Homoclinic and heteroclinic connections

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Mission statements

- Stress the common elements of most separatrix splitting problems.
- Write down some explicit formulae for the conservative discrete case.
- Present the problem under several point of views.
- List a broad bibliography, but never complete (snif!, snif!).

Plan and warnings

Plan:

- Explain the simplest example: a “fish-shaped” separatrix.
- Present a (hopefully, not very messy) miscellany of geometric, analytic, dynamic and numerical complications taken from not-so-simple examples.
- Explain a difficult example: the scattering map.

Warnings:

- At some points, the exposition will become informal and rough.
- I will try not to cheat too much.
- I will follow a “shaken, not stirred” style, because it is difficult to understand (ahem, or to explain) simultaneously two different things. Unfortunately, any interesting problem is “shaken”, “stirred”, “bended”, etc.

Fish-shaped separatrix (Unperturbed setup)

We begin with a "fish-shaped" homoclinic connection to a saddle fixed point of an area-preserving map.

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a (smooth) area-preserving map.
- A point $m_\infty \in \mathbb{R}^2$ is a *saddle point* of f when it is *fixed*: $f(m_\infty) = m_\infty$ and *hyperbolic*: $\operatorname{spec}[\mathrm{d}f(m_\infty)] = \{\lambda, \lambda^{-1}\}$ with $|\lambda| > 1$. We assume that $\lambda > 1$.
- The *stable* and *unstable invariant curves* of the saddle point are

$$W^\pm = W^\pm(m_\infty) = \left\{ m \in \mathbb{R}^2 : \lim_{k\to\pm\infty} f^k(m) = m_\infty \right\}.$$

  (Note: Minus sign means unstable curve, plus sign means stable curve.)
- If a branch of the unstable curve coincides with another branch of the stable one, they form a "fish-shaped" *separatrix* Γ. By convention, $m_\infty \notin \Gamma$.
- The term "separatrix" comes from the fact that the dynamics inside and outside Γ has usually different behaviours, and Γ "separates" them.

Fish-shaped separatrix (Questions & Answers)

Basic Question: What happens under small area-preserving perturbations?
Generic answer: The separatrix splits in a transverse way.

- Q: There exists maps with fish-shaped homoclinic connections?
  A: Yes, but they are the exception, not the rule.
- Q: Do persist some homoclinic orbits?
  A: Yes. Although, generically, only a finite number (probably, two) of primary homoclinic orbits persist.
- Q: Assuming f was integrable: Is the integrability preserved?
  A: Generically, it is destroyed.
- Q: What quantities are used to measure the splitting size?
  A: Splitting distances, splitting angles, Lazutkin invariants, lobe areas, etc.
- Q: Is there a method to measure the splitting size?
  A: Yes. It is called (ahem!) Melnikov method.
  Already known by Poincaré and others.
- Q: How is the perturbed dynamics?
  A: Generically, "chaotic" (Smale horseshoes, stochastic layers, etc.).

Fish-shaped separatrix (Existence)

- The time-1 flow of the Hamiltonian equation $x'' - x + x^2 = 0$ has a fish-shaped homoclinic connection to the origin.
- The integrable McMillan map

$$f(x, y) = \Big(y, -x + y^2/(1-y)\Big)$$

has a fish-shaped homoclinic connection to the saddle point $\left(\frac{2}{3}, \frac{2}{3}\right)$.

- *(V. F. Lazutkin)* If $f : \mathbb{C}^2 \to \mathbb{C}^2$ is entire, it has no “clinic” connections. (The Standard map, the Hénon map, and all polynomial maps are entire.)

References:

1. E. M. McMillan, A problem in the stability of periodic systems, in *Topics in Modern Physics*, E. Brittin and H. Odabasi, eds., Colorado Assoc. Univ. Press, Boulder, pp. 219–244 (1971).
2. V. F. Lazutkin, Splitting of complex separatrices, *Funct. Anal. Appl.*, **22**:154–156 (1998).

Fish-shaped separatrix (Persistence)

- The area-preserving property implies that the perturbed invariant curves intersect at some homoclinic points.
- A diffeomorphism $f : M \to M$ is *reversible* when there exists a diffeomorphism $R : M \to M$ such that $f \circ R = R \circ f^{-1}$, and then R is called a *reversor* of the map. Usually, R is an involution: $R^2 = \mathrm{I}$.
- If R is a reversor, the points in $\operatorname{Fix} R = \{m \in M : R(m) = m\}$ are *symmetric*. Usually, $\operatorname{Fix} R$ is a smooth curve and then it is called a *symmetry line*.
- Symmetric homoclinic points associated to transverse intersections of invariant curves and symmetry lines persist under reversible perturbations.
- We have a persistence result from the geometry, and a more refined location result from reversibilities.

References:

1. E. Zehnder, Homoclinic points near elliptic fixed points, *Comm. Pure Appl. Math.*, **26**:131–182 (1973).
2. R. L. Devaney, Reversible diffeomorphisms and flows, *Trans. Amer. Math. Soc.*, **218**:89–113 (1976).

