The Version of Record of this manuscript has been published and is available in
Journal of Mathematics and Music, April 2023
https://doi.org/10.1080/17459737.2023.2194301

On the divisions of the octave in generalized Pythagorean scales and their bidimensional representation

Rafael Cubarsi

Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain

For well-formed generalized Pythagorean scales it is explained how to fill in a bidimensional table, referred to as scale keyboard, to represent the scale tones, arranged bidimensionally as iterates and cardinals, together with the elementary intervals between them. In the keyboard, generalized diatonic and chromatic intervals are easily identified. Two factor decompositions of the scale tones, which are particular cases of duality, make evident several properties on the sequence of intervals composing the octave, such as the number of repeated adjacent intervals and the composition of the generic step-intervals. The keyboard is associated with two matrix forms. When they are mutual transpose, the keyboard is reversible, as in the 12-tone Pythagorean scale. In this case, the relationship between the two main factor decompositions is given by an involutory matrix.

Keywords: generated scale; closure condition; Pythagorean tuning; generated tone system; well-formed scales; scale temperament; generalized diatonic scales; Myhill's property

2010 Mathematics Subject Classification: 05C12; 11A05; 54E35

1. Introduction

In the 12-tone Pythagorean (chromatic) scale generated by iterations of the third harmonic, the ratio between two consecutive scale tones[1] is either the chromatic semitone A or the diatonic semitone B. By excluding the fundamental, the minimal and maximal extreme tones of the scale are the 7-th and 5-th iterates[2], respectively. The first iterate has cardinal 7 within the octave, i.e., its scale-order index is 7, and the scale note with cardinal 1 is the seventh iterate. Among the 12 elementary intervals composing the octave, 5 are chromatic and 7 diatonic, ordered as $ABABABBABABB$. The chromatic semitones, A, come alone; the diatonic semitones, B, corresponding to the notes of the 7-tone Pythagorean (diatonic) scale, can be single or double.

Nevertheless, the impossibility to provide a complete set of justly intoned concords for each of the seven diatones has historically led to complete the scale with more than five accidentals. Several examples of keyboards subdividing the octave with more chromatic notes are described by Lindley (1980). For instance, Marin Mersenne, in his Harmonie Universelle published in 1636, urged the adoption of elaborate keyboards comprising 18 pitch classes: "the most economical way to provide all possible pure concords among the naturals themselves is to have two D's, one pure with F and A, and the other, a comma higher, pure with G and B. Then these eight diatonic notes may be surrounded with chromatic notes giving each natural all six of its possible concords". Similarly, Guillaume Costeley in 1558 and Francisco de Salinas in 1577 used temperaments with 19 and even 24 notes per octave.

Email: rafael.cubarsi@upc.edu

[1] Scale tone refers to a scale note in the frequency domain $[1, 2)$, i.e., the representative of a frequency class (FC) in the octave, while in the $\log_2$ space it is a pitch class (PC) in $[0, 1)$.

[2] We will also refer to one iteration as a fifth, since it corresponds to the musical interval perfect fifth, 5 being the number of notes of the diatonic scale comprising the interval.

We propose to study how is the process of subdividing the octave in well-formed scales with an arbitrary number of tones and an arbitrary generator. The above and other properties concerning the distribution of the scale tones are analyzed in the current work by using the approach for generalized Pythagorean scales described in Cubarsi (2020) developed in the specific level of the frequency domain. In particular, we are interested in to represent all these features in a simple and organized way.

These are properties associated with (non-degenerate) well-formed scales of *one generator* (Carey and Clampitt 1989, 2012, 2017), a particular family of generalized Pythagorean scales generated by a positive real tone h other than a rational power of 2 (otherwise we meet the degenerate case of an equal temperament scale). Following Cubarsi (2020), such a n-tone scale will be referred to as a h-cyclic scale[3] E_n^h. Under this approach, its tones are

$$\nu_k = \frac{h^k}{2^{[\![k]\!]}}; \quad k = 0, \ldots, n-1 \tag{1}$$

with $[\![k]\!] = \lfloor k \log_2 h \rfloor$ (integer part). When the scale tones are ordered from the lowest to the highest pitch in $(1, 2)$ we find two *extreme tones*, the minimum tone ν_m and the maximum tone ν_M, which determine the two elementary factors $U = \nu_m$ (from the fundamental up) and $D = \frac{2}{\nu_M}$ (from the fundamental down) associated with the generic widths of the step interval according to the Myhill's property (Clough and Myerson 1985, 1986). The indices satisfy $n = m + M$, all of them coprime.

