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On Cayley conditions for billiards inside ellipsoids

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Abstract
Billiard trajectories inside an ellipsoid of $\mathbb{R}^n$ are tangent to $n-1$ quadrics of the pencil of confocal quadrics determined by the ellipsoid. The quadrics associated with periodic trajectories verify certain algebraic conditions. Cayley found them for the planar case. Dragović and Radnović generalized them to any dimension. We rewrite the original matrix formulation of these *generalized Cayley conditions* as a simpler polynomial one. We find several algebraic relations between caustic parameters and ellipsoidal parameters that give rise to non-singular periodic trajectories. These relations become remarkably simple when the *elliptic period* is minimal. We study the caustic types, the winding numbers and the ellipsoids of such minimal periodic trajectories. We also describe some non-minimal periodic trajectories.

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1. Introduction

One of the best-known discrete integrable systems is the billiard inside ellipsoids. All the segments (or their continuations) of a billiard trajectory inside an ellipsoid of $\mathbb{R}^n$ are tangent to $n-1$ quadrics of the pencil of confocal quadrics determined by the ellipsoid [1–3]. This situation is fairly exceptional. Quadrics are the only smooth hypersurfaces of $\mathbb{R}^n$, $n \geqslant 3$, that have caustics [4, 5]. A *caustic* is a smooth hypersurface with the property that a billiard trajectory, once tangent to it, stays tangent after every reflection. Caustics are a geometric manifestation of the integrability of billiards inside ellipsoids.

Periodic trajectories are the most distinctive trajectories, so their study is the first task. There exist two remarkable results concerning periodic billiard trajectories inside ellipsoids: the *generalized Poncelet theorem* and the *generalized Cayley conditions*.

A classical geometric theorem of Poncelet [6,7] implies that if a billiard trajectory inside an ellipse is periodic, then all the trajectories sharing its caustic are also periodic. Its generalization to the spatial case was proved by Darboux [8]. The extension of this result to arbitrary dimensions can be found in [9–12]. The generalized Poncelet theorem can be stated as follows. If a billiard trajectory inside an ellipsoid is closed after m_0 bounces and has length L_0, then all trajectories sharing the same caustics are also closed after m_0 bounces and have length L_0. Thus, a natural question arises. Which caustics give rise to periodic trajectories? The planar case was solved by Cayley [13, 14] in the 19th century. Still for the planar case, Halphen [15] gave explicit algebraic conditions for the periodic caustics up to period 11. Dragović and Radnović [16, 17] found some *generalized Cayley conditions* for billiards inside ellipsoids 15 years ago. They have also stated similar conditions for other billiard frameworks; see [18–23].

For simplicity, let us focus on the spatial case. Let $Q : x^2/a + y^2/b + z^2/c = 1$ be the triaxial ellipsoid with ellipsoidal parameters $0 < c < b < a$. Any billiard trajectory inside Q has as caustics two elements of the family of confocal quadrics

$$Q_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1 \right\}.$$

We restrict our attention to non-singular trajectories, that is, trajectories with two different caustics which are ellipsoids: $0 < \lambda < c$, hyperboloids of one sheet: $c < \lambda < b$, or hyperboloids of two sheets: $b < \lambda < a$. The singular values $\lambda \in \{a, b, c\}$ are discarded. There exist some restrictions on the caustics Q_{λ_1} and Q_{λ_2}. The only feasible caustic types are EH1, H1H1, EH2 and H1H2; see [24, 25]. The meaning of this notation is obvious.

The generalized Cayley condition in this context can be expressed as follows. The billiard trajectories inside the triaxial ellipsoid Q sharing the caustics Q_{λ_1} and Q_{λ_2} are periodic with *elliptic period* $m \geqslant 3$ if and only if

$$\operatorname{rank} \begin{pmatrix} f_{m+1} & \cdots & f_4 \\ \vdots & & \vdots \\ f_{2m-1} & \cdots & f_{m+2} \end{pmatrix} < m - 2,$$

where $f(t) = \sum_{l\geqslant 0} f_l t^l := \sqrt{\prod_{i=1}^5 (1-\gamma_i t)}$, $\{\gamma_1, \ldots, \gamma_5\} = \{1/a, 1/b, 1/c, 1/\lambda_1, 1/\lambda_2\}$. We deal with non-singular trajectories inside triaxial ellipsoids, so the inverse quantities $\gamma_1, \ldots, \gamma_5$ are pairwise distinct, and we can assume that $0 < \gamma_5 < \cdots < \gamma_1$. Most results related to generalized Cayley conditions become simpler when expressed in terms of the inverse quantities γ_i, instead of the original ellipsoidal parameters a, b, c, and the caustic parameters λ_1, λ_2. Proposition 12 is a paradigmatic sample.

The elliptic period is defined in section 2. Roughly speaking, the difference between the period m_0 and the elliptic period m of the periodic billiard trajectories sharing two given caustics is that *all* of those trajectories close in Cartesian (respectively, elliptic) coordinates after exactly m_0 (respectively, m) bounces. We will see that either $m = m_0/2$ or $m = m_0$.

The previous matrix formulation is nice from a theoretical point of view, but it has strong limitations from a computational point of view. We will see in section 3 that it can be written as a system of two homogeneous symmetric polynomial equations with rational coefficients of degrees $m^2 - 2$ and $m^2 - 1$ in the variables $\gamma_1, \ldots, \gamma_5$. Thus, both degrees grow *quadratically* with the elliptic period m, which turns this approach into a tough challenge. In particular, to our knowledge, the caustic parameters λ_1, λ_2 have never been explicitly expressed in terms of the ellipsoidal parameters a, b, c for any $m \geqslant 3$.

We will rewrite this matrix formulation as a computationally more appealing one which gives rise to (non-symmetric) homogeneous polynomial equations whose degrees are smaller than the elliptic period m. We will find the following remarkable algebraic relations between caustic and ellipsoidal parameters using the polynomial formulation. The billiard trajectories inside the ellipsoid Q sharing the caustics Q_{λ_1} and Q_{λ_2} are periodic with:

- elliptic period 3 if and only if $\gamma_1 + \gamma_4 + \gamma_5 = \gamma_2 + \gamma_3$ and $\gamma_1^2 + \gamma_4^2 + \gamma_5^2 = \gamma_2^2 + \gamma_3^2$;
- elliptic period 3 and caustic type H1H1 if and only if λ_1, λ_2 are the roots of $t^3 - (t - a)(t - b)(t - c)$;
- elliptic period 4 if (but not only if) $\exists d \in \mathbb{R}$ such that a, b, c are the roots of $t^4 - (t - d)^2(t - \lambda_1)(t - \lambda_2)$;
- elliptic period 5 if (but not only if) the roots of $t^5 - (t - c)(t - b)(t - a)(t - \lambda_1)(t - \lambda_2)$ are double.

Thus, it is easy to find simple examples of periodic trajectories with elliptic period 3. For instance, since $1 + 2 + 6 = 4 + 5$ and $1^2 + 2^2 + 6^2 = 4^2 + 5^2$, the billiard trajectories:

- inside the ellipsoid $Q : x^2 + 2y^2 + 5z^2 = 1$ with caustic parameters $\lambda_1 = \frac{1}{6}$ and $\lambda_2 = \frac{1}{4}$;
- inside the ellipsoid $Q : x^2 + 2y^2 + 6z^2 = 1$ with caustic parameters $\lambda_1 = \frac{1}{5}$ and $\lambda_2 = \frac{1}{4}$;
- inside the ellipsoid $Q : x^2 + 4y^2 + 5z^2 = 1$ with caustic parameters $\lambda_1 = \frac{1}{6}$ and $\lambda_2 = \frac{1}{2}$;
- inside the ellipsoid $Q : x^2 + 4y^2 + 6z^2 = 1$ with caustic parameters $\lambda_1 = \frac{1}{5}$ and $\lambda_2 = \frac{1}{2}$

are periodic with elliptic period 3, and caustic types EH1, H1H1, EH2 and H1H2, respectively.

Let us compare the matrix and polynomial formulations when the elliptic period is equal to 5: $m = 5$. Then the two homogeneous symmetric polynomial equations obtained from the matrix formulation have degrees 23 and 24 in the variables $\gamma_1, \ldots, \gamma_5$. On the other hand, $t^5 - (t - c)(t - b)(t - a)(t - \lambda_1)(t - \lambda_2)$ is a polynomial of degree 4 in a single variable. The polynomial formulation leads to a simpler problem. Nevertheless, the matrix formulation determines *all* periodic billiard trajectories with elliptic period $m = 5$, whereas we find just *some* of such trajectories using the polynomial $t^5 - (t - c)(t - b)(t - a)(t - \lambda_1)(t - \lambda_2)$.

Another natural question concerning periodic billiard trajectories is the following one. Which are the triaxial ellipsoids of $\mathbb{R}^3$ that display periodic billiard trajectories with a fixed caustic type and a fixed (elliptic) period? A numerical approach to that question was considered in [26], where the authors computed several bifurcations in the space of ellipsoidal parameters. We will find the algebraic relations that define the bifurcations associated with small elliptic periods. For instance, we will see that there exist periodic billiard trajectories with elliptic period $m = 3$ and caustic type EH1 if and only if $c < ab/(a + b + \sqrt{ab})$.

For brevity, we will not depict billiard trajectories inside triaxial ellipsoids of $\mathbb{R}^3$. The reader interested in 3D graphical visualizations is referred to [27], where several periodic billiard trajectories with small periods are displayed from different perspectives.

We complete this introduction with a note on the organization of the article. In section 2 we review briefly some well-known results concerning billiards inside ellipsoids, recalling the matrix formulation of the generalized Cayley conditions obtained by Dragović and Radnović. We also introduce the concept of elliptic period. The practical limitations of the matrix formulation are exposed in section 3. We present the polynomial formulation in section 4. In section 5 we carry out a detailed analysis for minimal elliptic periods, and the study of more general elliptic periods is postponed to section 6. The previous results are adapted to billiards inside ellipses of $\mathbb{R}^2$ and inside triaxial ellipsoids of $\mathbb{R}^3$ in sections 7 and 8, respectively.

2. Preliminaries

In this section we recall several classical results and their modern generalizations for billiards inside ellipsoids that go back to Jacobi, Chasles, Poncelet, Darboux and Cayley.

We consider the billiard dynamics inside the ellipsoid

$$Q = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i^2}{a_i} = 1 \right\}, \qquad 0 < a_1 < \cdots < a_n. \quad (1)$$

The degenerate cases in which the ellipsoid has some symmetry of revolution are not considered here. This ellipsoid is an element of the family of confocal quadrics

$$Q_\lambda = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i^2}{a_i - \lambda} = 1 \right\}, \qquad \lambda \in \mathbb{R}.$$

We note that $Q_\lambda = \emptyset$ for $\lambda > a_n$. Thus, there are exactly n different geometric types of non-singular quadrics in this family, which correspond to the cases

$$\lambda \in (-\infty, a_1), \quad \lambda \in (a_1, a_2), \quad \ldots, \quad \lambda \in (a_{n-1}, a_n).$$

For instance, the confocal quadric Q_λ is an ellipsoid if and only if $\lambda \in (-\infty, a_1)$. On the other hand, the meaning of Q_λ in the singular cases $\lambda \in \{a_1, \ldots, a_n\}$ is

$$Q_{a_j} = H_j = \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j = 0 \right\}.$$

The following theorems of Jacobi and Chasles can be found in [1–3].

Theorem 1 (Jacobi). *Any generic point $x \in \mathbb{R}^n$ belongs to exactly n distinct non-singular quadrics $Q_{\mu_0}, \ldots, Q_{\mu_{n-1}}$ such that $\mu_0 < a_1 < \mu_1 < a_2 < \cdots < a_{n-1} < \mu_{n-1} < a_n$.*

We denote by $\mu = (\mu_0, \ldots, \mu_{n-1}) \in \mathbb{R}^n$ the *Jacobi elliptic coordinates* of the point $x = (x_1, \ldots, x_n)$. Cartesian and elliptic coordinates are linked by relations

$$x_j^2 = \frac{\prod_{i=0}^{n-1}(a_j - \mu_i)}{\prod_{i \neq j}(a_j - a_i)}, \qquad j = 1, \ldots, n.$$

Hence, a point has the same elliptic coordinates as its orthogonal reflections with respect to the coordinate subspaces of $\mathbb{R}^n$. A point is *generic*, in the sense of theorem 1, if and only if it is outside all coordinate hyperplanes. When a point tends to the coordinate hyperplane H_j, some of its elliptic coordinates tend to a_j.

Theorem 2 (Chasles). *Any line in $\mathbb{R}^n$ is tangent to exactly $n - 1$ confocal quadrics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$.*

It is known that if two lines obey the reflection law at a point $x \in Q$, then both lines are tangent to the same confocal quadrics. Thus, all lines of a billiard trajectory inside the ellipsoid Q are tangent to exactly $n - 1$ confocal quadrics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$, which are called *caustics* of the trajectory, whereas $\lambda_1, \ldots, \lambda_{n-1}$ are the *caustic parameters* of the trajectory. We will say that a billiard trajectory inside Q is *non-singular* when it has $n - 1$ distinct non-singular caustics. We only deal with non-singular billiard trajectories in this paper.

The caustic parameters cannot take arbitrary values. For instance, a line cannot be tangent to two different confocal ellipsoids, and all caustic parameters must be positive. The following complete characterization was given in [24, 25].

