{m+1})\big) \quad \textit{as } \epsilon \to 0.$$

(A)$_m$ *There is a 'length' $L(\epsilon) > 0$ such that the side function $\varphi = \varphi(t;\epsilon)$ satisfies*

$$A_\epsilon\{\varphi\} - L(\epsilon) = \mathrm{O}(\epsilon^{m+1}) \quad \textit{as } \epsilon \to 0.$$

If the resonant caustic $\mathrm{O}(\epsilon^m)$-persists, but not $\mathrm{O}(\epsilon^{m+1})$-persists, we say that deformation Γ_ϵ $\mathrm{O}(\epsilon^{m+1})$-breaks *the caustic.*

Since $\mathrm{O}(\epsilon^m)$-persistence is the main concept of this work, some comments are in order. If $\varphi_0(t)$, $\psi_0(t)$ and $\theta_0(t)$ are the unperturbed side, normal and incidence-reflection functions, then

$$\sigma\{h' \circ \psi_0 \cdot \sin\theta_0\} - \delta\{h \circ \psi_0 \cdot \cos\theta_0\} = \mathrm{O}(\epsilon) \quad \text{as } \epsilon \to 0,$$

since $h = h_0 + \mathrm{O}(\epsilon)$. Therefore, the unperturbed caustic always $\mathrm{O}(\epsilon^0)$-persists. We do not need to check *all* three Conditions **(F)**$_m$, **(P)**$_m$ and **(A)**$_m$, because there are logical dependencies among them. We prove in Proposition 6 that Conditions **(F)**$_m$ and **(P)**$_m$ are equivalent and both imply Condition **(A)**$_m$. We will only check Condition **(F)**$_m$ in our computations. We have included the other conditions as part of our definition to present a broader view of the problem.

The three conditions look similar, but they have different characteristics. On the one hand, Conditions **(F)**$_m$ and **(P)**$_m$ are stated in terms of a *single* iteration of the perturbed map f_ϵ. On the other hand, Condition **(A)**$_m$ requires to consider all the shifts

$$\bar{\varphi}_j = \bar{\varphi}_j(t;\epsilon) = \tau^j\{\varphi(t;\epsilon)\} = \varphi(t + j\omega;\epsilon), \quad j = 0, \ldots, q.$$

Hence, Conditions **(F)**$_m$ and **(P)**$_m$ are easier to deal with from a computational point of view.

Condition **(F)**$_m$ means that there is a reparametrization $\varphi_\epsilon(t) = \varphi(t;\epsilon)$ of the original angle φ in terms of a new dynamical parameter t such that

$$\partial_2 S_\epsilon(\varphi_\epsilon(t-\omega), \varphi_\epsilon(t)) + \partial_1 S_\epsilon(\varphi_\epsilon(t), \varphi_\epsilon(t+\omega)) = \mathrm{O}(\epsilon^{m+1}) \quad \text{as } \epsilon \to 0,$$

which stands out its Lagrangian nature. It was inspired by the proofs of the existence of (resonant and nonresonant) rotational invariant curves contained in [34, 31]. Condition **(A)**$_m$ follows the variational approach in [17].

Remark 4. Following [41, 39], we could also have considered a fourth condition defined in terms of $\varphi \in \mathbb{T}$ instead of $t \in \mathbb{R}$. However, such approach forces us to deal with the power map f_ϵ^q, which is technically impractical. We have not pursued it. That discarded condition is:

(G)$_m$ There are smooth functions $g_\epsilon^\bullet, g_\epsilon^\star : \mathbb{T} \to \mathbb{R}$ such that $f_\epsilon^q(\varphi, g_\epsilon^\bullet(\varphi)) = (\varphi, g_\epsilon^\star(\varphi))$, $g_\epsilon^\star, g_\epsilon^\bullet = g_0 + \mathrm{O}(\epsilon)$ and $g_\epsilon^\star - g_\epsilon^\bullet = \mathrm{O}(\epsilon^{m+1})$ as $\epsilon \to 0$.

It means that f_ϵ^q projects $G_\epsilon^\bullet = \operatorname{graph}(g_\epsilon^\bullet)$ onto $G_\epsilon^\star = \operatorname{graph}(g_\epsilon^\star)$ in the vertical direction. None of these two graphs have to coincide with $G_\epsilon = c_\epsilon(\mathbb{T})$, but all of them are $\mathrm{O}(\epsilon^{m+1})$-close.

Let us prove the logical dependencies among these $\mathrm{O}(\epsilon^m)$-persistence conditions and how the dominant terms in their $\mathrm{O}(\epsilon^{m+1})$-errors are related.

Proposition 6. *Conditions* **(F)**$_m$ *and* **(P)**$_m$ *are equivalent and both imply Condition* **(A)**$_m$*. Let* $\varphi_0(t) = \varphi(t;0)$ *be the unperturbed side function. If the* p/q*-resonant caustic* $\mathrm{O}(\epsilon^m)$-persists *and we follow the notations introduced in Definition 2, then there is a smooth* 2π*-periodic function* $\mathcal{E}_{m+1}(t)$ *and a smooth* $2\pi/q$*-periodic function* $\mathcal{L}_{m+1}(t)$ *such that*

(9a) $$\sigma\{h'\circ\psi\cdot\sin\theta\} - \delta\{h\circ\psi\cdot\cos\theta\} = \epsilon^{m+1}\tau\{\mathcal{E}_{m+1}\} + \mathrm{O}(\epsilon^{m+2}),$$

(9b) $$f_\epsilon\circ c_\epsilon - \tau\{c_\epsilon\} = \big(0, \epsilon^{m+1}\tau\{\mathcal{E}_{m+1}\} + \mathrm{O}(\epsilon^{m+2})\big),$$

(9c) $$A_\epsilon\{\varphi\} - L(\epsilon) = \epsilon^{m+1}q\mathcal{L}_{m+1} + \mathrm{O}(\epsilon^{m+2}),$$

as $\epsilon \to 0$*. Besides,* $\mu\{\mathcal{E}_{m+1}\varphi_0'\} = \mathcal{L}_{m+1}'$ *and* $\int_{\mathbb{T}}\mathcal{E}_{m+1}\varphi_0' = 0$*.*

Proof. Firstly, we check that **(F)**$_m \Rightarrow$ **(P)**$_m$ & **(A)**$_m$, $\mathcal{L}_{m+1}' = \mu\{\mathcal{E}_{m+1}\varphi_0'\}$ and $\int_{\mathbb{T}}\mathcal{E}_{m+1}\varphi_0' = 0$.

Let $\varphi(t;\epsilon)$, $\psi(t;\epsilon)$ and $\theta(t;\epsilon)$ be the functions described in **(F)**$_m$ and $\mathcal{E}_{m+1}(t)$ be the function determined by (9a). Set $\varphi_1(t;\epsilon) = \varphi(t+\omega;\epsilon)$, so $\varphi_1 = \tau\{\varphi\}$. We consider the smooth 2π-periodic functions $\lambda(t;\epsilon)$ and $\lambda_1(t;\epsilon)$ given by

$$\lambda = h\circ\psi\cdot\cos\theta - h'\circ\psi\cdot\sin\theta, \qquad \lambda_1 = h\circ\psi\cdot\cos\theta + h'\circ\psi\cdot\sin\theta.$$

Set $c_\epsilon(t) = \big(\varphi(t;\epsilon),\lambda(t;\epsilon)\big)$ and $d_\epsilon(t) = \big(\varphi_1(t;\epsilon),\lambda_1(t;\epsilon)\big)$. Implicit equations (5) imply that $f\circ c_\epsilon = d_\epsilon$. Besides, $\lambda_1 - \tau\{\lambda\} = \sigma\{h'\circ\psi\cdot\sin\theta\} - \delta\{h\circ\psi\cdot\cos\theta\} = \epsilon^{m+1}\tau\{\mathcal{E}_{m+1}\} + \mathrm{O}(\epsilon^{m+2})$, which is equivalent to estimate (9b). This proves **(P)**$_m$.

