Existence and non-existence of (convex) caustics

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Mission statements

- Define caustics in the frame of convex billiard tables.
- Give some (qualitative and quantitative) negative results due to Mather, Gutkin and Katok: There are no convex caustics if the boundary of the billiard table has a flat point.
- Explain the string construction and introduce the Lazutkin parameter.
- State the positive result of Lazutkin: There exist infinitely many convex caustics close to the border of any sufficiently smooth and strictly convex billiard table.
- Clarify the situation in the three-dimensional case: Berger.

Convex curves

Let Γ be a smooth closed curve of the plane $\mathbb{R}^2$ of length L.

- *Arc length parameterization:* $c : [0, L] \to \Gamma$, $c = c(s)$, counterclockwise.
- *Unit tangent vector:* $t(s) = c'(s)$.
- *Unit inward normal vector:* $n(s)$.
- *Curvature and radius of curvature:* $c''(s) = \kappa(s) n(s)$, and $\rho(s) = 1/\kappa(s)$.
- *Convexity:* $\kappa(s) \geq 0$.
- *Strict convexity:* $\kappa(s) > 0$.
- *Flat points:* Γ is flat at $c_0 = c(s_0)$ if and only if $\kappa(s_0) = 0$.
- *Examples:*
  1. The curvature of a circumference is constant: $\kappa \equiv 1/r$, r is the radius.
  2. A straight line has zero curvature.
  3. The region $\Omega = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 \leq 1\}$ is convex, but its boundary has four flat points: $(\pm 1, 0)$ and $(0, \pm 1)$.

Convex billiards

- *Billiard table:* $\Omega \subset \mathbb{R}^2$ is a convex region, $\Gamma = \partial\Omega$, and $L = \text{Length}(\Gamma)$.
- *Billiard dynamics:* The angle of incidence equals the angle of reflection.
- *Configuration space:* $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$.
- *Phase space:* $\mathbb{A} = \mathbb{T} \times (0, \pi)$.
- *Billiard coordinates:* Each point $(s, \theta) \in \mathbb{A}$ determines the impact point $c = c(s)$ and the angle of incidence-reflection θ.
- *Billiard map:* $f : \mathbb{A} \to \mathbb{A}$, $f(s, \theta) = (s', \theta')$, which is C^r if Γ is C^{r+1}.
- *Fermat's principle:* Rays of light follow paths of stationary length.
- *Lagrangian:* $h : \mathbb{T}^2 \setminus \{s = s'\} \to \mathbb{R}_+$, $h(s, s') = |c(s) - c(s')|$.
- *Lagrangian formulation:* There exists a billiard trajectory from $c_- = c(s_-)$ to $c_+ = c(s_+)$ passing through $c = c(s)$ if and only if

$$\partial_2 h(s_-, s) + \partial_1 h(s, s_+) = 0.$$

- *Twist character:* $\frac{\partial s'}{\partial \theta} = \frac{h(s,s')}{\sin \theta'} > 0$, so $(0, \pi) \ni \theta \mapsto s' \in \mathbb{T} \setminus \{s\}$ is a diffeo.

Convex caustics and invariant circles

Let γ be a smooth convex closed *caustic* of a billiard table Ω. That is, a billiard trajectory, once tangent to γ, stays tangent after every reflection. Then:

- The billiard map $f : \mathbb{A} \to \mathbb{A}$ has two *invariant circles* $\hat{\gamma}^+, \hat{\gamma}^- \subset \mathbb{A}$. More precisely, there exists a Lipschitz function $\delta : \mathbb{T} \to (0, \pi)$ such that

  $$\hat{\gamma}^+ = \{(s, \delta(s)) : s \in \mathbb{T}\}, \qquad \hat{\gamma}^- = \{(s, \pi - \delta(s)) : s \in \mathbb{T}\}$$

  are graphs invariant under the billiard f.
- We denote $\hat{\gamma}_0^+ = \{(s, 0) : s \in \mathbb{T}\}$ and $\hat{\gamma}_0^- \{(s, \pi) : s \in \mathbb{T}\}$.
- There exist a smooth diffeomorphism $g : \mathbb{T} \to \mathbb{T}$ of degree one such that

  $$f(s, \delta(s)) = \big(g(s), \delta(g(s))\big).$$
- Using that $g'(s) > 0$, we deduce that $\gamma \subset \Omega$.
- The phase space $\mathbb{A}$ can be decomposed into three invariant regions with non-empty interior, so the billiard map f is not ergodic.

