Multiple precision computation of singular splittings for planar maps

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Mission statements

- Present the exponentially small splitting problem for analytic area-preserving maps.
- Explain the computational challenges of this problem.
- Give some general principles to improve the efficiency of any computation that requires the use of a multiple precision arithmetic.
- Learn how to compute the Lazutkin homoclinic invariant in the general case.
- Implement explicitely the simplest case: the Hénon map.

Basic definitions (1/2)

- A surface M is *symplectic* when it has a non-degenerate two-form Ω. The simplest example is $M = \mathbb{R}^2$ and $\Omega = \mathrm{d}x \wedge \mathrm{d}y$.
- A map $f : M \to M$ is *area-preserving* when $f^*\Omega = \Omega$.
- Classical examples of area-preserving maps are the *standard maps*

$$f(x,y) = (x_1 = x + y_1, y_1 = y + \epsilon p(x)), \qquad \epsilon > 0$$

  where $p(x)$ is a polynomial, trigonometric polynomial or rational function.
- A point $m_\infty \in \mathbb{R}^2$ is a *saddle point* of f when it is *fixed*: $f(m_\infty) = m_\infty$ and *hyperbolic*: $\mathrm{spec}[\mathrm{d}f(m_\infty)] = \{\lambda, \lambda^{-1}\}$ with $|\lambda| > 1$.
- We assume that the *characteristic multiplier* λ is bigger than one.
- The *stable* and *unstable invariant curves* of the saddle point are

$$W^\pm = W^\pm(m_\infty) = \left\{ m \in \mathbb{R}^2 : \lim_{n \to \mp\infty} f^n(m) = m_\infty \right\}.$$

  (Note: Minus sign means stable curve, plus sign means unstable curve.)

Basic definitions (2/2)

- If the map is analytic, its invariant curves are analytic and there exists some analytic *natural parameterizations* $m_\pm : \mathbb{R} \to W^\pm$ such that $m_\pm(0) = m_\infty$ and

$$f(m_\pm(r)) = m_\pm(\lambda^{\pm 1} r).$$

  They are uniquely defined up to substitutions of the form $r \mapsto cr$ with $c \neq 0$.
- Given any $r_1 > 0$, $D_\pm = m_\pm([\lambda^{-1} r_1, r_1))$ is a *fundamental domain* of $W^\pm$. The iterations $\{f^n(D_\pm) : n \in \mathbb{Z}\}$ cover the "positive" branch of $W^\pm$.
- An orbit $O = (m_n)_{n \in \mathbb{Z}}$ is *homoclinic* to m_∞ when $\lim_{n \to \pm\infty} m_n = m_\infty$.
- The *Lazutkin homoclinic invariant* of a homoclinic point m_0 is the quantity

$$\omega = \omega(m_0) := r_- r_+ \Omega\big(m'_-(r_-), m'_+(r_+)\big).$$

  where $r_\pm \in \mathbb{R}$ are the parameters such that $m_\pm(r_\pm) = m_0$. It does not depend on the point of the homoclinic orbit: $\omega(m_n) = \omega(m_0)$ for all n, so that we can write $\omega = \omega(O)$. It is invariant by symplectic changes of variables and is proportional to the splitting angle.

Reversors

In general, the search of homoclinic points of planar maps is a two-dimensional problem, but in some symmetric cases. For instance, in the reversible case.

- A diffeomorphism $f : M \to M$ is *reversible* when there exists a diffeomorphism $R : M \to M$ such that $f \circ R = R \circ f^{-1}$, and then R is called a *reversor* of the map. Usually, R is an involution: $R^2 = \mathrm{I}$.
- If R is a reversor, the points in $\mathrm{Fix}\, R = \{m \in M : R(m) = m\}$ are *symmetric*. Usually, $\mathrm{Fix}\, R$ is a smooth curve and then it is called a *symmetry line*.
- Let f be a R-reversible diffeomorphism with a saddle point $m_\infty \in \mathrm{Fix}\, R$. Let m be a natural parameterization of its unstable invariant curve W^+. Then:
  - $R \circ m$ is a natural parameterization of the stable invariant curve W^-.
  - If $m_0 = m(r_0) \in \mathrm{Fix}\, R$, then m_0 is a (symmetric) homoclinic point whose Lazutkin homoclinic invariant is

$$\omega(m_0) = (r_0)^2 \Omega\big(\mathrm{d}R(m_0) m'(r_0), m'(r_0)\big).$$

  - To find r_0, it suffices to solve the one-dimensional problem $m(r) \in \mathrm{Fix}\, R$.