Set $\bar\varphi_j = \tau^j\{\varphi\}$, $\bar\psi_j = \tau^j\{\psi\}$ and $\bar\theta_j = \tau^j\{\theta\}$ for all $j\in\mathbb{Z}$. Note that $\bar\varphi_{j+q}(t;\epsilon) = \bar\varphi_j(t;\epsilon) + 2\pi p$, $\bar\psi_{j+q}(t;\epsilon) = \bar\psi_j(t;\epsilon) + 2\pi p$ and $\bar\theta_{j+q}(t;\epsilon) = \bar\theta_j(t;\epsilon)$. If we derive the expression that defines the action and we recall that $\varphi = \varphi_0 + \mathrm{O}(\epsilon)$, then we obtain the estimate

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big[A_\epsilon\{\varphi\}\big] &= \textstyle\sum_{j=0}^{q-1}\big[\partial_1 S_\epsilon(\bar\varphi_j,\bar\varphi_{j+1})\bar\varphi_j' + \partial_2 S_\epsilon(\bar\varphi_j,\bar\varphi_{j+1})\bar\varphi_{j+1}'\big]\\
&= \textstyle\sum_{j=0}^{q-1}\big[\partial_2 S_\epsilon(\bar\varphi_{j-1},\bar\varphi_j) + \partial_1 S_\epsilon(\bar\varphi_j,\bar\varphi_{j+1})\big]\bar\varphi_j'\\
&= \textstyle\sum_{j=0}^{q-1}\big[\sigma\{h'\circ\bar\psi_{j-1}\cdot\sin\bar\theta_{j-1}\} - \delta\{h\circ\bar\psi_{j-1}\cdot\cos\bar\theta_{j-1}\}\big]\bar\varphi_j'\\
&= \epsilon^{m+1}\textstyle\sum_{j=0}^{q-1}\tau^j\{\mathcal{E}_{m+1}\varphi'\} + \mathrm{O}(\epsilon^{m+2})\\
&= \epsilon^{m+1}q\mu\{\mathcal{E}_{m+1}\varphi'\} + \mathrm{O}(\epsilon^{m+2})\\
&= \epsilon^{m+1}q\mu\{\mathcal{E}_{m+1}\varphi_0'\} + \mathrm{O}(\epsilon^{m+2}),
\end{aligned}$$

which, by integration, is equivalent to estimate (9c) for any smooth $2\pi/q$-periodic function $\mathcal{L}_{m+1}$ such that $\mathcal{L}_{m+1}' = \mu\{\mathcal{E}_{m+1}\varphi_0'\}$. This proves **(A)**$_m$ and the relation between $\mathcal{L}_{m+1}$ and $\mathcal{E}_{m+1}$. The operator μ is the projection onto the $q\mathbb{Z}$-harmonics, so the zero-th harmonics of $\mathcal{E}_{m+1}\varphi_0'$ and $\mathcal{L}_{m+1}'$ coincide, so $\int_{\mathbb{T}}\mathcal{E}_{m+1}\varphi_0' = \int_{\mathbb{T}}\mathcal{L}_{m+1}' = 0$.

Secondly, we check that **(P)**$_m \Rightarrow$ **(F)**$_m$. Let $c_\epsilon(t) = \big(\varphi(t;\epsilon),\lambda(t;\epsilon)\big)$ be the parametrization described in **(P)**$_m$ and $\mathcal{E}_{m+1}(t)$ be the error function given in (9b). Property $\varphi' > 0$ holds because $c_\epsilon : \mathbb{T}\to G_\epsilon \subset \mathbb{A}_{\Gamma_\epsilon}$ is a parametrization and G_ϵ is a graph. Functions φ, $\psi = \sigma\{\varphi\}/2$

and $\theta = \delta\{\varphi\}/2$ satisfy relations (6). Set $\big(\varphi_1(t;\epsilon), \lambda_1(t;\epsilon)\big) = f_\epsilon(c_\epsilon(t))$. We deduce from implicit equations (5) that

$$\sigma\{h' \circ \psi \cdot \sin\theta\} - \delta\{h \circ \psi \cdot \cos\theta\} = \lambda_1 - \tau\{\lambda\} = \epsilon^{m+1}\tau\{\mathcal{E}_{m+1}\} + \mathrm{O}(\epsilon^{m+2}),$$

which is exactly estimate (9a). This proves **(F)**$_m$. □

Remark 5. We can prove that $\int_{\mathbb{T}} \mathcal{E}_{m+1}\varphi_0' = 0$ in another way. The flux of the exact twist map f_ϵ across the graph $G_\epsilon = c_\epsilon(\mathbb{T})$, which is a rotational curve, is zero [38, §V]. This flux is

$$\begin{aligned}\int_{f_\epsilon(G_\epsilon)} \lambda \,\mathrm{d}\varphi - \int_{G_\epsilon} \lambda \,\mathrm{d}\varphi &= \int_{\mathbb{T}} (\lambda_1\varphi_1' - \lambda\varphi') \\ &= \int_{\mathbb{T}} \delta\{\lambda\varphi'\} + \epsilon^{m+1}\int_{\mathbb{T}} \tau\{\mathcal{E}_{m+1}\varphi'\} + \mathrm{O}(\epsilon^{m+2}) \\ &= \epsilon^{m+1}\int_{\mathbb{T}} \mathcal{E}_{m+1}\varphi_0' + \mathrm{O}(\epsilon^{m+2}) \qquad \text{as } \epsilon \to 0.\end{aligned}$$

We have used that if $c_\epsilon(t) = \big(\varphi(t;\epsilon), \lambda(t;\epsilon)\big)$ and $(\varphi_1, \lambda_1) = f(\varphi, \lambda)$, then $\varphi_1 = \tau\{\varphi\}$ and $\lambda_1 = \tau\{\lambda\} + \epsilon^{m+1}\tau\{\mathcal{E}_{m+1}\} + \mathrm{O}(\epsilon^{m+2})$. We have also used that $\varphi = \varphi_0 + \mathrm{O}(\epsilon)$.

Definition 3. *The $2\pi/q$-periodic function $\mathcal{L}_{m+1}(t)$ is the p/q-*resonant potential of order $m+1$. *The 2π-periodic function $\mathcal{E}_{m+1}(t)$ is the p/q-*resonant error of order $m+1$.

We will check in the next section that if the p/q-resonant caustic $\mathrm{O}(\epsilon^m)$-persists under a deformation of the unit circle, then the Melnikov potential $\mathcal{L}_{m+1}(t)$ is uniquely determined from previously computed objects. See the second item in Proposition 8.

4. High-order persistence in deformed circles

Let us apply the previous high-order persistence theory to smooth deformations of circles. The main goal is to check that the smooth 2π-periodic coefficients of the Taylor expansions in powers of the perturbative parameter ϵ of the perturbed side, normal and incidence-reflection functions can be computed recursively order by order as long as some compatibility conditions hold. Such compatibility conditions have to do with the inversion of the difference operator δ, so they boil down to the fact that certain smooth 2π-periodic functions have no $q\mathbb{Z}$-resonant harmonics. We look for a practical way to find the exact order at which a given resonant caustic is destroyed, so we write down explicit formulas for all recursive computations.

To begin with, we assume the following hypothesis from here on.

(H') Let $p/q \in (0, 1/2)$ be a rational rotational number such that $\gcd(p,q) = 1$. Let Γ_0 be the unit circle centered at the origin, see Example 1. The unperturbed side, normal and incidence-reflection functions are

$$\varphi_0(t) = t - \omega/2, \quad \psi_0(t) = t, \quad \theta_0(t) = \omega/2, \quad \omega = 2\pi p/q.$$

Let Γ_ϵ, with $\epsilon \in [-\epsilon_0, \epsilon_0]$, be a deformation of Γ_0 with smooth support function (1).

We look for some perturbed side, normal and incidence-reflection functions

$$(10) \qquad \varphi(t;\epsilon) \asymp \sum_{k\geq 0} \epsilon^k \varphi_k(t), \quad \psi(t;\epsilon) \asymp \sum_{k\geq 0} \epsilon^k \psi_k(t), \quad \theta(t;\epsilon) \asymp \sum_{k\geq 0} \epsilon^k \theta_k(t)$$

that satisfy Condition **(F)**$_m$ for an order $m \in \mathbb{N} \cup \{0\}$ as high as possible.

Notation 1. If $a(t;\epsilon) \asymp \sum_{k\geq 0} a_k(t)\epsilon^k$ as $\epsilon \to 0$, then $a_{\leq m}(t;\epsilon) = \sum_{k=0}^m a_k(t)\epsilon^k$. Symbol $a_{\leq m}(t;\epsilon)$ may be used even when there are no coefficients $a_k(t)$ with $k > m$. Symbol $a_{<m}(t;\epsilon)$ has a similar meaning. An expression like $\mathcal{C}_m = \mathcal{C}_m[\![a_{\leq m}, b_{<m}]\!]$ means that $\mathcal{C}_m$ is a smooth 2π-periodic function that can be written as a *differential* expression in some smooth 2π-periodic functions a_k for $1 \leq k \leq m$ and b_k for $1 \leq k < m$. The term differential means that the derivatives of functions a_k and b_k can appear in those expressions.

The study of the approximate difference equation (8) requires to recursively compute the asymptotic expansions of $\mathcal{R} := h \circ \psi \cdot \cos\theta$ and $\mathcal{Q} := h' \circ \psi \cdot \sin\theta$ as $\epsilon \to 0$.