Fish-shaped separatrix (Integrability)

- (R. Cushman) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an analytic planar map with a saddle point whose invariant curves have a topological crossing (that is, a finite-order contact), then f is nonintegrable (that is, it has no analytic first integral).
- The proof is based on the Moser Normal Form Theorem around saddle points of analytic area-preserving maps.
- There exist some higher-dimensional generalizations by S.A. Dovbysh and J. Cresson, but too technical to be explained here.

References:

1. R. Cushman, Examples of nonintegrable analytic Hamiltonian vector fields with no small divisors, *Trans. Am. Math. Soc.*, **238**:45–55 (1978).
2. S. A. Dovbysh, Transversal intersection of separatrices and branching of solutions as obstructions to the existence of an analytic integral in many-dimensional systems. I. Basic result: separatrices of hyperbolic periodic points, *Collect. Math.*, **50**:119–197 (1999).
3. J. Cresson, Hyperbolicity, transversality and analytic first integrals, *J. Differential Equations*, **196**:289–300 (2004).

Fish-shaped separatrix (Splitting quantities)

- The *splitting distance* along the separatrix (measured with respect some transverse directions). It has big oscillations close to the fixed point.
- The *splitting angle* at a homoclinic point. It is not invariant along the orbit.
- The *Lazutkin homoclinic invariant* of a homoclinic point m_0 is defined by using the natural parameterizations of the invariant curves. It does not depend on the point of the homoclinic orbit. Besides, it is invariant by symplectic changes of variables. It is proportional to the splitting angle.
- Given two homoclinic points such that the pieces of the invariant curves between them do no intersect, these pieces enclose a region called a *lobe*. The *area* of this lobe is another symplectic invariant.

References:

1. V. G. Gelfreich, V. F. Lazutkin, and N. V. Svanidze, A refined formula for the separatrix splitting for the standard map, *Phys. D*, **71**:82–101 (1994).
2. V. G. Gelfreich and V. F. Lazutkin, Splitting of separatrices: perturbation theory and exponential smallness, *Russian Math. Surveys*, **56**:499–558 (2001).

Fish-shaped separatrix (Melnikov method, 1/4)

- The goal of any *Melnikov method* is to find an object (a *Melnikov function*, a *Melnikov potential*, a *Melnikov form*, etc.) which computes the rate at which the distance between the pertubed invariant manifolds changes with the perturbation.
- It is just a first order method in the perturbative parameter. Therefore, all Melnikov objects have a similar look because:
  - The first order term in any perturbation theory has to be linear in the first order term of the perturbation;
  - The first order perturbative term has to depend only on the values of the perturbation on the unperturbed dynamics; and
  - Splitting problems are global, not local, so the first order perturbative term has to take into account the values of the perturbation on the whole unperturbed homoclinic connection.
  - Finally, if the Melnikov object is, in some sense, canonical, it must be invariant along any unperturbed orbit.

Fish-shaped separatrix (Melnikov method, 2/4)

- Let $f_\epsilon = f + \mathcal{O}(\epsilon)$ be a smooth area-preserving perturbation of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with a fish-shaped separatrix Γ.
- Then there exists a smooth function $M : \Gamma \to \mathbb{R}$ such that:
  - If $M \not\equiv 0$, then the separatrix splits.
  - It is invariant by the unperturbed map: $M \circ f = M$.
  - The simple (non-degenerate) zeros of M are associated to transverse intersections of the perturbed invariant curves.
  - Splitting angles and lobe areas are approximated in first order by derivatives and integrals of M, respectively.

Fish-shaped separatrix (Melnikov method, 3/4)

- Let us assume that the unperturbed map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is *twist*. That is, we assume that there exists a *generating function* $\mathcal{L} : \mathbb{R}^2 \to \mathbb{R}$ such that

$$f(x,y) = (X,Y) \iff y = -\partial_1 \mathcal{L}(x,X), \quad Y = \partial_2 \mathcal{L}(x,X).$$

  Standard-like maps (and, ahem, convex billiard maps) are twist maps. Twist maps can be studied by looking at only half of the coordinates.
- Let $f_\epsilon = f + \mathcal{O}(\epsilon)$ be a twist perturbation with generating function

$$\mathcal{L}_\epsilon = \mathcal{L} + \epsilon \mathcal{L}_1 + \mathcal{O}(\epsilon^2).$$

- Then, the function defined by the absolutely convergent series

$$L : \Gamma \to \mathbb{R}, \quad L(p) = \sum_{k \in \mathbb{Z}} \mathcal{L}_1(x_k, x_{k+1}), \quad (x_k, y_k) = f^k(p)$$

  is a *Melnikov potential*. That is, its "derivative" is a Melnikov function (in the sense that it has the above-mentioned properties).