The tone $\nu_n = \frac{h^n}{2^{[\![n]\!]}}$ does not belong to the scale E_n^h, but provides the closure condition (either $\nu_n \to 1^+$ or $\nu_n \to 2^-$) determining the n-order comma $\kappa_n = \min(\nu_n, \frac{2}{\nu_n})$, i.e., the error in closing the scale near the fundamental $\nu_0 = 1$ with no other scale notes between them. However, the comma itself does not provide the information about whether ν_n closes above or below the fundamental. By using the index[4]

$$N = [\![m]\!] + [\![M]\!] + 1 \tag{2}$$

it is possible to define the *scale closure* $\gamma_n = \frac{h^n}{2^N}$, which is a value close to 1, either above or below. Then, the *scale digit* $\delta = N - [\![n]\!]$, taking values 0 or 1, gives such an information, so that $\delta = 0 \iff \gamma_n > 1$ $(\nu_n \to 1^+, \gamma_n = \kappa_n)$ or $\delta = 1 \iff \gamma_n < 1$ $(\nu_n \to 2^-, \gamma_n = \kappa_n^{-1})$. The value $|\log_2 \gamma_n| = \log_2 \kappa_n$ multiplied by 1200 measures the distance from ν_n to 1 in cents.

A n-tone cyclic scale is associated with fractions $\frac{N}{n}$ providing convergent and semi-convergent continued fractions expansions of $\log_2 h$. In particular, *optimal* scales are those providing the best approximations from both sides, which is tantamount of getting the best estimations $\gamma_n \approx 1$ for the scale closure, among other properties discussed in Cubarsi (2020).

The family of cyclic scales can be determined by initializing $m = M = 1$ (for $n = 2$), so that the indices of the extreme tones of two consecutive cyclic scales $E_n^h \subset E_{n^+}^h$ satisfy

$$\begin{pmatrix} m^+ \\ M^+ \end{pmatrix} = \begin{pmatrix} 1 & 1-\delta \\ \delta & 1 \end{pmatrix} \begin{pmatrix} m \\ M \end{pmatrix}$$

[3] According to this paper, cyclic scales are well-formed scales that can be obtained for any real positive value h, by defining the octaves of the fundamental frequency ratio 1 from the monogenous group $\{\omega^k, k \in \mathbb{Z}\}$, for any real number $\omega > 1$, and not only for $\omega = 2$. This is a particular subset of the cyclic scales as defined in Jedrzejewski (2006, pp. 169-172), which are not restricted to well-formed scales. For well-formed scales, the partition of the octave induced by the scale notes has exactly two sizes of scale steps and each number of generic intervals occurs in two different sizes, while the other scales have three sizes of scale steps and generic intervals. In both cases, the generator h can be assigned to specific real numbers, such as the Golden number, e, π, the Euler constant, etc., as this author explains.

[4] It can also be expressed as $N = [\![n-k]\!] + [\![k]\!] + 1$, for $0 < k < n$.

In addition, for two consecutive cyclic scales $E^h_{n^-} \subset E^h_n$, one of the following cases takes place,

$$(\imath) \quad m > M \Rightarrow E^h_{n^-} = E^h_m \; ; \qquad (\imath\imath) \quad m < M \Rightarrow E^h_{n^-} = E^h_M \tag{3}$$