Lemma 7. *Set $c = \cos(\omega/2)$ and $s = \sin(\omega/2)$. The coefficients of the asymptotic expansions*

$$\mathcal{R} = h \circ \psi \cdot \cos\theta \asymp c + \sum_{k\geq 1} \mathcal{R}_k \epsilon^k, \qquad \mathcal{Q} = h' \circ \psi \cdot \sin\theta \asymp \sum_{k\geq 1} \mathcal{Q}_k \epsilon^k$$

have the form

$$\mathcal{Q}_k = \mathcal{Q}_k[\![h'_{\leq k}, \psi_{<k}, \theta_{<k}]\!] = sh'_k + \tilde{\mathcal{Q}}_k[\![h'_{<k}, \psi_{<k}, \theta_{<k}]\!],$$
$$\mathcal{R}_k = \mathcal{R}_k[\![h_{\leq k}, \psi_{<k}, \theta_{\leq k}]\!] = \tilde{\mathcal{R}}_k[\![h_{\leq k}, \psi_{<k}, \theta_{<k}]\!] - s\theta_k.$$

Besides, $\tilde{\mathcal{Q}}_1 = 0$, $\tilde{\mathcal{R}}_1 = ch_1$, $\tilde{\mathcal{Q}}_2 = sh''_1\psi_1 + ch'_1\theta_1$ and $\tilde{\mathcal{R}}_2 = ch_2 + ch'_1\psi_1 - sh_1\theta_1 - c\theta_1^2/2$.

The proof of Lemma 7 is postponed to Appendix B. The first coefficients are obtained from the explicit recurrences for $\tilde{\mathcal{Q}}_k$ and $\tilde{\mathcal{R}}_k$ given in Lemma 15.

Once we know that $\mathcal{Q}_{<k}$ and $\mathcal{R}_{<k}$ only depend on $\psi_{<k}$ and $\theta_{<k}$, we deduce that if $\varphi_{<k}(t;\epsilon)$, $\psi_{<k}(t;\epsilon)$ and $\theta_{<k}(t;\epsilon)$ satisfy Condition **(F)**$_{k-1}$, then $\varphi_{\leq k}(t;\epsilon)$, $\psi_{\leq k}(t;\epsilon)$ and $\theta_{\leq k}(t;\epsilon)$ satisfy Condition **(F)**$_k$ provided coefficients $\varphi_k(t)$, $\psi_k(t)$ and $\theta_k(t)$ are chosen in such a way that $\sigma\{\mathcal{Q}_k\} = \delta\{\mathcal{R}_k\}$. We prove below that these k-th coefficients can be found if and only if $\mathcal{Q}_k$ has no $q\mathbb{Z}$-resonant harmonics, in which case all three k-th coefficients are uniquely determined provided that they have no $q\mathbb{Z}$-resonant harmonics either.

Proposition 8. *Assume that the p/q-resonant caustic $\mathrm{O}(\epsilon^{k-1})$-persists for some order $k \in \mathbb{N}$.*

a) $\bar{\mathcal{Q}}_k := \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{Q}_k = 0$.

b) The p/q-resonant Melnikov potential of order k is completely determined from previously computed objects: $\mathcal{L}_k = \mathcal{L}_k[\![h_{\leq k}, \psi_{<k}, \theta_{<k}]\!]$.

c) The p/q-resonant caustic $\mathrm{O}(\epsilon^k)$-persists if and only if

$$\mu\{\mathcal{Q}_k\} = 0, \tag{11}$$

in which case $\theta_k(t)$ is the unique smooth 2π-periodic solution of

$$s\delta\{\theta_k\} = \delta\{\tilde{\mathcal{R}}_k\} - \sigma\{\mathcal{Q}_k\}, \qquad \mu\{\theta_k\} = 0, \tag{12}$$

and then $\varphi_k(t)$ and $\psi_k(t)$ are uniquely determined from $\theta_k(t)$ by

$$\delta\{\varphi_k\} = 2\theta_k, \qquad \mu\{\varphi_k\} = 0, \qquad \psi_k = \varphi_k + \theta_k. \tag{13}$$

d) If condition (11) *fails, then it is satisfied for any support function $h^\star = h + \epsilon^k\eta_k$ such that*

$$\mu\big\{s\eta'_k + \mathcal{Q}_k\big\} = 0, \tag{14}$$

so the p/q-resonant caustic $\mathrm{O}(\epsilon^k)$-persists under a corrected deformation $\Gamma^\star_\epsilon = \Gamma_\epsilon + \mathrm{O}(\epsilon^k)$. We can choose the correction η_k in such a way that it only contains $(q\mathbb{Z}\setminus\{0\})$-harmonics.

Proof. We use several times along this proof that $\mu \circ \tau = \mu$, $\mu \circ \sigma = 2\mu$ and $\mu \circ \delta = 0$ on the space of smooth 2π-periodic functions. We also use that if $b(t)$ is a smooth 2π-periodic function, $\delta\{a\} = b$ has a smooth 2π-periodic solution $a(t)$ if and only if $\mu\{b\} = 0$, in which case the solution is unique under the additional condition $\mu\{a\} = 0$. See Lemma 12.

a) The asymptotic expansions in Lemma 7 imply that

$$\sigma\{h' \circ \psi \cdot \sin\theta\} - \delta\{h \circ \psi \cdot \cos\theta\} = \sum_{j=1}^{k} \big(\sigma\{\mathcal{Q}_j\} - \delta\{\mathcal{R}_j\}\big)\epsilon^j + \mathrm{O}(\epsilon^{k+1}).$$

The $\mathrm{O}(\epsilon^{k-1})$-persistence hypothesis means that $\sigma\{\mathcal{Q}_j\} = \delta\{\mathcal{R}_j\}$ for $j = 1, \ldots, k-1$. If $\mathcal{E}_k$ and $\mathcal{L}_k$ are the p/q-resonant error and potential of order k, respectively, then $\tau\{\mathcal{E}_k\} = \sigma\{\mathcal{Q}_k\} - \delta\{\mathcal{R}_k\}$ and $\mathcal{L}'_k = \mu\{\mathcal{E}_k\} = 2\mu\{\mathcal{Q}_k\}$, since $\varphi'_0(t) = 1$. Finally, $\bar{\mathcal{Q}}_k = \frac{1}{2\pi}\int_{\mathbb{T}} \mathcal{Q}_k = \frac{1}{4\pi}\int_{\mathbb{T}} \mathcal{E}_k = 0$. (See the last claims in Proposition 6.)

b) It is a direct consequence of properties $\mathcal{L}'_k = 2\mu\{\mathcal{Q}_k\}$ and $\mathcal{Q}_k = \mathcal{Q}_k[\![h'_{\leq k}, \psi_{<k}, \theta_{<k}]\!]$.

c) Condition (11) is necessary: $\mathrm{O}(\epsilon^k)$-persistence means $\sigma\{\mathcal{Q}_k\} = \delta\{\mathcal{R}_k\}$, so $\mu\{\mathcal{Q}_k\} = 0$. Condition (11) is sufficient: If $\mu\{\mathcal{Q}_k\} = 0$, then

$$\mu\big\{\delta\{\tilde{\mathcal{R}}_k\} - \sigma\{\mathcal{Q}_k\}\big\} = -2\mu\{\mathcal{Q}_k\} = 0,$$

so (12) has a unique smooth 2π-periodic solution $\theta_k(t)$. Then we find φ_k and ψ_k by means of relations (13). Finally, the $\mathrm{O}(\epsilon^k)$-terms of $\sigma\{h' \circ \psi \cdot \sin\theta\}$ and $\delta\{h \circ \psi \cdot \cos\theta\}$ are $\sigma\{\mathcal{Q}_k\}$ and $\delta\{\tilde{\mathcal{R}}_k - s\theta_k\}$, respectively. See Lemma 7. Both terms coincide when the first relation in (12) holds. Property $\varphi'(t;\epsilon) = 1 + \mathrm{O}(\epsilon) > 0$ holds for $\epsilon \in [-\epsilon_0, \epsilon_0]$ if $0 < \epsilon_0 \ll 1$. Relations (6) follow from condition $\mu\{\theta_k\} = 0$ and relations (13), since $\psi_k = \varphi_k + \theta_k$ implies that $\mu\{\psi_k\} = 0$ and $\psi_k = \sigma\{\varphi_k\}/2$. This proves that Condition $(\mathbf{F})_k$ holds.

d) We deduce from Lemma 7 that if $\mathcal{Q}^\star_k$ is the $\mathrm{O}(\epsilon^k)$-coefficient of $(h^\star)' \circ \psi_{<k} \cdot \sin\theta_{<k}$, then $\mathcal{Q}^\star_k = s\eta'_k + \mathcal{Q}_k$. The existence of the correction η_k follows from property $\bar{\mathcal{Q}}_k = 0$. ☐

Finally, we prove Theorem 1 by recursively applying Proposition 8 for $k = 1, \ldots, m$.