An exponentially small upper bound

- We shall deal with maps whose stable and unstable invariant curves are exponentially close with respect to some small parameter.
- In order to derive simple expressions, the best parameter is the *characteristic exponent* of the saddle point: $h = \ln \lambda > 0$.
- (Fontich & Simó) Let $f_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be a diffeomorphism such that:
  - It is area-preserving and analytic in a big enough complex region;
  - It is $\mathcal{O}(h)$-close to the identity map;
  - The origin is a saddle point of f_h;
  - Its characteristic exponent at the origin is h; and
  - It has a homoclinic orbit to the origin for small enough h.

  Then, there exists $d_* > 0$ such that:

$$\text{splitting size} \leq \mathcal{O}(\mathrm{e}^{-2\pi d/h}) \qquad (h \to 0^+)$$

  for any $d \in (0, d_*)$. Besides, d_* is the analyticity width of the separatrix of certain limit Hamiltonian flow. Sometimes, it can be analytically computed.

The Standard map

- The first example is the *Standard map*

$$SM : \mathbb{T}^2 \to \mathbb{T}^2, \qquad SM(x,y) = (x + y + \epsilon \sin x, y + \epsilon \sin x).$$

- If $\epsilon > 0$, the origin is hyperbolic and $\epsilon = 4 \sinh^2(h/2)$.
- The map $R(x,y) = (2\pi - x, y + \epsilon \sin x)$ is a reversor, and $\operatorname{Fix} R = \{x = \pi\}$.
- (Gelfreich) Let ω be the Lazutkin homoclinic invariant of the symmetric homoclinic orbit passing through the first intersection of W^+ with $\operatorname{Fix} R$. Then

$$\omega \asymp 4\pi h^{-2} \mathrm{e}^{-\pi^2/h} \sum_{j \geq 0} \omega_j h^{2j} \qquad (h \to 0^+).$$

  This asymptotic expansion was proved using an approach suggested by Lazutkin.
- The first asymptotic coefficient $\omega_0 \approx 1118.827706$ is the *Lazutkin constant*.
- Simó conjectured that the series $\sum_{j \geq 0} \omega_j h^{2j}$ is Gevrey-1 of type $1/2\pi^2$.

The Hénon map

- The second example is the *Hénon map*

$$HM:\mathbb{R}^2\to\mathbb{R}^2,\quad HM(x,y)=\big(x+y+\epsilon x(1-x),y+\epsilon x(1-x)\big).$$

- If $\epsilon>0$, the origin is hyperbolic and $\epsilon=4\sinh^2(h/2)$.
- The map $R(x,y)=(x-y,-y)$ is a reversor, and $\operatorname{Fix}R=\{y=0\}$.
- Let ω be the Lazutkin homoclinic invariant of the symmetric homoclinic orbit passing through the first intersection of W^+ with $\operatorname{Fix}R$. Then

$$\omega\asymp 4\pi h^{-6}\mathrm{e}^{-2\pi^2/h}\sum_{j\ge 0}\omega_j h^{2j}\qquad (h\to 0^+).$$

  I do not know any complete proof of this asymptotic expansion.
- The first coefficient $\omega_0\approx 2474425.5935525105384$ was "approximated" by Chernov and "computed" by Simó. Gelfreich & Sauzin proved that $\omega_0\neq 0$.
- Gelfreich & Simó conjectured that $\sum_{j\ge 0}\omega_j h^{2j}$ is Gevrey-1 of type $1/2\pi^2$.

Polynomial standard maps

- The Hénon map is a particular case of the *polynomial standard maps*

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \qquad f(x,y) = (x + y + \epsilon p(x), y + \epsilon p(x))$$

  for some polynomial $p(x) = \sum_{k=1}^{n} p_k x^k$ such that $p_1 = 1$ and $p_n < 0$.
- If $\epsilon > 0$, the origin is hyperbolic and $\epsilon = 4\sinh^2(h/2)$.
- The map $R(x,y) = (x - y, -y)$ is a reversor, and $\mathrm{Fix}\, R = \{y = 0\}$.
- Let ω be the Lazutkin invariant of the primary symmetric homoclinic orbit associated to the reversor R. Gelfreich & Simó conjectured that:
  - The expansion $\omega \asymp \mathrm{e}^{-c/h} \sum_{k \geq k_0} c_k h^k$ *does not* hold for most $p(x)$.
  - There exist alternative asymptotic expansions with logarithmic terms and/or rational powers of h.
  - Sometimes, the series involved in these expansions are Gevrey-1.
  - If $n \geq 4$, ω can oscillate periodically in h^{-1}. If $n \geq 6$, the oscillations can be quasi-periodic.