Convex caustics and the mirror equation

- *Mirror equation:* Let A and B be the signed distances from the support points a and b to the impact point x. By convention, $A > 0$ if the incoming beam focuses before the reflection, and $B > 0$ if the reflected beam focuses after the reflection. Then
$$\frac{1}{A} + \frac{1}{B} = \frac{2\kappa}{\sin\theta}.$$
- *Example:* If Γ is a straight line, then $\kappa = 0$ and $B = -A$.
- *Important:* If γ is a convex caustic of a convex curve Γ, then $A, B \geq 0$.
- *Proof:* We can assume, without loss of generality, that $x = c(0)$, where $c : \mathbb{T} \to \Gamma$ is the arc length parameterization of $\Gamma = \partial\Omega$.
Next, we consider the length function
$$D(s) = |c(s) - a| + |c(s) - b|.$$
Rays of light follow paths of stationary length, so $D'(0) = 0$.
Infinitesimally close rays from a also reflect to rays through b, so $D''(0) = 0$, which is equivalent to the mirror equation. QED.

Non-existence of convex caustics: Glancing orbits

- *Theorem (Mather):* If the border of the convex billiard table has some flat point, then there are no smooth convex caustics inside the table.
- *Proof:* It is a corollary of the mirror equation, although Mather used another method based on the Lagrangian formulation.
- *Glancing trajectories:* A billiard trajectory is positively (resp., negatively) ϵ-glancing if, for some bounce, the angle of reflection with the positive (resp., negative) tangent vector is smaller than ϵ. Mather deduced, under the same flat point assumption, the existence of billiard trajectories that are both positively and negatively ϵ-glancing for any $\epsilon > 0$.
- *Open problem:* To bound the number of impacts $n = n(\epsilon)$ of such glancing billiard trajectories between its positive and negative ϵ-bounces as $\epsilon \to 0$.

The string construction and the Lazutkin parameter

- *Questions:* How can be constructed a billiard table Ω with a prefixed smooth convex caustic γ? How many of such tables do exist?
- *String construction:* For any $S > \text{Length}(\gamma)$, let us wrap a closed inelastic string of length S around γ, pull it tight at a point and move the point around γ to enclose a billiard table Ω. Hence, $\Gamma = \partial\Omega$ is an *involute* of γ, whereas γ is an *evolute* of Γ.
- *Example:* If γ is the segment with endpoints a and b, then Γ is the ellipse with foci a and b whose major axis is equal to $S - |a-b|$.
- *Theorem:* The billiard tables obtained through the string construction are the only ones with γ as caustic.
- *Lazutkin's parameter:* $\text{Lz}(\gamma;\Gamma) := S - \text{Length}(\gamma) > 0$. Clearly,
  1. $\Gamma \to \gamma$ as $\text{Lz}(\gamma;\Gamma) \to 0^+$; and
  2. Γ looks like a "big circumference centered at γ" as $\text{Lz}(\gamma;\Gamma) \to +\infty$.
- *Rotation number:* $\text{Rot}(\gamma;\Gamma) \in (0, 1/2]$ is the number of turns (in average) around γ per bounce. Clearly, $\text{Rot}(\gamma;\Gamma) \to 0^+$ as $\gamma \to \Gamma$.

Non-existence of convex caustics: Quantitative results

Let Ω be a smooth convex billiard table and $\Gamma = \partial\Omega$.

- Let $\underline{\kappa} = \min \kappa(s)$ and $\overline{\kappa} = \max \kappa(s)$, where $\kappa(s)$ is the curvature of Γ.
- Let L be the length of the curve Γ.
- Let d, w, and r be the the *diameter*, the *width*, and the *inradius* of the table Ω.
- *Theorem (Gutkin & Katok):* If some of the following geometric conditions holds, then the table Ω contains a region Ω_0 free of convex caustics.

Condition	Description of Ω_0
$\sqrt{2}\underline{\kappa}d^2 \leq r$	A disc of radius r_0 such that $r_0 > r - \sqrt{2}\underline{\kappa}d^2$
$\sqrt{2}\underline{\kappa}d^2 \leq w/3$	A disc of radius r_0 such that $r_0 > w/3 - \sqrt{2}\underline{\kappa}d^2$
$\sqrt{2}\underline{\kappa}\overline{\kappa}d^2 \leq 1$	A disc of radius r_0 such that $\overline{\kappa}r_0 > 1 - \sqrt{2}\underline{\kappa}\overline{\kappa}d^2$
$\sqrt{2}\underline{\kappa}\overline{\kappa}d^2 \leq 1$	A convex set such that $\mathrm{Area}(\Omega \setminus \Omega_0) \leq \sqrt{2}\underline{\kappa}d^2 L$

- *Example 1:* If Γ has a flat point, then $\underline{\kappa} = 0$, so $\Omega_0 = \Omega$.
- *Example 2:* If Ω is an ellipse with semiaxes $a > b$, then $\underline{\kappa} = b/a^2$, $\overline{\kappa} = a/b^2$, $d = 2a$, $w = 2b$ and $r = b$, so none of these conditions hold.