Several basic properties of cyclic scales are now reviewed, which will be taken as a starting point for the current work. Any pair of *pseudo-complementary* tones ν_k and ν_{n-k} satisfy

$$\nu_k \, \nu_{n-k} = 2\gamma_n; \quad k = 1, \ldots, n-1 \tag{4}$$

The pair $(\nu_k, \frac{2}{\nu_{n-k}})$ in the $\log_2$ space is the *spectrum* of a step-interval corresponding to a number of k h-iterates (or iterates, to simplify), with constant spectrum width $|\log_2 \gamma_n|$ for all the step-intervals. In particular, for the unit step-interval, we get $\gamma_n = \frac{\nu_m \, \nu_M}{2} = \frac{U}{D}$. Each pair of pseudo-complementary tones induces a partition in a set of j octaves. They satisfy

$$\nu_k^{n-k} \left(\frac{2}{\nu_{n-k}}\right)^k = 2^j; \quad k \in \{1, \ldots, n-1\} \tag{5}$$

where the value calculated from one of the following expressions,

$$j = (\llbracket n-k \rrbracket + 1) \; k - \llbracket k \rrbracket \; (n-k) = Nk - \llbracket k \rrbracket n \tag{6}$$

is the cardinal of the scale tone ν_k. In particular, for $k = m$, equation (6) becomes[5]

$$1 = (\llbracket M \rrbracket + 1) \; m - \llbracket m \rrbracket \; M \tag{8}$$

Another particular case gives the cardinal of the first iterate[6] ν_1,

$$\mu = N - \llbracket 1 \rrbracket n \tag{9}$$

On the ground of the foregoing properties, we shall analyze, for a generic cyclic scale, how the tones and elementary intervals are distributed along the octave. To this purpose, the scale tones will be expressed as the product of integer powers of two factors. It will be referred to as *factor decomposition* (FD). FD's are particular cases of duality (Regener 1973; Carey and Clampitt 1996; Clampitt and Noll 2011), here worked out in the space of the frequency ratios instead of note intervals. Two main FD's are studied, although we discuss other possible FD's.

The *first* FD is the one provided by the generator/co-generator pair. The reduction of the iterates in equation (1) to the reference octave $\Omega_0 = [1, 2)$ is interpreted as follows: it consists in either increasing the tones by the factor corresponding to the first iteration of the generator, i.e., ν_1 (the fifth, in the case of the 12-tone Pythagorean scale), or decreasing by its complementary $\frac{2}{\nu_1}$ (the fourth).

The *second* FD is obtained from the step/co-step factors (e.g., Noll 2015), that is, it consists in to determine the partition of the octave from iterations of the two elementary factors associated with the unit step-interval, U and D, which are the generalization of the chromatic and diatonic intervals of the Pythagorean scale.

We study the relationship between both main FD's and analyze the structure of the octave in a general case of a cyclic scale E^h_n. Examples for the Pythagorean scales of 12 and 53 tones are provided, where the scale temperament is represented in a table and in matrix form, leading

[5] This equation is equivalent to the following ones, with n, N coprime,

$$N\,m - n\,\llbracket m \rrbracket = 1, \quad n\,(\llbracket M \rrbracket + 1) - N\,M = 1 \tag{7}$$

[6] If the generator is a FC, i.e., it satisfies $1 < h < 2$, then $\llbracket 1 \rrbracket = \lfloor \log_2 h \rfloor = 0$ and N is just the cardinal μ of the generator.

to the concept of scale keyboard, where the notes of the scale are arranged bidimensionally according to both main FD's, together with the corresponding temperament. The singular case of reversible keyboards for the 12-tone Pythagorean tuning is also described, where the matrices associated with the keyboards are mutual transpose.

2. Factor decompositions

2.1. *Pairs of factors*

Although the iteration index of any tone $\nu_k \neq \nu_0$ of the cyclic scale E_n^h can be expressed from two coprime indices a and b as $k = ar - bs$, for certain positive integers r, s satisfying $0 < r \leq b$, $0 \leq s < a$, it is not always possible to write the scale tones as $\nu_k = \nu_a^r \left(\frac{2}{\nu_b}\right)^s$, since the powers of 2 on both sides may not match.

In a general case, we will write the scale tones as satisfying

$$\nu_k = \left(\frac{h^p}{2^{p'}}\right)^\alpha \left(\frac{2^{q'}}{h^q}\right)^\beta \tag{10}$$

with indices in $\mathbb{Z}$, and $p, q > 0$ coprime. Without loss of generality we may assume $\alpha \geq 0$. We also ask the above FD to include the fundamental at the endpoints of the octave. Since for $\alpha = \beta = 0$ we get $\nu_0 = 1$, in addition, for certain integers α_0 and β_0, it must be fulfilled

$$\left(\frac{h^p}{2^{p'}}\right)^{\alpha_0} \left(\frac{2^{q'}}{h^q}\right)^{\beta_0} = 2 \tag{11}$$

