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Choosing the representative tones of an abstract Pythagorean scale with regard to just intonation

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Hellegouarch (1999a) relies on a remark that Euler made to state that the way of choosing the representatives of an abstract Pythagorean scale should be based on the fact that the sense of hearing tends to identify with a single ratio all the ratios which are only slightly different from it. For this reason, the representatives were chosen as the simpler ratios among the possible tones of the Pythagorean scale. However, this could be a misinterpretation of Euler's remark, since there may be single ratios much simpler in the vicinity of the Pythagorean ratios. This paper is only a brief contribution on the discussion about how to select these representatives and shows that if Pythagorean tuning is to emulate or replace syntonic just tuning, it is better to adapt the scale to the key.

Keywords: generated scale; Pythagorean tuning; just intonation; scale temperament; Pythagorean comma; Syntonic comma

2010 Mathematics Subject Classification: 05C12; 11A05; 54E35

1. Introduction

The 12-tone Pythagorean scale is a particular case of well-formed scale (e.g., Wilson 1974, 1975; Carey and Clampitt 1989, 2012, 2017; Hellegouarch 1999a,b, 2002; Kassel and Kassel 2010; Cubarsi 2020, 2024) consisting of 12 consecutive perfect fifths, i.e., generated by the 3rd harmonic, reduced to one octave, since the octave (the interval corresponding to a frequency ratio 2:1) is the interval associated with maximum consonance. The starting tone (say frequency ν_0) and the twelfth pure fifth (frequency $\frac{3^{12}}{2^{19}}\,\nu_0$) are identified as if they had matched perfectly, but actually between them there is the small melodic distance[1] corresponding to a Pythagorean comma $\kappa = \frac{3^{12}}{2^{19}}$, equivalent to 23 cents. Thus, the distance between the eleventh fifth and the starting tone is a Pythagorean comma narrower. It is the "false, narrow, wolf" fifth[2].

Mathematically, such a construction gives rise to 12 tonal classes of an *abstract* scale, meaning that any two tones α, β such that $\beta = \frac{3^{12}}{2^{19}}\,\alpha$ belong to the same tonal class.

Each tonal class ν in the multiplicative space of frequencies is associated with a pitch class $\log_2 \nu$ in the additive space of notes. In the frequency space, the octave can be thought of as the interval $[1, 2)$ with the ends identified. Instead, the notes take values on a unit circle, i.e. in $[0, 1)$ also with the ends identified, or in cents in $[0, 1200)$.

One representative of each tonal class must be chosen in order to gather the twelve

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[1] The distance between two scale tones α, β is measured as $d(\alpha, \beta) = \min(|\log_2 \frac{\alpha}{\beta}|, 1 - |\log_2 \frac{\alpha}{\beta}|)$, although it is usually given multiplied by a factor 1200 in cents (¢).

[2] In the 12-tone Pythagorean scale, perfect fifths span 3 chromatic semitones $U = \frac{3^7}{2^{11}}$ (113.69¢) and 4 diatonic semitones $D = \frac{2^8}{3^5}$ (90.22¢), while narrow fifths span 2 chromatic semitones U and 5 diatonic semitones D.

consecutive fifths composing the 12 tones of a *concrete* Pythagorean scale. This paper is just a brief contribution on the discussion about how to select these representatives.

The process of choosing the representatives of the tonal classes is explained in detail in Hellegouarch (1999a,b, 2002) and Kassel and Kassel (2010). Hellegouarch determines successive Pythagorean scales with increasing number of tones from a quotient group G/H, where G is the group generated by the frequency ratios 2 and 3, and H is the group generated by the comma r_n. If $\frac{p_n}{q_n}$ is a convergent of the continued fraction expansion of $\log_2 3$, the comma $r_n = \left(\frac{3^{q_n}}{2^{p_n}}\right)^{(-1)^n}$ satisfies $r_n \to 1$ as $\frac{p_n}{q_n} \to \log_2 3$. The representative of each tonal class is chosen as the simplest irreducible fraction allowing an isomorphism φ between the q_n tones in $[1, 2)$ and the set $\{0, 1, \ldots, q_n - 1\}$ so that $\varphi(2) = q_n$.

The simplest irreducible fraction means the one providing the minimum value of the height $h(\frac{a}{b}) = \sup(a, b)$ for a positive irreducible fraction $\frac{a}{b}$. To this, Kassel and Kassel (2010) add a condition that ensures the existence of a unique scale, even if G is a free abelian group of rank 2.

Although it could seem that the above procedure is a criterion based on mathematical simplicity, Hellegouarch (1999a) roots it on the following remark made by Euler in 1766: "The sense of hearing is accustomed to identify with a *single ratio*, all the ratios which are only slightly different from it, so that the difference between them be almost imperceptible". Hence, he chooses the simplest ratios to represent the tonal classes of the abstract scale. He proposes to take as representatives the tonal classes generated by the comma $\frac{3^{12}}{2^{19}}$ as $\frac{3^p}{2^q}$ $(p, q \in \mathrm{Z})$, the simplest irreducible fractions satisfying that $\sup(3^p, 2^q)$ be minimal. This gives rise to the concrete scale

$$\frac{2^8}{3^5}, \ldots, \frac{2^2}{3^1}, \mathbf{1}, \frac{3}{2}, \ldots, \frac{3^6}{2^9} \quad (\mathrm{D}^\flat\text{-}\mathrm{A}^\flat\text{-}\mathrm{E}^\flat\text{-}\mathrm{B}^\flat\text{-F-}\mathbf{C}\text{-G-D-A-E-B-}\mathrm{F}^\sharp) \tag{1}$$

However, the above criterion may still admit some discussion. The present work will suggest that, in tonal music, it is better to adapt the scale to the key. The musical arguments favouring such a proposal can be easily converted into a simple mathematical method based on elementary linear algebra to use in more general cases.

In order to choose the representatives of the abstract scale, one has several options. For example, the 12 consecutive fifths may start from the fundamental tone 1 onward. When they are reduced to one octave, this gives rise to the following tones (and notes), which are written as consecutive fifths (not in pitch order),

$$\mathbf{1}, \frac{3}{2}, \ldots, \frac{3^{11}}{2^{17}} \quad (\mathbf{C}\text{-G-D-A-E-B-}\mathrm{F}^\sharp\text{-}\mathrm{C}^\sharp\text{-}\mathrm{G}^\sharp\text{-}\mathrm{D}^\sharp\text{-}\mathrm{A}^\sharp\text{-}\mathrm{E}^\sharp) \tag{2}$$

or, for example, one may consider forward and backward fifths around 1, as

$$\frac{2^{10}}{3^6}, \frac{2^8}{3^5}, \ldots, \frac{2^2}{3^1}, \mathbf{1}, \frac{3}{2}, \ldots, \frac{3^5}{2^7} \quad (\mathrm{G}^\flat\text{-}\mathrm{D}^\flat\text{-}\mathrm{A}^\flat\text{-}\mathrm{E}^\flat\text{-}\mathrm{B}^\flat\text{-F-}\mathbf{C}\text{-G-D-A-E-B}) \tag{3}$$

Thus, in the above examples, the tones $\frac{3^{11}}{2^{17}}$ and $\frac{2^2}{3^1}$ belong to the same tonal class.

Therefore, there are 12 possible ways of choosing the sequence of 12 consecutive fifths. In other words, the 12 scale notes can be drawn from any sequence of 12 consecutive tones, that includes the fundamental tone 1, drawn from the following set

$$\frac{2^{18}}{3^{11}}, \frac{2^{16}}{3^{10}}, \ldots, \frac{2^2}{3}, 1, \frac{3}{2}, \ldots, \frac{3^{10}}{2^{15}}, \frac{3^{11}}{2^{17}} \tag{4}$$

All of these scales will have the same mathematical properties derived from being built from 12 consecutive fifths, but with tones that may differ by a Pythagorean comma up or down, and with the narrow fifth between different tones. Therefore, these scales will show different properties with regard to tonality.

2. Sound perception

From a practical point of view, these questions may seem uninteresting to a musician used to play in 12-tone equal temperament (TET) or that uses electronic instruments or programs capable of virtually perfect just intonation, but, taking for granted the interest for mathematical theory of music, they are important for musicians who play in just intonation (JI), i.e., in intervals based on the overtone series expressed as simple frequency ratios, with acoustic fretless string instruments and with some wind instruments, which are many of them tuned by fifths, i.e., according to Pythagorean tuning.

From the Renaissance to the middle of the 18th century, just tuning, including several variants of meantone temperament, was a common practice, but gradually 12-TET was adopted, which firstly applied to the piano became the standard tuning. As Yanakiev (2018) says, "This attitude of limiting the hearing to the pitch categories of the 12-TET scale does not expand the musicians' capabilities of pitch discrimination towards their limits." Nevertheless, in the second half of the 20th century, on the one hand modern practice of historically informed approach, and on the other hand microtonal music (in particular, spectral music, whose material is often derived from the harmonic series), tended towards recovering performance in JI.

The question that arises is how do we distinguish such small differences in tuning?

From the point of view of categorical perception, i.e., the process whereby continuous acoustic variation is transformed into a discrete set of auditory events, the 12-TET scale presents some good properties in order to be categorized, such as that it is a *proper* scale (Rothenberg 1977a,b, 1978), i.e., there is some interval class which is the same size as the next larger one, but none which is strictly larger[3]. In a specifically musical context, categorical perception consists in basically two tasks, identification of music intervals and discrimination of pairs of intervals presented at different pitch levels (e.g., Burns and Ward 1974; Siegel and Siegel 1977; Burns and Campbell 1978; Burns 1999). In summary, their conclusions are that highly trained musicians with excellent relative pitch typically have learned just enough about tonal intervals to differentiate the musically relevant cues, although their ability to differentiate different examples of the same musical interval is extremely limited. Untrained subjects show no evidence of categorical perception.