Perturbed weakly hyperbolic integrable maps (1/2)

- The *perturbed McMillan maps* are

$$f:\mathbb{R}^2\to\mathbb{R}^2,\qquad f(x,y)=\Big(y,-x+2\mu_0 y/(1+y^2)+\epsilon V'(y)\Big)$$

  where $\epsilon V'(y)$ is an odd entire perturbation.
- If $\mu=\mu_0+\epsilon V''(0)>1$, the origin is hyperbolic and $\cosh h=\mu$.
- The map $R(x,y)=(y,x)$ is a reversor, and $\operatorname{Fix} R=\{y=x\}$.
- (Delshams & RRR; Martín, Sauzin & Seara) Let ω be the Lazutkin homoclinic invariant of the symmetric homoclinic orbit passing through the first intersection of W^+ with $\operatorname{Fix} R$. Let $\widehat{V}(\xi)$ be the Borel transform of $V(y)$. Then

$$\omega=16\pi^3\epsilon h^{-2}\mathrm{e}^{-\pi^2/h}\big(\widehat{V}(2\pi)+\mathcal{O}(h^2)\big)\qquad(\epsilon,h\to 0^+).$$

- Conjecture 1: $\omega\asymp 16\pi^3\epsilon h^{-2}\mathrm{e}^{-\pi^2/h}\sum_{j\geq 0}\omega_j(\epsilon)h^{2j}$ as $h\to 0^+$ (ϵ fixed).
- Conjecture 2: The series $\sum_{j\geq 0}\omega_j(\epsilon)h^{2j}$ is Gevrey-1 of type $1/2\pi^2$.

Perturbed weakly hyperbolic integrable maps (2/2)

- Let $f : \mathbb{T} \times (0, \pi) \to \mathbb{T} \times (0, \pi)$ be the area-preserving map that models the *billiard motion* inside the perturbed ellipses

$$C = \left\{ (x, y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{1 - e^2} + \epsilon (ey)^{2n} = 1 \right\}.$$

  Here, $e \in (0, 1)$ is the *eccentricity* of the unperturbed ellipse, ϵ is the *perturbative parameter*, and $2n$ is the degree of the perturbation.

- The map has a two-periodic hyperbolic orbit such that $e = \tanh(h/2)$.
- The map is reversible, due to the axial symmetries of the curves.
- RRR conjectured that the Lazutkin invariant of the corresponding symmetric heteroclinic orbit verifies the asymptotic expansion

$$\omega \asymp 2\pi^2 h^{-2} \epsilon \mathrm{e}^{-\pi^2/h} \sum_{j \geq 0} \omega_j(\epsilon) h^{2j} \qquad (h \to 0^+, \epsilon \text{ fixed})$$

  and the series $\sum_{j \geq 0} \omega_j(\epsilon) h^{2j}$ is Gevrey-1 of type $1/2\pi^2$.

First numerical problem: slow dynamics

- Let f be a R-reversible area-preserving map with a saddle point m_∞ whose unstable curve intersects the symmetry line $\operatorname{Fix} R$.
- Let $m(r)$ be a natural parameterization of the unstable invariant curve W^+.
- Let $r_0 > 0$ be the first positive parameter such that $m_0 = m(r_0) \in \operatorname{Fix} R$.
- To find numerically m_0, we solve the one-dimensional equation

$$f^N(m(r)) \in \operatorname{Fix} R, \qquad \lambda^{-1} r_1 \le r < r_1$$

where:

  - The fundamental domain $D = m([\lambda^{-1} r_1, r_1))$ must be chosen in such a way that the natural parameterization $m(r)$ can be computed with a given precision P for any $r \in (0, r_1)$. [precision P means error $\le 10^{-P}$.]
  - N is the smallest integer such that $f^N(D) \cap \operatorname{Fix} R \neq \emptyset$. Thus,

$$N \approx \frac{\log(r_0/r_1)}{h}.$$

Second numerical problem: cancellations (1/2)

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map preserving the standard area $\Omega = \mathrm{d}x \wedge \mathrm{d}y$.
- Let $m_\pm : \mathbb{R} \to W^\pm$ be some natural parameterizations of the stable and unstable invariant curves.
- Let $r_\pm \in \mathbb{R}$ be parameters such that $m_+(r_+) = m_0 = m_-(r_-)$.
- Let $m'_\pm = (x'_\pm, y'_\pm) = m'(r_\pm)$.
- If the Lazutkin homoclinic invariant

$$\omega = \omega(m_0) = r_- r_+ \Omega(m'_-, m'_+) = r_- r_+ (x'_- y'_+ - x'_+ y'_-)$$

is exponentially small in h, then the invariants $\omega_+ := r_- r_+ x'_- y'_+$ and $\omega_- := r_- r_+ x'_+ y'_-$ are exponentially close in h. Thus, the computation of their difference $\omega = \omega_+ - \omega_-$ produces a *big cancellation* of significant digits, even for moderate values of h.