LEMMA 2.1 *The existence of $\alpha_0, \beta_0 \in \mathbb{Z}$ in the above equation is the necessary and sufficient condition to obtain the FD of equation (10).*

Proof. It is obviously necessary in order to determine the extreme 2. Let us prove that it is sufficient. On the one hand, with regard to the powers of the generator h, it is required $p\alpha_0 - q\beta_0 = 0$. Hence, since p, q are coprime, $\alpha_0 = qx$ for certain $x > 0$, and $\beta_0 = px$. On the other hand, with regard to the powers of 2, $p'\alpha_0 - q'\beta_0 = -1$, hence $p'qx - q'px = -1$. Then, since p, q are coprime, it is required $x = 1$. Therefore, it must be fulfilled

$$d \equiv pq' - qp' = 1 \tag{12}$$

and $\alpha_0 = q$, $\beta_0 = p$. Equation (12) guarantees that the indices α, β in equation (10) are integers, since this involves the fulfillment of the following two systems,

$$\begin{pmatrix} k \\ [\![k]\!] \end{pmatrix} = \begin{pmatrix} p & -q \\ p' & -q' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{d} \begin{pmatrix} -q' & q \\ -p' & p \end{pmatrix} \begin{pmatrix} k \\ [\![k]\!] \end{pmatrix} \tag{13}$$

The value $d = 1$ (determinant of the first system) also guarantees that there is an isomorphism between the pairs of indices $(k, [\![k]\!])$ and (α, β). ∎

We point out several particular cases of pairs of factors generating the scale tones:

(1) A unique tone $\nu_p = \frac{h^p}{2^{[\![p]\!]}}$ generates the scale tones if, for certain $x \in \mathbb{Z}$, it satisfies $\nu_p^p \left(\frac{2^x}{\nu_p}\right)^p = 2$.

That is, $\left(\frac{h^p}{2^{[\![p]\!]}}\right)^p \left(\frac{2^{[\![p]\!]+x}}{h^p}\right)^p = 2$. Hence $x = p = 1$. Thus, any scale tone can be expressed as

$$\nu_k = \nu_1^r \left(\frac{\nu_1}{2}\right)^s$$

which we call *first* FD.

(2) There is only one case in equation (5), for $j = 1$, $k = m$, satisfying the condition of equation (10). It provides the *second* FD,

$$\nu_k = \nu_m^r \left(\frac{2}{\nu_M}\right)^s$$

(3) The extreme tones of any cyclic scale $E_{n'}^h \subset E_n^h$ satisfy $\nu_{m'}^{M'} \left(\frac{2}{\nu_{M'}}\right)^{m'} = 2$. Therefore, the condition of equation (10) is fulfilled. Hence, the scale tones of E_n^h can be expressed as

$$\nu_k = \nu_{m'}^r \left(\frac{2}{\nu_{M'}}\right)^s$$

Notice that the condition of coprime indices is not sufficient for such a specific FD. If further requires that they correspond to extreme tones of a cyclic scale.

2.2. *First FD*

According to the first case in Section 2.1, starting by $\nu_0 = 1$, a tone $\nu_k \in E_n^h$ can be obtained from ν_{k-1} either *increasing* by a factor $\nu_1 > 1$ if $\nu_1 \, \nu_{k-1} < 2$, or *decreasing* by a factor $\frac{2}{\nu_1} < 1$ if $\nu_1 \, \nu_{k-1} \geq 2$. Thus, from $k = 1$ to $n - 1$,

$$\nu_k = \begin{cases} \nu_{k-1} \, \nu_1 \, , & \nu_{k-1} < \frac{2}{\nu_1} \\ \nu_{k-1} \left(\frac{2}{\nu_1}\right)^{-1} , & \nu_{k-1} \geq \frac{2}{\nu_1} \end{cases} \tag{14}$$

Hence, we can write

$$\nu_k = \Psi(\alpha, \beta) \equiv (\nu_1)^\alpha \left(\frac{2}{\nu_1}\right)^{-\beta} ; \; \alpha, \beta \in \mathbb{N}; \; 0 \leq \alpha + \beta = k < n \tag{15}$$