In view of these results[4], it seems that categorical perception works in the opposite direction we are wishing to, i.e., it seems that we should not worry too much about distinguishing whether the notes belong to a 12-TET scale, a Pythagorean scale, or a JI scale. Consequently, our analysis must be rooted in other criteria.

Pierce (1999) gives an overview of the nature of musical sound, from which we highlight

[3]For instance, for the diatonic scale, the one step intervals are the semitone and the tone (2 semitones), the two step intervals are the minor third (3 semitones) and major third (4 semitones), the three step intervals are the fourth (5 semitones) and the tritone (6 semitones), the four step intervals are the fifth (7 semitones) and the tritone (6 semitones), etc. This scale is proper but not strictly proper because the three step intervals and the four step intervals share an interval size (the tritone), causing ambiguity.

[4]Siegel and Siegel (1977) conclude that "in music, categorical perception allows one to recognize the melody, even when the notes are out of tune, and to be blissfully unaware of the poor intonation that is characteristic of good musical performance." This is consistent with some explanations about possessors of absolute pitch by Ward (1999), when he mentions Abraham (1901), "...a given tune may be heard in many different keys. How can a child develop absolute recognition of a particular frequency, say 261 Hz, if it is called "do" today and "re" tomorrow or if it is heard when he presses the white key just left of the two black keys in the middle of the piano at home but a completely different key (perhaps even a black one) at grandfather's house? Considering all the factors that conspire to enhance the development of relative pitch at the expense of absolute pitch". Thus, Ward says that perhaps an inborn potential for developing absolute pitch was relatively widespread, but that it is simply trained out of most of us. Similarly, quoting Watt (1917), "... perhaps a highly favoured auditory disposition gives them the power to maintain their absoluteness of ear in spite of the universality of musical relativity. In that case we should all naturally possess absolute ear and then proceed to lose it or to lose the power to convert it into absolute nomenclature." Therefore, musical categorical perception would provide the auditory system with an adaptive mechanism to recognize diverse realizations of one musical entity which is never presented exactly in the same way.

the following statements, which are appropriate for the present study.

"What we hear is out there, not inside our heads." "The very fact that it is useful to represent musical sounds by a spectrum, a sum of sine waves, tells us a good deal about musical instruments, musical sounds, and about the mechanism of hearing." "Still, the vibrating strings and air columns of musical instruments and the early mechanical stages of the human auditory system are linear enough to make sine waves and the representation of sound waveforms by spectra (by collections of sine waves, or partials) useful in studies of musical sound." "In instruments in which the vibration is forced, the partials are nearly harmonic, and the chief components of the spectrum must be at least approximately harmonics of the fundamental frequency."

Finally, he vindicates the explanations of Helmholtz (1863) and Plomp (1966) of consonance and dissonance: "The association of dissonance with interaction of sine waves close in frequency is thus a plausible explanation of musical consonance."

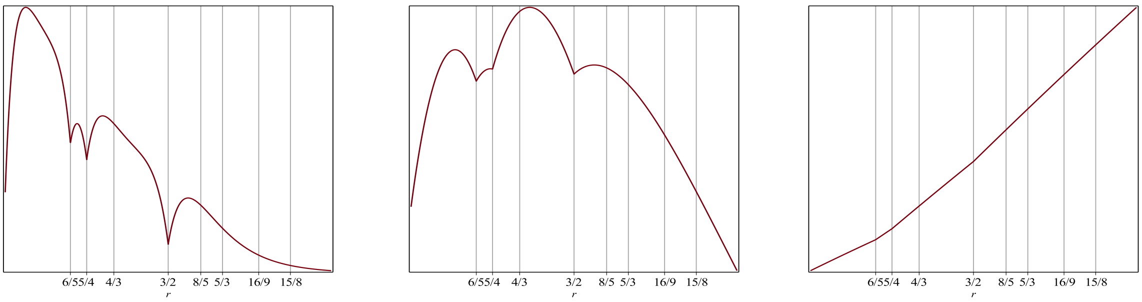

Figure 1. Sensory dissonances (left) $\Delta_S(400, 500, 600)$, (middle) $\Delta_S(40, 50, 60)$, (right) $\Delta_S(4, 5, 6)$.

Based on Plomp and Levelt (1965), sensory dissonance[5] (Sethares 1998) allows to understand some aspects of consonance and timbre. It accounts for the sensation of dissonance associated with the beats produced in interfering a weighted mixture of single waves. However, since the beats produced depend on the pitch height, the sensory dissonance is not appropriate to characterize the *abstract* concept of relative dissonance or consonance between simple frequency ratios and pitch classes (Cubarsi 2019). In Fig. 1, sensory dissonances for the frequency sets $(1, \frac{5}{4}, \frac{3}{2}) \cdot x$, for $x = 400, 40, 4$ are compared. It is clear that they depend on the actual frequencies and when they are calculated for single harmonics (or for small frequency ratios), the sensory dissonance loses its meaning, since it is not intended for this purpose.

3. Tuning to single ratios

Fortunately, Langner (2015, p180) gives us a clue. "The harmonic structure of tones as generated by musical instruments is not a product of careful manufacturing. Instead, it arises from resonances where the delays due to travel times in tubes or on strings are

[5]The sensory dissonance of a spectrum $F = \{\nu_1, \ldots, \nu_n\}$ is computed as

$$\Delta_S(F) = \frac{1}{2} \sum_{i \neq j} \min(a_i, a_j) \left(e^{-b_1 s_{ij} |\nu_i - \nu_j|} - e^{-b_2 s_{ij} |\nu_i - \nu_j|} \right), \tag{5}$$

with $s_{ij} = \frac{0.24}{0.0207 \min(\nu_i, \nu_j) + 18.96}$ and the constants $b_1 = 3.51, b_2 = -5.75$. It measures the sensation of dissonance produced by the union $F \cup rF$ of two spectra composed of tones that maintain mutually a frequency ratio r and it allows to detect which are the ratios producing a minimum of sensory dissonance. Generally, since the scale of dissonance is arbitrary, it will be appropriate to compare relative dissonances. In particular, more than the value itself, it is important to know where the relative minima are placed. The curve of Eq. 5 for a set F of n tones has up to $2n(n-1)$ relative minima, corresponding to ratios that provide a greater sensation of consonance.

compensated by the periods of the harmonics (depending on boundary conditions, integer multiples of the delays or the periods may have the same effect). Similarly, the integer relationships of horizontal harmony arise –in the spirit of Pythagoras– as inevitable mathematical consequences of the neuronal correlation analysis. Our coincidence neuron would prefer certain harmonically related tones even if such relationships were not also observable in our acoustic environment. In other words, our harmonic sense for pitch relations has mathematical reasons and is not, or at least not primarily, due to the adaptation of our auditory system to the physical conditions of our environment. As a result, our brain, or at least our hearing system, reacts almost like a musical instrument and therefore obeys the same mathematical laws as the physical environment."

It is worth noting that categorical perception was first developed to describe the results of speech experiments that used synthetic speech tokens that varied along a single acoustic continuum. But in music played by an instrument in which the vibration is forced, the sound produced by a tonal chord in perfect tuning is not like any other sound in the continuous space of possible sounds, but a singular and clearly identifiable sound. This is a physical fact beyond categorical perception. For instance, "the first and third degrees (together) and the first and fifth degrees (together) generate difference tones that reinforce the first degree" (Rothenberg 1977a, p230). It is easily seen in Fig. 2, with the pitch classes composing a C major chord, with frequency ratios $(1, \frac{5}{4}, \frac{3}{2})$. In this example, the mediant reinforces the tonic and the dominant.

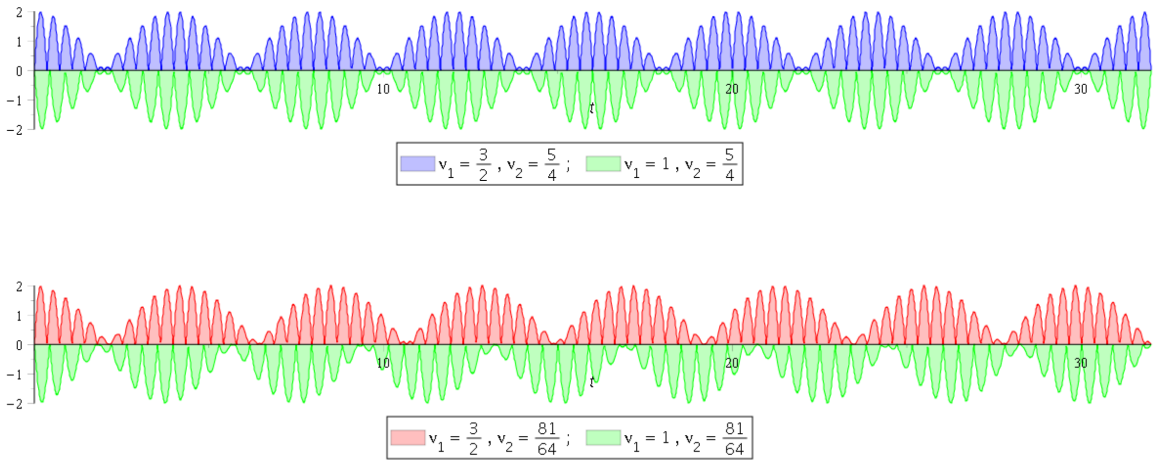

Figure 2. Absolute value $|f(t)|$ for a superposition of pure waves associated with the first and third degrees in JI (top, blue) and in Pythagorean tuning (bottom, red), and (upside down) for the first and fifth degrees (green).