Second numerical problem: cancellations (2/2)

- For sample, if $h = 1/7$, then

$$\begin{aligned} \omega_+ &\approx -0.005798965148971595791562099032310981683639488\textcolor{red}{269378} \\ \omega_- &\approx -0.005798965148971595791562099032310981683639488\textcolor{red}{305137} \end{aligned}$$

  for the primary homoclinic point of Hénon map on the x-axis.
- Therefore, *44* decimal digits are lost when we compute the difference

$$\omega = \omega_+ - \omega_- \approx 3.5759 \times 10^{-48}.$$

- The above computation is beyond single, double, and quadruple precisions. The use of a multiple precision arithmetic (MPA) is mandatory.
- In general, if we "know" that $\omega \asymp \mathrm{e}^{-c/h}$, then the number of decimal digits lost by the cancellation in the differences is approximately equal to

$$S = S(h) = \frac{c}{h \log(10)} = \mathcal{O}(1/h).$$

These problems are a bad combination

- Let $\overline{m}(r)$ be our numerical approximation to the parameterization $m(r)$. Assume that, if r is small enough, we have a bound for the error of the form

$$|m(r) - \overline{m}(r)| \leq Cr^{K+1}$$

for some constant $C > 0$ and some fixed order $K \geq 1$.

- Problem 2 implies that we must work with precision $P \geq S = \mathcal{O}(1/h)$.
- Then, we must choose $r_1 > 0$ such that

$$|m(r_1) - \overline{m}(r_1)| \leq C(r_1)^{K+1} \leq 10^{-P}.$$

That is, $r_1 = \mathcal{O}(10^{-P/(K+1)})$, and so: $-\log r_1 = \mathcal{O}\big(P/(K+1)\big) = \mathcal{O}(1/h)$.

- Besides, r_0 tends to some non-zero value as $h \to 0^+$.
- Finally, Problem 1 implies that, if K is fixed, then the number of iterations is

$$N \approx \frac{\log(r_0/r_1)}{h} = \mathcal{O}\big(P/(K+1)h\big) = \mathcal{O}(1/h^2).$$

And that's not all, folks!

- If we fix the order K of the error in the computation of $m(r)$, then the number of iterations is $N = \mathcal{O}(1/h^2)$. One could think that a quadratic increase in the number of operations is not very dramatic, but stay tuned!
- Besides, the precision $P = \mathcal{O}(1/h)$ also grows. We assume that the cost of one product is quadratic in P. Other operations like the evaluation of transcendental functions are worse.
- (There exist asymptotically faster implementations of MPAs (for instance, using the Karatsuba multiplication), but they become useful only for extremely high values of P.)
- Finally, the number of iterations to solve a 1D nonlinear equation with precision P by any standard iterative method (Newton's, Brent's, Ridders', etc.) grows logarithmically in P.
- Hence, the CPU time to solve the 1D nonlinear equation $f^N(m(r)) \in \operatorname{Fix} R$ is at least $\mathcal{O}(N \times P^2 \times \log(P)) = \mathcal{O}(h^{-4}|\log h|)$. Bad.

Promises, promises, ...

- We shall explain in the second hour how to deal with these numerical problems.
- For instance, we shall describe some (big, little and silly) tricks that give rise to an algorithm that takes an $\mathcal{O}\left(h^{-10/3}\right)$ time to compute the Lazutkin homoclinic invariant for the Hénon map.
- Besides, we shall show a short and simple GP-program, called `Henon.gp`, to compute this Lazutkin homoclinic invariant in a fast way. We shall also explain what means “GP-program”.

The principles of multiple precision computation

- Time is money.
- Empty your mind.
- Search & compare.
- Don't be too obsessive.
- Don't be too transcendental.
- Sometimes, be rational.