Proposition 3. *Let $\lambda_1 < \cdots < \lambda_{n-1}$ be some real numbers such that*

$$\{a_1, \ldots, a_n\} \cap \{\lambda_1, \ldots, \lambda_{n-1}\} = \emptyset.$$

Set $a_0 = 0$. Then there exist non-singular billiard trajectories inside the ellipsoid Q sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$ if and only if

$$\lambda_k \in (a_{k-1}, a_k) \cup (a_k, a_{k+1}), \qquad k = 1, \ldots, n-1. \tag{2}$$

Definition 1. The *caustic type* of a non-singular trajectory is the vector $\varsigma = (\varsigma_1, \ldots, \varsigma_{n-1}) \in \mathbb{Z}^{n-1}$ such that

$$\lambda_k \in (a_{\varsigma_k}, a_{\varsigma_k+1}), \qquad k = 1, \ldots, n-1.$$

We know from proposition 3 that $\varsigma_k \in \{k-1, k\}$ for $k = 1, \ldots, n-1$. Hence, there are exactly 2^{n-1} different caustic types. The two caustic types in the planar case correspond to ellipses: $\varsigma_1 = 0$, and hyperbolas: $\varsigma_1 = 1$. The four caustic types in the spatial case correspond to EH1: $\varsigma = (0, 1)$, H1H1: $\varsigma = (1, 1)$, EH2: $\varsigma = (0, 2)$ and H1H2: $\varsigma = (1, 2)$. The EH1, H1H1, EH2, H1H2 notation was described in the introduction.

Next, we recall a result concerning periodic billiard trajectories inside ellipsoids.

Theorem 4 (The generalized Poncelet theorem). *If a non-singular billiard trajectory is closed after m_0 bounces and has length L_0, then all trajectories sharing the same caustics are also closed after m_0 bounces and have length L_0.*

Poncelet proved this theorem for conics [6]. Darboux generalized it to triaxial ellipsoids of $\mathbb{R}^3$; see [8]. Later on, this result was generalized to any dimension in [9–12].

The periodic billiard trajectories sharing the same caustics also have the same winding numbers. In order to introduce these numbers, we set

$$\{c_1, \ldots, c_{2n-1}\} = \{a_1, \ldots, a_n\} \cup \{\lambda_1, \ldots, \lambda_{n-1}\}, \tag{3}$$

and $f(t) = \sqrt{\prod_{i=1}^{2n-1}(1 - t/c_i)}$. We deal with non-singular billiard trajectories inside ellipsoids without symmetries of revolution, so the parameters $c_1, \ldots, c_{2n-1}$ are pairwise distinct, and we can assume that $c_0 := 0 < c_1 < \cdots < c_{2n-1}$.

Theorem 5 (Winding numbers). *The non-singular billiard trajectories inside the ellipsoid Q sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$ are periodic with period m_0 if and only if there exist some positive integer numbers $m_1, \ldots, m_{n-1}$ such that*

$$\sum_{j=0}^{n-1} (-1)^j m_j \int_{c_{2j}}^{c_{2j+1}} \frac{t^i}{f(t)} \mathrm{d}t = 0, \qquad \forall i = 0, \ldots, n-2. \tag{4}$$

Each of these periodic billiard trajectories has m_j points at $Q_{c_{2j}}$ and m_j points at $Q_{c_{2j+1}}$. Besides this, $\{c_{2j}, c_{2j+1}\} \cap \{a_1, \ldots, a_n\} \neq \emptyset \Rightarrow m_j$ even. Finally, $\gcd(m_0, \ldots, m_{n-1}) \in \{1, 2\}$.

Let L_0 be the common length of these periodic billiard trajectories. Let $x(t)$ be an arc-length parameterization of any of these trajectories. Let $\mu(t) = (\mu_0(t), \ldots, \mu_{n-1}(t))$ be the corresponding parameterization in elliptic coordinates. Then:

(i) $c_{2j} \leqslant \mu_j(t) \leqslant c_{2j+1}$ *for all $t \in \mathbb{R}$.*
(ii) Functions $\mu_j(t)$ are smooth everywhere, except $\mu_0(t)$, which is non-smooth at impact points—that is, when $x(t_\star) \in Q$—in which case $\mu_0'(t_\star+) = -\mu_0'(t_\star-) \neq 0$.
(iii) If $\mu_j(t)$ is smooth at $t = t_\star$, then $\mu_j'(t_\star) = 0 \Leftrightarrow \mu_j(t_\star) \in \{c_{2j}, c_{2j+1}\}$.
(iv) $\mu_j(t)$ makes exactly m_j complete oscillations (round trips) inside the interval $[c_{2j}, c_{2j+1}]$ along one period $0 \leqslant t \leqslant L_0$.
(v) $\mu(t)$ has period $L = L_0/\gcd(m_0, \ldots, m_{n-1})$.

Definition 2. *The numbers $m_0, \ldots, m_{n-1}$ are called winding numbers. Theorem 5 contains three equivalent definitions for them: by means of the property regarding hyperelliptic integrals given in (4), as a geometric description of how the periodic billiard trajectories fold in $\mathbb{R}^n$, and as the number of oscillations of the elliptic coordinates along one period.*

Most of the statements of theorem 5 can be found in [19, 21], except the one concerning the even character of some winding numbers and the ones regarding $\gcd(m_0, \ldots, m_{n-1})$. The first statement is trivial; it suffices to realize that a periodic billiard trajectory can only have an even number of crossings with any coordinate hyperplane. The second ones follow from the oscillating behavior of elliptic coordinates along billiard trajectories described in theorem 5; it suffices to note that all elliptic coordinates make an integer number of complete oscillations inside their corresponding intervals along one half-period $L_0/2$ when $\gcd(m_0, \ldots, m_{n-1}) = 2$.

The following conjecture was stated in [26], where it was numerically tested.

Conjecture 1. *Winding numbers are always ordered in a strictly decreasing way; that is,*

$$2 \leqslant m_{n-1} < \cdots < m_1 < m_0.$$

It is known that the conjecture holds in the planar case. If this conjecture holds, then any non-singular periodic billiard trajectory inside Q has period at least $n + 1$. By the way, there are periodic billiard trajectories of smaller periods, but all of them are singular—they are contained in some coordinate hyperplane or in some ruled quadric of the confocal family.

In light of the last item of theorem 5, we present the following definitions.

Definition 3. *The elliptic period m and the elliptic winding numbers $\widetilde{m}_0, \ldots, \widetilde{m}_{n-1}$ of the non-singular periodic billiard trajectories with period m_0 and winding numbers $m_0, \ldots, m_{n-1}$ are*

$$m = m_0/d, \qquad \widetilde{m}_j = m_j/d,$$

where $d = \gcd(m_0, \ldots, m_{n-1})$.

Roughly speaking, the difference between the period m_0 and the elliptic period m is that periodic billiard trajectories close in Cartesian (respectively, elliptic) coordinates after exactly m_0 (respectively, m) bounces. In order to clarify this difference, let us consider the six planar periodic trajectories shown in figure 1; see section 7. Only the trajectory in figure 1(c) verifies that $m = m_0$. In contrast, the trajectories in figures 1(a), (b) and (e) (respectively, figure 1(d), figure 1(f)) have even period m_0 and any of their impact points becomes its reflection with respect to the origin (respectively, the vertical axis, the horizontal axis) after $m_0/2$ bounces, so they have elliptic period $m = m_0/2$.

It turns out that given any ellipsoid of the form (1) and any proper coordinate subspace of $\mathbb{R}^n$, there exist infinitely many sets of $n-1$ distinct non-singular caustics such that their tangent trajectories are periodic with even period, say m_0, and any of their impact points becomes its reflection with respect to that coordinate subspace after $m_0/2$ bounces. We will not prove this claim, since the proof requires some convoluted ideas developed in [26, 27].

It is natural to look for caustics giving rise to periodic billiard trajectories inside that ellipsoid. Such caustics can be found by means of certain algebraic conditions, called *generalized Cayley conditions*. They are found by working in elliptic coordinates, so they depend on the elliptic period m, not on the (Cartesian) period m_0.

Theorem 6 (Generalized Cayley conditions). *The non-singular billiard trajectories inside the ellipsoid Q sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$ are periodic with elliptic period m if and* *only if $m \geqslant n$ and*

$$\operatorname{rank}\begin{pmatrix} f_{m+1} & \cdots & f_{n+1} \\ \vdots & & \vdots \\ f_{2m-1} & \cdots & f_{m+n-1} \end{pmatrix} < m - n + 1,$$

where $f(t) = \sum_{l \geqslant 0} f_l t^l := \sqrt{\prod_{i=1}^{2n-1}(1 - t/c_i)}$.

Cayley proved this theorem for conics [13]. Later on, this result was generalized to any dimension by Dragović and Radnović in [16,17]. These authors have also given similar Cayley conditions in many other billiard frameworks; see [18–23].

Definition 4. *$\mathcal{C}(m, n)$ denotes the generalized Cayley condition that characterizes billiard trajectories of elliptic period m inside ellipsoids of $\mathbb{R}^n$ given in theorem 6.*

3. On the matrix formulation of the generalized Cayley conditions

The matrix formulation of the generalized Cayley condition stated in theorem 6 is nice from a theoretical point of view, but has limitations from a practical point of view. Let us describe them.

The function $f(t)$ is symmetric in the inverse quantities $\gamma_i = 1/c_i$. In order to exploit it, we introduce some notation for symmetric polynomials. Let $\mathbb{Q}_l^{\text{hom,sym}}[x_1, \ldots, x_s]$ be the vectorial space over $\mathbb{Q}$ of all homogeneous symmetric polynomials with rational coefficients of degree l in the variables $x_1, \ldots, x_s$. Let $\mathrm{e}_l(x_1, \ldots, x_s)$ be the *elementary symmetric polynomial* of degree l in the variables $x_1, \ldots, x_s$. That is, $\prod_{i=1}^{s}(1 + x_i t) = \sum_{l \geqslant 0} \mathrm{e}_l(x_1, \ldots, x_s) t^l$, so $\mathrm{e}_l(x_1, \ldots, x_s) = 0$ for all $l > s$. Clearly, $\mathrm{e}_l = \mathrm{e}_l(x_1, \ldots, x_s) \in \mathbb{Q}_l^{\text{hom,sym}}[x_1, \ldots, x_s]$.

We stress that $f_l = f_l(\gamma_1, \ldots, \gamma_{2n-1}) \in \mathbb{Q}_l^{\text{hom,sym}}[\gamma_1, \ldots, \gamma_{2n-1}]$, which is one of the reasons for the introduction of the inverse quantities $\gamma_i = 1/c_i$. Indeed, using that $f^2(t) = \prod_{i=1}^{2n-1}(1 - \gamma_i t)$, we get the recursive relations

$$f_0 = 1, \qquad 2 f_l = (-1)^l \mathrm{e}_l(\gamma_1, \ldots, \gamma_{2n-1}) - \sum_{k=1}^{l-1} f_k f_{l-k}, \quad \forall l \geqslant 1.$$

Hence, it is possible to compute recursively all Taylor coefficients f_l, although their expressions are rather complicated when l is big. Nevertheless, the computation of the Taylor coefficients $f_{n+1}, \ldots, f_{2m-1}$ is the simplest step in the practical implementation of the generalized Cayley condition $\mathcal{C}(m, n)$. Next, we must impose that all $(m - n + 1) \times (m - n + 1)$ minors of the matrix that appear in theorem 6 vanish. For simplicity, let us consider the minors formed by the first $m - n$ rows and the $(m - n + l)$th row of that matrix, for $l = 1, \ldots, n - 1$. Then the Cayley condition $\mathcal{C}(m, n)$ can be written as the system of $n - 1$ polynomial equations

$$M_{m,n,l} = M_{m,n,l}(\gamma_1, \ldots, \gamma_{2n-1}) := \begin{vmatrix} f_{m+1} & \cdots & f_{n+1} \\ \vdots & & \vdots \\ f_{2m-n} & \cdots & f_m \\ f_{2m-n+l} & \cdots & f_{m+l} \end{vmatrix} = 0, \qquad 1 \leqslant l \leqslant n - 1. \tag{5}$$

One can check that $M_{m,n,l}(\gamma_1, \ldots, \gamma_{2n-1}) \in \mathbb{Q}_{(m-n+2)m-n+l}^{\text{hom,sym}}[\gamma_1, \ldots, \gamma_{2n-1}]$ from the Leibniz formula for determinants. This implies that the resolution of system (5) is a formidable challenge, even from a purely numerical point of view and for relatively small values of m.