By taking into account equation (1), it is satisfied

$$\begin{pmatrix} k \\ [\![k]\!] \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ [\![1]\!] & 1 + [\![1]\!] \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{16}$$

The first FD defines univocally the scale tones from the indices (α, β). Thus, for $0 \leq \alpha < (1 + [\![1]\!])n - [\![n]\!]$ and $0 \leq \beta < [\![n]\!] - [\![1]\!]n$, the scale tones have *coordinates*

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 + [\![1]\!] & -1 \\ -[\![1]\!] & 1 \end{pmatrix} \begin{pmatrix} k \\ [\![k]\!] \end{pmatrix} \tag{17}$$

In addition, the values $\beta = -\alpha = [\![n]\!] - [\![1]\!]n$ determine the frequency 2.

Let us bear in mind that, when characterizing a tone by the couple $(k, [\![k]\!])$ according to equation (1), the latter value is not informative, since each scale tone corresponds to a different value of k. Nevertheless, when representing a tone with the pair (α, β), it is not possible to determine the tone just from one of the components, since it is possible that different tones share the same value of any of the previous components. Also notice that not all possible factor products give rise to scale tones.

The next iteration to $\nu_k \equiv \Psi(\alpha, \beta)$, with $k = \alpha + \beta$, is

$$\nu_{k+1} = \begin{cases} \Psi(\alpha + 1, \beta) \iff \nu_k < \frac{2}{\nu_1} \\ \Psi(\alpha, \beta + 1) \iff \nu_k \geq \frac{2}{\nu_1} \end{cases} \tag{18}$$

2.3. *Second FD*

According to the second case in Section 2.1, we express an arbitrary tone $\nu_k \in E_n^h$ in the following form,

$$\nu_k = \Phi(p, q) \equiv \nu_m^p \left(\frac{2}{\nu_M}\right)^q, \quad 0 \leq k < n \tag{19}$$

As justified in Cubarsi (2020), the factors ν_m and $\frac{2}{\nu_M}$ are the only two possible ratios (of value greater than 1) between two consecutive scale tones introducing the non-equal temperament.

The next tone to $\Phi(p, q)$, by ordering the scale as increasing frequencies, is either $\Phi(p + 1, q)$ or $\Phi(p, q + 1)$. Therefore, the value $j = p + q$ is the *cardinal* of ν_k counted from the fundamental $(p = q = 0)$.

By taking into account equation (1), we get the relationship between the corresponding indices,

$$\begin{pmatrix} k \\ [\![k]\!] \end{pmatrix} = \begin{pmatrix} m & -M \\ [\![m]\!] & -([\![M]\!] + 1) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{20}$$

for[7] $0 < p \leq M$, $0 \leq q < m$, and $0 \leq k < n = m + M$. The extreme tones correspond to $\Phi(1, 0)$ and $\Phi(M, m - 1)$. In addition, $\Phi(M, m) = 2$.

2.4. *Relationship between both FD's*

For the tone $\nu_k \in E_n^h$, making $\Psi(\alpha, \beta) = \Phi(p, q)$, according to equations (15) and (19) we have

$$\begin{pmatrix} 1 & 1 \\ [\![1]\!] & 1 + [\![1]\!] \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} m & -M \\ [\![m]\!] & -([\![M]\!] + 1) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \tag{21}$$

Let us bear in mind that these expressions are valid for $\alpha + \beta < n$ and $p + q < n$, by excluding the indices that satisfy $\alpha + \beta = p + q = n$. These FD's give the coordinates of the scale notes in two reference systems: one from the generator/co-generator pair and the other one from the step/co-step factors. Since the determinants of these matrices are non-null, the transformations are isomorphic.

[7] Since m, M are coprime we may recall Bézout's lemma stating that there are two integers a and b that satisfy $1 = ma - Mb$. This equation admits infinitely many pairs of solutions, although there are two solutions satisfying $|a| < |M|$ and $|b| < |m|$. Furthermore, since m and M are positive, there exists a unique couple of values (a, b) satisfying $0 < a < M$, $0 \leq b < m$. The three pairs (a, b), (a, m), (M, b) are also coprime numbers. The values for (a, b) are those given in equation (8). If we multiply the above equation by an integer $k > 0$ and write $p = ka, q = kb$, we get the Diophantine equation $k = mp - Mq$, corresponding to the first component in equation (20). Then, there only exists one pair of values (p, q) satisfying $0 < p \leq M$, $0 \leq q < m$, although such an equation is also valid for the value $k = 0$ in the trivial case $p = q = 0$. It is possible to determine directly the values (p, q) by using the appropriate algorithm by working only with the first component in equation (20); however, the linear system provides them directly.