In the top panel, the wave $f(t) = \sin 2\pi\nu_1 t + \sin 2\pi\nu_2 t$ (absolute value) produced by the superposition of waves associated with the dominant and mediant, with frequencies $\nu_1 = \frac{3}{2}$ and $\nu_2 = \frac{5}{4}$ (blue) is compared to the superposition of tonic and mediant, with frequencies $\nu_1 = 1$ and $\nu_2 = \frac{5}{4}$ (green, upside down). In the bottom panel, the compared sounds are the superpositions with frequencies $\nu_1 = \frac{3}{2}$ and the Pythagorean mediant $\nu_2 = \frac{3^4}{2^6}$ (red) to frequencies $\nu_1 = 1$ and $\nu_2 = \frac{3^4}{2^6}$ (green). It is clear that the respective amplitude-modulated sine waves, despite some very similar carrier frequencies in the blue and red curves, have a different gear with the green curves in both panels. In the former case both sounds reinforce each other, while in the latter case the interference is much less constructive. In acoustic resonant instruments such a difference elicits distinguishable vibrations that should be clearly perceived, not only by the performer, but for the surrounding audience, since instrument's timbre and resonance become improved.

Thus, we need a measure for consonance for sets of single frequency ratios not depending on the pitch height, intensities, etc. In other words, a more *symbolic* concept of consonance.

At this primary level, it suffices to estimate how a set of pure sine waves reinforces or annihilate each other.

Helmholtz's (1863, 187) Table listed the ratios $\frac{p}{q}$ (for p, q coprime) of the main harmonics and their *intensity of influence* (in %) given by

$$C_{\mathrm{h}}(p, q) = \frac{100}{pq} \tag{6}$$

This parameter, referred to as harmonic consonance, is the relative strength of the beats resulting from the mistuning of the corresponding interval, obtained by modelling the strength of the sympathetic vibration produced in the Corti's organ (ibid. Appendix 15, 415). According to Helmholtz, the lower the product pq, the greater the degree of beatings in mistuning the interval, i.e., the mistuning is more noticeable. This is interpreted as both harmonics being more consonant (before mistuning).

The harmonic consonance is related to Tenney's harmonic distance (Tenney 2015, 240-279) between the fundamental tone 1 and the pitch $\frac{p}{q}$, $\Delta_{\mathrm{h}}(1, \frac{p}{q}) = \Delta_{\mathrm{h}}(p, q) = \log_2(pq)$, which is indicative of the degree of beatings, since it is inversely correlated with the harmonic consonance, $C_{\mathrm{h}} = 100 \cdot 2^{-\Delta_{\mathrm{h}}}$.

Harmonic distance informs us of how far away two sine waves of irreducible frequency ratios of α and β match at the lowest common harmonic $\mathrm{LCH}(\alpha, \beta)$ where they become immersed as a fused sound image, and how far should we go back to find their common fundamental, i.e., their lowest common ancestor $\mathrm{LCA}(\alpha, \beta)$, which has the period of the whole superposition wave. The value $\Delta_{\mathrm{h}}(\alpha, \beta) = \log_2 \frac{\mathrm{LCH}(\alpha,\beta)}{\mathrm{LCA}(\alpha,\beta)}$ is their harmonic distance, which for an arbitrary set of frequency ratios was generalized as harmonic dissonance in Cubarsi (2019). Therefore, it is a measure of how constructive a superposition of waves is.

Now, under the acoustic context explained in the previous section and according to the harmonic concept of consonance, is how we will henceforth interpret Tenney's words in *John Cage and the Theory of Harmony* (Tenney 2015, 280-304, reproducing a 1983 paper), which say that in a scale close to JI, such as equal temperament and Pythagorean, the ear tends to resolve towards the frequency of the nearest JI, i.e., a simple frequency ratio. The existence of such a vicinity, each one dominated by a single frequency ratio, implies that there is a finite number of intervals which can be tuned by ear (not as absolute pitch, but in the reference frame of a tonic, although it also depends on the ear training). Therefore, we meet again Euler's remark (e.g., Blaine and Ferré 2021, §20).

This fact is consistent with the fusion theory (Stumpf 1890; Ebeling 2008; Langner 2015). According to Moore, Peters, and Glasberg (1985), the low harmonics may be out of tune around the 1-3% respect to their nominal value and still be perceived as belonging to the same harmonic series. However, the threshold decreases for higher harmonics, and also depends on conditions such as the intensity of the harmonics and whether the duration of the sound allows pulsations to be appreciated. As a variation of the 1.5% in the frequency is roughly equivalent to a Pythagorean comma, we can consider that harmonic distances of a JI system replace that of the equal temperament or Pythagorean system.

Simple frequency ratios are always more consonant than those of Pythagorean tuning (or equal consonant if they match). When comparing harmonic distances between tones in JI and Pythagorean scales it results that for Pythagorean scales the harmonic distance increases nearly in a linear way in terms of the fifth index (Cubarsi 2019). This produces some inconsistencies, such as that, in Expression 2, the harmonic distance between C and E$^\sharp$ (11 perfect fifths) is much greater than between C and G, whereas the size of the interval E$^\sharp$-C is smaller than between C and G. All these considerations lead to the fact that the estimation of consonances should be done with JI notes close to the Pythagorean ones, instead of those of the Pythagorean ones themselves.

Conversely, we should ask the performer to play as close as possible to these simple ratios.

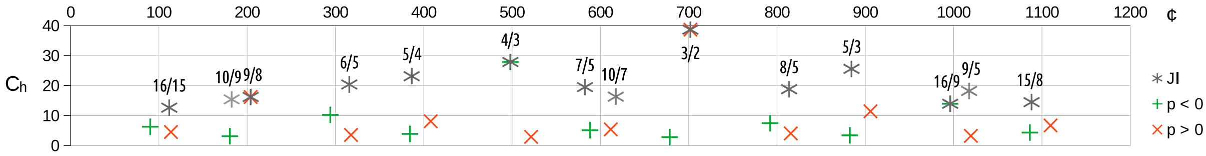

Figure 3. Harmonic consonance C_h and cents within the octave for JI notes (balck marks) and Pythagorean fifths $\frac{3^p}{2^q}$, either iterated from 1 onward ($p > 0$) (red marks) or backwards ($p < 0$) (green marks).

Figure 3 shows the harmonic consonance C_h of Eq. 6 and position (in cents) within the octave for the notes of a JI scale and the notes of a Pythagorean scale with frequency ratios $\frac{3^p}{2^q}$, where red marks are for forward iterates $p = 1, 2, \ldots$ and green marks are for backward iterates $p = -1, -2, \ldots$.

4. Tonality matters

From a tonal point of view, it is clear that not all notes are of equal importance. Let us consider that we start the sequence of fifths at the note F (assuming the fundamental tone is C), and form the scale

$$\tfrac{4}{3}, \mathbf{1}, \tfrac{3}{2}, \ldots, \tfrac{3^{10}}{2^{15}} \quad (\text{F-}\mathbf{C}\text{-G-D-A-E-B-F}^\sharp\text{-C}^\sharp\text{-G}^\sharp\text{-D}^\sharp\text{-A}^\sharp) \tag{7}$$

This scale is made up of the diatones, which are the notes of the 7-tone Pythagorean scale, the one just preceding the 12-tone scale in the chain of Pythagorean scales, i.e., made up of the first seven fifths F-C-G-D-A-E-B (the white keys of the piano), plus the accidentals, which, in fact, are a refinement of the previous ones, and are interspersed between them. Namely, F$^\sharp$, C$^\sharp$, G$^\sharp$, D$^\sharp$, A$^\sharp$ (the black piano keys).

The diatones are relevant because they contain the fundamental tone, its previous and next fifths, F-C-G, and each one is accompanied with the tonal class of its 3-rd overtone (its fifth) and its 5-th overtone (the major third). In a JI scale they would be F-A$_J$-C ($\frac{4}{3}$-$\frac{5}{3}$-1), C-E$_J$-G (1-$\frac{5}{4}$-$\frac{3}{2}$), G-B$_J$-D ($\frac{3}{2}$-$\frac{15}{8}$-$\frac{9}{8}$). Thus, with respect to the fundamental, in a diatonic scale there appear the most significant degrees of a major scale, i.e., subdominant, tonic, dominant, supertonic, submediant, mediant and leading tone. Therefore, the C major diatonic scale is formed by the sequence of fifths F-C-G-D, plus the class of the 5-th overtone of the first three previous notes, i.e., A-E-B.

On the other hand, if we invert the previous sequence of fifths, D-G-C-F, according to the 3-rd undertone, and we add the class of the 5-th undertone of the first three of them, i.e., A$^\sharp_J$, D$^\sharp_J$, G$^\sharp_J$ ($\frac{9}{5}$,$\frac{6}{5}$,$\frac{8}{5}$), we get the natural C minor scale[6] C-D-D$^\sharp_J$-F-G-G$^\sharp_J$-A$^\sharp_J$, containing the minor third, the minor sixth and the minor seventh degrees of the fundamental. Both major and minor diatonic scales are depicted in black in Table 1.