Proof of Theorem 1. We denote the Fourier coefficients of smooth 2π-periodic functions with a hat, so we write $h_1(t) = \sum_{l\in\mathbb{Z}} \hat{h}_{1,l} \mathrm{e}^{\mathrm{i}lt}$ and $\theta_1(t) = \sum_{l\in\mathbb{Z}} \hat{\theta}_{1,l} \mathrm{e}^{\mathrm{i}lt}$.

a) We know that $\mathcal{Q}_1 = sh'_1$ from Lemma 7. Hence, the necessary and sufficient condition (11) for $\mathrm{O}(\epsilon)$-persistence becomes $\mu\{h'_1\} = 0$ or, equivalently, $\mu_{q\mathbb{Z}^\star}\{h_1\} = 0$.

b) We know that $\mathcal{Q}_2 = sh'_2 + \tilde{\mathcal{Q}}_2$ and $\tilde{\mathcal{Q}}_2 = sh''_1\psi_1 + ch'_1\theta_1$ from Lemma 7. Let us check that $\mu\{\tilde{\mathcal{Q}}_2\} = s\mu\{\theta_1\theta'_1\}$. In order to do that, we need three properties.

Firstly, once we know that the resonant caustic $\mathrm{O}(\epsilon)$-persists, we determine the first-order coefficient $\theta_1(t)$ by solving (12) with $k = 1$. That is,

$$s\delta\{\theta_1\} = \delta\{\tilde{\mathcal{R}}_1\} - \sigma\{\mathcal{Q}_1\} = \delta\{ch_1\} - \sigma\{sh'_1\}, \qquad \mu\{\theta_1\} = 0. \tag{15}$$

Secondly, we determine $\varphi_1(t)$ and $\psi_1(t)$ by solving (13) with $k = 1$. In particular,

$$\theta_1 = \delta\{\varphi_1\}/2, \qquad \psi_1 = \sigma\{\varphi_1\}/2.$$

Thirdly, we recall that if $a(t)$ and $b(t)$ are 2π-periodic functions, then

$$\mu\big\{a\delta\{b\}\big\} = -\mu\big\{\delta\{a\}\tau\{b\}\big\}, \qquad \mu\big\{a\sigma\{b\}\big\} = \mu\big\{\sigma\{a\}\tau\{b\}\big\}.$$

Next, we use these three properties to get the formula we are looking for:

$$\begin{aligned}\mu\{\tilde{\mathcal{Q}}_2\} &= \mu\{sh_1''\psi_1 + ch_1'\theta_1\} = \mu\big\{sh_1''\sigma\{\varphi_1\} + ch_1'\delta\{\varphi_1\}\big\}/2 \\ &= \mu\big\{\big[\sigma\{sh_1''\} - \delta\{ch_1'\}\big]\tau\{\varphi_1\}\big\}/2 = \mu\big\{\big[\sigma\{sh_1'\} - \delta\{ch_1\}\big]'\tau\{\varphi_1\}\big\}/2 \\ &= -s\mu\big\{\delta\{\theta_1'\}\tau\{\varphi_1\}\big\}/2 = s\mu\big\{\delta\{\varphi_1\}\theta_1'\big\}/2 = s\mu\{\theta_1\theta_1'\}.\end{aligned}$$

Thus, condition (11) for $\mathrm{O}(\epsilon^2)$-persistence becomes $\mu\big\{h_2' + \theta_1\theta_1'\big\} = 0$ or, equivalently,

$$\mu_{q\mathbb{Z}^\star}\{h_2 + \theta_1^2/2\} = 0.$$

Set $e_l^\pm = \mathrm{e}^{\mathrm{i}l\omega} \pm 1$. The Fourier coefficients of the unique smooth 2π-periodic solution of (15) are easily determined: $\hat{\theta}_{1,l} = 0$ for all $l \in q\mathbb{Z}$ and

$$se_l^-\hat{\theta}_{1,l} = \big(ce_l^- - \mathrm{i}lse_l^+\big)\hat{h}_{1,l}, \qquad \forall l \notin q\mathbb{Z}.$$

The RHS of this identity vanishes, and $\hat{\theta}_{1,l}$ too, when $l = \pm 1$. If $l \notin q\mathbb{Z} \cup \{-1, 1\}$, then

$$\hat{\theta}_{1,l} = \left(\frac{c}{s} - \mathrm{i}l\frac{e_l^+}{e_l^-}\right)\hat{h}_{1,l} = \begin{cases} \dfrac{\tan(l\omega/2) - l\tan(\omega/2)}{\tan(\omega/2)\tan(l\omega/2)}\hat{h}_{1,l}, & \text{if } 2l \notin q\mathbb{Z}, \\ \dfrac{\hat{h}_{1,l}}{\tan(\omega/2)}, & \text{if } 2l \in q\mathbb{Z}. \end{cases}$$

We have used that $e_l^+ = 0$ when $2l \in q\mathbb{Z}$ and $l \notin q\mathbb{Z}$, whereas $e_l^- \neq 0$ for all $l \notin q\mathbb{Z}$. The formulas above are equivalent to the expression of $\theta_1(t)$ given in Theorem 1.

c) We know from Lemma 7 that $\mathcal{Q}_m = sh_m' + \tilde{\mathcal{Q}}_m$, where $\tilde{\mathcal{Q}}_m$ is a smooth 2π-periodic function, only depending on $h_1, \ldots, h_{m-1}$ —since approximations $\varphi_{<k}$, $\psi_{<k}$ and $\theta_{<k}$ are uniquely determined from $h_{<k}$ for any $k = 1, \ldots, m-1$— that can be explicitly computed from recurrences given in Appendix B. We also know that $\int_{\mathbb{T}} \mathcal{Q}_m = 0$, and so $\int_{\mathbb{T}} \tilde{\mathcal{Q}}_m = 0$. This means that there is a smooth 2π-periodic function ζ_m such that $s\zeta_m' = \tilde{\mathcal{Q}}_m$. Finally, the necessary and sufficient condition for $\mathrm{O}(\epsilon^m)$-persistence becomes $\mu\{h_m' + \zeta_m'\} = 0$ or, equivalently, $\mu_{q\mathbb{Z}^\star}\{h_m + \zeta_m\} = 0$. □

Functions ζ_m can be recursively computed from the formulas listed in Appendix B. For instance, we have already seen that $\zeta_1 = 0$ and $\zeta_2 = \theta_1^2/2$ along the proof of Theorem 1. After some tedious computations by hand that we do not include here, we get that

$$\zeta_3 = \theta_1\theta_2 + h_1\theta_1^2/2 - h_1''\psi_1^2/2 + c\theta_1^3/3s - ch_1'\theta_1\psi_1/s,$$

which will allow us to analyze some $\mathrm{O}(\epsilon^3)$-persistence problems in the future. See Section 6. We stress that further expressions for $\zeta_4, \zeta_5, \ldots$ can be obtained using a symbolic algebra system, but we have doubts about their practical usefulness.

5. Polynomial deformations of the unit circle

We tackle two problems in this section, both related with polynomial deformations of the unit circle. First, to prove Theorem 2. Second, to prove that the support function (1) of any polynomial deformation (4) with $P(x, y; \epsilon) = 1 + \epsilon P_1(x, y)$ and $P_1(x, y) \in \mathbb{R}_n[x, y]$ satisfies condition (3); see Lemma 10. This second result allows us to explain with Theorem 2 some of the numerical experiments about the polynomial deformations (4) with $P(x, y; \epsilon) = 1 - \epsilon y^n$ performed in [37], which was the original motivation of this work.

To begin with, we check that, once fixed the order $m \in \mathbb{N}$ and the degree $n \geq 3$, all resonant caustics of high enough period $\mathrm{O}(\epsilon^m)$-persists under polynomial deformations of degree $\leq n$. We explicitly quantify how high this period q should be. Some resonant caustics become more persistent in the presence of symmetries.