The respective coordinates are related according to the following matrices

$$\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)=A\left(\begin{array}{c}p\\ q\end{array}\right);\quad A=\left(\begin{array}{cc}(1+[\![1]\!])m-[\![m]\!] & [\![M]\!]+1-(1+[\![1]\!])M\\ [\![m]\!]-[\![1]\!]m & [\![1]\!]M-[\![M]\!]-1\end{array}\right) \tag{22}$$

$$\left(\begin{array}{c}p\\ q\end{array}\right)=A^{-1}\left(\begin{array}{c}\alpha\\ \beta\end{array}\right),\quad A^{-1}=\left(\begin{array}{cc}[\![M]\!]-[\![1]\!]M+1 & [\![M]\!]+1-(1+[\![1]\!])M\\ [\![m]\!]-[\![1]\!]m & [\![m]\!]-(1+[\![1]\!])m\end{array}\right) \tag{23}$$

The tone iterate $k=\alpha+\beta$ has cardinal is $j=p+q$. Then, by taking into account equations (2) and (9), they can be computed as

$$\alpha+\beta=m\,p-M\,q\,;\quad p+q=\mu\,\alpha-(n-\mu)\,\beta \tag{24}$$

2.5. *Involution*

In Section 5.4 we shall discuss the meaning of the particular case where $A^{-1}=A$, i.e., when A is an involutory matrix. Taking into account equations (22) and (23), this is satisfied if $(1+[\![1]\!])m-[\![m]\!]=[\![M]\!]-[\![1]\!]M+1$. By rearranging terms, we get $(1+[\![1]\!])m+[\![1]\!]M=[\![m]\!]+[\![M]\!]+1$. Bearing in mind equations (2) and (9), this condition becomes

$$m=\mu \tag{25}$$

That is, the index of the minimum tone m matches the cardinal μ of the first iterate.

In this case, the role of the pairs (α,β) and (p,q) is interchangeable, and will give rise to the concept of reversible keyboards, where the iterates and cardinals, according to equation (24), follow a similar scheme

$$k=m\,p-M\,q\,;\quad j=m\,\alpha-M\,\beta \tag{26}$$

i.e., if the pair of coordinates refers to the tone cardinal then the number we get is the tone iteration, but if the same pair refers to the iterate then we get the tone cardinal.

For example, for the Pythagorean scale ($h=3$), we have $n=12$, $N=19$, $m=7$, $M=5$, $[\![1]\!]=1$, and $\mu=7$. Hence the matrices A and A^{-1} are the same one:

$$A=\left(\begin{array}{cc}3 & -2\\ 4 & -3\end{array}\right),\quad A^{-1}=A$$

Then, iterates and cardinals satisfy $k=7\,p-5\,q$, $j=7\,\alpha-5\,\beta$.

3. Relationship between coordinates

3.1. *Tone cardinals*

By inverting the system of equation (20), and by taking into account equation (8), we obtain the following relationship

$$\begin{aligned}p&=([\![M]\!]+1)k-M[\![k]\!]\\ q&=[\![m]\!]k-m[\![k]\!]\end{aligned} \tag{27}$$

Adding both equations, since $n = m + M$, $N = [\![M]\!] + [\![m]\!] + 1$ and $j = p + q$, we get

$$j = Nk - n[\![k]\!] \tag{28}$$

so that,

$$j = kN \bmod n, \quad 0 \leq k < n \tag{29}$$

Since N and n are coprime, for values of k from 0 to $n-1$, j also takes all possible values from 0 to $n-1$. Equation (29) provides a way for counting the cardinal of the tone iterates ν_k within the scale, that we shall notate as

$$\vartheta_j \equiv \nu_k = \Phi(p, q) \tag{30}$$

Thus, the cardinal of ν_1 is

$$\mu = N \bmod n \tag{31}$$

More precisely, according to the equations (28) and (31), as already pointed out as a particular case of equation (6), we get

$$\mu = N - [\![1]\!]n \tag{32}$$

Hence, the cardinal μ of the first iterate is coprime with n. This is the condition ensuring that a well-formed scale has similar symmetry properties to a n-TET scale.