The central row is shared by both diatonic scales. With the top row we get the C major JI scale (circles), with the wolf fifth between D and A$_J$. According to Sara Cubarsi-Fernandez (2023), this is "the major scale in syntonic just intonation, which was unambiguously illustrated in Galeazzi's fingering charts in his *Elementi teorico-pratici della musica, con*

[6]The notes of this scale are usually referred to as C-D-E$^\flat$-F-G-A$^\flat$-B$^\flat$, however, to fix a criterion that does not depend on the key the notes are named according to Table A1. Simple frequency ratios are labelled by adding a subscript J to the closest Pythagorean, although read in a particular key the name may not correspond to its degree.

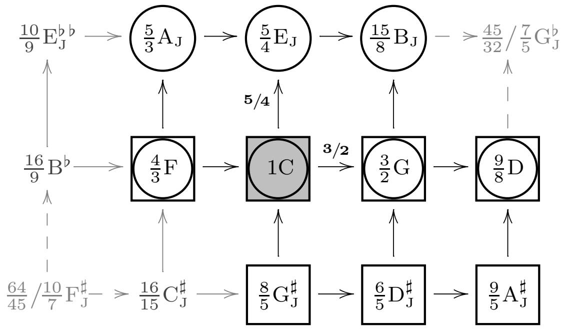

Table 1. The circles contain the notes of the C major diatonic JI scale and squares are for the C minor scale. The scheme is completed with their missing inverses (dark gray) together with the notes close to the tritone (light gray). Factors between notes operates in the group of frequency classes.

un saggio sopra l'arte di suonare il violino (1791) and also documented and named as *Gamme grecque*[7] by physicist Jacques-Alexandre Charles in *Course de physique* (1802)".

Similarly, by considering the bottom row, we form the C minor JI scale (squares), with the wolf fifth between $A^\sharp_J$ and F.

In order that each note has its inverse tonal class in the whole system (radial symmetry relative to the fundamental C), the above set of tonal classes may be completed by adding $\frac{16}{9}$, $\frac{16}{15}$ and $\frac{10}{9}$, corresponding to the notes $B^\flat$, $C^\sharp_J$ and $E^{\flat\flat}_J$. In this way, we are able to form three major JI scales (C, F, and $G^\sharp$) and three minor JI scales (c, f, and a).

Thus, in Pythagorean tuning the diatones match or come close to (indicated with brackets) the simple ratios $1, \frac{9}{8} \vee [\frac{10}{9}], [\frac{5}{4}], \frac{4}{3}, \frac{3}{2}, [\frac{5}{3}], [\frac{15}{8}]$, while the accidentals come close to $[\frac{16}{15}], [\frac{6}{5}], [\frac{8}{5}], \frac{16}{9} \vee [\frac{9}{5}]$.

However, to get a JI system covering the twelve notes of the Pythagorean, we must still add two new single ratios close to the tritone $F^\sharp$. In order to maintain the consistency with the just intonation viewpoint, we choose $\frac{7}{5}$ and $\frac{10}{7}$ (noted as $G^\flat_J$ and $F^\sharp_J$), which are mutually inverse and easy to identify by ear (Sara Cubarsi-Fernandez, personal communication).

The ratios $\frac{45}{32}$ and $\frac{64}{45}$, although less simple than the previous ones, are also an interesting alternative, since they keep the fifth's relationship in the first and third row of Table 1, and like the other alternative pairs of tones they can be considered as enharmonic equivalents to within a syntonic comma.

Notice that, as shown in Fig. 4, the small interval between $\frac{7}{5}$ and $\frac{3}{2}$ is exactly measurable in terms of the amplitude-modulated waves produced by the intervals associated with the fifth relative to the tonic and the minor third relative to the dominant, while $\frac{45}{32}$ is not.

Nevertheless, in the current study these notes are less relevant since they does not belong to the diatonic scales of the tonic. With the tritones we may form another major scale ($C^\sharp$) and another minor scale (e).

Therefore, with the exception of the ratios that already belong to a Pythagorean scale, the other ratios are simpler and very close to those of the Pythagorean scale.

According to Figure 3, since the ear will tend to resolve towards the more consonant ratio in the vicinity (which we may consider as a Pythagorean comma, since it is the error allowed for the Pythagorean scale), the ear will tend to perceive the tone $\frac{3^4}{2^6}$ (408¢) corresponding to the note E as it were $\frac{5}{4}$ (E_J, 386¢), or the one of note A, $\frac{3^3}{2^4}$ (906¢), as it were $\frac{5}{3}$ (A_J, 884¢), or the one of note B, $\frac{3^5}{2^7}$ (1110¢), as it were $\frac{15}{8}$ (B_J, 1088¢). Nevertheless, we can make it easier for the ear to do this work of retuning the respective intervals in which they differ –equivalent to the syntonic comma $\frac{3^4}{2^4 5}$ (21¢)– , by preferring the Pythagorean

[7] If one more fifth (A) is added in the central row instead of the corresponding note of the upper row, we get the *Gamme européenne*, with the wolf fifth between A and E, which prioritizes expressive Pythagorean tuning over JI.

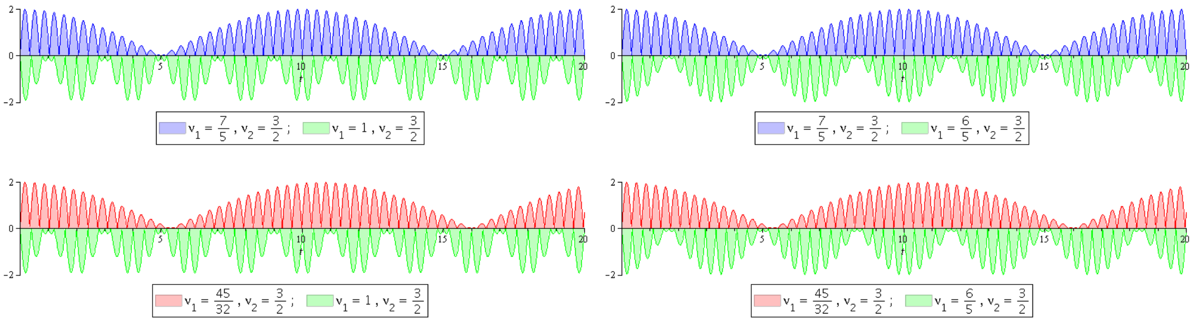

Figure 4. Absolute value $|f(t)|$ for a superposition of sine waves by using $\nu_1 = \frac{7}{5}$ (top, blue) and $\nu_1 = \frac{45}{32}$ (bottom, red) together with the dominant, compared (upside down, green) with the specified superpositions.

representatives $\frac{2^{13}}{3^8}$ ($F^\flat$, 384¢), $\frac{2^{15}}{3^9}$ ($B^{\flat\flat}$, 882¢) and $\frac{2^{12}}{3^7}$ ($C^\flat$, 1086¢), respectively, since they almost match with those of JI (they differ in 1.95¢, i.e., one schisma), despite having lower harmonic consonance than their Pythagorean alternative representatives.

On the other hand, it seems logical that if the ear has to identify the tones $\frac{2^5}{3^3}$ ($E^\flat$, 294¢), $\frac{3^9}{2^{14}}$ ($D^\sharp$, 318¢) or $\frac{6}{5}$ ($D^\sharp_J$, 316¢, ca. $D^\sharp$), it will do it towards the latter, since $D^\sharp$ is only about 2¢ away from $\frac{6}{5}$. It certainly will not resolve towards $E^\flat$ (294¢), which is what Hellegouarch proposes instead of $D^\sharp$. The same would happen if one had to choose between $\frac{2^7}{3^4}$ ($A^\flat$), $\frac{3^8}{2^{12}}$ ($G^\sharp$) and $\frac{8}{5}$($G^\sharp_J$, ca. $G^\sharp$), or between $\frac{2^4}{3^2}$ ($B^\flat$), $\frac{3^{10}}{2^{15}}$($A^\sharp$) and $\frac{9}{5}$($A^\sharp_J$, ca. $A^\sharp$), or between $\frac{2^8}{3^5}$($D^\flat$), $\frac{3^7}{2^{11}}$ ($C^\sharp$) and $\frac{16}{15}$($C^\sharp_J$, ca. $C^\sharp$).

Therefore, by taking into account Euler's remark, the representatives of the tonal classes should be chosen by taking in mind the JI system close to the Pythagorean scale.

5. Pythagorean scales close to just intonation

It is easy to compensate for the syntonic comma associated with the consecutive fifths A-E-B with the Pythagorean comma of the false closing of the circle of fifths, so that the main degrees of the scale are maintained with regard to a JI scale. These fifths, as major thirds of the principal subdominant, tonic, and dominant triads in C major, i.e., the Pythagorean scale members four pure fifths forward from F, C, G, respectively, are a syntonic comma too high. To compensate, to within 2¢, it suffices to replace them by their enharmonic equivalents a Pythagorean comma (−12 fifths) flat: $B^{\flat\flat}$, $F^\flat$, $C^\flat$. That is, by taking the Pythagorean chromatic scale with $B^{\flat\flat}$ initiating the line of perfect fifths, ending with D, gives a good approximation of the just major scale. As in the actual JI scale the one deficient fifth is on the second scale degree, here D, by closing the circle of fifths D-$B^{\flat\flat}$ it is possible to tune the fifths corresponding to all the diatones correctly or with an error of one schisma. It is well known that this is sufficient to tune the notes of the C major scale, but the nice point here is that also two of the chromatic notes come close to the just ratios. The tritone from the tonic is necessarily $G^\flat$, given the initial $B^{\flat\flat}$, within 6¢ of $\frac{7}{5}$ (2¢ of $\frac{45}{32}$). In addition, the fifth corresponding to the tone $\frac{16}{9}$ ($B^\flat$) is also just. In total there are 9 matches with JI notes, while starting with $D^\flat$, as Hellegouarch does, there are only 6 of them (with the tritone at $\frac{10}{7}$, $F^\sharp$).