Theorem 9. *Under the hypotheses of Theorem 2, the p/q-resonant caustic $\mathrm{O}(\epsilon^m)$-persists if any of the following three conditions are met:*
a) $q > nm$*;*
b) Γ_ϵ *is centrally symmetric,* q *is odd and* $2q > nm$*; or*
c) Γ_ϵ *is anti-centrally symmetric,* $q \neq m \mod 2$*,* $m \geq 2$ *and* $q > n(m-1)$*.*

Lemma 10. *Any deformation of the unit circle expressed in Cartesian coordinates as* (4) *for some* $P(x,y;\epsilon) = 1 + \epsilon P_1(x,y)$ *with* $P_1(x,y) \in \mathbb{R}_n[x,y]$ *is a polynomial deformation of degree* $\leq n$ *in the sense of Definition 1. If* P_1 *is even (respectively, odd), then this deformation is centrally (respectively, anti-centrally) symmetric.*

We have postponed the proofs of Theorem 9 and Lemma 10 to Appendix C, since they are technically similar. The main difficulty is to check that some recursively computed 2π-periodic trigonometric polynomials have degrees $\leq mn$ by induction on a recursive index $m \geq 1$.

Theorem 2 is a direct consequence of Theorem 9.

Proof of Theorem 2. Let $\chi = \chi(\Gamma_\epsilon, q) \in \mathbb{N}$ be the exponent defined in Theorem 2. Let $m = \chi - 1$. We deduce from Theorem 9 that the p/q-resonant caustic $\mathrm{O}(\epsilon^m)$-persists because:

- If Γ_ϵ is anti-centrally symmetric, then $nm < q$ or $n(m-1) < q$ with $m \neq q \mod 2$ and $m \geq 2$;
- If Γ_ϵ is centrally symmetric and q is odd, then $nm < 2q$; and
- Otherwise, $nm < q$. □

The experiments described in [37, Numerical Result 5] suggest that the p/q-resonant caustic does not $\mathrm{O}(\epsilon^\chi)$-persist under the polynomial deformation (4) with $P(x,y;\epsilon) = 1 - \epsilon y^n$. To prove it, we should check that $h_\chi + \zeta_\chi$ has some non-zero $(q\mathbb{Z} \setminus \{0\})$-harmonic.

6. Open problems

We describe three open problems that have arisen during the development of our high-order perturbation theory. We have not addressed them here. Each of them is a nontrivial research challenge. They are work in progress.

As a general principle, we claim that many billiard computations are greatly simplified when working with the Bialy-Mironov generating function, so its discovery opens the door to the resolution of many billiard problems that seemed almost intractable.

We also stress that both our high-order perturbation theory and our list of open problems can be extended to the setting of dual, symplectic, wire, pensive and coin billiards, provided we deal with those billiards in deformed circles.

6.1. **Co-preservation of resonant caustics.** Tabachnikov asked if there are convex domains, other than circles and ellipses, that possess resonant caustics with different rotation numbers. See [13, Question 4.7]. Theorem 1 and Cyr's result on Gutkin's equation provide an excellent framework to study the co-preservation of resonant caustics under deformations of the unit circle. We mention two problems about such co-preservation.

Firstly, we consider the co-preservation of caustics with different rotation numbers but equal periods. Let $q \neq 2, 3, 4, 6$ be a fixed period, so

$$\mathcal{P}_q = \{p \in \mathbb{N} : p/q < 1/2,\ \gcd(p, q) = 1\}$$

has several elements. If the deformation of the unit circle with support function (1) preserves all p/q-resonant caustics with $p \in \mathcal{P}_q$, then none of the smooth 2π-periodic functions h_1 and $h_2 + \big(\theta_1^{[p/q]}\big)^2/2$ with $p \in \mathcal{P}_q$ have resonant $(q\mathbb{Z} \setminus \{0\})$-harmonics, where

$$\theta_1^{[p/q]}(t) = \sum_{l \notin q\mathbb{Z} \cup \{-1,1\}} \hat{\theta}_{1,l}^{[p/q]} \mathrm{e}^{\mathrm{i}lt}, \qquad \hat{\theta}_{1,l}^{[p/q]} = \nu_l(p/q) \hat{h}_{1,l},$$

for some functions $\nu_l : (0, 1/2) \to \mathbb{R}$ defined in (2), which have no rational roots when $|l| \geq 2$. The above necessary conditions for $\mathrm{O}(\epsilon^2)$-persistence impose rather stringent restrictions on the first-order coefficient h_1. We do not state any specific result because we have not yet found one satisfactory enough.

Secondly, we consider the co-preservation of caustics with rotation numbers $1/2$ and $1/q$ for some fixed period $q \geq 3$ under deformations of the unit circle. The preservation of the $1/2$-resonant caustic implies that all $(2\mathbb{Z} \setminus \{0\})$-harmonics of the support function (1) are zero, which greatly simplifies the problem. The case $q = 3$ for centrally symmetric deformations was studied by J. Zhang [44]. His method consists in a careful analysis of the obstructions for $\mathrm{O}(\epsilon^2)$-persistence of the $1/3$-resonant caustic. His computations are more involved than the ones outlined here because he uses the standard generating function for billiards (that is, the minus distance between consecutive impacts) and he considers deformations of the unit circle written in polar coordinates. We plan to extend Zhang's results to some periods $q > 3$, starting with periods $q = 4, 5$ for which it is necessary to study the obstructions for $\mathrm{O}(\epsilon^3)$-persistence.

6.2. **A convergence problem.** We recall that the function ζ_m in Theorem 1 only depends on $h_1, \dots, h_{m-1}$. Hence, if the p/q-resonant caustic $\mathrm{O}(\epsilon^m)$-persists, but not $\mathrm{O}(\epsilon^{m+1})$-persists, and we fix any higher order $r > m$, then there is a smooth 2π-periodic function $\eta_\epsilon^{[r]} = \mathrm{O}(\epsilon^{m+1})$ such that the resonant caustic $\mathrm{O}(\epsilon^r)$-persists under the new deformation $\Gamma_\epsilon^{[r]} = \Gamma_\epsilon + \mathrm{O}(\epsilon^{m+1})$ with support function $h_\epsilon^{[r]} = h_\epsilon + \eta_\epsilon^{[r]}$. This function $\eta_\epsilon^{[r]}$ can also be recursively computed and only contains $(q\mathbb{Z} \setminus \{0\})$-harmonics. See the last item in Proposition 8.

A natural question is: Does $h_\epsilon^{[\infty]} := \lim_{r \to +\infty} h_\epsilon^{[r]}$ exist? That is, we look for a corrected support function $h_\epsilon^{[\infty]} = h_\epsilon + \mathrm{O}(\epsilon^{m+1})$ such that the p/q-resonant caustic persists *at all orders* under the deformation $\Gamma_\epsilon^{[\infty]}$ with support function $h_\epsilon^{[\infty]}$. This problem is closely related to the density property established in [31].

The first author, in collaboration with V. Kaloshin and K. Zhang, is working on a more ambitious version of this problem. Namely, once *fixed* a rational rotation number $1/q$, the goal is to construct a functional

$$\mathcal{H}_{\mathbb{Z} \setminus q\mathbb{Z}} \ni h \overset{\mathcal{F}_{1/q}}{\mapsto} \eta \in \mathcal{H}_{q\mathbb{Z} \setminus \{0\}}$$

such that: 1) $\mathcal{H}_A$ is a neighborhood of zero in a suitable space of 2π-periodic functions with only A-harmonics; 2) $\mathcal{F}_{1/q}(0) = 0$; 3) $\mathcal{F}_{1/q}$ is as regular as possible at $h = 0$, and 4) the convex domain with support function $1 + h + \mathcal{F}_{1/q}[h]$ has a $1/q$-resonant caustic for any $h \in \mathcal{H}_{\mathbb{Z} \setminus q\mathbb{Z}}$.

6.3. **Exponentially small phenomena.** Let Γ_ϵ be a polynomial deformation of the unit circle of degree $\leq n$ in the sense of Definition 1. Let $q \geq 3$ be a *fixed* period. Theorem 2 implies that the p/q-resonant caustics $\mathrm{O}(\epsilon^{\chi-1})$-persist under Γ_ϵ for all $p \in \mathcal{P}_q$, so there are perturbed normal functions $\psi_\epsilon^{[p/q]}(t)$ and perturbed incidence-reflection functions $\theta_\epsilon^{[p/q]}(t)$ such that the error

$$\mathcal{E}_\epsilon^{[p/q]}(t) := \sigma\big\{h'_\epsilon(\psi_\epsilon^{[p/q]}(t)) \cdot \sin\theta_\epsilon^{[p/q]}(t)\big\} - \delta\big\{h_\epsilon(\psi_\epsilon^{[p/q]}(t)) \cdot \cos\theta_\epsilon^{[p/q]}(t)\big\}$$

is $\mathrm{O}(\epsilon^\chi)$ as $\epsilon \to 0$. We recall that $\chi \asymp 2q/n$ as odd $q \to +\infty$ for centrally symmetric deformations, and $\chi \asymp q/n$ as $q \to +\infty$ otherwise. Let us focus on the second case, which is the generic one. It is natural to ask whether, in this generic case,

$$\mathcal{E}_\epsilon^{[p/q]}(t) = \mathrm{O}(\epsilon^\chi) \simeq \mathrm{O}(\epsilon^{q/n}) = \mathrm{O}(\mathrm{e}^{-q|\log\epsilon|/n})$$

as $q \to +\infty$ for *fixed* $\epsilon \in [-\epsilon_0, \epsilon_0]$. This is a hard problem. It was partially addressed in [36], where the authors establish an exponentially small upper bound on the difference of lengths of $1/q$-periodic billiard trajectories in analytic strictly convex domains as $q \to +\infty$. The numerical experiments described in [37] suggest that these upper bounds can be improved to exponentially small asymptotic formulas for some deformations. To be precise, we deduce from the computations about the polynomial deformations (4) with $P(x, y; \epsilon) = 1 - \epsilon y^n$, that there are constants $\xi(\Gamma_\epsilon)$ with a finite limit as $\epsilon \to 0$ such that the error $\mathcal{E}_\epsilon^{[1/q]}(t)$ has 'size' $q^{-3}\mathrm{e}^{-q[|\log\epsilon|/n)+\xi(\Gamma_\epsilon)]}$ as $q \to +\infty$ for any fixed $\epsilon \in [-\epsilon_0, \epsilon_0]$.