Fish-shaped separatrix (Melnikov method, 4/4)

- Now, let us assume that the unperturbed map f has a nondegenerate first integral $I : \mathbb{R}^2 \to \mathbb{R}$. This is a rather usual situation.
- Let $f_\epsilon = (\mathrm{I} + \epsilon\kappa) \circ f$ be the perturbed map for some $\kappa : \mathbb{R}^2 \to \mathbb{R}^2$.
- Then, the function defined by the absolutely convergent series

$$M : \Gamma \to \mathbb{R}, \qquad M = \sum_{k \in \mathbb{Z}} \langle \nabla I, \kappa \rangle \circ f^k$$

is a Melnikov function. That is, it has the above-mentioned properties.

References:

1. M. L. Glasser, V. G. Papageorgiou, and T. C. Bountis, Melnikov's function for two-dimensional mappings, *SIAM J. Appl. Math.*, **49**:692–703 (1989).
2. A. Delshams and R. Ramírez-Ros, Poincaré-Melnikov-Arnold method for analytic planar maps, *Nonlinearity*, **9**:1–26 (1996).

Fish-shaped separatrix ("chaotic" dynamics)

- *(Smale)* Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism with a saddle point whose invariant curves has a transverse intersection. Then some power f^N has an invariant Cantor set C, the *Smale horseshoe*. Besides, the restriction of that power to C is topologically conjugated to the *Bernoulli shift* defined by

$$\{0,1\}^{\mathbb{Z}} \ni (\sigma_n)_{n\in\mathbb{Z}} \mapsto (\sigma_{n+1})_{n\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}.$$

- The Bernoulli shift (and so, the map $f^N|_C$) has the following properties:
  - It has a countable amount of periodic orbits;
  - Its set of periodic points (and its complementary) is dense;
  - Any couple of periodic orbits have infinitely many heteroclinic orbits;
  - It has dense orbits; and
  - Its topological entropy is positive; namely, it is equal to $\log 2$.

Reference:

1. S. Smale, Diffeomorphisms with many periodic points, in *Differential and Combinatorial Topology*, S. S. Cairns, ed., Princeton Univ. Press, pp. 63–80 (1963).

The general splitting problem: Unperturbed setup

- Let $f : M \to M$ be a diffeomorphism with two *normally hyperbolic invariant manifolds* N^- and N^+ whose *stable* and *unstable invariant manifolds*

$$W^- = W^{\mathrm{u}}(N^-), \qquad W^+ = W^{\mathrm{s}}(N^+)$$

  are *doubled*. That is, there exists a *separatrix* $\Gamma \subset W^+ \cap W^-$ such that

$$T_p\Gamma = T_pW^- = T_pW^+, \qquad \forall p \in \Gamma.$$

  (The case $\dim \Gamma < \dim W^\pm$ can also be studied: Bolotin, Delshams & RRR.)
- The normal hyperbolicity is required for the persistence of the invariant manifolds under small perturbations.
  (There are other ways to ensure persistence. For instance, the study of "diophantine whiskered" tori: Eliasson; Treschev; Simó; Rudnev and Wiggins; Lochak, Marco, and Sauzin; Delshams, Gutiérrez, and Seara; etc.)
- The coincidence of the tangent spaces allows us to express the perturbed invariant manifolds as deformations of the unperturbed separatrix.

The general splitting problem: Difficulties

- It is more difficult to define splitting quantities which are geometric and homoclinic invariants. (An example: We could consider the volume of a 3D-"lobe", but some 3D maps have no lobes in the usual sense.)
- The heteroclinic case: $N^- \neq N^+$, is more subtle. For instance, the persistence results require additional hypotheses.
- The numbers $\dim N^\pm$, $\dim M - \dim \Gamma$, and $\dim \Gamma - \dim N^\pm$ are important, because:
  - If $N^\pm$ are not points, their intern dynamics must be taken into account. For instance, to prove the (absolute) convergence of "Melnikov-like" objects. This is crucial for the construction of scattering maps.
  - If the separatrix has codimension greater than one, we need, in general, a *Melnikov vector* instead of a *Melnikov function*.
  - If there are several hyperbolic directions: $\dim W^\pm - \dim N^\pm > 1$, then the tangent spaces of $W^\pm$ can do some surprising things. (An example: There exist 4D symplectic maps such that

$$p \in (W^- \cap W^+) \setminus (N^- \cup N^+) \not\Rightarrow T_p W^+ = T_p W^-.)$$

Melnikov: Discrete vs. continuous

- Let $H_\epsilon(x,y,t) = H(x,y) + \epsilon H_1(x,y,t)$ be a time-periodic Hamiltonian perturbation of a Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ with a fish-shaped separatrix Γ to some saddle point.
- Then the Melnikov potential of this problem is

$$L : \Gamma \to \mathbb{R}, \qquad L(p) = -\int_{\mathbb{R}} H_1(\Phi^t(p), t)\mathrm{d}t$$

and the Melnikov function is

$$M : \Gamma \to \mathbb{R}, \qquad M(p) = \int_{\mathbb{R}} \{H(\Phi^t(p), H_1(\Phi^t(p), t)\}\mathrm{d}t$$

where Φ^t is the unperturbed Hamiltonian flow.