Therefore, the cardinal j of ν_k can also be expressed as

$$j = k\mu \bmod n, \quad 0 \leq k < n \tag{33}$$

with values $0 \leq j < n$. The one-to-one mapping between the indices of the scale notes given by equation (33) is one of Carey and Clampitt's (1989) characterizations of well-formed scales with regard to preserving rotational symmetry.

3.2. *Tone iterates*

We shall determine the k-th iteration in terms of the corresponding j-th cardinal. For the first component in equation (20), since $n = m + M$, the k-th iteration satisfies

$$k = jm - qn; \quad 0 \leq j < n, \quad 0 \leq q < m \tag{34}$$

As m and n are coprime, the bijection between the number of iteration k and the cardinal of the j-th note is given by the relationship

$$k = j\,m \bmod n, \quad 0 \leq j < n \tag{35}$$

Similarly, for the second component, since $N = [\![m]\!] + [\![M]\!] + 1$, we get

$$[\![k]\!] = j[\![m]\!] - qN; \quad 0 \leq j < n, \quad 0 \leq q < m \tag{36}$$

Since $[\![m]\!]$ and N are coprime, the bijection between the values $[\![k]\!]$ and j, is given by

$$[\![k]\!] = j\,[\![m]\!] \bmod N, \quad 0 \leq j < n \tag{37}$$

3.3. *Tones next and previous*

Next tone refers to the scale tones ordered by pitch in $[1, 2]$. Let us determine what is the iterate corresponding to each one of the two possible next tones to $\nu_k = \Phi(p, q)$ (for $0 \leq k < n$), namely $\Phi(p+1, q)$ and $\Phi(p, q+1)$.

For $\nu_k = \Phi(p, q)$ the first equation of equation (20) is satisfied. Then, for $\Phi(p+1, q)$ we write

$$(p+1)m - qM = k + m, \quad 0 \leq k + m < m + M$$

Therefore $k < M$, which is equivalent to $p < \frac{M}{m}(q+1)$.

For $\Phi(p, q+1)$, we have

$$pm - (q+1)M = k - M, \quad 0 \leq k - M < m + M$$

In such a case $k \geq M$, which is equivalent to $p \geq \frac{M}{m}(q+1)$.

Therefore, the next tone to $\nu_k = \Phi(p, q)$ is given by

$$\Phi^+(p, q) = \begin{cases} \Phi(p+1, q) = \nu_{k+m}; & \text{if } p < \frac{M}{m}(q+1), \ k < M \\ \Phi(p, q+1) = \nu_{k-M}; & \text{if } p \geq \frac{M}{m}(q+1), \ k \geq M \end{cases} \tag{38}$$

Thus, the tone ν_k increases by a fraction of either $U = \nu_m$ or $D = \frac{2}{\nu_M}$ depending on whether its index is or is not lower than the index M of the maximum tone[8]. In particular, the next tone to $\nu_M = \Phi(M, m-1)$, computed according to the previous second case, is $\Phi(M, m) = 2\nu_0$, so that $\Phi(M, m)$ and $\Phi(0, 0)$ represent the fundamental at both endpoints of the octave.

Similarly, the previous tone to $\nu_k = \Phi(p, q)$ $(0 < k < n)$ is determined as

$$\Phi^-(p, q) = \begin{cases} \Phi(p-1, q) = \nu_{k-m}; & \text{if } q \leq \frac{m}{M}(p-1), \ k \geq m \\ \Phi(p, q-1) = \nu_{k+M}; & \text{if } q > \frac{m}{M}(p-1), \ k < m \end{cases} \tag{39}$$

Thus, the tone ν_k decreases by a fraction of either $D = \frac{2}{\nu_M}$ or $U = \nu_m$ depending on whether its index is or is not lower than the index m of the minimum tone. In particular, the previous tone to ν_0 can be computed as the previous to $\Phi(M, m)$, according to the above second case, which is $\Phi(M, m-1) = \nu_M$.