Time is money

Current best price: less than 1$ per GFLOP. Explanations are unnecessary.[a]

[a] "Nowadays people know the price of everything and the value of nothing", Oscar Wilde (The Picture of Dorian Gray, 1891)

Empty your mind

- There are some principles that are good for single precision arithmetic, but a disaster in MPA.
- You must think carefully about how the MPA affects your algorithm.
- *Example of bad principle:* "A product is more expensive that a sum, but not MUCH more." This is clearly false in MPA. We are talking about different orders of magnitude.
- *Example of good idea:* Complex multiplication can be reduced to a sequence of ordinary operations on real numbers, but there are two ways.
  - "Expensive" way (using 4 real multiplications and 2 real additions):

$$(a + b\mathrm{i}) \times (c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}.$$

  - "Cheap" way (using 3 real multiplications and 5 real additions):

$$(a + b\mathrm{i}) \times (c + d\mathrm{i}) = (p - q) + (p + q + r)\mathrm{i}$$

    where $p = ac$, $q = bd$, and $r = (a - b)(d - c)$.

Search & compare

- Try several different methods and compare them. First and/or lazy choices are usually not the best ones.
- *Example:* To solve the nonlinear one-dimensional equation to compute the primary homoclinic point of the Hénon map, I compared the following possibilities: Newton's method, Ridders's method, secant method, and the GP-routine `solve`.
  The last one was my first try, but it was the worse one.
  Newton's method was the best choice.

Don’t be too obsessive

- *First Rule:* If at some moment, you have to work a couple of (human) days to win a couple of (CPU or GPU) seconds, something is wrong.
- *Second Rule:* Don’t forget the First Rule.

Don't be too transcendental

Transcendental operations must be avoided as much as possible. I have used the following tricks in several splitting problems:

- *Working with the Hénon map:* If $\lambda = \mathrm{e}^h$ and $\epsilon = 4\sinh^2(h/2)$, then $\epsilon = \lambda - 2 + \lambda^{-1}$.
- *Computing the lobe area of some perturbed McMillan maps:*

$$\sum_{n=1}^{N} \log(x_n) = \log\left(\Pi_{n=1}^{N} x_n\right).$$

- *Computing Melnikov functions of some volume-preserving maps:* If $r = \mathrm{e}^t$, $\lambda = \mathrm{e}^h$, and $\mu = \mathrm{e}^{\omega \mathrm{i}}$, then

$$E(t) := \sum_{k\in\mathbb{Z}} \frac{\cos(\omega k)}{\cosh(t + kh)} = \sum_{k\in\mathbb{Z}} \frac{\mu^k + \mu^{-k}}{\lambda^k r^2 + \lambda^{-k}} r.$$

Sometimes, be rational

If we are working with a MPA, rational numbers have two good properties:

- They are *cheap*. A `rational` $\times$ `real` product is peccadillo with respect to a `real` $\times$ `real` one when the numerator and denominator are not too big integers.
  Example: If we perform an heuristic study on some "continuous" property for the Hénon map
  $$(x, y) \mapsto (x + y + \epsilon x(1 - x), y + \epsilon x(1 - x))$$
  in the range $\epsilon \in (a, b)$ that requires the computation of many iterates with a very high precision, take $\epsilon \in (a, b) \cap \mathbb{Q}$.
- They are *exact*. For instance, they are not affected by changes in the precision and they can not be the weak link in any computation.

Bibliography

The following works contain multiple precision computations related to exponentially small phenomena in analytic area-preserving maps.

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Software options

There are several choices to carry out a multiple precision computation.

- *Hand-made.* Write your own implementation starting from scratch. It is a hard and long way, but it is highly educative. It can be useful to read the Knuth's book about this subject. The choice of real men. [a]
- *Commercial packages (Mapple, Mathematica,...).* I don't like this option [b], but as a first approach or for some toy problems. Nevertheless, this option has been succesfully used in some recent research papers.
- *PARI/GP (`http://pari.math.u-bordeaux.fr/`).* A free computer algebra system designed for fast computations in number theory. It can be used as a C library (called `PARI`) or in a interactive shell (called `gp`) giving access to the PARI functions. The second one is my current choice, because it provides a readable code. See the attached GP-program for the Hénon map.
- *GMP (`http://gmplib.org/`).* A free library for arbitrary precision arithmetic. It is the fastest option (with my apologies to real men).

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[a] "Write your own programs, be a man", Carles Simó (s'Agaró, June 2nd 2006)

[b] "Software is like sex: it's better when it's free", Linus Torvalds

Notations (1/2)

- M is the bi-dimensional phase space
- Ω is the area form.
- $f : M \to M$ is the analytic weakly-hyperbolic area-preserving map.
- $R : M \to M$ is the reversor.
- $\operatorname{Fix} R = \{m \in M : G(m) = 0\}$ is the symmetry line of the reversor.
- $m_\infty \in M$ is the saddle point.
- $\lambda \gtrsim 1$ is the characteristic multiplier.
- $h = \log \lambda \ll 1$ is the characteristic exponent.
- $W^\pm$ are the stable and unstable invariant curves of the saddle point.
- $m : \mathbb{R} \to W^+$ is the natural parameterization of the unstable curve.
- $m_0 = m(r_0)$, $r_0 > 0$, is the primary symmetric homoclinic point on $\operatorname{Fix} R$.
- $\omega = (r_0)^2 \Omega\big(\mathrm{d}R(m_0) m'(r_0), m'(r_0)\big)$ is the Lazutkin homoclinic invariant.