- *Homoclinic sum:* $\Sigma_{k\in\mathbb{Z}}\bigstar(k)$ becomes $\int_{\mathbb{R}}\bigstar(t)\mathrm{d}t$;
- *First order term of the perturbation:* $\mathcal{L}_1$ becomes $-H_1$; and
- *Unperturbed dynamics:* $f^k(p)$ becomes $\Phi^t(p)$.

Melnikov: Volume-preserving maps (1/2)

- Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a integrable volume-preserving map with two hyperbolic fixed points a and b whose invariant manifolds are *completely doubled* giving rise to a two-dimensional separatrix

$$\Gamma = W^{\mathrm{u}}(a) \setminus \{a\} = W^{\mathrm{s}}(b) \setminus \{b\}.$$

- Let $f_\epsilon = (\mathrm{I} + \epsilon\kappa) \circ f$ be the perturbed map for some $\kappa : \mathbb{R}^3 \to \mathbb{R}^3$. (Exercise: f_ϵ is volume-preserving iff $\mathrm{d}\kappa$ is nilpotent everywhere.)
- Then, the function defined by the absolutely convergent series

$$M : \Gamma \to \mathbb{R}, \qquad M = \sum_{k \in \mathbb{Z}} \langle \nabla I, \kappa \rangle \circ f^k$$

is a Melnikov function that verifies the following properties:
  - Its nondegenerated zeros are associated to transverse intersections; and
  - If 0 is a regular value of M, the perturbed manifolds have transverse intersections along smooth curves of $\mathbb{R}^3$ that are $\mathcal{O}(\epsilon)$-close to $M^{-1}(0)$.

Melnikov: Volume-preserving maps (2/2)

Some applications:

- Study the shape and bifurcations of the level set $M^{-1}(0) \subset \Gamma$.
- Bound the topological complexity of the level set $M^{-1}(0)$.
- Obtain sufficient conditions for the splitting of separatrices under some kind of perturbations.

References:

1. H. E. Lomelí and J. D. Meiss, Heteroclinic primary intersections and codimension one Melnikov method for volume-preserving maps, *Chaos*, **10**:109–121 (2000).
2. H. E. Lomelí and R. Ramírez-Ros, Separatrix splitting in 3D volume-preserving maps, to appear in *SIAM J. Appl. Dyn. Syst.* (2008).

Melnikov: Twist maps (1/2)

- Let $M = T^*Q$ be the cotangent bundle of some n-dimensional space Q.
- A map $f : M \to M$ is *twist* when there exists a *generating function* $\mathcal{L} : Q \times Q \to \mathbb{R}$ such that

$$f(x, y) = (X, Y) \Longleftrightarrow Y\mathrm{d}X - y\mathrm{d}x = \mathrm{d}\mathcal{L}(x, X).$$

- Let $f : M \to M$ be a twist map with a hyperbolic fixed point m_∞ whose invariant manifolds are *doubled* giving rise to a n-dimensional separatrix

$$\Gamma \subset W^{\mathrm{u}}(m_\infty) \cap W^{\mathrm{s}}(m_\infty).$$

- Let $f_\epsilon = f + \mathcal{O}(\epsilon)$ be a twist perturbation with generating function

$$\mathcal{L}_\epsilon = \mathcal{L} + \epsilon\mathcal{L}_1 + \mathcal{O}(\epsilon^2).$$

- Then the *Melnikov potential* of this problem is

$$L : \Gamma \to \mathbb{R}, \quad L(p) = \sum_{k\in\mathbb{Z}} \mathcal{L}_1(x_k, x_{k+1}), \quad (x_k, y_k) = f^k(p).$$

Melnikov: Twist maps (2/2)

Some Applications:

- We consider the twist maps $f_\epsilon : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by

$$f_\epsilon(x,y) = \left(y, -x + \frac{2\mu y}{1+|y|^2} + \epsilon \nabla V(y) \right), \qquad \mu > 1, \quad \epsilon \ll 1.$$

  The unperturbed map $f = f_0$ has a separatrix to the origin.
  If the potential $V : \mathbb{R}^n \to \mathbb{R}$ is entire but non-constant, the perturbation splits the separatrix.

- Find transverse homoclinic orbits in the 4D billiard maps associated to some perturbations of an ellipsoid in $\mathbb{R}^3$.

References:

1. A. Delshams and R. Ramírez-Ros, Melnikov potential for exact symplectic maps, *Comm. Math. Phys.*, **190**:213–45 (1997).
2. A. Delshams, Yu. Fedorov, and R. Ramírez-Ros, Homoclinic billiard orbits inside symmetrically perturbed ellipsoids, *Nonlinearity*, **14**:1141–95 (2001).

