Existence and non-existence of (convex) caustics

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We consider the billiard dynamics inside a planar domain —a billiard table— whose border is a smooth closed convex curve: a particle follows straight lines inside the billiard table and it is reflected at the border following the rule "the angle of incidence equals the angle of reflection". From now on, the term *convex* means that the border of the table has curvature everywhere non-negative, the term *strict* means that it has no *flat* points —points at which the curvature vanishes—, and the term *smooth* means that it admits a sufficiently high number of continuous derivatives, the number being different on each result.

We recall that a smooth curve inside the table is a *caustic* if a billiard trajectory, once tangent to it, stays tangent after every reflection. We refer to the books [9, 10] for a background on billiards and caustics.

There exist several negative and positive results about convex caustics. First, we shall describe some qualitative and quantitative non-existence theorems, which go back to Mather, Gutkin and Katok. Next, we shall state the classical existence result of Lazutkin, whose regularity was later improved by R. Douady. Finally, we shall present a negative result for higher dimensional tables found by Berger.

Theorem 1 (Mather [6])**.** *If the border of the table is C^2 and has some flat point, then there are no smooth convex caustics inside the table.*

This result follows from a formula known in geometrical optics as the mirror equation, see [10]. Mather used another method of proof based on the Lagrangian formulation of billiard dynamics. Both proofs are elementary.

Gutkin and Katok obtained the following quantitative versions of Mather's theorem. Let d, w, and r be the the diameter, the width, and the inradius of the billiard table. Let $\underline{\kappa}$ and $\overline{\kappa}$ be the minimal and maximal values of the curvature of the border of the billiard table, and let L be its length.

Theorem 2 (Gutkin & Katok [4])**.** *If some of the following quantitative geometric conditions holds, the billiard table Ω contains a region Ω' free of convex caustics.*

Condition	*Description of* Ω'
$\sqrt{2}\underline{\kappa}d^2 \leq r$	*A disc of radius r' such that $r' > r - \sqrt{2}\underline{\kappa}d^2$*
$\sqrt{2}\underline{\kappa}d^2 \leq w/3$	*A disc of radius r' such that $r' > w/3 - \sqrt{2}\underline{\kappa}d^2$*
$\sqrt{2}\underline{\kappa}\overline{\kappa}d^2 \leq 1$	*A disc of radius r' such that $\overline{\kappa}r' > 1 - \sqrt{2}\underline{\kappa}\overline{\kappa}d^2$*
$\sqrt{2}\underline{\kappa}\overline{\kappa}d^2 \leq 1$	*A convex set such that* $\mathrm{Area}(\Omega \setminus \Omega') \leq \sqrt{2}\underline{\kappa}d^2 L$

We note that if the border of the table has a flat point, then $\underline{\kappa} = 0$ and $\Omega' = \Omega$, so we recover Mather's theorem. In particular, if we have a one-parameter family of strictly convex billiard tables Ω_t whose minimal curvature approaches zero at some critical parameter $t = t_*$ of the family, while the global shape of the table remains essentially unchanged, then the convex caustics are gradually pushed out to the boundary in the limit $t \to t_*$. An example of this situation is given by the strictly convex tables

$$\Omega_t = \left\{(x, y) \in \mathbb{R}^2 : x^2 + ty^2 + y^4 \leq 1\right\}, \qquad t > t_* = 0.$$

A key step in the proof of this quantitative theorem is to establish a suitable set of upper and lower bounds on the Lazutkin parameter that arises in the string construction. This construction is a geometric method —similar to the gardener's method to draw ellipses with prefixed foci— to draw all billiard tables with a prefixed smooth convex closed caustic. These billiard tables are parameterized by the Lazutkin parameter, which quantifies the distance between the caustic and the border of the table. Small Lazutkin parameters correspond to caustics close to the border. Gutkin and Katok showed that too big Lazutkin parameters are incompatible with the geometric hypotheses of their theorem.

The non-existence of convex caustics implies the existence of billiard trajectories whose past and future behaviours differ dramatically. To be more precise, we say that a billiard trajectory is positively (respectively, negatively) ϵ-glancing if, for some bounce, the angle of reflection with the positive (respectively, negative) tangent vector is smaller than ϵ. Mather established, under the non-existence of smooth convex caustics, the existence of infinitely many billiard trajectories that are both positively and negatively ϵ-glancing for any $\epsilon > 0$. To bound the number of impacts $n = n(\epsilon)$ of such glancing billiard trajectories between its positive and negative ϵ-bounces as $\epsilon \to 0$ is an open problem, similar to bound the speed of Arnold diffusion in Hamiltonian Systems.