Notations (2/2)

- $c > 0$ is the constant such that $\omega = \mathcal{O}(\mathrm{e}^{-c/h})$ as $h \to 0^+$.
- $D = m([r_1/\lambda, r_1))$, $0 < r_1 < r_0$, is a fundamental domain of W^+.
- $\overline{m}(r)$ is our numerical approximation to the natural parametrerization $m(r)$.
- $K + 1$ is the order of the error in the previous approximation:

$$|m(r) - \overline{m}(r)| = \mathcal{O}\left(r^{K+1}\right).$$

- N is the smallest integer such that $f^N(D) \cap \operatorname{Fix} R \neq \emptyset$.
- $\bar{r}_0 \in \left[r_1/\lambda, r_1\right)$ is the root of the one-dimensional equation

$$Z(r) := G(f^N(m(r))) = 0.$$

First big trick: Don't fix the order

- In order to control the number of iterations $N = \mathcal{O}\big(P/(K+1)h\big)$, the order K must increase when $h \to 0^+$.
- Orders below hundreds do not serve in edge scenarios. For sample, we shall see that the optimal choice in the Hénon map with $h = 0.02$ is $K \approx 100$.
- Therefore, we must find a recursive algorithm to determine the Taylor coefficients up to any given (but arbitrary!) order.
- It is easier to find a good algorithm for maps that have explicit expressions: the Hénon map, the Standard map, polynomial standard maps, perturbed McMillan maps, etc.
- Implicit maps can also be dealt with, although they require more work. For instance, there is a nice algorithm for the billiard maps previously introduced.

A sample: the Hénon map

- Let $x(r) = \sum_{k\geq 1} x_k r^k$ and $y(r) = \sum_{k\geq 1} y_k r^k$ be the Taylor expansions of the natural parameterization $m(r) = (x(r), y(r))$ of the Hénon map

$$x_1 = x + y_1, \qquad y_1 = y + \epsilon x(1-x).$$

- The relation $f(m(r)) = m(\lambda r)$ is equivalent to the functional equations

$$x(\lambda r) - x(r) = y(\lambda r), \qquad y(\lambda r) - y(r) = \epsilon x(r)\big(1 - x(r)\big).$$

- We get from relation $x(\lambda r) - (2+\epsilon)x(r) + x(r/\lambda) = -\epsilon x(r)^2$ that

$$d_k x_k = -\epsilon \textstyle\sum_{j=1}^{k-1} x_j x_{k-j}, \qquad \forall k \geq 1$$

  where $d_k = \lambda^k - (2+\epsilon) + \lambda^{-k}$ and $d_k = 0 \Leftrightarrow k = \pm 1$.
- Hence, x_1 is free and we normalize it by taking $x_1 = 1$.
- Next, we can compute recursively x_k for all $k \geq 2$.
- Finally, $y(\lambda r) = x(\lambda r) - x(r) \Longrightarrow y_k = (1 - \lambda^{-k}) x_k$ for any $k \geq 1$.

A couple of little tricks

- Evaluate the Taylor expansions using the *Horner's rule*.
- The computational effort to perform the *convolution*

$$\sum_{j=a}^{b-a} x_j x_{b-j} = x_a x_{b-a} + x_{a+1} x_{b-a-1} + \cdots + x_{b-a-1} x_{a+1} + x_{b-a} x_a$$

can be reduced by half using the formulae

$$\sum_{j=a}^{b-a} x_j x_{b-j} = \begin{cases} 2\sum_{j=a}^{(b-1)/2} x_j x_{b-j} & \text{if } b \text{ is odd} \\ 2\sum_{j=a}^{b/2-1} x_j x_{b-j} + \left(x_{b/2}\right)^2 & \text{if } b \text{ is even} \end{cases} .$$

Second big trick: Don't fix the precision

- In order to find, with a high precision P, the root of a function $Z : (a,b) \to \mathbb{R}$ such that $Z(a)$ and $Z(b)$ have opposite signs, we shall apply the following algorithm:
  1. Refine the interval (a,b) with some secure method (bisection, Brent's) in "single" precision.
  2. Choose some fast iterative method (Newton's, Brent's, Ridders') and *increase the precision by a factor equal to its order of convergence after each iteration*. For instance, doubling the precision in Newton's method.
  3. Stop the iterations when we exceed the given precision P.
  4. Don't check the error.
- This method rocks! Really.