Melnikov: Normally hyperbolic manifolds (1/2)

- Let $f_\epsilon : M \to M$ be a smooth family of diffeomorphisms such that the unperturbed map $f = f_0$ has a separatrix Γ between two compact r-normally hyperbolic invariant manifolds N^- and N^+.
- Let $\nu(\Gamma) \equiv T_\Gamma M / T\Gamma$ be the algebraic normal bundle of the separatrix.
- Then there exists a canonical C^{r-1} section $\mathcal{D} : \Gamma \to \nu(\Gamma)$, called the *Melnikov displacement*, that measures the splitting in first order.
- It turns out, from deformation calculus, that this Melnikov displacement is given by the absolutely convergent series

$$\mathcal{D} = \sum_{k \in \mathbb{Z}} (f_0^*)^k \mathcal{F}_0$$

where $\mathcal{F}_\epsilon$ is the vector field defined by

$$\frac{\partial}{\partial \epsilon} f_\epsilon = \mathcal{F}_\epsilon \circ f_\epsilon$$

and f_0^* denotes the pullback of the unperturbed map.

Melnikov: Normally hyperbolic manifolds (2/2)

References:

1. M. Baldomà and E. Fontich, Poincaré-Melnikov theory for n-dimensional diffeomorphisms, *Appl. Math. (Warsaw)*, **25**:129–152 (1998).
2. H. E. Lomelí, J.D. Meiss, and R. Ramírez-Ros, Canonical Melnikov theory for diffeomorphisms, *Nonlinearity*, **21**:485–508 (2008).

Singular splitting: The basic problem

- Let $f_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be an area-preserving map with a separatrix Γ_h to a saddle point m_∞ such that $\mathrm{d}f_h(m_\infty) = \{\mathrm{e}^{\pm h}\}$. That is, the saddle point is weakly hyperbolic if h is small.
- Now, given a perturbation $f_{h,\epsilon} = f_h + \mathcal{O}(\epsilon)$ of the previous map, we want to measure the splitting size using a Melnikov method.
- Suppose, for instance, that we can compute the first order Melnikov approximation A_1 of certain splitting quantity A; that is,

$$A = A(h,\epsilon) = \epsilon A_1(h) + \mathcal{O}(\epsilon^2), \qquad \text{for any fixed } h > 0.$$

- In many (analytic) cases, there exist a constant $c > 0$ such that

$$A_1(h) = \mathcal{O}(\mathrm{e}^{-c/h}) \qquad h \to 0^+.$$

- Question: What can be said about A when both ϵ and h are small?
  For instance, when $\epsilon = \epsilon(h) = h^p$ for some exponent $p > 0$?
  Answer: A priori, nothing!

Singular splitting: An upper bound

- (Fontich & Simó) Let $g_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be a diffeomorphism such that:
  - It is area-preserving and analytic in a big enough complex region;
  - It is $\mathcal{O}(h)$-close to the identity map;
  - The origin is a saddle point of g_h;
  - Its characteristic exponent at the origin is h; and
  - It has a homoclinic orbit to the origin for small enough h.

  Then, there exists $d_* > 0$ such that:

$$\text{splitting size} \le \mathcal{O}(\mathrm{e}^{-2\pi d/h}) \qquad (h \to 0^+)$$

  for any $d \in (0, d_*)$. Besides, d_* is the analyticity width of the separatrix of certain limit Hamiltonian flow. Sometimes, it can be analytically computed.

Reference:

1. E. Fontich and C. Simó, The splitting of separatrices for analytic diffeomorphisms, *Ergodic Theory Dyn. Sist.*, **10**:295–318 (1990).

Singular splitting: The standard map (1/2)

- The most celebrated example is the *Standard map*

$$SM : \mathbb{T}^2 \to \mathbb{T}^2, \qquad SM(x,y) = (x + y + \epsilon \sin x, y + \epsilon \sin x).$$

- If $\epsilon > 0$, the origin is hyperbolic and $\epsilon = 4\sinh^2(h/2)$.
- The map $R(x,y) = (2\pi - x, y + \epsilon \sin x)$ is a reversor, and $\operatorname{Fix} R = \{x = \pi\}$.
- *(Gelfreich)* The Lazutkin homoclinic invariant of the symmetric homoclinic orbit passing through the first intersection of W^+ with $\operatorname{Fix} R$ verifies

$$\omega \asymp 4\pi h^{-2} \mathrm{e}^{-\pi^2/h} \sum_{j \geq 0} \omega_j h^{2j} \qquad (h \to 0^+).$$

  This expansion was proved using an approach suggested by Lazutkin.
- The first asymptotic coefficient $\omega_0 \approx 1118.827706$ is the *Lazutkin constant*.
- *(Gelfreich & Simó)* It is conjectured that the series $\sum_{j \geq 0} \omega_j h^{2j}$ is Gevrey-1 of type $1/2\pi^2$. Detailed numerical experiments support the conjecture.