We want to write down the solutions of system (5) in an explicit algebraic way. Let us focus on the planar case $n = 2$, when condition $\mathcal{C}(m, 2)$ becomes a single homogeneous symmetric polynomial equation of degree $m^2 - 1$ in three unknowns, namely,

$$M_m = M_m(\gamma_1, \gamma_2, \gamma_3) := \begin{vmatrix} f_{m+1} & \cdots & f_3 \\ \vdots & & \vdots \\ f_{2m-1} & \cdots & f_{m+1} \end{vmatrix} = 0.$$

For instance, condition $\mathcal{C}(2, 2)$ can be easily solved, since

$$\begin{aligned} -16M_2 &= \gamma_1^3 + \gamma_2^3 + \gamma_3^3 - \gamma_1^2\gamma_2 - \gamma_1^2\gamma_3 - \gamma_2^2\gamma_1 - \gamma_2^2\gamma_3 - \gamma_3^2\gamma_1 - \gamma_3^2\gamma_2 + 2\gamma_1\gamma_2\gamma_3 \\ &= (\gamma_1 - \gamma_2 - \gamma_3)(\gamma_3 - \gamma_1 - \gamma_2)(\gamma_2 - \gamma_3 - \gamma_1). \end{aligned}$$

The inverse quantities $\gamma_i = 1/c_i$ verify that $0 < \gamma_3 < \gamma_2 < \gamma_1$, since $0 < c_1 < c_2 < c_3$. Therefore, only the first factor of the above formula provides a feasible solution, and so condition $\mathcal{C}(2, 2)$ has a unique solution:

$$\gamma_1 = \gamma_2 + \gamma_3.$$

The computations for condition $\mathcal{C}(3, 2)$ are much harder, so we have implemented them using a computer algebra system. We got the factorization $-16\,384M_3 = q_0q_1q_2q_3$, where

$$\begin{aligned} q_0 &= \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 2\gamma_1\gamma_2 - 2\gamma_1\gamma_3 - 2\gamma_2\gamma_3, \\ q_k &= 3\gamma_k^2 - 2(\gamma_i + \gamma_j)\gamma_k - (\gamma_i - \gamma_j)^2, \qquad \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

The first factor q_0 can, in its turn, be factored as

$$q_0 = (\sqrt{\gamma_1} - \sqrt{\gamma_2} - \sqrt{\gamma_3})(\sqrt{\gamma_1} + \sqrt{\gamma_2} - \sqrt{\gamma_3})(\sqrt{\gamma_1} - \sqrt{\gamma_2} + \sqrt{\gamma_3})(\sqrt{\gamma_1} + \sqrt{\gamma_2} + \sqrt{\gamma_3}).$$

The factor q_0 provides a unique feasible solution: $\sqrt{\gamma_1} = \sqrt{\gamma_2} + \sqrt{\gamma_3}$, because $0 < \gamma_3 < \gamma_2 < \gamma_1$. Next, we consider the factor q_k as a second-order polynomial in the variable γ_k with coefficients in $\mathbb{Z}^{\mathrm{sym}}[\gamma_i, \gamma_j]$. Then we get the solutions

$$\gamma_k = \gamma_k^{\pm}(\gamma_i, \gamma_j) := \frac{\gamma_i + \gamma_j}{3} \pm \frac{2}{3}\sqrt{\gamma_i^2 + \gamma_j^2 - \gamma_i\gamma_j}.$$

It turns out that $\gamma_k^- \leqslant 0 < \max(\gamma_i, \gamma_j) < \gamma_k^+$, so only the factor q_1 gives a solution compatible with the ordering $0 < \gamma_3 < \gamma_2 < \gamma_1$, namely, $\gamma_1 = \gamma_1^+(\gamma_2, \gamma_3)$. Hence, $\mathcal{C}(3, 2)$ has only two solutions:

$$\sqrt{\gamma_1} = \sqrt{\gamma_2} + \sqrt{\gamma_3} \qquad \text{and} \qquad 3\gamma_1 = \gamma_2 + \gamma_3 + 2\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_2\gamma_3}. \tag{6}$$

We have tried to write down explicitly the solutions of system (5) in other cases, but we did not succeed, even after implementing the computations in a computer algebra system. This shows the limitations of the matrix formulation.

4. A polynomial formulation of the generalized Cayley conditions

Let us present a polynomial formulation of the generalized Cayley condition $\mathcal{C}(m, n)$. The key idea goes back to Halphen [15, section XIV, p 600], but he only studied the planar case $n = 2$.

Theorem 7. *Let* $r(t) = \prod_{i=1}^{2n-1}(1 - t/c_i)$ *and* $f(t) = \sqrt{r(t)}$. *The generalized Cayley condition* $\mathcal{C}(m, n)$ *is equivalent to each of the following two conditions:*

(i) There exists a non-zero polynomial $s(t) \in \mathbb{R}_{m-n}[t]$ *such that*

$$\left.\frac{\mathrm{d}^l}{\mathrm{d}t^l}\right|_{t=0} \{s(t)f(t)\} = 0, \qquad l = m + 1, \ldots, 2m - 1. \tag{7}$$

(ii) There exist $\alpha \neq 0$, $s(t) \in \mathbb{R}_{m-n}[t]$, and $q(t) \in \mathbb{R}_{m-1}[t]$ such that $s(0) = q(0) = 1$ and

$$s^2(t)r(t) = (\alpha t^m + q(t))q(t). \tag{8}$$

Proof. We split the proof into three steps.

Step 1. $\mathcal{C}(m,n) \Leftrightarrow$ (i). $\mathcal{C}(m,n)$ means that the $m-n+1$ columns of the matrix given in theorem 6 are linearly dependent, so there exist $s_0, \ldots, s_{m-n} \in \mathbb{R}$, not all zero, such that

$$s_0 \times (\text{first column}) + \cdots + s_{m-n} \times (\text{last column}) = 0,$$

which is equivalent to condition (7) when $s(t) = \sum_{l=0}^{m-n} s_l t^l \in \mathbb{R}_{m-n}[t]$.

Step 2. (i) $\Rightarrow$ (ii). If $s(t) \in \mathbb{R}_{m-n}[t]$ verifies (7), then $g(t) = \sum_{l\geqslant 0} g_l t^l := s(t)f(t)$ verifies that $g_l = 0$ for $l = m+1, \ldots, 2m-1$. Hence,

$$g(t) = q(t) + \alpha t^m/2 + \mathrm{O}(t^{2m}),$$

where $q(t) = g_0 + \cdots + g_{m-1}t^{m-1} \in \mathbb{R}_{m-1}[t]$ and $\alpha = 2g_m$. Therefore,

$$s^2 r = s^2 f^2 = g^2 = q^2 + \alpha t^m q + \mathrm{O}(t^{2m}) = (q + \alpha t^m)q + \mathrm{O}(t^{2m}),$$

and so $s^2 r = (q + \alpha t^m)q$, since $\deg[s^2 r] \leqslant 2m-1$ and $\deg[(q+\alpha t^m)q] \leqslant 2m-1$. Besides this, $\alpha \neq 0$, because $\deg[r]$ is odd and $s(t) \not\equiv 0$.

Let $h(t) = \sum_{l\geqslant 0} h_l t^l := g^2(t) = s^2(t)r(t) \in \mathbb{R}_{2m-1}[t]$. Then

$$0 = h_{2m} = \sum_{l=0}^{2m} g_l g_{2m-l} = (g_m)^2 + 2g_0 g_{2m} \Rightarrow g_0 g_{2m} = -\alpha^2/8 \neq 0.$$

From this property, we deduce that $q(0) = g_0 \neq 0$ and $s^2(0) = q^2(0)/r(0) \neq 0$. Thus, we can normalize $s(t)$ by imposing $s(0) = 1$, since condition (7) only determines $s(t)$ up to a multiplicative constant. This implies that $q^2(0) = s^2(0)r(0) = 1$, so $q(0) = \pm 1$. We can assume, without loss of generality, that $q(0) = 1$. Otherwise, we substitute $q(t)$ and α in the identity $s^2 r = (\alpha t^m + q)q$, by $-q(t)$ and $-\alpha$, respectively.

Step 3. (ii) $\Rightarrow$ (i). If there exist $\alpha \neq 0$, $s(t) \in \mathbb{R}_{m-n}[t]$, and $q(t) \in \mathbb{R}_{m-1}[t]$ such that $s(0) = q(0) = 1$ and relation (8) holds, we set $g(t) = \sum_{l\geqslant 0} g_l t^l := s(t)f(t)$. Then,

$$g^2 = s^2 f^2 = s^2 r = (q + \alpha t^m)q = (1 + \alpha t^m/q)q^2.$$

The last operation is well defined for small values of $|t|$, because $q(0) \neq 0$. Hence,

$$g = \pm q\sqrt{1 + \frac{\alpha t^m}{q}} = \pm q\left(1 + \frac{\alpha t^m}{2q} + \mathrm{O}(t^{2m})\right) = \pm q \pm \frac{\alpha t^m}{2} + \mathrm{O}(t^{2m}),$$

so $g_l = 0$ for $l = m+1, \ldots, 2m-1$. $\square$

Next, we present three examples of the results that can be obtained from this formulation.

Theorem 8. *The non-singular billiard trajectories inside the ellipsoid (1) sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$ are periodic with:*

- *elliptic period $m = n$ if the roots of $t^n - \prod_{j=1}^n (t - a_j)$ are the caustic parameters;*
- *elliptic period $m = n+1$ if there exists $d \in \mathbb{R}$ such that the roots of $t^{n+1} - (t-d)^2 \prod_{k=1}^{n-1}(t - \lambda_k)$ are the ellipsoidal parameters; and*
- *elliptic period $m = 2n-1$ if the roots of $t^{2n-1} - \prod_{j=1}^n (t - a_j) \prod_{k=1}^{n-1}(t - \lambda_k)$ are double.*

Proof. It suffices to find $\alpha \neq 0$, $s(t) \in \mathbb{R}_{m-n}[t]$, and $q(t) \in \mathbb{R}_{m-1}[t]$ such that

$$s^2(t)r(t) = (\alpha t^m + q(t))q(t), \qquad s(0) = q(0) = 1,$$

for $m = n$, $m = n+1$ and $m = 2n-1$, respectively; see theorem 7. We recall that $r(t) = \prod_{i=1}^{2n-1}(1 - t/c_i)$ with $\{c_1, \ldots, c_{2n-1}\} = \{a_1, \ldots, a_n\} \cup \{\lambda_1, \ldots, \lambda_{n-1}\}$.

Case $m = n$. If the caustic parameters are the roots of $t^n - \prod_{j=1}^n (t - a_j)$, then

$$t^n - \prod_{j=1}^{n}(t - a_j) = \kappa \prod_{k=1}^{n-1}(t - \lambda_k),$$

for some factor $\kappa \in \mathbb{R}$. Indeed, $\kappa = \prod_{j=1}^n a_j \prod_{k=1}^{n-1} \lambda_k^{-1}$. We take $\alpha = (-1)^n \prod_{j=1}^n a_j^{-1}$, $s(t) \equiv 1$, and $q(t) = \prod_{k=1}^{n-1}(1 - t/\lambda_k)$. Clearly, $\alpha \neq 0$, $s(t) \in \mathbb{R}_0[t]$, $q(t) \in \mathbb{R}_{n-1}[t]$, and $s(0) = q(0) = 1$. Besides this,

$$s^2(t)r(t) = \prod_{j=1}^{n}(1 - t/a_j) \prod_{k=1}^{n-1}(1 - t/\lambda_k) = (\alpha t^n + q(t))q(t),$$

since $\prod_{j=1}^n (1 - t/a_j) = \alpha \prod_{j=1}^n (t - a_j) = \alpha t^n - \alpha\kappa \prod_{k=1}^{n-1}(t - \lambda_k) = \alpha t^n + q(t)$.

Case $m = n+1$. If $a_1, \ldots, a_n$ are the roots of $t^{n+1} - (t-d)^2 \prod_{k=1}^{n-1}(t - \lambda_k)$, then

$$t^{n+1} - (t-d)^2 \prod_{k=1}^{n-1}(t - \lambda_k) = \kappa \prod_{j=1}^{n}(t - a_j),$$

for some $\kappa \in \mathbb{R}$. Indeed, $\kappa = d^2 \prod_{k=1}^{n-1} \lambda_k \prod_{j=1}^n a_j^{-1}$. We take $\alpha = (-1)^{n+1} d^{-2} \prod_{k=1}^{n-1} \lambda_k^{-1}$, $s(t) = (1 - t/d)$, and $q(t) = \prod_{j=1}^n (1 - t/a_j)$. Clearly, $\alpha \neq 0$, $s(t) \in \mathbb{R}_1[t]$, $q(t) \in \mathbb{R}_n[t]$, and $s(0) = q(0) = 1$. Besides this,

$$s^2(t)r(t) = (1 - t/d)^2 \prod_{j=1}^{n}(1 - t/a_i) \prod_{k=1}^{n-1}(1 - t/\lambda_k) = (\alpha t^{n+1} + q(t))q(t),$$

since $(1 - t/d)^2 \prod_{k=1}^{n-1}(1 - t/\lambda_k) = \alpha (t-d)^2 \prod_{k=1}^{n-1}(1 - t/\lambda_k) = \alpha t^{n+1} - \alpha\kappa \prod_{j=1}^n (t - a_j) = \alpha t^{n+1} + q(t)$.