This problem can be addressed by the direct approach used by Wang in [43] or the approach used by Kaloshin and Zhang in [31]. However, we feel that exponentially small asymptotic formulas can only be obtained with more refined techniques, like the extension of $\psi_\epsilon^{[p/q]}(t)$ and $\theta_\epsilon^{[p/q]}(t)$ to complex values of t, the analysis of their complex singularities, resurgence theory and complex matching. See [18, 21, 35] for examples of how these refined techniques are applied in the setting of discrete systems (maps).

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APPENDIX A. SHIFT, SUM, DIFFERENCE AND AVERAGE OPERATORS

Let $0 < p < q$ integers such that $\gcd(p, q) = 1$. Set $\omega = 2\pi p/q$. The shift, sum, difference and average operators act on smooth functions $a : \mathbb{T} \to \mathbb{R}$ as follows

$$\begin{aligned}
\tau\{a(t)\} &= a(t + \omega), \\
\sigma\{a(t)\} &= a(t + \omega) + a(t), \\
\delta\{a(t)\} &= a(t + \omega) - a(t), \\
\mu\{a(t)\} &= \tfrac{1}{q} \textstyle\sum_{j=0}^{q-1} a(t + j\omega).
\end{aligned}$$

They diagonalize in the Fourier basis and the average operator is the projection onto the resonant $q\mathbb{Z}$-harmonics.

Lemma 11. *If* $a(t) = \sum_{l\in\mathbb{Z}} \hat{a}_l e^{ilt}$ *is a smooth* 2π*-periodic function, then*

$$\begin{aligned}\tau\{a(t)\} &= \textstyle\sum_{l\in\mathbb{Z}} e^{il\omega}\hat{a}_l e^{ilt},\\ \sigma\{a(t)\} &= \textstyle\sum_{l\in\mathbb{Z}}(e^{il\omega}+1)\hat{a}_l e^{ilt},\\ \delta\{a(t)\} &= \textstyle\sum_{l\in\mathbb{Z}}(e^{il\omega}-1)\hat{a}_l e^{ilt},\\ \mu\{a(t)\} &= \textstyle\sum_{l\in q\mathbb{Z}} \hat{a}_l e^{ilt}.\end{aligned}$$

Hence, $\mu\circ\tau = \mu\circ\mu = \mu$, $\mu\circ\sigma = 2\mu$ *and* $\mu\circ\delta = 0$ *on the set of smooth* 2π*-periodic functions.*

Proof. The average $\frac{1}{q}\sum_{j=0}^{q-1} e^{ijl\omega}$ is equal to one when $l \in q\mathbb{Z}$ but it vanishes otherwise. □

We deduce from this lemma that persistence condition (11) holds if and only if all resonant $q\mathbb{Z}$-harmonics of $\mathcal{Q}_k$ are equal to zero.

Next, we invert operator δ, which is the key point in solving (12) and (13).

Lemma 12. *Let* $b(t)$ *be any smooth* 2π*-periodic function. Equation*

$$\delta\{a(t)\} = b(t)$$

has a smooth 2π*-periodic solution* $a(t)$ *if and only if* $\mu\{b(t)\} = 0$*, in which case the solution is unique under the additional condition* $\mu\{a(t)\} = 0$*. Analogously, equation*

$$\delta\{t + \tilde{a}(t)\} = b(t)$$

has a smooth 2π*-periodic solution* $\tilde{a}(t)$ *if and only if* $\mu\{b(t)\} \equiv \omega$*, in which case the solution is unique under the additional condition* $\mu\{\tilde{a}(t)\} = 0$*.*

Proof. If $a(t)$ is a 2π-periodic solution, then

$$\mu\{b(t)\} = \mu\big\{a(t+\omega) - a(t)\big\} = \frac{1}{q}\sum_{j=0}^{q-1}[a(t+(j+1)\omega) - a(t+j\omega)] = \frac{a(t+q\omega)-a(t)}{q} = 0.$$

Conversely, if $\mu\{b(t)\} = 0$, then $b(t) = \sum_{l\notin q\mathbb{Z}} \hat{b}_l e^{ilt}$, so that

$$\hat{a}_l = \hat{b}_l/(e^{il\omega} - 1), \qquad \forall l \notin q\mathbb{Z},$$

defines the Fourier coefficients of the smooth 2π-periodic solution $a(t) = \sum_{l\notin q\mathbb{Z}} \hat{a}_l e^{ilt}$ such that $\mu\{a(t)\} = 0$. Smoothness follows from the bounds $|\hat{a}|_l \leq \nu|\hat{b}_l|$ for all $l \notin q\mathbb{Z}$, where $\nu = 1/|e^{i2\pi/q} - 1| > 0$.

If $a(t) = t + \tilde{a}(t)$ is a solution such that $\tilde{a}(t)$ is 2π-periodic, then

$$\mu\{b(t)\} = \frac{a(t+q\omega)-a(t)}{q} = \frac{q\omega + \tilde{a}(t+q\omega) - \tilde{a}(t)}{q} = \omega.$$

The converse is obtained with the same argument as before. □

Appendix B. Computation of some asymptotic expansions

Recall that $\omega = 2\pi p/q$, $c = \cos(\omega/2)$ and $s = \sin(\omega/2) > 0$, where $p/q \in (0, 1/2)$ is some rational number such that $\gcd(p, q) = 1$. Let us compute the asymptotic expansions of

$$\mathcal{R} := h \circ \psi \cdot \cos\theta, \qquad \mathcal{Q} := h' \circ \psi \cdot \sin\theta,$$

from the asymptotic expansion (1) of the support function and the asymptotic expansions (10) of the side, normal and incidence-reflection functions. We assume that all coefficients $h_k(\psi)$, $\varphi_k(t)$, $\psi_k(t)$ and $\theta_k(t)$ are smooth 2π-periodic functions.

Lemma 13. *The coefficients of the asymptotic expansions*

$$\sin\theta \asymp \sum_{k\geq 0} \mathcal{S}_k \epsilon^k, \quad \cos\theta \asymp \sum_{k\geq 0} \mathcal{C}_k \epsilon^k \quad \text{as } \epsilon \to 0$$

can be computed from the initial values $\mathcal{S}_0 = s$ and $\mathcal{C}_0 = c$ by means of the recurrences

$$\mathcal{S}_k = c\theta_k + \frac{1}{k}\sum_{l=1}^{k-1} l\theta_l \mathcal{C}_{k-l}, \quad \mathcal{C}_k = -s\theta_k - \frac{1}{k}\sum_{l=1}^{k-1} l\theta_l \mathcal{S}_{k-l}, \quad \forall k \geq 1. \tag{16}$$

In particular, $\mathcal{S}_k = \mathcal{S}_k[\![\theta_{\leq k}]\!] = c\theta_k + \tilde{\mathcal{S}}_k[\![\theta_{<k}]\!]$ and $\mathcal{C}_k = \mathcal{C}_k[\![\theta_{\leq k}]\!] = -s\theta_k + \tilde{\mathcal{C}}_k[\![\theta_{<k}]\!]$.