Existence of convex caustics

Let Ω be a sufficiently smooth and strictly convex billiard table and $\Gamma = \partial\Omega$.

- *Theorem (Lazutkin):* There exists a collection of smooth convex caustics $\{\gamma_y : y \in \mathcal{C}\} \subset \Omega$, $\lim_{y \to 0^+} \gamma_y = \Gamma$, whose union has positive area.
- *Equivalent formulation:* The billiard map has 2 collections of invariant circles $\{\hat{\gamma}_y^\pm : y \in \mathcal{C}\} \subset \mathbb{A}$, $\lim_{y \to 0^+} \hat{\gamma}_y^\pm = \hat{\gamma}_0^\pm$, whose union has positive area.
- *Corollary:* The billiard map $f : \mathbb{A} \to \mathbb{A}$ is not ergodic.
- The original statement asked for C^{553} regularity; Douady reduced it to C^6.
- $\mathcal{C}$ is a Cantor subset of $\mathbb{R}$ with positive length and ∞ gaps of the form

$$\mathcal{C} = \mathcal{C}_{\lambda,\tau,y_*} := \{y \in (0, y_*) : |y - m/n| \geq \lambda n^{-\tau}, \quad \forall n \in \mathbb{N}, m \in \mathbb{Z}\}$$

  for some constants $\lambda > 0$, $\tau > 2$, and $0 < y_* \ll 1$. We note that $0 \in \overline{\mathcal{C}}$.
- $\mathrm{Rot}(\gamma_y; \Gamma) = y \in \mathcal{C}$, which implies that all the rotation numbers of these caustics are poorly approximated by rational numbers. (Generically, there are no caustics whose rotational numbers are close to rational values.)

Lazutkin's coordinates

- The billiard map $f : \mathbb{A} \to \mathbb{A}$, $f(s,\theta) = (s',\theta')$, verify the approximation

$$\begin{cases} s' &= s + 2\rho(s)\theta + 4\rho(s)\rho'(s)\theta^2/3 + \mathrm{O}(\theta^3) \\ \theta' &= \theta - 2\rho'(s)\theta^2/3 + \big(4(\rho'(s))^2/9 - 2\rho(s)\rho''(s)/3\big)\theta^3 + \mathrm{O}(\theta^4) \end{cases},$$

  where $\rho(s) = 1/\kappa(s)$ is the radius of curvature of Γ, for small values of θ.
- We introduce the coordinates $\xi = \xi(s,\theta) \in \mathbb{R}/\mathbb{Z}$, $\eta = \eta(s,\theta) > 0$ given by

$$\xi = K \int_0^s \kappa^{2/3}(s)\mathrm{d}s, \quad \eta = 4K\rho^{1/3}(s)\sin(\theta/2), \qquad K^{-1} = \int_0^L \kappa^{2/3}(s)\mathrm{d}s.$$

- These coordinates are well-defined for small angles of incidence θ. We note that $\eta(s,0) \equiv 0$. In particular, $0 < \theta \ll 1 \Leftrightarrow 0 < \eta \ll 1$.
- The billiard map in these new coordinates becomes really simple:

$$\begin{cases} \xi' &= \xi + \eta + \mathrm{O}(\eta^3) \\ \eta' &= \eta + \mathrm{O}(\eta^4) \end{cases}.$$

Invariant Curve Theorem (a "toy" KAM-like theorem)

- Let $f(\xi,\eta) = (\xi',\eta')$ be a sufficiently smooth map such that:
  1. It has the previous simple form for $\xi \in \mathbb{R}/\mathbb{Z}$ and $|\eta| < \eta_*$; and
  2. $f(\hat{\gamma}) \cap \hat{\gamma} \neq \emptyset$ for any closed circle $\hat{\gamma}$ homotopic —and sufficiently close— to the curve $\{\eta = 0\}$.
- *ICT (Kolmogorov, Arnold, Moser, Lazutkin):* Under these assumptions, there exists a close-to-the-identity smooth change of variables $(\xi,\eta) \mapsto (x,y)$ defined for $|y| < y_*$ such that the map in the new coordinates has the form

$$\begin{cases} x' &= & x + y + \mathrm{O}(y^3) \\ y' &= & y + \mathrm{O}(y^4) \end{cases},$$

  but both $\mathrm{O}(y^3)$ and $\mathrm{O}(y^4)$ terms vanish identically for all $y \in \mathcal{C}$.
- Then the curves $y = \text{constant} \in \mathcal{C}$ are invariant under the map $f(x,y) = (x',y')$, being y their rotational numbers. These curves are transformed under the changes $(x,y) \mapsto (\xi,\eta) \mapsto (s,\theta)$ into the invariant circles $\{\hat{\gamma}_y^+ : y \in \mathcal{C}\}$ close to $\theta = 0$ we were looking for. QED.