Table 2 displays the notes of the Pythagorean scale that match or do not differ in more than 2¢ (in parenthesis, 6¢) to the following notes of a JI scale, which we now order by increasing pitch,

$$1, \frac{16}{15}, \frac{10}{9}, \frac{9}{8}, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{16}{9}, \frac{9}{5}, \frac{15}{8} \quad (\text{C-}C^\sharp_J\text{-}E^{\flat\flat}_J\text{-D-}D^\sharp_J\text{-}E_J\text{-F-}G^\flat_J\text{-}F^\sharp_J\text{-G-}G^\sharp_J\text{-}A_J\text{-}B^\flat\text{-}A^\sharp_J\text{-}B_J)$$

first note	narrow fifth	p	**C** 1	$C^\sharp_J$ $\frac{16}{15}$	$E^{\flat\flat}_J$ $\frac{10}{9}$	**D** $\frac{9}{8}$	$D^\sharp_J$ $\frac{6}{5}$	$\mathbf{E}_J$ $\frac{5}{4}$	**F** $\frac{4}{3}$	$G^\flat_J$ $\frac{7}{5}$	$F^\sharp_J$ $\frac{10}{7}$	**G** $\frac{3}{2}$	$G^\sharp_J$ $\frac{8}{5}$	$\mathbf{A}_J$ $\frac{5}{3}$	$B^\flat$ $\frac{16}{9}$	$A^\sharp_J$ $\frac{9}{5}$	$\mathbf{B}_J$ $\frac{15}{8}$	
C	$E^\sharp$-C	0	●	●	○	●	●	○	○	○	(●)	●	●	○	○	●	○	
F	$A^\sharp$-F	-1	●	●	○	●	●	○	●	○	(●)	●	●	○	○	●	○	$G^\sharp$, c
$B^\flat$	$D^\sharp$-$B^\flat$	-2	●	●	○	●	●	○	●	○	(●)	●	●	○	●	○	○	$C^\sharp$, f
$E^\flat$	$G^\sharp$-$E^\flat$	-3	●	●	○	●	○	○	●	○	(●)	●	●	○	●	○	○	
$A^\flat$	$C^\sharp$-$A^\flat$	-4	●	●	○	●	○	○	●	○	(●)	●	○	○	●	○	○	
$D^\flat$	$F^\sharp$-$D^\flat$	-5	●	○	○	●	○	○	●	○	(●)	●	○	○	●	○	○	
$G^\flat$	B-$G^\flat$	-6	●	○	○	●	○	○	●	(●)	○	●	○	○	●	○	○	
$C^\flat$	E-$C^\flat$	-7	●	○	○	●	○	○	●	(●)	○	●	○	○	●	○	●	
$F^\flat$	A-$F^\flat$	-8	●	○	○	●	○	●	●	(●)	○	●	○	○	●	○	●	
$B^{\flat\flat}$	D-$B^{\flat\flat}$	-9	●	○	○	●	○	●	●	(●)	○	●	○	●	●	○	●	C, e
$E^{\flat\flat}$	G-$E^{\flat\flat}$	-10	●	○	●	○	○	●	●	(●)	○	●	○	●	●	○	●	F, a
$A^{\flat\flat}$	C-$A^{\flat\flat}$	-11	●	○	●	○	○	●	●	(●)	○	○	○	●	●	○	●	

Table 2. Coincidences of less than one schisma (in parenthesis, up to 6 ¢) between tones of Pythagorean and JI scales, where p is the power of 3 of the first fifth. In gray those with most matches. Notes between vertical lines are approximately one semitone apart. The column on the right indicates the diatonic JI scales that can be formed with a subset of them.

Depending on the choices for the alternative pairs $\frac{9}{8} \vee \frac{10}{9}$, $\frac{7}{5} \vee \frac{10}{7}$, and $\frac{16}{9} \vee \frac{9}{5}$ we obtain eight JI scales close to the 12-tone Pythagorean scale (listed as JI-1, ..., JI-8 in Table 3 together with the major or minor diatonic scales they match).

In Figure 5, diatonic scales formed according to the following criteria are compared: (P/F) Pythagorean scale with the fifths starting at F; (ET) equal temperament scale; (P/B♭♭) Pythagorean scale with the fifths starting at $B^{\flat\flat}$; (MT) quarter-comma meantone temperament, i.e., fitting well the class of the fifth harmonic in exchange for decreasing in 5¢ the accuracy of the third harmonic[8]; (JI) syntonic just intonation. The deviation from the diatonic JI scale is measured by the mean square error (MSE, expressed in ¢^{2}) of the seven tones. It is worth noticing that the scale P/B♭♭ is the one more close to JI, even more than the MT scale.

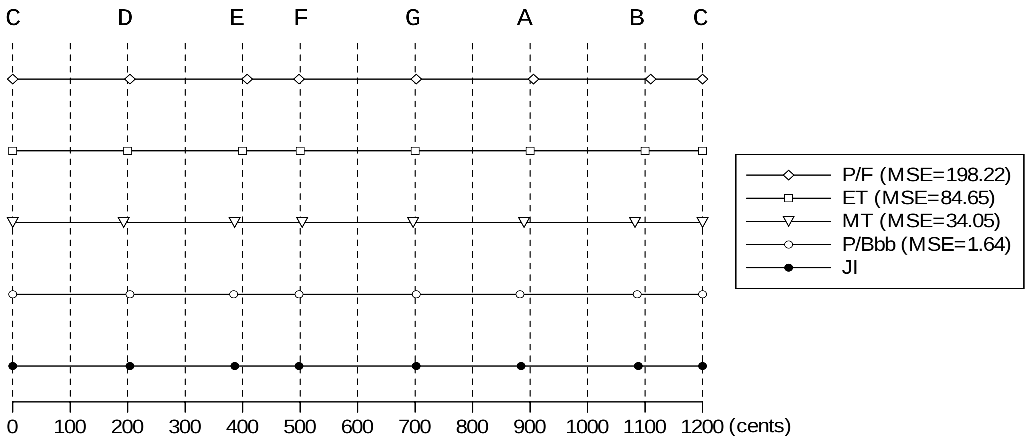

Figure 5. Diatonic scales according to several intonation systems (P/F=Pythagorean starting in F, ET=Equal Temperament, P/B♭♭=Pythagorean starting in $B^{\flat\flat}$, MT=Meantone Temperament, JI=Just Intonation). The value MSE (¢^{2}) is the mean square error relative to the tones of the diatonic JI scale.

[8] Quarter-comma meantone temperament is a well-formed scale with generator $\sqrt[4]{5}$.

6. Mathematical approach

The above procedure can be easily generalized. For $j = 1, \ldots, n$ let $\mathbf{p}(j)$ be the vectors which components are the tones of the n possible Pythagorean scales. That is,

$$\begin{gathered}
\mathbf{p}(1) = (\nu_0, \nu_1, \ldots, \nu_{n-1})^T \\
\mathbf{p}(2) = (\nu_{-1}, \nu_0, \ldots, \nu_{n-2})^T \\
\vdots \\
\mathbf{p}(n) = (\nu_{-(n-1)}, \nu_{-(n-2)}, \ldots, \nu_0)^T
\end{gathered}$$

Hence, by components,

$$p^i(j) = \nu_{i-j}\ ; \quad i = 1, \ldots, n; \quad j = 1, \ldots, n$$

Let $\mathbf{r}$ be the vector which components are the single ratios of an n-tone JI scale close to the Pythagorean one. We write its components as

$$r^i = \frac{p^{\alpha_i}}{q^{\beta_i}}\ ; \quad i = 1, \ldots, n$$

In order to account for a hierarchy based on tonality, we consider the harmonic distances of each tone relative to the fundamental,

$$\Delta_{\mathrm{h}}(1, r^i) = \Delta_{\mathrm{h}}(p^{\alpha_i}, q^{\beta_i}) = \alpha_i \log_2 p_i + \beta_i \log_2 q_i$$

Their associated harmonic consonances will be taken as the weights to ponder whether one note of the Pythagorean scale that matches (or is very close) to one of the JI scale must be taken more or less into consideration in order to evaluate the dissimilarity between both scales. Thus, once normalized, the weights are

$$w^i = \frac{C_{\mathrm{h}}(1, r^i)}{\sum_{j=1}^{12} C_{\mathrm{h}}(1, r^j)}; \quad i = 1, \ldots, n$$

They are used to define the inner product matrix $\mathbf{W} = \mathrm{diag}(\mathbf{w}) \in \mathcal{M}_{n\times n}$, which obviously is positive-definite.

Indeed, we may compare a Pythagorean scale with several JI scales having different intonation qualities, i.e., with different selections of the single ratios. In such a case, for $k = 1, \ldots, m$, there will be a set of $\mathbf{r}(k)$ vectors, with components $r^i(k)$ and weights $w^i(k)$, and each vector associated with an inner product matrix $\mathbf{W}(k)$.