Proof. These recurrences are directly obtained from identities

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\{\sin\theta\} = \frac{\mathrm{d}\theta}{\mathrm{d}\epsilon} \cdot \cos\theta, \quad \frac{\mathrm{d}}{\mathrm{d}\epsilon}\{\cos\theta\} = -\frac{\mathrm{d}\theta}{\mathrm{d}\epsilon} \cdot \sin\theta. \qquad \square$$

Definition 4. *If $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \ldots\}$ is a sequence of non-negative integers with a finite number of non-zero terms and $\boldsymbol{\psi} = \{\psi_1, \psi_2, \ldots\}$ is a sequence of smooth 2π-periodic functions, then*

$$|\boldsymbol{\alpha}| = \sum_{l\geq 1} \alpha_l, \quad \|\boldsymbol{\alpha}\| = \sum_{l\geq 1} l\alpha_l, \quad \boldsymbol{\alpha}! = \prod_{l\geq 1} \alpha_l!, \quad \boldsymbol{\psi}^{\boldsymbol{\alpha}} = \prod_{l\geq 1} \psi_l^{\alpha_l}.$$

Lemma 14. *The coefficients of the asymptotic expansion*

$$h \circ \psi \asymp 1 + \sum_{k\geq 1} \mathcal{H}_k \epsilon^k \quad \text{as } \epsilon \to 0$$

are given by $\mathcal{H}_k = \mathcal{H}_k[\![h_{\leq k}, \psi_{<k}]\!] = h_k + \tilde{\mathcal{H}}_k[\![h_{<k}, \psi_{<k}]\!]$ with

$$\tilde{\mathcal{H}}_k = \sum_{i=1}^{k-1}\sum_{j=1}^{k-i}\left(\sum_{|\boldsymbol{\alpha}|=j, \|\boldsymbol{\alpha}\|=k-i} \frac{\boldsymbol{\psi}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\right) h_i^{(j)}, \quad \forall k \geq 1. \tag{17}$$

Analogously, $h' \circ \psi \asymp \sum_{k\geq 1} \mathcal{H}_k[\![h'_{\leq k}, \psi_{<k}]\!]\epsilon^k$ as $\epsilon \to 0$.

Proof. Set $\Delta\psi := \psi - t \asymp \sum_{i\geq 1} \epsilon^i \psi_i$ as $\epsilon \to 0$. Formula (17) is a direct consequence of the asymptotic expansions

$$h_i(t + \Delta\psi) \asymp \sum_{j\geq 0} \frac{(\Delta\psi)^j}{j!} h_i^{(j)}, \qquad \frac{(\Delta\psi)^j}{j!} \asymp \sum_{|\boldsymbol{\alpha}|=j} \frac{\boldsymbol{\psi}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\epsilon^{\|\boldsymbol{\alpha}\|}$$

as $\Delta\psi \to 0$ and $\epsilon \to 0$, respectively. The first expansion is the Taylor theorem. The second expansion is the multinomial theorem.

On the one hand, if $|\boldsymbol{\alpha}| = j = 0$ and $\|\boldsymbol{\alpha}\| = k - i$, then $i = k$ and $\boldsymbol{\alpha} = \{0, 0, \ldots\}$. On the other hand, if $|\boldsymbol{\alpha}| = j \geq 1$ and $\|\boldsymbol{\alpha}\| = k - i$, then $i \leq k - 1$, $j \leq k - i$ and $\alpha_l = 0$ for all $l \geq k$. This justifies that $\mathcal{H}_k = \mathcal{H}_k[\![h_{\leq k}, \psi_{<k}]\!] = h_k + \tilde{\mathcal{H}}_k[\![h_{<k}, \psi_{<k}]\!]$. □

The next result is a more informative version of Lemma 7, whose proof was pending.

Lemma 15. *The coefficients of the asymptotic expansions*

$$\mathcal{R} = h \circ \psi \cdot \cos\theta \asymp c + \sum_{k \geq 1} \mathcal{R}_k \epsilon^k, \quad \mathcal{Q} = h' \circ \psi \cdot \sin\theta \asymp \sum_{k \geq 1} \mathcal{Q}_k \epsilon^k \quad \textit{as } \epsilon \to 0$$

are given by

$$\begin{aligned}
\mathcal{Q}_k &= \mathcal{Q}_k[\![h'_{\leq k}, \psi_{<k}, \theta_{<k}]\!] = s h'_k + \tilde{\mathcal{Q}}_k[\![h'_{<k}, \psi_{<k}, \theta_{<k}]\!], \\
\tilde{\mathcal{Q}}_k &= s\tilde{\mathcal{H}}_k[\![(h'_{<k}, \psi_{<k}]\!] + \textstyle\sum_{l=1}^{k-1} \mathcal{H}_l[\![h'_{\leq l}, \psi_{<l}]\!] \mathcal{S}_{k-l}[\![\theta_{\leq k-l}]\!], \\
\mathcal{R}_k &= \mathcal{R}_k[\![h_{\leq k}, \psi_{<k}, \theta_{\leq k}]\!] = \tilde{\mathcal{R}}_k[\![h_{\leq k}, \psi_{<k}, \theta_{<k}]\!] - s\theta_k, \\
\tilde{\mathcal{R}}_k &= c\mathcal{H}_k[\![h_{\leq k}, \psi_{<k}]\!] + \tilde{\mathcal{C}}_k[\![\theta_{<k}]\!] + \textstyle\sum_{l=1}^{k-1} \mathcal{H}_l[\![h_{\leq l}, \psi_{<l}]\!] \mathcal{C}_{k-l}[\![\theta_{\leq k-l}]\!].
\end{aligned}$$

Proof. It is a direct consequence of Lemmas 13 and 14. □

Finally, we compute the asymptotic expansion of the support function of the deformation of the unit circle given by (4) in Cartesian coordinates. We assume that $P(x, y; \epsilon) = 1 + \epsilon P_1(x, y)$ and $P_1(x, y) \in \mathbb{R}_n[x, y]$.

Lemma 16. *If $P(x, y; \epsilon) = 1 + \epsilon P_1(x, y)$ with $P_1(x, y) = \sum_{i,j \geq 0, i+j \leq n} p_{ij} x^i y^j$, then the coefficients of the asymptotic expansion* (1) *of the support function of the deformation* (4) *can be computed from recurrences*

$$2h_k + \tilde{\mathcal{G}}^\star_k[\![h_{<k}]\!] + \tilde{\mathcal{G}}^\bullet_k[\![h'_{<k}]\!] = \mathcal{G}^\diamond_{k-1}[\![h_{\leq k-1}, h'_{\leq k-1}]\!], \qquad \forall k \geq 1, \tag{18}$$

where

$$\begin{aligned}
\mathcal{G}^\star_k &= 2h_k + \tilde{\mathcal{G}}^\star_k = 2h_k + 2\textstyle\sum'_{|\boldsymbol{\alpha}|=2, \|\boldsymbol{\alpha}\|=k} \boldsymbol{h}^{\boldsymbol{\alpha}}/\boldsymbol{\alpha}!, \\
\tilde{\mathcal{G}}^\bullet_k &= 2\textstyle\sum'_{|\boldsymbol{\alpha}|=2, \|\boldsymbol{\alpha}\|=k} (\boldsymbol{h}')^{\boldsymbol{\alpha}}/\boldsymbol{\alpha}!, \\
\mathcal{G}^\diamond_k &= \textstyle\sum_{i,j \geq 0, i+j \leq n} p_{ij}\ i!\ j! \sum_{|\boldsymbol{\alpha}|=i, |\boldsymbol{\beta}|=j, \|\boldsymbol{\alpha}\|+\|\boldsymbol{\beta}\|=k} \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}/\boldsymbol{\alpha}!\boldsymbol{\beta}!, \\
\boldsymbol{x} &= \cos\psi \cdot \boldsymbol{h} - \sin\psi \cdot \boldsymbol{h}', \\
\boldsymbol{y} &= \sin\psi \cdot \boldsymbol{h} + \cos\psi \cdot \boldsymbol{h}'.
\end{aligned}$$

Here, $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots\}$, $\boldsymbol{\beta} = \{\beta_0, \beta_1, \ldots\}$, $\boldsymbol{h} = \{1, h_1, h_2, \ldots\}$ and $\boldsymbol{h}' = \{0, h'_1, h'_2, \ldots\}$. Symbol $\sum'_{|\boldsymbol{\alpha}|=2, \|\boldsymbol{\alpha}\|=k}$ means that we do not include the term with $\alpha_0 = 1$ and $\alpha_k = 1$.