Singular splitting: The standard map (2/2)

References:

1. V. F. Lazutkin, Splitting of separatrices for the Chirikov standard map (Translated from the Russian and with a preface by V. Gelfreich), *J. Math. Sci. (N. Y.)*, **128**:2687–2705 (2005).
   [This work was originally published as a VINITI preprint in 1984.]
2. V. G. Gelfreich, A proof of the exponentially small transversality of the separatrices for the standard map, *Comm. Math. Phys.*, **201**:155–216 (1999).
3. V. G. Gelfreich and C. Simó, High-precision computations of divergent asymptotic series and homoclinic phenomena, *Discrete Contin. Dyn. Syst. - Series B*, **10**:511–536 (2008).

On resurgence:

- Once we know that Gevrey functions appear in some singular splitting problems, it is natural to study them using resurgence. We could mention the works by Gelfreich and Sauzin; Baldomá and Seara; Martín and Sera, etc. I have skipped this part, since I haven't time.

Singular splitting: Numeric experiments (1/2)

- The computation of splitting quantities that can be $\mathcal{O}(10^{-10000})$ is not easy.
- *First problem (slow dynamics):* Homoclinic intersections are usually computed by iterating a small local part of the invariant manifolds. *Solution:* Expand the local manifolds up to high order.
- *Second problem (massive cancellations):* Splitting quantities are usually computed as the difference of two exponentially close quantities. *Solution:* Use a multiple precision arithmetic.
- *Key tricks:* Optimize dynamically the choice of:
  - The order of the local expansions; and
  - The precision of the arithmetic.

Singular splitting: Numeric experiments (2/2)

References:

1. C. Simó, On the analytical and numerical approximations of invariant manifolds, in *Modern Methods in Celestial Mechanics*, D. Benest and C. Froeschlé eds., Editions Frontières, pp. 285–329 (1990).
2. A. Delshams and R. Ramírez-Ros, Singular separatrix splitting and the Melnikov method: an experimental study, *Experiment. Math.*, **8**:29–48 (1999).
3. C. Simó, Analytical and numerical detection of exponentially small phenomena, in *Proceedings of the International Conference on Differential Equations*, B. Fiedler, K. Gröger and J. Sprekels, eds., World Sci. Publishing, pp. 967–976 (2000).
4. R. Ramírez-Ros, Exponentially small separatrix splittings and almost invisible homoclinic bifurcations in some billiard tables, *Phys. D*, **210**:149–179 (2005).
5. V. G. Gelfreich and C. Simó, High-precision computations of divergent asymptotic series and homoclinic phenomena, *Discrete Contin. Dyn. Syst. - Series B*, **10**:511–536 (2008).

Singular splitting: Eight-shaped separatrices

- *(Treschev)* Let $g_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be a diffeomorphism such that
  - It is area-preserving, analytic, and symmetric with respect to the origin;
  - It is $\mathcal{O}(h)$-close to the identity map;
  - The origin is a saddle point of g_h;
  - Its characteristic exponent at the origin is h; and
  - The invariant curves of the saddle point have an "eight"-shaped form.

  Then, the *width of the stochastic layer w* and the *splitting amplitude d* verify the asymptotic relation

$$w \asymp c_0 h^{-1} d \qquad (h \to 0^+)$$

  where $c_0 \approx 12.9332$ is a positive constant related to the Standard map.

References:

1. V. F. Lazutkin, On the width of the instability zone near the separatrices of a standard mapping, *Soviet. Math. Dokl.*, **313**:5–9 (1991).
2. D. Treschev, Width of stochastic layers in near-integrable two-dimensional symplectic maps, *Phys. D*, **116**:21–43 (1998).

Persistence: Exact maps (completely doubled case)

- Let M be an exact symplectic manifold with symplectic form $\omega = \mathrm{d}\alpha$.
- A map $f : M \to M$ is *exact* when the 1-form $f^*\alpha - \alpha$ is exact; that is, when $f^*\alpha - \alpha = \mathrm{d}S$ for some generating function $S : M \to \mathbb{R}$.
- Two points a and b are on the *same action level* of the exact map f when

$$S(a) = S(b).$$

- Let $f : M \to M$ be an exact map with two hyperbolic fixed points a and b whose invariant manifolds are *completely doubled* giving rise to the n-dimensional separatrix $\Gamma = W^{\mathrm{u}}(a) \setminus \{a\} = W^{\mathrm{s}}(b) \setminus \{b\}$
- *(Xia)* If the perturbed fixed points remain on the same action level, then the perturbed exact map has at least three (in the planar case, four) primary heteroclinic orbits close to Γ.

Reference:

1. Z. Xia, Homoclinic points and intersections of Lagrangian submanifolds, *Discrete Contin. Dyn. Syst.*, **6**:243–253 (2000).