Non-persistence of convex resonant caustics

- *Question:* Do convex resonant caustics exist/persist?
- *Answer:* Generically not, since they are too fragile objects.
- *Claim:* Let γ be a convex caustic such that $\mathrm{Rot}(\gamma;\Gamma) = n/m \in \mathbb{Q}$, and let $\hat{\gamma}^{\pm}$ be its associated invariant circles. Then the billiard map $f : \mathbb{A} \to \mathbb{A}$ verifies that $f^n = \mathrm{Id}$ on $\hat{\gamma}^{\pm}$. These invariant circles —called *resonant*, since they are composed of periodic points—, are easily destroyed under arbitrarily small perturbations of the billiard table.
- *Example (RRR):* Let Γ_0 be a circle of radius R_0, so its concentric circle γ_0 of radius $R_0 \cos(m\pi/n)$ is a convex caustic with rotation number m/n. Let Γ_ϵ be the perturbed circle that in polar coordinates (r,φ) has the form

$$r = R_\epsilon(\varphi) = R_0 + \epsilon S(\varphi) + \mathrm{O}(\epsilon^2), \qquad S(\varphi) = \sum_{j\in\mathbb{Z}} \hat{S}_j \mathrm{e}^{\mathrm{i}j\varphi}.$$

If there exists some $j \in n\mathbb{Z} \setminus \{0\}$ such that $\hat{S}_j \neq 0$, then the caustic γ_0 does not persist. That is, there does not exist a "perturbed" convex caustic γ_ϵ such that $\mathrm{Rot}(\gamma_\epsilon;\Gamma_\epsilon) = m/n$ for all ϵ small enough.

On the 3D case

- Suppose that a smooth surface σ is a caustic of another smooth surface Σ.
- Then the tangent cone to σ from any point $x \in \Sigma$ is a symmetric cone whose axis is perpendicular to Σ at x.
- Let $a_0, b_0 \in \sigma$ and $x_0 \in \Sigma$ be three points such that:
  1. The line l_0 from a_0 to x_0 is tangent to σ at a_0;
  2. The line m_0 from x_0 to b_0 is tangent to σ at b_0; and
  3. The line l_0 is reflected onto the line m_0 at x_0.
- Let $\mathcal{A}$ and $\mathcal{B}$ be the sets of lines tangent to σ at points close to a_0 and b_0.
- Using the reflection at Σ, we construct a one-to-one correspondence $\mathcal{A} \ni l \mapsto m = g(l) \in \mathcal{B}$ such that $l \cap g(l) \in \Sigma$.
- Hence, $\dim\{l \cap g(l) : l \in \mathcal{A}\} \leq \dim \Sigma = 2$, which is hard to accomplish since $\dim \mathcal{A} = 3$.
- *Theorem (Berger):* This degenerate situation can take place if and only if Σ and σ are pieces of confocal quadrics. This is a local result: the existence of just two pieces of caustic already has strong consequences on Σ.

Existence and non-existence of caustics: References

1. M. Berger, Seules les quadriques admettent des caustiques, *Bulletin Soc. Math. France*, **123**:107–116 (1995).
2. E. Gutkin and A. Katok, Caustics for inner and outer billiards, *Comm. Math. Phys.*, **173**:101–133 (1995).
3. V. F. Lazutkin, The existence of caustics for a billiard problem in a convex domain, *Math. USSR Izvestija*, **7**:185–214 (1973).
4. J. Mather, Glancing billiards, *Ergodic Theory Dyn. Syst.*, **2**:397–403 (1982).
5. P. M. Gruber, Only ellipsoids have caustics, *Math. Ann.*, **303**:185–194 (1995).
6. J. Pöschel, A lecture on the classical KAM theorem. (arXiv: 0908.2234v1)
7. R. Ramírez-Ros, Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables, *Phys. D*, **214**:78–87 (2006).
8. C. L. Siegel and J. K. Moser, *Lectures On Celestial Mechanics*, Springer-Verlag, New York-Heidelberg, 1971.
9. S. Tabachnikov, *Billiards*, Panor. Synth. **1**, Soc. Math. France, Paris, 1995.
10. S. Tabachnikov, *Geometry and billiards*, Amer. Math. Soc., Providence, 2005.