Proof. We know that $z(\psi; \epsilon) = \big(x(\psi; \epsilon), y(\psi; \epsilon)\big)$, where

$$\begin{aligned}
x &= x(\psi; \epsilon) = \cos\psi \cdot h(\psi; \epsilon) - \sin\psi \cdot h'(\psi; \epsilon), \\
y &= y(\psi; \epsilon) = \sin\psi \cdot h(\psi; \epsilon) + \cos\psi \cdot h'(\psi; \epsilon),
\end{aligned}$$

is a normal parametrization of Γ_ϵ, so the support function satisfies the implicit equation

$$h^2 + (h')^2 = x^2 + y^2 = 1 + \epsilon P_1(x, y).$$

Therefore, recurrence (18) is a direct consequence of the asymptotic expansions

$$h^2 \asymp 1 + \sum_{k\geq 1} \mathcal{G}^{\star}_k \epsilon^k, \quad (h')^2 \asymp \sum_{k\geq 2} \tilde{\mathcal{G}}^{\bullet}_k \epsilon^k, \quad P_1(x,y) \asymp \sum_{k\geq 0} \mathcal{G}^{\diamond}_k \epsilon^k$$

as $\epsilon \to 0$, all of which follow from the multinomial theorem. □

APPENDIX C. PROOFS OF THEOREM 9 AND LEMMA 10

There are two main tools for both proofs. Firstly, the explicit formulas for the asymptotic coefficients $\mathcal{S}_k = c\theta_k + \tilde{\mathcal{S}}_k$, $\mathcal{C}_k = \tilde{\mathcal{C}}_k - s\theta_k$, $\mathcal{H}_k = h_k + \tilde{\mathcal{H}}_k$, $\mathcal{R}_k = \tilde{\mathcal{R}}_k - s\theta_k$, $\mathcal{Q}_k$, $\tilde{\mathcal{G}}^{\star}_k$, $\tilde{\mathcal{G}}^{\bullet}_k$ and $\mathcal{G}^{\diamond}_k$ given in Appendix B. Secondly, the following elementary properties:

- $T_k[t]$ is a real vector space;
- $a(t) \in T_k[t],\ b(t) \in T_l[t] \Rightarrow a(t)b(t) \in T_{k+l}[t],\ a'(t) \in T_k[t]$;
- $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \ldots\}$ and $\boldsymbol{a} = \{a_0, a_1, \ldots\}$ with $a_j(t) \in T_{nj}[t] \Rightarrow \boldsymbol{a}^{\boldsymbol{\alpha}}(t) \in T_{n\|\boldsymbol{\alpha}\|}[t]$;
- $a(t) \in T_k[t] \Rightarrow \sigma\{a(t)\}, \delta\{a(t)\}, \mu\{a(t)\} \in T_k[t]$; and
- $b(t) \in T_k[t],\ \mu\{b(t)\} = 0 \Rightarrow \exists! a(t) \in T_k[t]$ s. t. $\delta\{a(t)\} = b(t)$ and $\mu\{a(t)\} = 0$.

We will use these properties without any explicit mention in what follows.

Lemma 17. *If condition* (3) *holds and $q > nm$, then we can compute the $\mathrm{O}(\epsilon^m)$-corrections $\theta_{\leq m}(t;\epsilon)$, $\varphi_{\leq m}(t;\epsilon)$ and $\psi_{\leq m}(t;\epsilon)$ by solving the compatible equations* (12) *and* (13) *for $k = 1, \ldots, m$. Compatibility is guaranteed because all necessary and sufficient persistence conditions* (11) *hold for $k = 1, \ldots, m$.*

Proof. It suffices to check that

i) $\mathcal{H}_k = h_k + \tilde{\mathcal{H}}_k \in T_{nk}[t]$;
ii) $\tilde{\mathcal{R}}_k, \mathcal{Q}_k \in T_{nk}[t]$ —so condition (11) holds because $q > nm$ and $\bar{\mathcal{Q}}_k = 0$—;
iii) $\theta_k, \varphi_k, \psi_k \in T_{nk}[t]$; and
iv) $\mathcal{S}_k, \mathcal{C}_k \in T_{nk}[t]$;

for $k = 1, \ldots, m$. We prove it by induction over m.

The base case $m = 1$ is trivial. Namely, $\mathcal{H}_1 = h_1 \in T_n[t]$ by condition (3), $\tilde{\mathcal{R}}_1 = ch_1 \in T_n[t]$, $\mathcal{Q}_1 = sh_1' \in T_n[t]$, $\theta_1 \in T_n[t]$ is the unique solution of problem (15), $\varphi_1 \in T_n[t]$ is the unique solution of equation $\delta\{\varphi_1\} = 2\theta_1$ such that $\mu\{\varphi_1\} = 0$, $\psi_1 = \varphi_1 + \theta_1 \in T_n[t]$, $\mathcal{S}_1 = c\theta_1 \in T_n[t]$ and $\mathcal{C}_1 = -s\theta_1 \in T_n[t]$.

Next, let us assume that properties i)–iv) hold for $k = 1, \ldots, m-1$. We need to prove that they also hold for $k = m$. Property i) follows from (17) and condition (3). Property ii) follows from the recurrences given in Lemma 15. Then $\theta_m \in T_{nm}[t]$ is the unique solution of equation $s\delta\{\theta_m\} = \delta\{\tilde{\mathcal{R}}_m\} - \sigma\{\mathcal{Q}_m\}$ such that $\mu\{\theta_m\} = 0$, $\varphi_m \in T_{nm}[t]$ is the unique solution of equation $\delta\{\varphi_m\} = 2\theta_m$ such that $\mu\{\varphi_m\} = 0$, and $\psi_m = \varphi_m + \theta_m \in T_{nm}[t]$. This proves property iii). Property iv) follows from recurrences (16). □

Lemma 18. *If condition* (3) *holds, Γ_ϵ is centrally symmetric, q is odd and $2q > nm$, then we can compute all $\mathrm{O}(\epsilon^m)$-corrections too.*

Proof. If Γ_ϵ is centrally symmetric, then its support function $h(\psi;\epsilon)$ is π-periodic in ψ. In that case, it is not hard to prove by induction over m that objects i)–iv) listed in the proof of the previous lemma are also π-periodic for $k = 1, \ldots, m$. In particular, if q is odd and $2q > nm$, then persistence condition (11) holds because all resonant $q\mathbb{Z}$-harmonics of the π-periodic trigonometric polynomial $\mathcal{Q}_k \in T_{nk}[t]$ are equal to zero for $k = 1, \dots, m$. $\square$

Lemma 19. *If condition* (3) *holds,* Γ_ϵ *is anti-centrally symmetric,* $q \neq m \mod 2$*,* $m \geq 2$ *and* $q > n(m-1)$*, then we can compute all* $\mathrm{O}(\epsilon^m)$*-corrections too.*

Proof. We already know from Lemma 17 that we can compute the $\mathrm{O}(\epsilon^{m-1})$-corrections, since $q > n(m-1)$. Therefore, we only need to check that the last persistence condition

$$\mu\{\mathcal{Q}_m\} = 0$$

holds. That is, we need to check that all resonant $q\mathbb{Z}$-harmonics of $\mathcal{Q}_m$ are equal to zero.

If the perturbation Γ_ϵ is anti-centrally symmetric, then its support function satisfies that $h(\psi; \epsilon) = h(\psi + \pi; -\epsilon)$ and its asymptotic coefficients satisfy that

$$h_k(\psi + \pi) = (-1)^k h_k(\psi), \quad \forall k \in \mathbb{N}.$$

In that case, it is not hard to prove by induction over m that objects i)–iv) listed in the proof of Lemma 17 satisfy the same property. Namely, that they are π-periodic and π-antiperiodic functions for even and odd indexes k, respectively. In particular, if m is even (respectively, odd), then $\mathcal{Q}_m$ only contains even (respectively, odd) harmonics. Hence, $\mathcal{Q}_m$ does not contain harmonics $\mathrm{e}^{\pm \mathrm{i} qt}$, because $q \neq m \mod 2$, and it does not contain any harmonic $\mathrm{e}^{\pm \mathrm{i} lqt}$ with $l \geq 2$ either, because $lq \geq 2q > 2n(m-1) \geq nm$ and $\mathcal{Q}_m \in T_{nm}[t]$. We have used that $m \geq 2$ in the last inequality. $\square$

Theorem 9 is a direct consequence of the previous three lemmas.

Proof of Lemma 10. We have to check that $\tilde{\mathcal{G}}_k^\star, \tilde{\mathcal{G}}_k^\bullet, \mathcal{G}_{k-1}^\diamond, h_k \in T_{nk}$ for all $k \geq 1$. We prove it by induction over k. The base case $k = 1$ is trivial, because $\tilde{\mathcal{G}}_1^\star = 0$, $\tilde{\mathcal{G}}_1^\bullet = 0$, $\mathcal{G}_0^\diamond = P_1(\cos\psi, \sin\psi)$ and $h_1 = \frac{1}{2} P_1(\cos\psi, \sin\psi)$. The induction step follows from the explicit formulas given in Lemma 16 and the elementary properties listed at the beginning of this appendix. The claims about symmetries are trivial. $\square$

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