Two approaches are proposed to quantify the degree of coincidence of two scales $\mathbf{p}(j)$ and $\mathbf{r}(k)$ ($j = 1, \ldots, n; k = 1, \ldots, m$).

(1) Starting from the error vector $\mathbf{e}(j,k) = \mathbf{p}(j) - \mathbf{r}(k)$, the quadratic norm associated with the inner product provides a measure of the degree of coincidence of the notes of the respective scales,

$$\|\mathbf{e}(j,k)\|_W^2 = (\mathbf{p}(j) - \mathbf{r}(k))^T\, \mathbf{W}(k)\, (\mathbf{p}(j) - \mathbf{r}(k)) \tag{8}$$

To compare the deviation of the j-th Pythagorean scale relative to the k-th JI scale

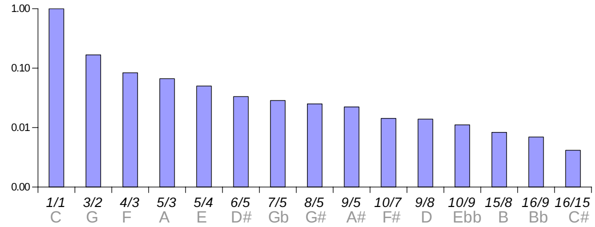

Figure 6. Relative harmonic consonances in logarithmic scale for JI tones.

we may use the mean square error (MSE),

$$E(j,k) = \frac{1}{n}\,\|\mathbf{e}(j,k)\|_W^2 \tag{9}$$

(2) Alternatively, it is possible to count the number of coincidences or matches between the notes of the respective scales. We will say that two tones in the i-th component of the scales $\mathbf{p}(j)$ and $\mathbf{r}(k)$ coincide at level ε if $|e^i(j,k)| = |p^i(j) - r^i(k)| \le \varepsilon$. Then, the components of the *vector of coincidences* $\mathbf{m}_\varepsilon(j,k)$ and the number of matches at level ε of both scales are given as

$$m^i_\varepsilon(j,k) = \begin{cases} 1, & \text{if} \quad |e^i(j,k)| \le \varepsilon \\ 0, & \text{if} \quad |e^i(j,k)| > \varepsilon \end{cases} \qquad m_\varepsilon(j,k) = \|\mathbf{m}_\varepsilon(j,k)\|^2 = \sum_{i=1}^{n} m^i_\varepsilon(j,k) \tag{10}$$

The latter norm is unweighted.

7. Results

The tones composing the JI scales, i.e., the twelve components of vectors $\mathbf{r}(k)$ for $k = 1,\ldots,8$ are represented as black dots in Table 3.

	C 1	$C^\sharp_J$ $\frac{16}{15}$	$E^{\flat\flat}_J$ $\frac{10}{9}$	**D** $\frac{9}{8}$	$D^\sharp_J$ $\frac{6}{5}$	$\mathbf{E}_J$ $\frac{5}{4}$	**F** $\frac{4}{3}$	$G^\flat_J$ $\frac{7}{5}$	$F^\sharp_J$ $\frac{10}{7}$	**G** $\frac{3}{2}$	$G^\sharp_J$ $\frac{8}{5}$	$\mathbf{A}_J$ $\frac{5}{3}$	$B^\flat$ $\frac{16}{9}$	$A^\sharp_J$ $\frac{9}{5}$	$\mathbf{B}_J$ $\frac{15}{8}$	
JI-1	●	●	●	○	●	●	●	●	○	●	●	●	●	○	●	F,f,a
JI-2	●	●	●	○	●	●	●	●	○	●	●	●	○	●	●	G♯,a
JI-3	●	●	●	○	●	●	●	○	●	●	●	●	●	○	●	F,f,a
JI-4	●	●	●	○	●	●	●	○	●	●	●	●	○	●	●	G♯,a
JI-5	●	●	○	●	●	●	●	●	○	●	●	●	●	○	●	C,f
JI-6	●	●	○	●	●	●	●	●	○	●	●	●	○	●	●	C,G♯,c
JI-7	●	●	○	●	●	●	●	○	●	●	●	●	●	○	●	C,f
JI-8	●	●	○	●	●	●	●	○	●	●	●	●	○	●	●	C,G♯,c

Table 3. Twelve-tone just intonation scales depending on the choice of the alternative pairs of tones. The column on the right indicates the JI diatonic scales that can be formed with a subset of them.

The first approach uses the normalized weights corresponding to the harmonic consonances shown in Figure 6. It leads to the results detailed in Figure 7, showing how close each Pythagorean scale $\mathbf{p}(j)$ $(j = 1,\ldots,12)$ is to the JI scale $\mathbf{r}(k)$ $(k = 1,\ldots,8)$, according to Eq. 9. It is summarized in Figure 8, where the average MSE for each Pythagorean scale (relative to all of the JI scales) together with the JI scale which is best fitted (indicated just below the label of the corresponding Pythagorean scale) are shown.

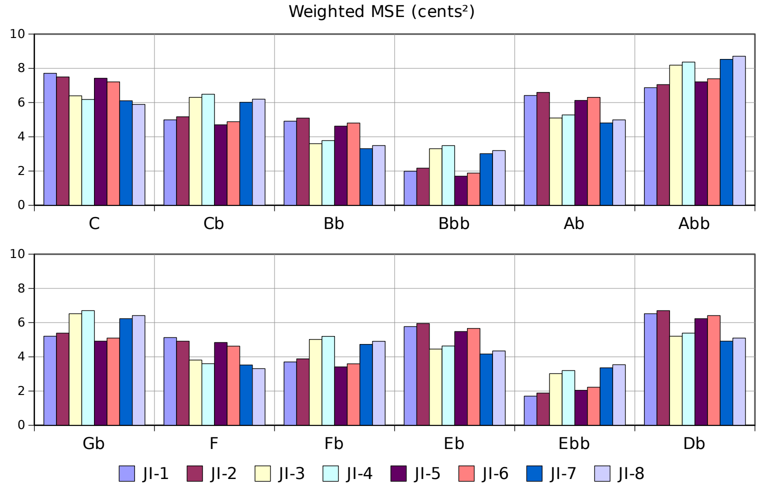

Figure 7. MSE of the twelve Pythagorean scales relative to the eight JI scales.

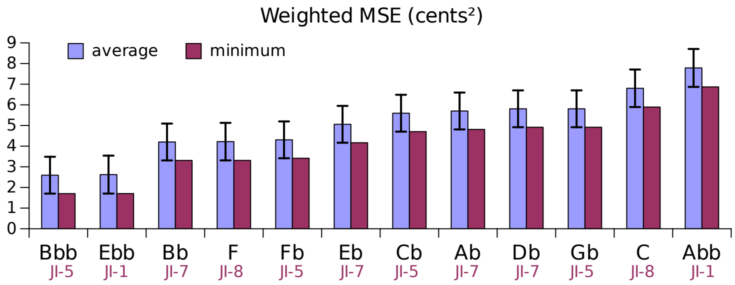

Figure 8. Pythagorean scales in order of best MSE and best fitted JI scale.

There are two Pythagorean scales providing the best fit of one JI scale. One starts the fifth iterations at $B^{\flat\flat}$, with the narrow fifth between D and A ($B^{\flat\flat}$), similarly as the *Gamme grecque*. The best fit is with scale JI-5, matching, in particular, all the notes of the C major diatonic scale (see Table 3). Scales J-6, J-1, and J-2 also have very low MSE values.

The other one is the Pythagorean scale starting at $E^{\flat\flat}$, with the wolf fifth between G and D ($E^{\flat\flat}$). The best fitting scale is JI-1, matching all the notes of the F major diatonic scale. Scales J-2, J-5, and J-6 also have very low MSE values.

The following two better fits are the Pythagorean's $B^{\flat}$ (with scale JI-7, matching the F minor diatonic scale) and F (with scale JI-8, matching C minor diatonic scale).

The second approach does not take into account weights. It only accounts for the number of matches regardless the consonance level of the tones. Therefore, it is a method independent of tonality, as it would be the first approach if the weights were not taken into account. It leads to the results shown in Figure 7. The maximum number of matches, either at coincidence levels $\varepsilon = 2¢$ or $6¢$, is reached by the four scales that in the previous approach led to the lowest MSE's. Let us remember that the 6¢-level becomes a 2¢-level by using the less JI tritones $\frac{45}{32}$ and $\frac{64}{45}$ instead of $\frac{7}{5}$ and $\frac{10}{7}$.

As shown in Table 4, the Pythagorean scale starting at $B^{\flat}$ has 9 coincidences with J-7 at level $\varepsilon = 6¢$. We know by Table 2 that the former scale matches Pythagorean F minor and $C^{\sharp}$ major diatonic scales. Also with 9 coincidences, the scales starting at $B^{\flat\flat}$ (matching C major and $F^{\flat}$ minor) with J-5, the scale starting at F (matching C minor and $G^{\sharp}$ major) with J-8, and the one starting at $E^{\flat\flat}$ (matching F major and $B^{\flat\flat}$ minor) with J-1.