Case $m = 2n-1$. If $t^{2n-1} - \prod_{i=1}^{2n-1}(t - c_i)$ has double roots $d_1, \ldots, d_{n-1}$, then

$$t^{2n-1} - \prod_{i=1}^{2n-1}(t - c_i) = \kappa \prod_{l=1}^{n-1}(t - d_l)^2,$$

for some $\kappa \in \mathbb{R}$. Indeed, $\kappa = \prod_{i=1}^{2n-1} c_i \prod_{l=1}^{n-1} d_l^{-2}$. We take $\alpha = -\prod_{i=1}^{2n-1} c_i^{-1}$, $s(t) = \prod_{l=1}^{n-1}(1 - t/d_l)$, and $q(t) = s^2(t)$. Clearly, $\alpha \neq 0$, $s(t) \in \mathbb{R}_{n-1}[t]$, $q(t) \in \mathbb{R}_{2n-2}[t]$, and $s(0) = q(0) = 1$. Besides this,

$$s^2(t)r(t) = \prod_{l=1}^{n-1}(1 - t/d_l)^2 \prod_{i=1}^{2n-1}(1 - t/c_i) = (\alpha t^{2n-1} + q(t))q(t),$$

since $\prod_{i=1}^{2n-1}(1 - t/c_i) = \alpha \prod_{i=1}^{2n-1}(t - c_i) = \alpha t^{2n-1} - \alpha\kappa \prod_{l=1}^{n-1}(t - d_l)^2 = \alpha t^{2n-1} + q(t)$. □

Several questions arise concerning the periodic trajectories found in the previous theorem. Let us mention just three. Which are their caustic types, their (Cartesian) periods, and their winding numbers? Inside what ellipsoids do they exist? Are there other non-singular periodic billiard trajectories with elliptic period n, $n+1$ or $2n-1$?

We will give some partial answers in the following sections.

Some technicalities become simpler after the change of variables $t = 1/x$. Thus, we state another polynomial formulation of the generalized Cayley condition $\mathcal{C}(m,n)$.

Proposition 9. *Let* $R(x) = x\prod_{i=1}^{2n-1}(x-\gamma_i)$*, where* $\gamma_i = 1/c_i$*. The generalized Cayley condition* $\mathcal{C}(m,n)$ *holds if and only if there exist two monic polynomials* $S(x), P(x) \in \mathbb{R}[x]$ *such that* $\deg[S] = m-n$*,* $\deg[P] = m$*,* $P(0) \neq 0$*, and*

$$S^2(x)R(x) = P(x)\big(P(x) - P(0)\big). \tag{9}$$

Furthermore, if such polynomials $S(x)$ *and* $P(x)$ *exist, the following properties hold:*

(i) $S(x)$ *has no multiple roots;*
(ii) all the real roots of $S(x)$ *are contained in* $\{x \in \mathbb{R} : R(x) < 0\}$*;*
(iii) all the roots of $S(x)$ *are real when* $m \leqslant n+3$*; and*
(iv) $P(x)$ *and* $P(x) - P(0)$ *have the same number of real roots (counted with multiplicity).*

Proof. The 'if and only if' follows directly from the change of variables $t = 1/x$. Concretely, the relation between the objects of identities (8) and (9) is

$$R(x) = x^{2n}r(1/x), \qquad S(x) = x^{m-n}s(1/x), \qquad P(x) = \alpha + x^m q(1/x).$$

Then $P(0) \neq 0$ if and only if $\alpha \neq 0$, $s(0) = 1$ if and only if $S(x)$ is a monic polynomial of degree $m-n$, and $q(0) = 1$ if and only if $P(x)$ is a monic polynomial of degree m.

To prove the first two properties, it suffices to prove that $\gcd[S, RS'] = 1$ and

$$l_+ := \#\{x \in \mathbb{R} : S(x) = 0 < R(x)\} = 0.$$

If $W(x) = P(x)(P(x) - P(0))$ and $T(x) = P(x) - P(0)/2$, we get from (9) that

$$\begin{aligned} W(x) &= S^2(x)R(x) = T^2(x) - P^2(0)/4, \\ W'(x) &= S(x)(S(x)R'(x) + 2R(x)S'(x)) = 2T(x)P'(x). \end{aligned}$$

We consider the factorization $W'(x) = 2mW_-(x)W_0(x)W_+(x)W_*(x)$, where if $z \in \mathbb{C}$ is a root of multiplicity β of $W'(x)$ such that $W(z) < 0$, $W(z) = 0$, $W(z) > 0$, or $W(z) \notin \mathbb{R}$, then $(x-z)^\beta$ is included in the monic factor $W_-(x)$, $W_0(x)$, $W_+(x)$, or $W_*(x)$, respectively. Next, we find some lower bounds of the degrees of these factors.

First, T is divisor of W_-, because W takes the negative value $-P^2(0)/4$ at each root of T. Hence, $\deg[W_-] \geqslant \deg[T] = m$. Second, $S\gcd[S, RS']$ is a divisor of W_0, because W vanishes at each root of S. Thus, $\deg[W_0] \geqslant m - n + l_0$, where l_0 denotes the degree of $\gcd[S, RS']$. Third,

$$\deg[W_+] \geqslant \#\left\{ (a,b) \subset \mathbb{R} : \begin{array}{c} W(a) = W(b) = 0 \\ R(x) > 0 \text{ for all } x \in (a,b) \\ S(x) \neq 0 \text{ for all } x \in (a,b) \end{array} \right\} = n - 1 + l_+.$$

To understand the above inequality, we realize that if (a,b) is an open bounded interval that satisfies the above three properties, then $W(x) = S^2(x)R(x) > 0$ for all $x \in (a,b)$, and $W'(x)$ vanishes at some point $c \in (a,b)$, by Rolle's theorem. Therefore, $\deg[W_+]$ is at least the number of such intervals. We combine these three lower bounds:

$$2m - 1 = \deg[W'] \geqslant \deg[W_-] + \deg[W_0] + \deg[W_+] \geqslant 2m - 1 + l_0 + l_+.$$

This implies that $l_0 = l_+ = 0$. Indeed, $W_- = T$, $W_0 = S$, $W_* = 1$, and $\gcd[S, RS'] = 1$.

Next, we prove the property concerning the number of roots of $P(x)$ and $P(x) - P(0)$. Let z be a root of the derivative P'. Since $W' = 2TP'$ and $W_- = T$, we deduce that $W(z)$ cannot be a negative number. This implies that if $P(z)$ is a real value between 0 and $P(0)$, then $P'(z) \neq 0$, since $W(z) = P(z)(P(z) - P(0)) < 0$. In particular, we deduce that the number of real roots (counted with multiplicity) of the polynomial $P(x) - \eta$ does not change when the constant $\eta \in \mathbb{R}$ moves from 0 to $P(0)$.

Finally, we prove that $S(x)$ has only real roots when $m \leqslant n+3$. Let us suppose that $z \notin \mathbb{R}$ is a root of $S(x)$. Then $\bar{z}$ is also a root of $S(x)$, so $(x - z)(x - \bar{z}) \mid S(x)$. Using the identity $S^2(x)R(x) = P(x)(P(x) - P(0))$, we get that $(x - z)^2(x - \bar{z})^2$ is either a divisor of $P(x)$ or a divisor of $P(x) - P(0)$, since $P(x)$ and $P(x) - P(0)$ have no common factors. But $P(x)$ and $P(x) - P(0)$ have the same number of real roots, so there exists another $w \notin \mathbb{R} \cup \{z, \bar{z}\}$ such that $(x - w)^2(x - \bar{w})^2$ is a divisor of $P(x)(P(x) - P(0))$. This implies that $S(x)$ has at least four different complex roots, and so $m - n = \deg[S] \geqslant 4$. $\square$

There are some theoretical arguments against the existence of non-real roots of polynomial $S(x)$, although we have not been able to give a proof.

Conjecture 2. *Let $R(x) = x\prod_{i=1}^{2n-1}(x - \gamma_i)$ with $0 < \gamma_{2n-1} < \cdots < \gamma_1$. If relation (9) holds for some polynomials $S(x)$, $P(x) \in \mathbb{R}[x]$ such that $P(0) \neq 0$, then $S(x)$ has only real roots.*

5. Generalized Cayley conditions in the minimal case

Let us consider the case of minimal elliptic periods; that is, $m = n$.

We begin with a technical lemma for describing how the roots of the polynomials of the form $P(x)(P(x) - P(0))$ with $P(0) \neq 0$ are ordered in the real line, assuming that all these roots—except the trivial one—are positive and have multiplicity at most 2.

Lemma 10. *Let $P(x) \in \mathbb{R}[x]$ be a monic polynomial of degree m such that $P(0) \neq 0$ and all the roots of $P(x)(P(x) - P(0))$—except a simple root at $x = 0$—are positive and have multiplicity at most 2. Let $\alpha_m \leqslant \cdots \leqslant \alpha_1$ be the positive roots of $P(x)$. Let $\beta_{m-1} \leqslant \cdots \leqslant \beta_1$ be the positive roots of $P(x) - P(0)$.*

If m is odd, then $\beta_{2l-1}, \beta_{2l} \in (\alpha_{2l}, \alpha_{2l-1})$ for all $l = 1, \ldots, (m-1)/2$; so

$$0 < \alpha_m \leqslant \alpha_{m-1} < \beta_{m-1} \leqslant \beta_{m-2} < \alpha_{m-2} \leqslant \alpha_{m-3} < \cdots < \alpha_3 \leqslant \alpha_2 < \beta_2 \leqslant \beta_1 < \alpha_1.$$

If m is even, then $\beta_1 > \alpha_1$ and $\beta_{2l}, \beta_{2l+1} \in (\alpha_{2l+1}, \alpha_{2l})$ for all $l = 1, \ldots, (m-2)/2$; so

$$0 < \alpha_m \leqslant \alpha_{m-1} < \beta_{m-1} \leqslant \beta_{m-2} < \alpha_{m-2} \leqslant \alpha_{m-3} < \cdots < \beta_3 \leqslant \beta_2 < \alpha_2 \leqslant \alpha_1 < \beta_1.$$

Proof. Let $\eta \in \mathbb{R}$. Using that the only critical points of $P(x)$ are non-degenerate local maxima or non-degenerate local minima, we deduce that the polynomial $P(x) - \eta$ has m real roots (counted with multiplicity) if and only if $\underline{\eta} \leqslant \eta \leqslant \overline{\eta}$, where

$$\overline{\eta} = \min\{P(\overline{x}) : \overline{x} \text{ is a non-degenerate local maximum of } P(x)\},$$
$$\underline{\eta} = \max\left\{P(\underline{x}) : \underline{x} \text{ is a non-degenerate local minimum of } P(x)\right\}.$$

Therefore, $\underline{\eta} \leqslant \min(0, P(0))$ and $\overline{\eta} \geqslant \max(0, P(0))$.

We begin with the case m odd, so $P(0) = (-1)^m\prod_{j=1}^m \alpha_j < 0$, $\underline{\eta} \leqslant P(0)$, and $\overline{\eta} \geqslant 0$. The roots of $P(x)$ and $P(x) - P(0)$ can be viewed as the abscissas of the intersections of the graph $\{y = P(x)\}$ with the horizontal lines $\{y = 0\}$ and $\{y = P(0)\}$, respectively. Double roots correspond to tangential intersections. We know that $P(\overline{x}) \geqslant \overline{\eta} \geqslant 0$ at the local maxima, and $P(\underline{x}) \leqslant \underline{\eta} \leqslant P(0)$ at the local minima. This means that the intersections of the graph $\{y = P(x)\}$ with the lines $\{y = 0\}$ and $\{y = P(0)\}$ have the following pattern from left to right. First, the graph crosses $\{y = P(0)\}$ at the abscissa $x = 0$; second, it intersects $\{y = 0\}$ at two abscissas α_m and α_{m-1}, which may coincide giving rise to a double root of $P(x)$; third, it intersects $\{y = P(0)\}$ at two abscissas β_{m-1} and β_{m-2}, which may coincide giving rise to a double root of $P(x) - P(0)$; fourth, it intersects $\{y = 0\}$ at two abscissas α_{m-2} and α_{m-3}, which may coincide giving rise to a double root of $P(x)$; and so on. The last intersection corresponds to the abscissa $x = \alpha_1$.

The proof for m even is similar. We skip the details. □

We emphasize that ellipsoidal parameters $0 < a_1 < \cdots < a_n$ and non-singular caustic parameters $\lambda_1 < \cdots < \lambda_{n-1}$ verify restrictions (2); then the parameters $0 < c_1 < \cdots < c_{2n-1}$ are defined in (3); next the inverse quantities $0 < \gamma_{2n-1} < \cdots < \gamma_1$ are given by $\gamma_i = 1/c_i$; and finally, $R(x) = x\prod_{i=1}^{2n-1}(x - \gamma_i)$. We will make use of these orderings and conventions, and this notation, throughout the paper without any explicit mention.

Corollary 11. *Let* $\{1, \ldots, 2n-1\} = J_n \cup K_n$ *be the decomposition defined by*

$$J_1 = \{1\}, \qquad J_2 = \{2, 3\}, \qquad J_n = J_{n-2} \cup \{2n-2, 2n-1\}, \qquad K_n = J_{n-1}.$$

If $P(x) \in \mathbb{R}[x]$ *is a monic polynomial of degree* n *such that* $P(0) \neq 0$ *and*

$$R(x) = P(x)(P(x) - P(0)),$$

then $P(x) = \prod_{j\in J_n}(x - \gamma_j) = P(0) + x\prod_{k\in K_n}(x - \gamma_k)$.