A silly trick: Choose the optimal “single” precision

- This previous algorithm can give the root at the cost of just

$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{n \geq 0} 4^{-n} = 4/3$$

  evaluations of the function $Z(r) := G(f^N(m(r)))$ with precision P.
- The idea is silly, but effective: to determine the optimal “single” precision p from a certain limited range that gives the “final” precision P with the minimum computational effort.
- Example with Newton’s method: To reach $P = 4000$ from a “single” precision $p \leq 18$, we see that
  - $p = 18, 36, 72, 144, 288, 576, 1152, 2304, 4608, 9216, \ldots$
  - $p = 17, 34, 68, 136, 272, 544, 1088, 2176, 4352, 8704, \ldots$
  - $p = 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, \ldots$
  - $p = 15, 30, 60, 120, 240, 480, 960, 1920, 3840, 7680, \ldots$
  - Et cetera.

  Thus, $p = 16$ is the optimal “single” precision and $p = 15$ is the worst one.

Where are we now?

- The main numerical difficulties that appear during the study of the singular splitting of our maps are the computation of:
  - The map f and its differential with an arbitrary precision P;
  - The Taylor expansion of $m(r)$ up to an arbitrary order K; and
  - The Lazutkin homoclinic invariant ω with an arbitrary precision Q.
- Clearly, the precision Q is an input of the algorithm.
- On the contrary, P and K must be determined in an automatic way when the computation begins.

The choice of P

- We assume that $\omega = \mathcal{O}(\mathrm{e}^{-c/h})$ for some constant $c > 0$. For instance, we recall that

map	Hénon	Standard	polynomial	"McMillan"	"Billiard"
c	$2\pi^2$	π^2	variable	π^2	π^2

- Let $S \approx \frac{c}{h \log(10)}$ be the number of digits lost by cancellation.
- For the sake of safety, set $P = 1.1(Q + S)$.

The choice of r_1

- Let $\overline{m}_K(r) = \sum_{k=0}^{K} m_k r^k$ be the Taylor polynomial of degree K of the natural parameterization $m(r)$ of the unstable curve.
- Once fixed an order $K \geq 1$ and a precision P, we need a parameter $r_1 > 0$, as biggest as possible, such that

$$|m(r) - \overline{m}_K(r)| \leq 10^{-P}, \qquad \forall r \in (0, r_1).$$

- If the sequence $(m_k)_{k\geq 0}$ is alternate and $|m_k| \leq C\rho^k$ for some constants $C, \rho > 0$, then it suffices to set r_1 by means of the relation

$$C(\rho r_1)^{K+1} = 10^{-P}.$$

- These hypotheses hold for the Hénon map with $C = 1$ and $\rho = 1/5$, so we can set $r_1 = 5 \times 10^{-P/(K+1)}$.
- If the map is entire (as the Hénon map), the coefficients m_k decrease asymptotically at a factorial speed. Nevertheless, this factorial behaviour appears only at very high orders and so, it is not so useful.

The choice of K

- The order K is chosen to minimize the computation time.
- In order to determine it, we estimate the number of products $T = T(k)$, where the variable k runs over the range of possible orders.
- This number of products $T(k)$ is approximated by a sum of three terms related to: 1) computing the Taylor expansions, 2) solving the nonlinear equation $Z(r) = 0$, and 3) computing the Lazutkin homoclinic invariant ω.
- For instance, using Newton's method in the Hénon map, we have that

$$T(k) \approx k^2/4 + 4N + 3N \approx k^2/4 + 7P\log(10)/kh$$

because $N \approx h^{-1}\log(r_0/r_1) = h^{-1}(\log r_0 - \log 5 + P\log(10)/(k+1)) \approx h^{-1}P\log(10)/(k+1)$. Therefore, the optimal order is

$$K \approx \sqrt[3]{14P\log(10)/h} = \mathcal{O}\left(P^{1/3}h^{-1/3}\right) = \mathcal{O}\left(h^{-2/3}\right)$$

and $N = \mathcal{O}(P/Kh) = \mathcal{O}\left(P^{2/3}h^{-2/3}\right) = \mathcal{O}\left(h^{-4/3}\right)$, since $P = \mathcal{O}\left(h^{-1}\right)$.