Persistence: Twist maps (general case, 1/2)

- Let $M = Q \times Q$ be the square of some n-dimensional configuration space Q.
- $f : M \to M$ is *twist* when there exists some *Lagrangian* $\mathcal{L} : M \to \mathbb{R}$ such that

$$f(q, q') = (q', q'') \iff \partial_2 \mathcal{L}(q, q') + \partial_1 \mathcal{L}(q', q'') = 0.$$

- Two points a and b are on the *same action level* of the twist map f when

$$\mathcal{L}(a) = \mathcal{L}(b).$$

- Let $f : M \to M$ be a twist map with two hyperbolic fixed points a and b whose invariant manifolds $W^+ = W^{\mathrm{u}}(a)$ and $W^- = W^{\mathrm{s}}(b)$ have the following properties:
  - W^- and W^+ have a clean intersection along an invariant manifold N;
  - $\overline{N} = N \cup \{a, b\}$ is compact in the topology of M; and
  - Given any neighbourhood V of the fixed points a and b, there exists $k_0 > 0$ such that $f^k(N \setminus V) \subset V$ for all integer k with $|k| \leq k_0$.
    That is, the topologies in N induced from M, W^+, and W^- coincide.

Persistence: Twist maps (general case, 2/2)

- *(Bolotin, Delshams & RRR)* If the perturbed fixed points remain on the same action level, then the perturbed twist map has at least $\operatorname{cat}(N/f)$ primary heteroclinic orbits close to N.

Applications:

- The billiard map f associated to a prolate ellipsoid of $\mathbb{R}^3$ is twist and has a two-periodic hyperbolic trajectory whose invariant manifolds are completely doubled.
- The billiard map f associated to a generic ellipsoid of $\mathbb{R}^3$ is twist and has a two-periodic hyperbolic trajectory whose invariant manifolds verify the previous hypotheses for some manifold N such that

$$\operatorname{cat}(N/f) = 8.$$

Reference:

1. S. Bolotin, A. Delshams, and R. Ramírez-Ros, Persistence of homoclinic orbits for billiards and twist maps, *Nonlinearity*, **17**:1153–1177 (2004).

Scattering map: Definitions

- Let $f : M \to M$ be an exact symplectic diffeomorphism with a normally hyperbolic invariant manifold Λ whose stable and unstable invariant manifolds $W^- = W^{\mathrm{u}}(\Lambda)$ and $W^+ = W^{\mathrm{s}}(\Lambda)$ have a transverse and clean intersection along some *homoclinic channel* Γ:
- We introduce the *wave operators*

$$W^\pm \ni x \xmapsto{\Omega_\pm} x_\pm \in \Lambda$$

as the "projections" of the point x in the stable or unstable manifold onto the base point of its "fiber".

- Assume that the restrictions $(\Omega_\pm)|_\Gamma : \Gamma \to \Omega_\pm(\Gamma) \subset \Lambda$ of the wave operators to the homoclinic channel are diffeomorfisms.
- Then, we can define the *scattering map*

$$\sigma = \sigma^\Gamma = \Omega_+ \circ (\Omega_-)^{-1} : \Omega_-(\Gamma) \to \Omega_+(\Gamma).$$

- This construction depends on the homoclinic channel Γ chosen.

Scattering map: Dynamics, regularity, geometry, . . .

- *Dynamics:* $\sigma(x_-) = x_+$ means that there exists some $x \in \Gamma$ such that

$$\lim_{k \to \pm\infty} f^k(x) = x_\pm.$$

- *Regularity:* The scattering map is less regular than the original map. Its regularity depends on the ratio of the (uniform) hyperbolic exponents.
- *Invariance:* $f \circ \sigma^\Gamma = \sigma^{f(\Gamma)} \circ f$.
  (Warning: σ^Γ and $\sigma^{f(\Gamma)}$ are different scattering maps.
  This is crucial for diffusion problems.)
- *Geometry:* If f and Λ are (exact) symplectic, then σ is (exact) symplectic.

Reference:

1. A. Delshams, R. de la Llave, and T. M. Seara, Geometric properties of the scattering map of a normally hyperbolic invariant manifold, *Adv. Math.*, **217**:1096–1153 (2008).

Scattering map: Exactness

- Let M be an exact symplectic manifold with symplectic form $\omega = \mathrm{d}\alpha$.
- Let $f : M \to M$ be an exact map with generating (primitive) function $S : M \to \mathbb{R}$; that is,
$$f^*\alpha - \alpha = \mathrm{d}S.$$
- Assume that the normally hyperbolic manifold Λ is exact. That is, the restriction $\omega|_\Lambda$ is also symplectic.
- Then, the scattering map σ is an exact map with generating function
$$L = \sum_{k\in\mathbb{Z}} \Delta_k^{\mathrm{sign}(k)} = \lim_{n,m\to\infty} \left(\Delta_{-n}^- + \cdots + \Delta_{-1}^- + \Delta_0^+ + \Delta_1^+ + \cdots + \Delta_m^+\right)$$
where the terms $\Delta_k^\pm$ are defined as the differences
$$\Delta_k^- = S\circ f^k \circ (\Omega_-^\Gamma)^{-1} - S\circ f^k, \quad \Delta_k^+ = S\circ f^k\circ (\Omega_+^\Gamma)^{-1}\circ\sigma - S\circ f^k\circ\sigma.$$
- Exercise: Compare this formula with the ones about Melnikov potentials.