Proof. There exists a decomposition $\{1, \ldots, 2n-1\} = J' \cup K'$ such that $\#J' = n$, $\#K' = n - 1$, and $P(x) = \prod_{j\in J'}(x - \gamma_j) = P(0) + x\prod_{k\in K'}(x - \gamma_k)$. The polynomial $P(x)$ verifies the hypotheses stated in lemma 10, so the roots $\{\alpha_1, \ldots, \alpha_m\} = \{\gamma_j : j \in J'\}$ and $\{\beta_1, \ldots, \beta_{m-1}\} = \{\gamma_k : k \in K'\}$ obey the ordering described in that lemma. Therefore, $J' = J_n$ and $K' = K_n$. □

We now rewrite the generalized Cayley condition $\mathcal{C}(n, n)$ using the previous results. For brevity, we omit the dependence of the decomposition $\{1, \ldots, 2n-1\} = J \cup K$ on the index n. We note that $\#J = n$ and $\#K = n - 1$. The symbol e_l('a set of parameters') denotes the *elementary symmetric polynomial* of degree l in those parameters.

Proposition 12. $\mathcal{C}(n, n)$ *is equivalent to each of the following four conditions:*

(i) *If* $P(x) = \prod_{j\in J}(x - \gamma_j)$, *then* $P(x) - P(0) = x\prod_{k\in K}(x - \gamma_k)$.
(ii) $\prod_{j\in J}(\gamma_j - \gamma_k) = \prod_{j\in J}\gamma_j$, *for all* $k \in K$.
(iii) $\mathrm{e}_l(\{\gamma_j\}_{j\in J}) = \mathrm{e}_l(\{\gamma_k\}_{k\in K})$, *for all* $l = 1, \ldots, n-1$.
(iv) $\sum_{j\in J}\gamma_j^l = \sum_{k\in K}\gamma_k^l$, *for all* $l = 1, \ldots, n-1$.

Proof. We split the proof into four steps.

Step 1. $\mathcal{C}(n, n) \Leftrightarrow$ (i). Let us assume that $\mathcal{C}(n, n)$ holds. Then there exist a monic polynomial $P(x) \in \mathbb{R}[x]$ of degree n such that $P(0) \neq 0$ and

$$R(x) = P(x)(P(x) - P(0)).$$

Thus, condition (i) follows from corollary 11.

Reciprocally, if condition *(i)* holds, $P(x)(P(x) - P(0)) = x\prod_{i=1}^{2n-1}(x - \gamma_i)$, so $\mathcal{C}(n, n)$ holds.

Step 2. (i) $\Leftrightarrow$ (ii). If $P(x) = \prod_{j\in J}(x - \gamma_j)$ and $Q(x) = x\prod_{k\in K}(x - \gamma_k)$, then

$$Q(x) = P(x) - P(0) \Leftrightarrow P(\gamma_k) = P(0),\ \forall k \in K \Leftrightarrow \prod_{j\in J}(\gamma_j - \gamma_k) = \prod_{j\in J}\gamma_j,\ \forall k \in K.$$

Step 3. (i) ⇔ (iii). If $P(x) = \prod_{j\in J}(x-\gamma_j)$ and $Q(x) = x\prod_{k\in K}(x-\gamma_k)$, then

$$Q(x) = x^n + \sum_{l=1}^{n-1}(-1)^l \mathrm{e}_l(\{\gamma_k\}_{k\in K})x^{n-l}, \quad P(x) = x^n + \sum_{l=1}^{n-1}(-1)^l \mathrm{e}_l(\{\gamma_j\}_{j\in J})x^{n-l} + P(0).$$

Step 4. (iii) ⇔ (iv). It follows from Newton's identities connecting the elementary symmetric polynomials and the power sum symmetric polynomials; see [28]. □

The examples given in the introduction concerning billiard trajectories inside ellipsoids of $\mathbb{R}^3$ with elliptic period $m = 3$ follow from this proposition. Let us present some planar examples.

Example 1. *The quantities $\gamma_3 = 1$, $\gamma_2 = 2$, and $\gamma_1 = 3$ verify $\mathcal{C}(2,2)$, because $1+2=3$. This means that the billiard trajectories:*

- *inside the ellipse $Q : x^2 + 2y^2 = 1$ with caustic parameter $\lambda = 1/3$ or*
- *inside the ellipse $Q : x^2 + 3y^2 = 1$ with caustic parameter $\lambda = 1/2$*

are periodic with elliptic periodic $m = 2$ and caustic types $\varsigma = 0$ and $\varsigma = 1$, respectively.

Let us compare the system of homogeneous symmetric polynomial equations (5), which was obtained directly from the matrix formulation, with the system of homogeneous non-symmetric polynomial equations $\sum_{j\in J}\gamma_j^l = \sum_{k\in K}\gamma_k^l$, $1 \leqslant l \leqslant n-1$, obtained in the previous proposition. We are dealing with the case $m = n$, so the lth equation of the former system has degree $n+l$, whereas the lth equation of the new system has degree l. Besides this, the new system has a remarkably simple closed expression. This shows that the polynomial formulation simplifies the problem.

The beauty of the conditions regarding the elementary symmetric polynomials and the power sum symmetric polynomials given in proposition 12 has been the motivation for the introduction of the inverse quantities $\gamma_i = 1/c_i$. Nevertheless, we find it useful to state the following result in terms of the ellipsoidal parameters a_j, in order to answer some questions concerning the non-singular periodic billiard trajectories found in the first item of theorem 8.

Theorem 13. *There exist non-singular periodic billiard trajectories inside the ellipsoid (1) with elliptic period $m = n$ and caustic type*

$$\varsigma = \begin{cases} (1,1,3,3,\ldots,n-2,n-2) & \text{for } n \text{ odd} \\ (0,2,2,4,4,\ldots,n-2,n-2) & \text{for } n \text{ even} \end{cases} \tag{10}$$

if and only if all the roots of $t^n - \prod_{j=1}^n (t-a_j)$ are real and simple. These periodic billiard trajectories have the roots of $t^n - \prod_{j=1}^n (t-a_j)$ as caustic parameters, period $m_0 = 2n$, and even winding numbers $m_0,\ldots,m_{n-1}$. Indeed,

$$m_j = 2\widetilde{m}_j = 2(n-j), \qquad j = 0,\ldots,n-1, \tag{11}$$

provided Conjecture 1 on the strict decreasing ordering of winding numbers holds.

Proof. Let us assume that there exist non-singular periodic billiard trajectories with elliptic period n and caustic type (10). By definition of caustic type, the caustic parameters $\lambda_1 < \cdots < \lambda_{n-1}$ of such trajectories verify that:

- if n is odd, then $\lambda_{2l-1}, \lambda_{2l} \in (a_{2l-1}, a_{2l})$, for $l = 1,\ldots,(n-1)/2$;
- if n is even, then $\lambda_1 \in (0, a_1)$, and $\lambda_{2l}, \lambda_{2l+1} \in (a_{2l}, a_{2l+1})$, for $l = 1,\ldots,(n-2)/2$.

Hence, we can split the set $\{\gamma_i = 1/c_i : i = 1, \ldots, 2n-1\}$ as the disjoint union of the sets

$$\{\gamma_j : j \in J\} = \{1/a_1, \ldots, 1/a_n\}, \qquad \{\gamma_k : k \in K\} = \{1/\lambda_1, \ldots, 1/\lambda_{n-1}\},$$

where $\{1, \ldots, 2n-1\} = J \cup K$ is the decomposition described in corollary 11. Thus, we know from condition (ii) of proposition 12 that

$$\prod_{j=1}^{n} \left(\frac{1}{a_j} - \frac{1}{\lambda_k} \right) = \prod_{j=1}^{n} \frac{1}{a_j}, \qquad k = 1, \ldots, n-1.$$

This identity can be written as $\lambda_k^n = \prod_{j=1}^n (\lambda_k - a_j)$, for all $k = 1, \ldots, n-1$, which implies that the caustic parameters $\lambda_1, \ldots, \lambda_{n-1}$ are the roots of $t^n - \prod_{j=1}^n (t - a_j)$.

Reciprocally, let us assume that the roots of $q(t) = t^n - \prod_{j=1}^n (t - a_j)$ are real and simple. Let $\lambda_1 < \cdots < \lambda_{n-1}$ be these roots. None of them is zero, since $q(0) \neq 0$. Besides this, λ_k is a root of $q(t)$ if and only if $\beta_k := 1/\lambda_k \neq 0$ is a root of

$$Q(x) := \frac{(-1)^{n-1} x^n q(1/x)}{a_1 \cdots a_n} = P(x) - P(0),$$

where $P(x) = \prod_{j=1}^n (x - \alpha_j)$ with $\alpha_j = 1/a_j$. Therefore, the roots $\alpha_j = 1/a_j$ and $\beta_k = 1/\lambda_k$ are ordered as stated in lemma 10. The consequences are twofold. On the one hand, $\lambda_k \in (a_{\varsigma_k}, a_{\varsigma_k + 1})$, where $\varsigma = (\varsigma_1, \ldots, \varsigma_{n-1})$ is the caustic type given in (10). On the other hand, there exist non-singular billiard trajectories inside the ellipsoid Q sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$, because the existence conditions (2) hold. Thus, the trajectories sharing the caustics $Q_{\lambda_1}, \ldots, Q_{\lambda_{n-1}}$ are periodic with elliptic period n and caustic type ς, since the generalized Cayley condition $\mathcal{C}(n, n)$ holds; see proposition 12.

Next, we prove the claims on the (Cartesian) period and the winding numbers. The caustic parameters are located in the intervals delimited by the ellipsoidal parameters given at the beginning of the proof, which implies that

$$\{c_{2j}, c_{2j+1}\} \cap \{a_1, \ldots, a_n\} \neq \emptyset, \qquad j = 0, \ldots, n-1,$$

where $c_0 := 0 < c_1 < \cdots < c_{2n-1}$ are defined in (3). Thus, all winding numbers are even—see theorem 5—and so, by definition of elliptic period, $m_0 = 2m = 2n$.

Finally, let us assume that winding numbers are ordered as stated in conjecture 1, so $2 \leqslant m_{n-1} < \cdots < m_0 = 2n$ with $m_0, \ldots, m_{n-1}$ even. Then $m_j = 2\widetilde{m}_j = 2(n-j)$. □

Remark 1. If all the roots of $t^n - \prod_{j=1}^n (t - a_j)$ are real, but some of them are double, then we get singular periodic billiard trajectories. In that case, there are only two possible scenarios. Either n is odd and $\lambda_{2l-1} = \lambda_{2l}$ for some $l = 1, \ldots, (n-1)/2$; or n is even and $\lambda_{2l} = \lambda_{2l+1}$ for some $l = 1, \ldots, (n-2)/2$. In all of these cases, the singular periodic trajectories are formed by segments contained in some non-singular ruled confocal quadrics.

Remark 2. All of the periodic billiard trajectories mentioned in theorem 13 have caustic type (10). One may establish similar theorems for other caustic types. For instance, the versions EH1, H1H1, EH2 and H1H2 of theorem 13 in the spatial case will be listed in table 2.

6. Cayley conditions in the general case

Now that we have understood the minimal case $m = n$, we tackle the general case $m \geqslant n$.

Let us explain the fundamental question by means of an example. In section 3 we saw that condition $\mathcal{C}(3, 2)$ becomes a single homogeneous symmetric polynomial equation of degree 8 in the variables $\gamma_1, \gamma_2, \gamma_3$, with only two feasible solutions; namely, the ones given in (6).

Thus, it is natural to ask whether can we rewrite $\mathcal{C}(3, 2)$ as a set of two simpler conditions such that each one of them gives rise to one of the solutions given in (6).

By the way, we raise a question for any $m \geqslant n$. Can we rewrite $\mathcal{C}(m, n)$ as a set of 'simpler' conditions such that each one of them gives rise to just 'one' solution of $\mathcal{C}(m, n)$? We answer this question in the affirmative. Indeed, we parameterize these 'simpler' conditions with the elements of the set

$$\mathcal{T}(m, n) = \left\{(\tau_1, \ldots, \tau_n) \in \mathbb{Z}^n : \tau_1 + \cdots + \tau_n = m - n,\ \tau_1, \ldots, \tau_n \geqslant 0\right\}.$$

The cardinal of $\mathcal{T}(m, n)$ is the number of monomials of degree $m - n$ in n variables. Thus, $\#\mathcal{T}(m, n) = \binom{m-1}{n-1}$, which gives a precise estimate of the complexity of the Cayley condition $\mathcal{C}(m, n)$ when m grows. We will refer to the elements of $\mathcal{T}(m, n)$ as *signatures*. We set $\gamma_{2n} = 0$ in order to simplify some notation.

Definition 5. *Given any signature $\tau = (\tau_1, \ldots, \tau_n) \in \mathcal{T}(m, n)$, we say that condition $\mathcal{C}(m, n; \tau)$ holds if and only if there exist two monic polynomials $S(x), P(x) \in \mathbb{R}[x]$ such that* $\deg[S] = m - n$, $\deg[P] = m$, $P(0) \neq 0$, $S^2(x)R(x) = P(x)(P(x) - P(0))$, *and $S(x)$ has $m - n$ simple real roots $\delta_{m-n} < \cdots < \delta_1$ such that*

$$\#\big(\{\delta_1, \ldots, \delta_{m-n}\} \cap (\gamma_{2r}, \gamma_{2r-1})\big) = \tau_r, \qquad r = 1, \ldots, n. \tag{12}$$

Corollary 14. *If there exists $\tau \in \mathcal{T}(m, n)$ such that $\mathcal{C}(m, n; \tau)$ holds, then $\mathcal{C}(m, n)$ also holds. The reciprocal implication is true for $m \leqslant n + 3$ (or provided conjecture 2 holds).*

Proof. The first implication is obvious. For the reciprocal implication, we simply recall that $S(x)$ has only real roots when $m \leqslant n + 3$ and all its real roots are contained in $\{x \in \mathbb{R} : R(x) < 0\} = \bigcup_{r=1}^{n}(\gamma_{2r}, \gamma_{2r-1})$; see proposition 9. $\square$

Definition 6. *Given any signature $\tau \in \mathcal{T}(m, n)$, let $\{1, \ldots, 2n - 1\} = J_\tau \cup K_\tau$ and $\{1, \ldots, m - n\} = V_\tau \cup W_\tau$ be the decompositions determined as follows. If $\delta_{m-n} < \cdots < \delta_1$ is any ordered sequence verifying (12), then the elements of the multisets*

$$\begin{aligned} \{\alpha_1, \ldots, \alpha_m\} \quad &= \{\gamma_j : j \in J_\tau\} \cup \{\delta_v, \delta_v : v \in V_\tau\}, \\ \{\beta_1, \ldots, \beta_{m-1}\} &= \{\gamma_k : k \in K_\tau\} \cup \{\delta_w, \delta_w : w \in W_\tau\}, \end{aligned}$$

are ordered as in lemma 10.