		C	$C^\flat$	$B^\flat$	$B^{\flat\flat}$	$A^\flat$	$A^{\flat\flat}$	$G^\flat$	F	$F^\flat$	$E^\flat$	$E^{\flat\flat}$	$D^\flat$
ε=2¢	JI-1	5	5	7	7	5	7	4	6	6	6	8	4
	JI-2	6	4	6	6	4	6	3	7	5	5	7	3
	JI-3	5	5	7	7	5	7	4	6	6	6	8	4
	JI-4	6	4	6	6	4	6	3	7	5	5	7	4
	JI-5	6	6	8	8	6	6	5	7	7	7	7	5
	JI-6	7	5	7	7	5	5	4	8	6	6	6	4
	JI-7	6	6	8	8	6	6	5	7	7	7	7	5
	JI-8	7	5	7	7	5	5	4	8	6	6	6	4
ε=6¢	JI-1	5	6	7	8	5	8	5	6	7	6	9	4
	JI-2	6	5	6	7	4	7	4	7	6	5	8	3
	JI-3	6	5	8	7	6	7	4	7	6	7	8	5
	JI-4	7	4	7	6	5	6	3	8	5	6	7	4
	JI-5	6	7	8	9	6	7	6	7	8	7	8	5
	JI-6	7	6	7	8	5	6	5	8	7	6	7	4
	JI-7	7	6	9	8	7	6	5	8	7	8	7	6
	JI-8	8	5	8	7	6	5	4	9	6	7	6	5

Table 4. Number of matches between the JI scales (rows) and the Pythagorean scales (columns, indicating the first iterate) at level ε. In gray, maximum number of coincidences.

8. Conclusion

By following Euler's remark, and according to Hellegouarch (1999a,b, 2002) and Kassel and Kassel (2010), the way of choosing the representatives of a 12-tone abstract Pythagorean scale should be based on the fact that the sense of hearing tends to identify with a single ratio all the ratios which are only slightly different from it. For this reason they choose the representatives as the simpler ratios among the possible tones of the Pythagorean scale. However, this is a misinterpretation of Euler's remark, since there may be single ratios much simpler in the vicinity of the Pythagorean ratios.

Two viewpoints were adopted to discuss the election of the representatives. The first one was strictly based on the tonal qualities of the diatonic major and minor JI scales, which are composed of the simplest single ratios. For the second viewpoint, two mathematical approaches were proposed, which may substitute the previous analysis when the number of scale tones is greater or if an alternative selection of JI tones is made. Since there will always be a wolf fifth in any Pythagorean scale, and there will also be couples of single ratios that cannot be matched simultaneously, we will need a simple way to count the tones that match the largest possible number of single ratios while also prioritizing the highest possible harmonic consonances.

According to the first viewpoint, if the Pythagorean comma is compensated by the syntonic comma by using several iterates backwards from the fundamental, it is possible to match the single ratios corresponding to the diatones of a JI scale within a coincidence level of one schisma, and also two more tones of the Pythagorean chromatic scale, one of them the tritone. Since these single ratios have greater harmonic consonance than the Pythagorean tones (when they do not match), the ear will tend to identify the JI tones.

By taking the diatonic C major JI scale as reference, the concrete Pythagorean scale starting at the class of the fifth $B^{\flat\flat}$ (9 fifths before the fundamental) has 9 out of 12 matching notes. If an instrument were to be tuned by fifths and were to play a composition in that mode, by tuning it so that the narrow fifth is between D and A, it would be fully compatible with JI instruments. The resulting scale is even better than the quarter-comma meantone temperament. Of course, such a criterion is always relative to the tonic, i.e., according to a movable do system. Similarly, if an instrument using Pythagorean tuning were to play in C minor mode, to match the diatonic minor JI scale tones it should be tuned starting from the first fifth before the fundamental, with the narrow fifth between

A♯ and F. Table 5 shows these choices and the four notes associated with improved the harmonic consonance, since they become much closer to single ratios.

Pyth	C	D♭	D	E♭	F♭	F	G♭	G	A♭	B♭♭	B♭	C♭
B♭♭	$3^0/2^0$	$2^8/3^5$	$3^2/2^3$	$2^5/3^3$	$2^{13}/3^8$	$2^2/3^1$	$2^{10}/3^6$	$3^1/2^1$	$2^7/3^4$	$2^{15}/3^9$	$2^4/3^2$	$2^{12}/3^7$
¢	0.00	90.22	203.91	294.13	384.36	498.04	588.27	701.96	792.18	882.40	996.09	1086.31
J-5	1/1	16/15	9/8	6/5	5/4	4/3	7/5	3/2	8/5	5/3	16/9	15/8
¢	0.00	111.73	203.91	315.64	386.31	498.04	582.51	701.96	813.69	884.36	996.09	1088.27

Pyth	C	C♯	D	D♯	E	F	F♯	G	G♯	A	A♯	B
F	$3^0/2^0$	$3^7/2^{11}$	$3^2/2^3$	$3^9/2^{14}$	$3^4/2^6$	$2^2/3^1$	$3^6/2^9$	$3^1/2^1$	$3^8/2^{12}$	$3^3/2^4$	$3^{10}/2^{15}$	$3^5/2^7$
¢	0.00	113.69	203.91	317.60	407.82	498.04	611.73	701.96	815.64	905.87	1019.55	1109.78
J-8	1/1	16/15	9/8	6/5	5/4	4/3	10/7	3/2	8/5	5/3	9/5	15/8
¢	0.00	111.73	203.91	315.64	386.31	498.04	617.49	701.96	813.69	884.36	1017.60	1088.27

Table 5. Pythagorean scales with the closest JI scales (matching tones in light gray). Pyth B♭♭ matches the diatonic major JI scale and Pyth F the diatonic minor JI scale. In dark gray, notes that improve Hellegouarch's choice.

In the second viewpoint, both mathematical approaches have confirmed the previous analysis. The first approach quantified the dissimilarity of two scales (in the present case, one Pythagorean, the other JI), by means of the quadratic norm of the vector composed of the respective differences of their tones, in cents. Since tonality matters, we used weights to be proportional to the harmonic consonance of the JI scale tones. The method was applied to test the twelve possible concrete Pythagorean scales in front of height JI scales generated by the diatonic major and minor JI scales, which were completed up to get a set of twelve single ratios close to those of the Pythagorean scale. By taking into account the tonal hierarchy of the scale tones, it was possible to evaluate which are the closest scales. Two Pythagorean scales provided the lowest MSE's, by matching all the tones of two diatonic major JI scales (the maximum number of tones that can be heard as simple ratios is 9). The Pythagorean scale that begins the circle of fifths at B♭♭ contains a whole major diatonic JI scale in C and the scale that begins the circle of fifths at E♭♭ contains a whole diatonic major JI scale in F.

There are also two Pythagorean scales with low MSE's that coincide with 9 tones of two diatonic minor JI scales. One scale begins the fifth iterations at F and the other begins at B♭.

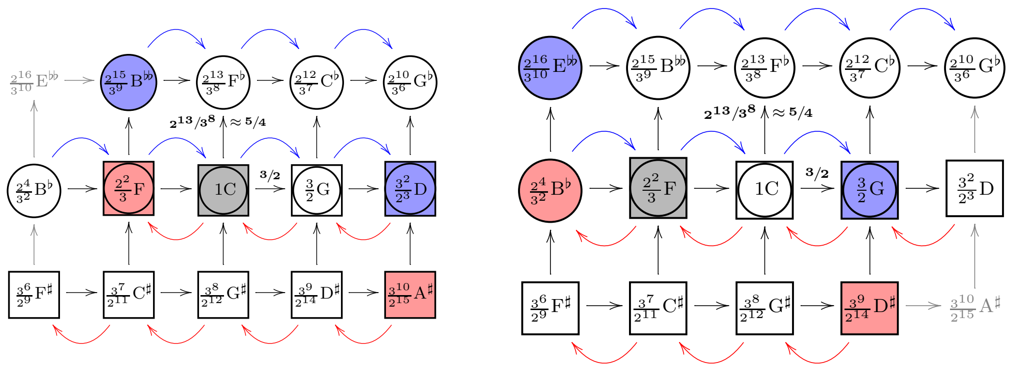

Table 6. (Left) The Pythagorean scale rising fifths from B♭♭ contains the notes between blue marks (as in reading direction), which match the C major diatonic JI scale. The Pythagorean scale rising fifths from F (descending fifths from A♯) (notes between red marks) matches the C minor scale. (Right) The Pythagorean scale rising fifths from E♭♭ (notes between blue marks) matches the F major scale. The Pythagorean scale rising fifths from B♭ (descending fifths from D♯) (notes between red marks) matches the F minor scale. All of these scales match 9 out of 12 JI notes.

According to Table 1, in the relation of major to minor scales, there is a dualism that connects the best Pythagorean B♭♭ for C major to the best Pythagorean F for C minor

(Table 6, left panel). In C major, the framework is rising fifths from $B^{\flat\flat}$ to D, i.e., from $\frac{2^{15}}{3^9}$ to $\frac{3^2}{2^3}$, using the notes $B^{\flat\flat}$-$F^{\flat}$-$C^{\flat}$ in the first row of Table 6 (left panel) to capture the syntonic corrections to A-E-B on the flat side. The tritone $G^{\flat}$ is therefore determined, as well as $B^{\flat}$ in the second row. The nine matching notes with those of JI are the two upper rows except $E^{\flat\flat}$ (between the notes marked in light blue, as in reading direction), with the narrow fifth in D-A.

For C minor, dually we take the chain of descending fifths (rising fourths), where the corrected tones are for the minor thirds of the principal triads on G-C-F (under major thirds of D-G-C), that is, $A^{\sharp}$-$D^{\sharp}$-$G^{\sharp}$. Again, the tritone comes for free $F^{\sharp}$, as well as $C^{\sharp}$. Thus the reversal, from $A^{\sharp}$ to F, i.e., from $\frac{3^{10}}{2^{15}}$ to $\frac{2^2}{3}$, i.e. forward fifths Pythagorean F. The nine matching notes with those of JI are the two lower rows except $B^{\flat}$ (between the notes marked in light red, as in reading direction), with the narrow fifth in $A^{\sharp}$-F.