On the CPU time for the Hénon map

- How many "products" takes the computation of the Taylor expansion up to order K in the previous Hénon example?
  Answer: $K^2/4 + \mathcal{O}(K)$, if we use the convolution trick.
- How many "products" takes Newton's method in the Hénon map?
  Answer: One evaluation of $\mathrm{d}f$ requires 3 products, so $4N = \frac{4}{3}3N$ (approximately).
- Once computed the root $\bar{r}_0 \in [r_1\lambda, r_1)$ that gives the homoclinic point: How many "products" takes the computation of ω in the Hénon map?
  Answer: One evaluation of $\mathrm{d}f$ requires 3 products, so $3N$ (approximately).
- Using all the previous (big, little and silly) tricks and assuming that products in our multiple precision arithmetic take a time quadratic in P, the order of the CPU time in the Hénon problem for fixed Q can be reduced to $\mathcal{O}\left(h^{-10/3}\right)$ from the original $\mathcal{O}\left(h^{-4}|\log h|\right)$.
- *Challenge:* Improve this algorithm without changing the multiple precision arithmetic.

Some results for the Hénon map

Let ω be the Lazutkin homoclinic invariant of the symmetric homoclinic orbit passing through the first intersection of W^+ with the symmetry line $\{y = 0\}$.

The GP-code written in the file `Henon.gp` gives rise to the following results.

h	0.5	0.05	0.005	0.0005
P	75	245	1942	18916
K	20	55	233	1069
N	18	206	3859	81778
ω	1.36×10^{-8}	7.02×10^{-157}	5.93×10^{-1694}	1.1×10^{-17118}
time (ms)	4	24	2046	1009735

(CPU = Intel Core 2 Duo at 3 GHz, RAM = 2 Gb.)

The general algorithm

Given the characteristic exponent h and the desired precision Q, follow the steps:

1. Compute the number of digits $S \approx \frac{c}{h \log(10)}$ lost by cancellation.
2. Set the precision $P = 1.1(Q + S)$, by safety.
3. Choose the order K by minimizing the function $T(k)$.
4. Compute the Taylor expansion $\overline{m}(r) = \sum_{k=0}^{K} m_k r^k$.
5. Choose the biggest $r_1 > 0$ such that $|m(r) - \overline{m}(r)| \leq 10^{-P}$ for all $r \in (0, r_1)$.
6. Find the smallest integer N such that $f^N(\overline{m}([r_1/\lambda, r_1)) \cap \operatorname{Fix} R \neq \emptyset$.
7. Find the root $\bar{r}_0$ of the equation $G(f^N(\overline{m}(r))) = 0$ in the interval $[r_1/\lambda, r_1)$.
8. Compute the Lazutkin homoclinic invariant

$$\begin{aligned} \omega &= (r_0)^2 \Omega\big(\mathrm{d}R(m_0) m'(r_0), m'(r_0)\big) \\ &\approx (\bar{r}_0)^2 \Omega\big(\mathrm{d}R(f^N(\overline{m}(\bar{r}_0)))\mathrm{d}f^N(\overline{m}(\bar{r}_0))\overline{m}'(\bar{r}_0), \mathrm{d}f^N(\overline{m}(\bar{r}_0))\overline{m}'(\bar{r}_0)\big). \end{aligned}$$

9. Enjoy! (optional).

Exercises (1/2)

Write the recursions to compute the Taylor expansions of the natural paremeterizations in the following maps (in increasing order of difficulty):

- *(DS & RRR, 1999)* The perturbed McMillan map

  $$f(x,y) = \left(y, -x + 2\mu_0 y/(1+y^2) + \epsilon y^{2n+1}\right)$$

  for several "small" values of $n \geq 1$.
- *(VG & CS, 2007)* The polynomial maps $(x,y) \mapsto (x + y + \epsilon p(x), y + \epsilon p(x))$ for several "simple" polynomials or rational functions $p(x)$.
- *(CS, 20??)* The Standard map $(x,y) \mapsto (x + y + \epsilon \sin x, y + \epsilon \sin x)$.
- *(RRR, 2005)* The billiard maps associated to the perturbed ellipses

  $$C = \left\{ (x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{1-e^2} + \epsilon (ey)^{2n} = 1 \right\}$$

  for several "small" values of $n \geq 2$.

Exercises (2/2)

- Estimate the order of the general algorithm for all of the previous maps.
- Implement this algorithm in some platform (GMP, PARI/GP, real men) for some of the previous maps.
- Write a paper describing and improving the general algorithm and estimate explicitely its cost in terms of the cost of one evaluation of the map and the multiple precision arithmetic used.
- Send me the preprint.