Multisets are a generalization of sets in which members are allowed to appear more than once; see [29]. In our case, the numbers $\delta_1, \ldots, \delta_{m-n}$ appear twice.

These decompositions are well defined. That is, they only depend on the signature τ, since any ordered sequence $\delta_{m-n} < \cdots < \delta_1$ verifying (12) gives rise to the same decomposition. The decomposition $\{1, \ldots, 2n - 1\} = J_n \cup K_n$ given in corollary 11 corresponds to the trivial signature $\tau = (0, \ldots, 0) \in \mathcal{T}(n, n)$.

Next, we generalize corollary 11 and proposition 12 to the case $m \geqslant n$.

Corollary 15. *Let $\delta_{m-n} < \cdots < \delta_1$ be an ordered sequence verifying (12) for some signature $\tau \in \mathcal{T}(m, n)$. If $P(x)$ is a monic polynomial of degree m such that $P(0) \neq 0$ and*

$$\prod_{u=1}^{m-n}(x - \delta_u)^2 \cdot R(x) = P(x)(P(x) - P(0)),$$

then $P(x) = \prod_{j\in J_\tau}(x - \gamma_j)\prod_{v\in V_\tau}(x - \delta_v)^2 = P(0) + x\prod_{k\in K_\tau}(x - \gamma_k)\prod_{w\in W_\tau}(x - \delta_w)^2$.

Proof. There exist two decompositions $\{1,\ldots,2n-1\} = J'\cup K'$ and $\{1,\ldots,m-n\} = V'\cup W'$ such that $P(x) = \prod_{j\in J'}(x-\gamma_j)\prod_{v\in V'}(x-\delta_v)^2 = P(0) + x\prod_{k\in K'}(x-\gamma_k)\prod_{w\in W'}(x-\delta_w)^2$. The polynomial $P(x)$ verifies the hypotheses stated in lemma 10, so the roots

$$\begin{aligned}\{\alpha_1,\ldots,\alpha_m\} &= \{\gamma_j : j\in J'\}\cup\{\delta_v,\delta_v : v\in V'\}\\ \{\beta_1,\ldots,\beta_{m-1}\} &= \{\gamma_k : k\in K'\}\cup\{\delta_w,\delta_w : w\in W'\}\end{aligned}$$

obey the ordering described in that lemma. Hence, $J' = J_\tau$, $K' = K_\tau$, $V' = V_\tau$, and $W' = W_\tau$. □

Proposition 16. *Condition $\mathcal{C}(m,n;\tau)$ holds if and only if there exists a sequence $\delta_{m-n} < \cdots < \delta_1$ verifying (12) such that the following three equivalent properties hold:*

(i) if $P(x) = \prod_{j\in J_\tau}(x-\gamma_j)\prod_{v\in V_\tau}(x-\delta_v)^2$, then $P(x) - P(0) = x\prod_{k\in K_\tau}(x-\gamma_k)\prod_{w\in W_\tau}(x-\delta_w)^2$;

(ii) $\mathrm{e}_l\big(\{\gamma_j\}_{j\in J_\tau}\cup\{\delta_v,\delta_v\}_{v\in V_\tau}\big) = \mathrm{e}_l\big(\{\gamma_k\}_{k\in K_\tau}\cup\{\delta_w,\delta_w\}_{w\in W_\tau}\big)$, for all $l = 1,\ldots,m-1$;

(iii) $\sum_{j\in J_\tau}\gamma_j^l + 2\sum_{v\in V_\tau}\delta_v^l = \sum_{k\in K_\tau}\gamma_k^l + 2\sum_{w\in W_\tau}\delta_w^l$, for all $l = 1,\ldots,m-1$.

Proof. We simply repeat the steps of the proof of proposition 12, but using corollary 15 instead of corollary 11. □

Example 2. *The quantities $\gamma_3 = 1$, $\gamma_2 = 4$, and $\gamma_1 = 9$ verify condition $\mathcal{C}(3,2;\tau)$ with $\tau = (1,0)$, because $1+4+9 = 2\cdot 7$, $1^2+4^2+9^2 = 2\cdot 7^2$, and $7\in(4,9)$. Hence, the billiard trajectories:*

- *inside the ellipse $Q : x^2+4y^2 = 1$ with caustic parameter $\lambda = 1/9$ or*
- *inside the ellipse $Q : x^2+9y^2 = 1$ with caustic parameter $\lambda = 1/4$*

are periodic with elliptic periodic $m = 3$. Their caustic types are $\varsigma = 0$ and $\varsigma = 1$, respectively.

All conditions $\mathcal{C}(m,n;\tau)$, $\tau\in\mathcal{T}(m,n)$, give rise to non-singular periodic billiard trajectories with elliptic period m, so we wondered what is the dynamical meaning of the signature τ. We believe that there exists a one-to-one correspondence between the elliptic winding numbers $\tilde{m}_0,\ldots,\tilde{m}_{n-1}$—see definition 3—and the signature $\tau = (\tau_1,\ldots,\tau_n)$.

Conjecture 3. *Set $\tilde{m}_n = 0$. Then $\tilde{m}_j = \tilde{m}_{j+1} + \tau_{j+1} + 1$ for all $j = 0,\ldots,n-1$.*

This conjecture follows from the interpretation of $\mathcal{C}(m,n;\tau)$ as a singular limit of $\mathcal{C}(m,m)$ when $m-n$ couples of simple roots collide, so they become double roots. Unfortunately, we have not been able to transform this argument into a rigorous proof, although all our analytical and numerical computations agree with the conjecture.

To end this section, we stress that if conjectures 2 and 3 hold, then the elliptic winding numbers $\tilde{m}_0,\ldots,\tilde{m}_{n-1}$ of any non-singular periodic billiard trajectory verify the above-mentioned relations for some signature $\tau = (\tau_1,\ldots,\tau_n)$ with non-negative entries, so the sequence $\tilde{m}_0,\ldots,\tilde{m}_{n-1}$ strictly decreases, and conjecture 1 holds.

7. The planar case

We adapt the previous setting of billiards inside ellipsoids of $\mathbb{R}^n$ to the planar case $n = 2$. To follow traditional conventions in the literature, we write the ellipse as

$$Q = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a} + \frac{y^2}{b} = 1 \right\}, \qquad a > b > 0. \tag{13}$$

Any non-singular billiard trajectory inside Q is tangent to one confocal caustic

$$Q_\lambda = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} = 1 \right\},$$

where $\lambda \in \Lambda = E \cup H$, with $E = (0, b)$ and $H = (b, a)$.

The names of the connected components of Λ come from the fact that Q_λ is a confocal ellipse for $\lambda \in E$ and a confocal hyperbola for $\lambda \in H$. The singular cases $\lambda = b$ and $\lambda = a$ correspond to the x-axis and y-axis, respectively. We say that the *caustic type* of a billiard trajectory is E or H when its caustic is an ellipse or a hyperbola (compare with definition 1). We also distinguish between E-caustics and H-caustics.

We recall some concepts related to periodic trajectories of billiards inside ellipses. These results can be found, for instance, in [9, 26]. To begin with, we introduce the function $\rho : \Lambda \to \mathbb{R}$ given by the quotient of elliptic integrals

$$\rho(\lambda) = \rho(\lambda; b, a) := \frac{\displaystyle\int_0^{\min(b,\lambda)} \frac{\mathrm{d}t}{\sqrt{(\lambda - t)(b - t)(a - t)}}}{\displaystyle 2\int_{\max(b,\lambda)}^{a} \frac{\mathrm{d}t}{\sqrt{(\lambda - t)(b - t)(a - t)}}}. \tag{14}$$

It is called the *rotation number* and characterizes the caustic parameters that give rise to periodic trajectories. To be precise, the billiard trajectories with caustic Q_λ are periodic if and only if

$$\rho(\lambda) = m_1/2m_0 \in \mathbb{Q}$$

for some integers $2 \leqslant m_1 < m_0$, which are the *winding numbers*. On the one hand, m_0 is the period. On the other hand, m_1 is twice the number of turns around the ellipse Q_λ for E-caustics, and the number of crossings of the y-axis for H-caustics. Thus, m_1 is always even. Besides this, all periodic trajectories with H-caustics have even period. (Compare with theorem 5.)

Proposition 17. *The winding numbers $2 \leqslant m_1 < m_0$, rotation number $\rho = m_1/2m_0$, signature $\tau = (\tau_1, \tau_2) \in \mathcal{T}(m, 2)$, caustic type (E or H), and caustic parameter λ of all non-singular periodic billiard trajectories inside the ellipse (13) with elliptic period $m \in \{2, 3\}$ are listed in table 1. The ellipses for which such trajectories take place are also listed.*

Proof. We split the proof into four steps.

Step 1. To find the solutions of $\mathcal{C}(m, 2)$ in terms of the inverse quantities γ_i. First, we saw in proposition 12 that $\mathcal{C}(2, 2)$ holds if and only if $\gamma_1 = \gamma_2 + \gamma_3$.

Next, we focus on the case $m = 3$. We note that $\mathcal{C}(3, 2)$ holds if and only if $\mathcal{C}(3, 2; \tau)$ holds for some $\tau = (\tau_1, \tau_2) \in \mathbb{Z}^2$ such that $\tau_1 + \tau_2 = 1$ and $\tau_1, \tau_2 \geqslant 0$; see corollary 14.

Let us begin with the signature $\tau = (1, 0)$. After a straightforward check, we get that the decompositions presented in definition 6 are $J_\tau = \{1, 2, 3\}$, $K_\tau = V_\tau = \emptyset$, and $W_\tau = \{1\}$. Thus, $\mathcal{C}(3, 2; \tau)$ holds if and only if there exists some $\delta_1 \in (\gamma_2, \gamma_1)$ such that

$$P(x) = (x - \gamma_1)(x - \gamma_2)(x - \gamma_3) = P(0) + x(x - \delta_1)^2,$$

Table 1. Algebraic formulas for the caustic parameter corresponding to non-singular periodic billiard trajectories with elliptic period $m \in \{2, 3\}$ in the planar case.

m	m_0	m_1	ρ	τ	Type	Ellipses	Caustic parameter
2	4	2	1/4	(0, 0)	E	Any	$\dfrac{ab}{a+b}$
2	4	2	1/4	(0, 0)	H	$2b < a$	$\dfrac{ab}{a-b}$
3	6	2	1/6	(1, 0)	E	Any	$\dfrac{ab}{a+b+2\sqrt{ab}}$
3	6	2	1/6	(1, 0)	H	$4b < a$	$\dfrac{ab}{a+b-2\sqrt{ab}}$
3	3	2	1/3	(0, 1)	E	Any	$\dfrac{3ab}{a+b+2\sqrt{a^2-ab+b^2}}$
3	6	4	1/3	(0, 1)	H	$4b < 3a$	$\dfrac{ab}{2\sqrt{a^2-ab}+b-a}$

or, equivalently, if and only if the discriminant of the polynomial

$$Q(x) = \frac{P(x) - P(0)}{x} = x^2 - \mathrm{e}_1(\gamma_1, \gamma_2, \gamma_3)x + \mathrm{e}_2(\gamma_1, \gamma_2, \gamma_3)$$

is equal to zero. The discriminant of $Q(x)$ is

$$\Delta = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 - 2\gamma_1\gamma_2 - 2\gamma_1\gamma_3 - 2\gamma_2\gamma_3.$$

We already saw in section 3 that the only feasible solution of $\Delta = 0$ is $\sqrt{\gamma_1} = \sqrt{\gamma_2} + \sqrt{\gamma_3}$.