A similar reasoning connects the best Pythagorean $E^{\flat\flat}$ for F major to the best Pythagorean $B^{\flat}$ for F minor. In F major, the framework is rising fifths from $E^{\flat\flat}$ to G, i.e., from $\frac{2^{16}}{3^{10}}$ to $\frac{3}{2}$, using the first row of Table 6 (right panel) to capture the syntonic corrections on the flat side. The tritone $C^{\flat}$ is therefore determined, as well as $G^{\flat}$ in the first row. The nine matching notes with those of JI are the two upper rows of Table 6 (right panel) except D, with the narrow fifth in G-D.

For F minor, we take the chain of descending fifths, from $D^{\sharp}$ to $B^{\flat}$, i.e., from $\frac{3^9}{2^{14}}$ to $\frac{2^4}{3^2}$, i.e. forward fifths Pythagorean $B^{\flat}$. The nine matching notes with those of JI are the two lower rows except $A^{\sharp}$, with the narrow fifth in $E^{\flat}$-$B^{\flat}$.

The second mathematical approach only considers the number of matching tones at a specific level of coincidence without taking consonance into account. It clearly identifies the previous four Pythagorean scales that have 9 matching tones at level $\varepsilon = 6¢$ and 8 matching tones at level $\varepsilon = 2¢$, although 9 matches at level $\varepsilon = 2¢$ by using the tritones $\frac{45}{32}, \frac{64}{45}$.

As displayed in Table 2, it is not possible to tune at the same time, according to just intonation, some pairs, such as $G^{\sharp}_{J}$-A_{J} and $G^{\sharp}_{J}$-B_{J}, which would be the ones needed to play in the harmonic minor or melodic minor just intonation modes. These are out of tune by a syntonic comma. Therefore, if Pythagorean tuning is to emulate or replace syntonic just tuning, one should always carry such a difference of one syntonic comma towards the three harmonically least relevant notes (according to the musical key) and keep it in tune just the other 9 notes.

In the end, such a just intonation criterion for choosing the notes of a Pythagorean scale, has followed another Hellegouarch's remark (Hellegouarch 2002): "players who are not slaves to a fixed pitch and can distinguish between $C^{\sharp}$ and $D^{\flat}$ will gain in expressivity".

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Appendix A. Notation used for the scale tones

$E_{53}^{\top}$			E_{53}^{3} (E_{12}^{3})					$E_{53}^{1/3}$ ($E_{12}^{1/3}$)					$E_{12}^{\top}$		supersubharmonics		
k	ν	¢	p	q	ν_p	¢		q'	p'	$\nu_{p'}$	¢		ν	¢	r	ν	¢
0	1.0000	0.00	0	0	1.0000	0.00	C	85	53	1.9958	1196.38		1.0000	0	1/1	1.0000	0.00
1	1.0132	22.64	12	19	1.0136	23.46		65	41	1.0115	19.84						
2	1.0265	45.28	24	38	1.0275	46.92		46	29	1.0253	43.30						
3	1.0400	67.92	36	57	1.0415	70.38		27	17	1.0393	66.76						
4	1.0537	90.57	48	76	1.0557	93.84		8	5	1.0535	90.22	D$^\flat$					
5	1.0676	113.21	7	11	1.0679	113.69	C$^\sharp$	73	46	1.0656	110.07		1.0595	100	16/15	1.0667	111.73
6	1.0816	135.85	19	30	1.0824	137.15		54	34	1.0802	133.53						
7	1.0959	158.49	31	49	1.0972	160.61		35	22	1.0949	156.99						
8	1.1103	181.13	43	68	1.1122	184.07		16	10	1.1099	180.45	E$^{\flat\flat}$			10/9	1.1111	182.40
9	1.1249	203.77	2	3	1.1250	203.91	D	81	51	1.1227	200.29		1.1225	200	9/8	1.1250	203.91
10	1.1397	226.42	14	22	1.1403	227.37		62	39	1.1380	223.75				8/7	1.1429	231.17
11	1.1547	249.06	26	41	1.1559	250.83		43	27	1.1535	247.21						
12	1.1699	271.70	38	60	1.1717	274.29		24	15	1.1692	270.67				7/6	1.1667	266.87
13	1.1853	294.34	50	79	1.1877	297.75		5	3	1.1852	294.13	E$^\flat$	1.1892	300			
14	1.2009	316.98	9	14	1.2014	317.60	D$^\sharp$	70	44	1.1988	313.98				6/5	1.2000	315.64
15	1.2167	339.62	21	33	1.2177	341.06		51	32	1.2152	337.44						
16	1.2328	362.26	33	52	1.2344	364.52		32	20	1.2318	360.90						
17	1.2490	384.91	45	71	1.2512	387.98		13	8	1.2486	384.36	F$^\flat$			5/4	1.2500	386.31
18	1.2654	407.55	4	6	1.2656	407.82	E	78	49	1.2630	404.20		1.2599	400			
19	1.2821	430.19	16	25	1.2829	431.28		59	37	1.2802	427.66				9/7	1.2857	435.08
20	1.2990	452.83	28	44	1.3004	454.74		40	25	1.2977	451.12						
21	1.3161	475.47	40	63	1.3181	478.20		21	13	1.3154	474.58						
22	1.3334	498.11	52	82	1.3361	501.66		2	1	1.3333	498.04	F	1.3348	500	4/3	1.3333	498.04
23	1.3509	520.75	11	17	1.3515	521.51	E$^\sharp$	67	42	1.3487	517.89						
24	1.3687	543.40	23	36	1.3700	544.97		48	30	1.3671	541.35				11/8	1.3750	551.32
25	1.3867	566.04	35	55	1.3887	568.43		29	18	1.3858	564.81				7/5	1.4000	582.51
26	1.4050	588.68	47	74	1.4076	591.89		10	6	1.4047	588.27	G$^\flat$			45/32	1.4063	590.22
27	1.4235	611.32	6	9	1.4238	611.73	F$^\sharp$	75	47	1.4209	608.11		1.4142	600	64/45	1.4222	609.78
28	1.4422	633.96	18	28	1.4433	635.19		56	35	1.4402	631.57				10/7	1.4286	617.49
29	1.4612	656.60	30	47	1.4629	658.65		37	23	1.4599	655.03						
30	1.4805	679.25	42	66	1.4829	682.11		18	11	1.4798	678.49	A$^{\flat\flat}$					
31	1.4999	701.89	1	1	1.5000	701.96	G	83	52	1.4969	698.34		1.4983	700	3/2	1.5000	701.96
32	1.5197	724.53	13	20	1.5205	725.42		64	40	1.5173	721.80						
33	1.5397	747.17	25	39	1.5412	748.88		45	28	1.5380	745.26						
34	1.5600	769.81	37	58	1.5622	772.34		26	16	1.5590	768.72				14/9	1.5556	764.92
35	1.5805	792.45	49	77	1.5836	795.80		7	4	1.5802	792.18	A$^\flat$	1.5874	800			
36	1.6013	815.09	8	12	1.6018	815.64	G$^\sharp$	72	45	1.5985	812.02				8/5	1.6000	813.69
37	1.6224	837.74	20	31	1.6237	839.10		53	33	1.6203	835.48				13/8	1.6250	840.53
38	1.6437	860.38	32	50	1.6458	862.56		34	21	1.6424	858.94						
39	1.6654	883.02	44	69	1.6683	886.02		15	9	1.6648	882.40	B$^{\flat\flat}$			5/3	1.6667	884.36
40	1.6873	905.66	3	4	1.6875	905.87	A	80	50	1.6840	902.25		1.6818	900			
41	1.7095	928.30	15	23	1.7105	929.33		61	38	1.7070	925.71				12/7	1.7143	933.13
42	1.7320	950.94	27	42	1.7339	952.79		42	26	1.7302	949.17						
43	1.7548	973.58	39	61	1.7575	976.25		23	14	1.7538	972.63				7/4	1.7500	968.83
44	1.7779	996.23	51	80	1.7815	999.71		4	2	1.7778	996.09	B$^\flat$	1.7818	1000	16/9	1.7778	996.09
45	1.8013	1018.87	10	15	1.8020	1019.55	A$^\sharp$	69	43	1.7983	1015.93				9/5	1.8000	1017.60
46	1.8250	1041.51	22	34	1.8266	1043.01		50	31	1.8228	1039.39						
47	1.8491	1064.15	34	53	1.8515	1066.47		31	19	1.8477	1062.85						
48	1.8734	1086.79	46	72	1.8768	1089.93		12	7	1.8729	1086.31	C$^\flat$			15/8	1.8750	1088.27
49	1.8981	1109.43	5	7	1.8984	1109.78	B	77	48	1.8945	1106.16		1.8877	1100			
50	1.9230	1132.08	17	26	1.9243	1133.24		58	36	1.9203	1129.62						
51	1.9484	1154.72	29	45	1.9506	1156.70		39	24	1.9465	1153.08						
52	1.9740	1177.36	41	64	1.9772	1180.16		20	12	1.9731	1176.54						
53	2.0000	1200.00	53	84	1.0021	3.62		0	0	2.0000	1200.00	C					

Table A1. Pythagorean scales E_{12}^{3}, $E_{12}^{1/3}$ (alternating stripes) and E_{53}^{3}. $\top$ indicates equal temperament scales.