The only positive result of this talk is the following one.

Theorem 3 (Lazutkin [5], Douady [2])**.** *If the border of the table is C^6 and strictly convex, then there exists a collection of smooth convex caustics close to the border of the table whose union has positive area.*

Originally Lazutkin asked for C^{553} regularity. Douady reduced it to C^6, and conjectured that C^4 regularity may suffice. There exist C^1 examples —C^2 except for a finite set of points— without caustics.

This result is deduced from an Invariant Curve Theorem for area-preserving twist maps on the annulus that was one of the first results in KAM theory. The reader is referred to the book [8] for a proof of the Invariant Curve Theorem in the analytic case; the differentiable case contained in [5, 2] is technically more involved, so it is not recommended as a first reading.

As a by-product of standard KAM-like results, all the caustics obtained in Lazutkin's theorem have two important properties. First, they persist under small enough C^6 perturbations of the table. Second, their rotation numbers are poorly approximated by rational numbers since they belong to a Cantor set of the form

$$\mathcal{C} = \mathcal{C}_{\lambda,\tau,y_*} := \left\{ y \in (0, y_*) : |y - m/n| \geq \lambda n^{-\tau}, \quad \forall n \in \mathbb{N}, m \in \mathbb{Z} \right\}$$

for some constants $\lambda > 0$, $\tau > 2$, and $y_* > 0$. This set can be viewed as the open interval $(0, y_*)$ with a countable number of small gaps centered at rational values.

On the contrary, *resonant* caustics —the ones whose tangent trajectories are closed polygons— have rational rotation numbers and can be destroyed under arbitrarily small perturbations of the billiard table. See the example in [7].

Finally, let us consider the higher dimensional case. That is, we deal with hypersurfaces of the Euclidean n-dimensional space instead of smooth curves of the plane, for any $n \geq 3$. We assume that we have three open hypersurfaces V_-, U, and V_+ of class C^2 with non-degenerate second fundamental form at their respective points y_-, x, and y_+. We also suppose that any line tangent to V_- at a point enough close to y_- intersects transversely U at a point close to x and its reflection is tangent to V_+ at a point close to y_+. All these hypotheses are local.

Theorem 4 (Berger [1])**.** *Under these hypotheses of regularity, non-degeneracy, and tangent reflection, the billiard hypersurface U is part of a quadric Q, and both caustic hypersurfaces V_- and V_+ are part of the same quadric Q' confocal to Q.*

The proof follows from a duality argument once a higher dimensional version of the planar mirror equation is established. In particular, the principal curvatures of the hypersurface U at points close to x play a role similar to the one played by the (planar) curvature in the planar case.

Gruber proved a similar theorem under weaker regularity hypotheses in [3].

References

[1] M. Berger, *Seules les quadriques admettent des caustiques*, Bulletin Soc. Math. France **123** (1995), 107–116.

[2] R. Douady, *Applications du théorème des tores invariantes*, PhD. These, Université Paris VII, 1982.

[3] P. M. Gruber, *Only ellipsoids have caustics*, Math. Ann. **303** (1995), 185–194.

[4] E. Gutkin and A. Katok, *Caustics for inner and outer billiards*, Comm. Math. Phys. **173** (1995), 101–133.

[5] V. F. Lazutkin, *The existence of caustics for a billiard problem in a convex domain*, Math. USSR Izvestija **7** (1973), 185–214.

[6] J. Mather, *Glancing billiards*, Ergodic Theory Dyn. Syst. **2** (1982), 397–403.

[7] R. Ramírez-Ros, *Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables*, Phys. D **214** (2006), 78–87.

[8] C. L. Siegel and J. K. Moser, *Lectures On Celestial Mechanics*, Springer-Verlag, New York-Heidelberg, 1971.

[9] S. Tabachnikov, *Billiards*, Panor. Synth. **1**, Soc. Math. France, Paris, 1995.

[10] S. Tabachnikov, *Geometry and Billiards*, Amer. Math. Soc., Providence, 2005.