When $\tau = (0, 1)$, the decompositions are $J_\tau = \{1\}$, $K_\tau = \{2, 3\}$, $V_\tau = \{1\}$, and $W_\tau = \emptyset$. Thus, $\mathcal{C}(3, 2; \tau)$ holds if and only if there exists some $\delta_1 \in (0, \gamma_3)$ such that

$$\gamma_1 + 2\delta_1 = \gamma_2 + \gamma_3, \qquad \gamma_1^2 + 2\delta_1^2 = \gamma_2^2 + \gamma_3^2,$$

or, equivalently, if and only if

$$\begin{aligned} 3\gamma_1^2 - 2(\gamma_2 + \gamma_3)\gamma_1 - (\gamma_2 - \gamma_3)^2 &= (\gamma_2 + \gamma_3 - \gamma_1)^2 - 2(\gamma_2^2 + \gamma_3^2 - \gamma_1^2) \\ &= (2\delta_1)^2 - 4\delta_1^2 = 0. \end{aligned}$$

And we already saw in section 3 that the only feasible solution of the above equation is

$$3\gamma_1 = \gamma_2 + \gamma_3 + 2\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_2\gamma_3}.$$

Step 2. To express the above solutions in terms of a, b and λ. If the caustic type is E, then $\lambda \in (0, b)$, $\gamma_1 = 1/\lambda$, $\gamma_2 = 1/b$ and $\gamma_3 = 1/a$. Thus,

$$\gamma_1 = \gamma_2 + \gamma_3 \Leftrightarrow \lambda = \frac{ab}{a+b},$$

$$\sqrt{\gamma_1} = \sqrt{\gamma_2} + \sqrt{\gamma_3} \Leftrightarrow \lambda = \frac{ab}{a+b+2\sqrt{ab}}$$

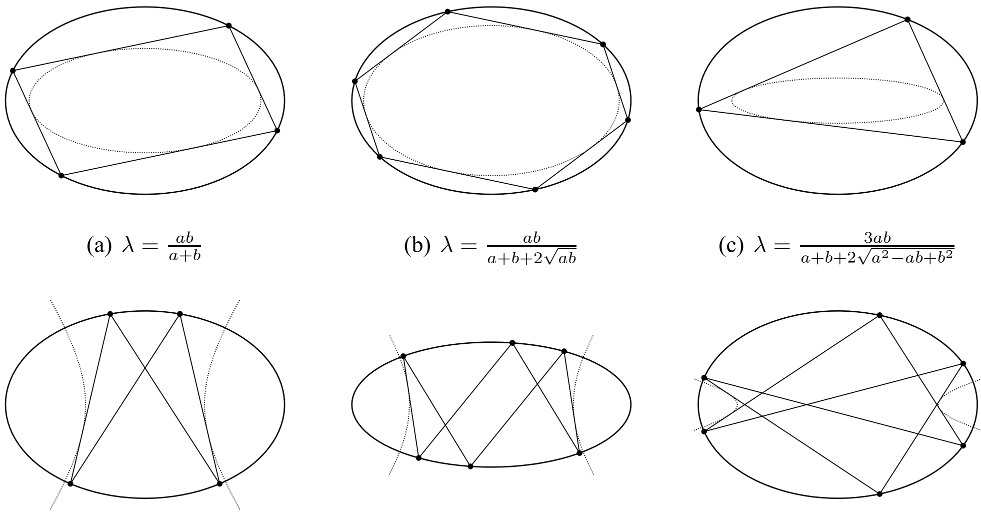
$$3\gamma_1 = \gamma_2 + \gamma_3 + 2\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_2\gamma_3} \Leftrightarrow \lambda = \frac{3ab}{a+b+2\sqrt{a^2-ab+b^2}}.$$

If the caustic type is H, then $\lambda \in (b, a)$, $\gamma_1 = 1/b$, $\gamma_2 = 1/\lambda$ and $\gamma_3 = 1/a$. Thus,

$$\gamma_1 = \gamma_2 + \gamma_3 \Leftrightarrow \lambda = \frac{ab}{a-b},$$

$$\sqrt{\gamma_1} = \sqrt{\gamma_2} + \sqrt{\gamma_3} \Leftrightarrow \lambda = \frac{ab}{a+b-2\sqrt{ab}}$$

$$3\gamma_1 = \gamma_2 + \gamma_3 + 2\sqrt{\gamma_2^2 + \gamma_3^2 - \gamma_2\gamma_3} \Leftrightarrow \lambda = \frac{ab}{2\sqrt{a^2-ab}+b-a}.$$



(a) $\lambda = \frac{ab}{a+b}$ (b) $\lambda = \frac{ab}{a+b+2\sqrt{ab}}$ (c) $\lambda = \frac{3ab}{a+b+2\sqrt{a^2-ab+b^2}}$

(d) $\lambda = \frac{ab}{a-b}$ (e) $\lambda = \frac{ab}{a+b-2\sqrt{ab}}$ (f) $\lambda = \frac{ab}{2\sqrt{a^2-ab+b^2}+b-a}$

Figure 1. Some periodic trajectories corresponding to the caustic parameters given in table 1. The ellipse for $\lambda = \frac{ab}{a+b-2\sqrt{ab}}$ is flatter, because it must satisfy the condition $4b < a$.

Step 3. To determine the ellipses for which such periodic billiard trajectories take place. We ask whether the caustic parameters found above belong to the interval $(0, b)$ for E-caustics, and to the interval (b, a) for H-caustics. The caustic type E does not give any restriction, because

$$0 < b < a \qquad \text{and} \qquad \lambda \in \left\{ \frac{ab}{a+b}, \frac{ab}{a+b+2\sqrt{ab}}, \frac{3ab}{a+b+2\sqrt{a^2-ab+b^2}} \right\}$$
$$\Rightarrow \lambda \in (0, b).$$

In contrast, the caustic type H gives rise to some restrictions. Namely,

$$b < \frac{ab}{a-b} < a \Leftrightarrow 2b < a,$$
$$b < \frac{ab}{a+b-2\sqrt{ab}} < a \Leftrightarrow 4b < a,$$
$$b < \frac{ab}{2\sqrt{a^2-ab}+b-a} < a \Leftrightarrow 4b < 3a.$$

Step 4. To find the winding numbers and the rotation number. The winding numbers $2 \leqslant m_1 < m_0$ and the rotation number $\rho(\lambda) = m_1/2m_0$ are obtained from geometric arguments. To be precise, we draw in figure 1 a billiard trajectory tangent to Q_λ for each of the caustic parameters listed in table 1. Then we recall that m_0 is the period and m_1 is twice the number of turns around the ellipse Q_λ for E-caustics, and the number of crossings of the y-axis for H-caustics. □

Halphen [15, p 377] obtained several algebraic equations related to the formulas given in table 1. Halphen looks for all caustics associated with some fixed period, whereas we look for the unique caustic associated with some fixed winding numbers and caustic type. Thus, Halphen's equations have many different solutions, whereas each one of our formulas gives rise to a single caustic.

Table 2. Algebraic formulas for the caustic parameters corresponding to the non-singular periodic trajectories with elliptic period $m = 3$ in the spatial case.

Type	Ellipsoids	Caustic parameters
EH1	$c < \dfrac{ab}{a+b+\sqrt{ab}}$	$c^3 = (c-\lambda_1)(b-c)(a-c)$ $1/\lambda_2 + 1/c = 1/a + 1/b + 1/\lambda_1$
H1H1	$c < \dfrac{ab}{a+b+2\sqrt{ab}}$	Roots of $t^3 - (t-a)(t-b)(t-c)$
EH2	$\begin{cases} c < \dfrac{a-2b}{2a-3b}a \\ 2b < a \end{cases}$	Roots of $(a-b)(a-c)t^2 + (bc - a(b+c))at + a^2bc$
H1H2	$\begin{cases} c < \dfrac{a-2b}{(a-b)^2}ab \\ b > \dfrac{ac}{a+c-\sqrt{ac}} \end{cases}$	$b^3 = (b-c)(\lambda_2 - b)(a-b)$ $1/\lambda_1 + 1/b = 1/a + 1/c + 1/\lambda_2$

It is interesting to realize that the results in table 1 agree with conjecture 3.

In the planar case $n = 2$, the caustic type (10) is $\varsigma = 0$ or, equivalently, E. Hence, the planar version of theorem 13 is shown in the first row of table 1, because $\lambda = ab/(a+b)$ is the root of $t^2 - (t-a)(t-b)$. This naive observation was the germ of this paper.

8. The spatial case

In order to study the spatial case $n = 3$, we consider the triaxial ellipsoid

$$Q = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1 \right\}, \qquad a > b > c > 0. \tag{15}$$

Any non-singular billiard trajectory inside Q is tangent to two distinct non-singular caustics Q_{λ_1} and Q_{λ_2}, with $\lambda_1 < \lambda_2$, of the confocal family

$$Q_\lambda = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1 \right\}. \tag{16}$$

The caustic Q_λ is an ellipsoid for $\lambda \in (0, c)$, a hyperboloid of one sheet when $\lambda \in (c, b)$, and a hyperboloid of two sheets if $\lambda \in (b, a)$. Not all combinations of non-singular caustics can take place: only the four caustic types EH1, H1H1, EH2 and H1H2.

Proposition 18. *The caustic type and caustic parameters of all non-singular periodic billiard trajectories inside the triaxial ellipsoid (15) with elliptic period $m = 3$ are listed in table 2. The ellipsoids for which such trajectories take place are also listed.*

Proof. If a, b, and c are the ellipsoidal parameters, and λ_1 and λ_2 are the caustic parameters, we set $\{c_1, c_2, c_3, c_4, c_5\} = \{a, b, c, \lambda_1, \lambda_2\}$, where $0 < c_1 < c_2 < c_3 < c_4 < c_5$. We also set $\gamma_i = 1/c_i$. Let $\{1, 2, 3, 4, 5\} = J \cup K$, with $J = \{1, 4, 5\}$ and $K = \{2, 3\}$, be the decomposition defined in corollary 11 when $n = 3$. From proposition 12 we know that

$$\begin{aligned} \mathcal{C}(3,3) &\Leftrightarrow \gamma_2 + \gamma_3 = \gamma_1 + \gamma_4 + \gamma_5 \text{ and } \gamma_2^2 + \gamma_3^2 = \gamma_1^2 + \gamma_4^2 + \gamma_5^2 \\ &\Leftrightarrow (\gamma_1 - \gamma_k)(\gamma_4 - \gamma_k)(\gamma_5 - \gamma_k) = \gamma_1\gamma_4\gamma_5, \text{ for } k = 2, 3 \\ &\Leftrightarrow c_k^3 = (c_k - c_1)(c_4 - c_k)(c_5 - c_k), \quad \text{for } k = 2, 3. \end{aligned}$$

In the rest of the proof, we study each caustic type separately.

Caustic type EH1. In this case $0 < \lambda_1 < c < \lambda_2 < b < a$, so

$$c_1 = \lambda_1, \qquad c_2 = c, \qquad c_3 = \lambda_2, \qquad c_4 = b, \qquad c_5 = a.$$

Thus the formula for λ_1 follows from the relation $c_2^3 = (c_2 - c_1)(c_4 - c_2)(c_5 - c_2)$, whereas the formula for λ_2 follows from the relation $\gamma_2 + \gamma_3 = \gamma_1 + \gamma_4 + \gamma_5$. Next, we look for ellipsoidal parameters such that the caustic parameters computed using these two formulas are placed in the right intervals: $\lambda_1 \in (0, c)$ and $\lambda_2 \in (c, b)$.

To begin with, we note that $\lambda_1 < c$, since $(c - \lambda_1)(b - c)(a - c) = c^3 > 0$. Besides this,

$$\lambda_1 = \frac{ab - (a + b)c}{(b - c)(a - c)} c > 0 \Leftrightarrow c < \frac{ab}{a + b}.$$

On the other hand, if $\lambda_1 \in (0, c)$, then

$$1/\lambda_2 = 1/a + 1/b + (1/\lambda_1 - 1/c) > 1/a + 1/b > 1/b,$$

so $\lambda_2 < b$. Finally,

$$\lambda_2 > c \Leftrightarrow \frac{1}{c} + \frac{c}{ab - (a + b)c} = \frac{1}{\lambda_1} = \frac{1}{\lambda_2} + \frac{1}{c} - \frac{1}{a} - \frac{1}{b} < \frac{2}{c} - \frac{1}{a} - \frac{1}{b}$$
$$\Leftrightarrow c < \frac{ab}{a + b + \sqrt{ab}}.$$

Therefore, $\lambda_1 \in (0, c)$ and $\lambda_2 \in (c, b)$ if and only if $c < ab/(a + b + \sqrt{ab})$.

Caustic type H1H1. If $n = 3$, then the caustic type (10) is $\varsigma = (1, 1)$ or, equivalently, H1H1. Hence, the study for the caustic type H1H1 was already carried out in theorem 13. It suffices to note that the polynomial

$$t^3 - (t - a)(t - b)(t - c) = (a + b + c)t^2 - (ab + ac + bc)t + abc$$

has two real simple roots if and only if its discriminant

$$\Delta = (ab + ac + bc)^2 - 4abc(a + b + c) = (a - b)^2 c^2 - 2ab(a + b)c + a^2 b^2$$

is positive. This discriminant is a second-degree polynomial in c whose roots are

$$c_\pm = \frac{ab(a + b) \pm 2ab\sqrt{ab}}{(a - b)^2} = \frac{ab}{a + b \mp 2\sqrt{ab}}.$$

We note that $0 < c_- < b < c_+$. Thus, using that $0 < c < b < a$, we get $\Delta > 0 \Leftrightarrow c < c_-$.