Scattering map: A simple 4D example

- The *PRTBP* is a two-degree-of-freedom Hamiltonian system with just one first integral: the Jacobi Constant $\mathcal{C}$.
- It has a normally hyperbolic invariant surface Λ foliated by invariant curves: the Lyapunov orbits around the L_1 collinear libration point.
- We fix an annular piece of Λ which topologically is $[\mathcal{C}_-, \mathcal{C}_+] \times \mathbb{T}$, so that its points are parametriced by some action-angle coordinates $(\mathcal{C}, \theta)$.
- Then the scattering map has the form of an integrable twist map; that is,

$$\sigma(\mathcal{C}, \theta) = \big(\mathcal{C}, \theta + \Delta(\mathcal{C})\big)$$

  (This integrable form is usual for unperturbed “simple” systems.)
- The rotation angle $\Delta(\mathcal{C})$ has been computed numerically.

Reference:

1. E. Canalias, A. Delshams, J. J. Masdemont, and P. Roldán, The scattering map in the planar restricted three body problem, *Celestial Mech. Dynam. Astronom.*, **95**:155–171 (2006).

Scattering map: A degenerate 4D example

- The *billiard dynamics inside a generic ellipsoid* in $\mathbb{R}^3$ is completely integrable, since any billiard trajectory inside any such ellipsoid is tangent to *two* fixed confocal quadrics.
- It has a normally hyperbolic invariant surface Λ foliated by invariant curves, which corresponds to the planar trajectories contained in the plane generated by the two biggest axis of the ellipsoid.
- The stable and unstable whiskers of any fixed invariant curve are completely doubled, due to the integrability.
- We fix a piece of Λ which topologically is $[I_-, I_+] \times \mathbb{T}$, so that its points are parametriced by some action-angle coordinates (I, θ).
- (Work in progress) In this problem, the “scattering map” is very degenerated. Namely,

$$\sigma(I, \theta) = (I, \theta + \pi).$$

Scattering map: Diffusion

- Let $f_0 : M \to M$ be a 4D exact symplectic map with a normally hyperbolic invariant surface Λ_0 such that:
  - Its stable and unstable invariant manifolds have a transverse and clean intersection along a homoclinic channel Γ_0;
  - It is foliated by invariant curves $\mathcal{T}_0(I)$, parameterized by some "action" coordinate $I \in [I_-, I_+]$; and
  - There exists some KAM result about the persistence of the (diophantine) invariant curves under conservative perturbations.
- Suppose that, given an exact symplectic perturbation $f_\epsilon = f_0 + \mathcal{O}(\epsilon)$, we are able to prove that there exists two exponents $m < \ell$ such that
  - The gap between adjacent perturbed KAM curves in Λ_ϵ is $= \mathcal{O}(\epsilon^m)$
  - The amplitude of the oscillations of $\sigma_\epsilon(\mathcal{T}_\epsilon)$, where $\mathcal{T}_\epsilon$ is any perturbed KAM curve and σ_ϵ is the perturbed scattering map, is $\geq \mathcal{O}(\epsilon^\ell)$.
- Then we can construct a transition chain of tori such that the unstable whisker of one intersects the stable whisker of the next. Eureka!

Scattering map: Applications

- Prove the existence of orbits with unbounded energy (and so, velocity) in perturbations of a generic geodesic flow in the two-dimensional torus by a generic periodic (in time) potential.
  Remark: It is possible to choose the metric and the potential arbitrarily close to the flat metric and zero, respectively.
- Extend the previous result to higher-dimensional compact manifolds (not necessarily tori) and to quasiperiodic (in time) potentials.
- Overcome the large gap problem in some diffusion problems.
- Study of homoclinic zero cost transfers from a Lyapunov orbit around a collinear libration point in the PRTBP to itself, but changing the phase.
- Study a similar problem for a three-degrees-of-freedom system: the spatial Hill's problem.

Scattering map: References

1. A. Delshams, R. de la Llave, and T. M. Seara, A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of T^2, *Comm. Math. Phys.*, **209**:353–392 (2000).
2. A. Delshams, R. de la Llave, and T. M. Seara, Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows, *Adv. Math.*, **202**:64–188 (2006).
3. A. Delshams, R. de la Llave, and T. M. Seara, A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model, *Mem. Amer. Math. Soc.*, **179**:viii+141 pp. (2006).
4. E. Canalias, A. Delshams, J. J. Masdemont, and P. Roldán, The scattering map in the planar restricted three body problem, *Celestial Mech. Dynam. Astronom.*, **95**:155–171 (2006).
5. A. Delshams, R. de la Llave, and T. Seara, Geometric properties of the scattering map of a normally hyperbolic invariant manifold, *Adv. Math.*, **217**:1096–1153 (2008).