Caustic type EH2. In this case $0 < \lambda_1 < c < b < \lambda_2 < a$, so

$$c_1 = \lambda_1, \qquad c_2 = c, \qquad c_3 = b, \qquad c_4 = \lambda_2, \qquad c_5 = a.$$

Using relations $\gamma_2^l + \gamma_3^l = \gamma_1^l + \gamma_4^l + \gamma_5^l$, with $l = 1, 2$, we know that

$$s_l := \frac{1}{\lambda_1^l} + \frac{1}{\lambda_2^l} = \frac{1}{c^l} + \frac{1}{b^l} - \frac{1}{a^l}, \qquad l = 1, 2.$$

Hence, $1/\lambda_1$ and $1/\lambda_2$ are the roots of the polynomial

$$(x - 1/\lambda_1)(x - 1/\lambda_2) = x^2 - s_1 x + \frac{s_1^2 - s_2}{2} = x^2 + \frac{bc - a(b + c)}{abc} x + \frac{(a - b)(a - c)}{a^2 bc}.$$

Thus, using the change of variables $t = 1/x$, we get that λ_1 and λ_2 are the roots of

$$Q(t) = (a - b)(a - c)t^2 + (bc - a(b + c))at + a^2 bc.$$

We look for ellipsoidal parameters such that $Q(t)$ has a root in $(0, c)$ and a root in (b, a). The root in $(0, c)$ always exists, since $Q(0) = a^2 bc > 0$ and $Q(c) = -c^3(a - b) < 0$. Besides this, $Q(b) = -b^3(a-c) < 0$ and $\lim_{t\to+\infty} Q(t) = +\infty$, so $Q(t)$ has a root in (b, a) if and only if

$$Q(a) = a^2\left(a^2 - 2a(b+c) + 3bc\right) > 0,$$

or, equivalently, if and only if $c < (a-2b)a/(2a-3b)$ and $2b < a$. We have used that $0 < c < b < a$ in the last equivalence.

Caustic type H1H2. In this case $0 < c < \lambda_1 < b < \lambda_2 < a$, so

$$c_1 = c, \qquad c_2 = \lambda_1, \qquad c_3 = b, \qquad c_4 = \lambda_2, \qquad c_5 = a.$$

Thus the formula for λ_2 follows from the relation $c_2^3 = (c_2 - c_1)(c_4 - c_2)(c_5 - c_2)$, whereas the formula for λ_1 follows from the relation $\gamma_2 + \gamma_3 = \gamma_1 + \gamma_4 + \gamma_5$. Next, we look for conditions on the ellipsoidal parameters such that the caustic parameters computed from the previous formulas are placed in the right intervals: $\lambda_1 \in (c, b)$ and $\lambda_2 \in (b, a)$.

To begin with, we note that $\lambda_2 > b$, because $(b-c)(\lambda_2 - b)(a-b) = b^3 > 0$. Besides this,

$$\frac{(a+c)b - ac}{(a-b)(b-c)} b = \lambda_2 < a \Leftrightarrow c < \frac{a-2b}{(a-b)^2} ab.$$

On the other hand, using that $0 < c < b < a$, we get that

$$\begin{aligned} c < \lambda_1 < b &\Leftrightarrow \frac{2}{b} - \frac{1}{a} - \frac{1}{c} < \frac{1}{\lambda_2} = \frac{1}{\lambda_1} + \frac{1}{b} - \frac{1}{a} - \frac{1}{c} < \frac{1}{b} - \frac{1}{a} \\ &\Leftrightarrow \frac{2}{b} - \frac{1}{a} - \frac{1}{c} < \frac{1}{b} - \frac{b}{(a+c)b - ac} < \frac{1}{b} - \frac{1}{a} \\ &\Leftrightarrow b > \frac{ac}{a + c - \sqrt{ac}}. \end{aligned}$$

Thus, $\lambda_1 \in (0, c)$ and $\lambda_2 \in (b, a)$ if and only if $c < (a-2b)ab/(a-b)^2$ and $b > ac/(a + c - \sqrt{ac})$. □

Let us look for the winding numbers of the trajectories described in the previous proposition. The winding numbers m_0, m_1 and m_2 describe how the periodic billiard trajectories fold in $\mathbb{R}^3$. The following results can be found in [26, table 1]. First, m_0 is the period. Second, m_1 is the number of xy-crossings and m_2 is twice the number of turns around the z-axis for EH1-caustics; m_1 is twice the number of turns around the x-axis and m_2 is the number of yz-crossings for EH2-caustics; m_1 is the number of tangential touches with each hyperboloid of one sheet caustic and m_2 is twice the number of turns around the z-axis for H1H1-caustics; m_1 is the number of xz-crossings and m_2 is the number of yz-crossings for H1H2-caustics. Besides this, all periodic trajectories with H1H1-caustics or H1H2-caustics have even period. Several periodic billiard trajectories with elliptic period $m = 3$ were depicted in [27, tables XV and XVII]. We conclude by direct inspection of those pictures that the non-singular periodic billiard trajectories inside a triaxial ellipsoid with elliptic period $m = 3$ have winding numbers

$$m_2 = 2, \qquad m_1 = 4, \qquad m_0 = 6.$$

This agrees with the formulas (11) given in theorem 13. We emphasize that those formulas were not rigorously proved, because their 'proof' was based on conjecture 1.

Next, we establish the algebraic formulas for the caustic parameters of other non-singular periodic billiard trajectories. We begin with a technical lemma concerning fourth-degree polynomials.

Lemma 19. *Let* $Q(x) = (x - \alpha_-)(x - \beta_-)(x - \beta_+)(x - \alpha_+)$ *for some* $\alpha_- < \beta_- < \beta_+ < \alpha_+$. *Let* $\nu_- \in (\alpha_-, \beta_-)$, $\nu \in (\beta_-, \beta_+)$, *and* $\nu_+ \in (\beta_+, \alpha_+)$ *be the three roots of* $Q'(x)$. *Then*

$$Q(\nu_+) < Q(\nu_-) \Leftrightarrow \alpha_- + \alpha_+ > \beta_- + \beta_+.$$

Proof. If we set $\eta = (\beta_+ + \beta_-)/2$ and $\xi = (\beta_+ - \beta_-)/2$, then

$$Q(\eta + s) - Q(\eta - s) = 2(s^2 - \xi^2)(\beta_- + \beta_+ - \alpha_- - \alpha_+)s, \qquad \forall s \in \mathbb{R}.$$

On the one hand, if $\alpha_- + \alpha_+ > \beta_- + \beta_+$, then $Q(\eta + s) < Q(\eta - s)$ for all $s > \xi$, which implies that $Q(\nu_+) < Q(\nu_-)$. On the other hand, if $\alpha_- + \alpha_+ < \beta_- + \beta_+$, then $Q(\eta + s) > Q(\eta - s)$ for all $s > \xi$, which implies that $Q(\nu_+) > Q(\nu_-)$. Finally, if $\alpha_- + \alpha_+ = \beta_- + \beta_+$, then $Q(\eta + s) = Q(\eta - s)$ for all $s \in \mathbb{R}$, which implies that $Q(\nu_+) = Q(\nu_-)$. □

We can now answer some questions concerning the non-singular periodic billiard trajectories found in the second item of theorem 8, although the study is restricted to the spatial case.

Proposition 20. *There exist periodic billiard trajectories inside the triaxial ellipsoid (15) with elliptic period* $m = 4$, *signature* $\tau = (0, 0, 1)$, *and caustic type H1H1 if and only if*

$$c < ab/(a + b).$$

Besides this, the caustic parameters λ_1 *and* λ_2 *of such periodic billiard trajectories are the roots of the quadratic polynomial* $(s_1^2 - s_2)t^2/2 - s_1 t + 1$, *where*

$$s_l = 1/a^l + 1/b^l + 1/c^l - 2/d^l, \qquad l = 1, 2, \tag{17}$$

and d *is the only root of the cubic polynomial* $t^3 - 2(a + b + c)t^2 + 3(ab + ac + bc)t - 4abc$ *in the interval* $(a, +\infty)$.

Proof. If $\tau = (0, 0, 1)$, the decompositions presented in definition 6 are $J_\tau = \{2, 3\}$, $K_\tau = \{1, 4, 5\}$, $V_\tau = \{1\}$ and $W_\tau = \emptyset$. Thus, $\mathcal{C}(4, 3; \tau)$ holds if and only if there exists some $\delta_1 \in (0, \gamma_5)$ such that the following two equivalent properties hold:

(i) $P(x) = (x - \delta_1)^2(x - \gamma_2)(x - \gamma_3) \Rightarrow Q(x) := x(x - \gamma_1)(x - \gamma_4)(x - \gamma_5) = P(x) - P(0)$.
(ii) $\gamma_2^l + \gamma_3^l + 2\delta_1^l = \gamma_1^l + \gamma_4^l + \gamma_5^l$, for $l = 1, 2, 3$.

If the caustic type is H1H1, then $0 < c < \lambda_1 < \lambda_2 < b < a$, so

$$\gamma_1 = 1/c, \qquad \gamma_2 = 1/\lambda_1, \qquad \gamma_3 = 1/\lambda_2, \qquad \gamma_4 = 1/b, \qquad \gamma_5 = 1/a.$$

Let $\mathrm{e}_l = \mathrm{e}_l(\gamma_1, \gamma_4, \gamma_5)$ for $l = 1, 2, 3$. We set $d = 1/\delta_1 > a$. Then $Q(x) = x^4 - \mathrm{e}_1 x^3 + \mathrm{e}_2 x^2 - \mathrm{e}_1 x$ and δ_1 is a root of $Q'(x) = 4x^3 - 3\mathrm{e}_1 x^2 + 2\mathrm{e}_2 x - \mathrm{e}_3$. Hence, d is a root of the cubic polynomial

$$q(t) = -abct^3 Q'(1/t) = t^3 - 2(a + b + c)t^2 + 3(ab + ac + bc)t - 4abc.$$

We note that $q(0) = -4abc < 0$, $q(b) = -b(b - a)(b - c) > 0$, $q(a) = -a(a - b)(a - c) < 0$, and $\lim_{t \to +\infty} q(t) = +\infty$. This shows that $q(t)$ has just one root in the interval $(a, +\infty)$.

From property (ii) above, we deduce that the sums $s_l := 1/\lambda_1^l + 1/\lambda_2^l$ verify relations (17). Besides this, $1/\lambda_1$ and $1/\lambda_2$ are the roots of $(x - 1/\lambda_1)(x - 1/\lambda_2) = x^2 - s_1 x + (s_1^2 - s_2)/2$, so λ_1 and λ_2 are the roots of the quadratic polynomial $(s_1^2 - s_2)t^2/2 - s_1 t + 1$.

We look for ellipsoidal parameters such that the previous periodic trajectories exist. From property (i) above, we deduce that such ellipsoidal parameters exist if and only the graph $\{y = Q(x)\}$ intersects the horizontal line $\{y = Q(\delta_1)\}$ at two different points $\gamma_2, \gamma_3 \in (\gamma_4, \gamma_5)$

or, equivalently, if and only if $Q(\delta_3) < Q(\delta_1)$, where $\delta_1 < \delta_2 < \delta_3$ are the three ordered roots of the derivative of the polynomial $Q(x) = x(x - \gamma_1)(x - \gamma_4)(x - \gamma_5)$. But

$$Q(\delta_3) < Q(\delta_1) \Leftrightarrow \gamma_1 > \gamma_4 + \gamma_5 \Leftrightarrow c < ab/(a + b),$$

according to lemma 19. □

As we have explained before, the period and winding numbers of any non-singular periodic billiard trajectory can be determined by direct inspection of its corresponding figure. A periodic billiard trajectory with elliptic period $m = 4$ and caustic type H1H1 whose caustic parameters verify the relations given in proposition 20 is displayed in [26, figure 13]. That trajectory has (Cartesian) period $m_0 = 4$ and winding numbers

$$m_2 = 2, \qquad m_1 = 3, \qquad m_0 = 4.$$

Hence, $\gcd(m_0, m_1, m_2) = 1$, so the elliptic winding numbers are $\widetilde{m}_2 = 2$, $\widetilde{m}_1 = 3$, and $\widetilde{m}_0 = 4$. This result reinforces conjecture 3. Besides this, these non-singular billiard trajectories of period 4 with caustic type H1H1 are quite interesting, because they display the minimal period among all non-singular periodic billiard trajectories; see [26, theorem 1].

We end the study at this point. We just mention that there exist similar results for when the signature or the caustic type do not coincide with the ones given in proposition 20. Analogously, the case $m = 5$ can be dealt with using the same techniques, although the final formulas become more complicated. For instance, it can be easily checked that the caustic parameters λ_1 and λ_2 of the billiard trajectories with elliptic period $m = 5$ and signature $\tau = (1, 1, 0)$ verify the homogeneous symmetric polynomial equations

$$8s_3 + s_1^3 = 6s_1 s_2, \qquad 16s_4 + s_1^4 = 4s_1^2 s_2 + 4s_2^2,$$

where $s_l = 1/a^l + 1/b^l + 1/c^l + 1/\lambda_2^l + 1/\lambda_1^l$ for $l = 1, 2, 3, 4$. Each of the other signatures $\tau \in \mathcal{T}(5, 3)$ gives rise to similar homogeneous—although not symmetric—polynomial equations of degrees 3 and 4 in the variables $1/a$, $1/b$, $1/c$, $1/\lambda_1$ and $1/\lambda_2$. We leave the details to the reader. Finally, we recall that the original matrix formulation of the generalized Cayley condition $\mathcal{C}(5, 3)$ gives rise to two homogeneous symmetric polynomial equations of degrees 23 and 24 in those five variables, as explained in section 3. This confirms, once again, that the polynomial formulation offers great computational advantages over the matrix formulation.

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