3.4. *Increasing by the same elementary factor*

We may also ask whether starting from $\nu_k = \Phi(p, q)$ it is possible to get a number of i consecutive tones by increasing only the first argument, i.e., what are the consecutive tones $\Phi(p+i, q)$ with $i > 1$? These tones, according to equation (19), will all increase by factors U. In such a case, by assuming the pair (p, q) as corresponding to the k-th iterate, it is satisfied

$$pm - qM = k, \quad 0 \leq k < n = m + M$$

[8] From the point of view of algebraic combinatorics of words, equation (38) defines the scale as a Christoffel word of the alphabet $\{U, D\}$ with slope $\frac{m}{M}$ and length n (e.g., Noll 2008). The letter in the position $j+1$ is U when the h-iteration increases from k to $k+m$, according to the case $k < M$; hence, owing to equation (35), $jm \bmod n < (j+1)m \bmod n$. Otherwise, the letter in the position $j+1$ is D when the h-iteration decreases from k to $k-M$, according to the case $k \geq M$; therefore $jm \bmod n > (j+1)m \bmod n$. This matches the definition of a Christoffel word.

Thus, for $\Phi(p+i, q)$ we get the identity $(p+i)m - qM = k + im$, and the condition $0 \leq k + im < n$. Therefore, since $i \geq 1$, the increment of i consecutive tones by a factor U is only possible if

$$im < n - k \tag{40}$$

Hence, for the iterates $k \geq M$ it is not possible to increase by a factor U. In particular, the maximum value of i is obtained for $k = 0$, so that

$$im < n \tag{41}$$

Similarly, we may also ask for how many consecutive tones it is possible to increase in the second argument. These tones, according to equation (19), will all increase by factors D. Then, for $\Phi(p, q + i)$ we get the identity $pm - (q + i)M = k - iM$ and the condition $0 \leq k - iM < n$. Therefore, since $i \geq 1$, the increment of i consecutive tones by a factor D is only possible if

$$iM \leq k \tag{42}$$

Hence, for the iterates $k < M$ it is not possible to increase by a factor D. In particular, the maximum value of i is obtained for $k = n$, so that

$$iM \leq n \tag{43}$$

In Section 5 we shall see how to use the equations (41) and (43). Therefore we have proved the following results:

Theorem 3.1 *Consider the scale tones in cyclic order as $\{\vartheta_0, \vartheta_1, \ldots\}$, where each tone is the result of increasing the previous one by one elementary factor, either $U = \nu_m$ or $D = \frac{2}{\nu_M}$. The maximum number of consecutive U factors is the highest integer i_U satisfying $i_U\, m < n$. The maximum number of consecutive D factors is the highest integer i_D satisfying $i_D\, M \leq n$.* ∎

Since $n = m + M$, if $m > M$, it is $2m > n$, which is inconsistent with equation (41) for $i = 2$. If $M > m$, it is $2M > n$, which is inconsistent with equation (43) for $i = 2$. Then,

Corollary 3.2 *If $m > M$, $i_U = 1$. If $M > m$, $i_D = 1$.* ∎

3.5. *Coarser and refined scales*

Since, in both cases, *one of the elementary factors comes alone*, i.e. *has no similar adjacents*, it is always possible to merge two consecutive factors U and D as a *whole step* and consider the remaining one as a *half step*. Notice that, by construction, the first step after 1 must be U, and the last step before 2 must be D, therefore, the whole step must be formed as UD. From this point of view, it is clear why an alternative approach for the refinement of generic scales satisfying Myhill's property consists in to use words in the letters U and D (e.g., Noll 2006, 2007) (generic scales, however, do not need to satisfy the requirement of beginning by U and ending by D, since this is a feature of cyclic scales). Thus, given a scale whose factors satisfy $U^p D^q = 2$, if $U < D$ then it can be refined by factorizing $D = UD'$, otherwise by factorizing $U = U'D$, and so on. Notice that, on the contrary, to factorize $D = D'U$ would yield a scale beginning and ending with the factor U, and to factorize $U = DU'$ would lead to a scale beginning and ending with the same factor D, which is inconsistent with the definition of cyclic scales.

3.6. *Generic step-intervals*

It is also possible to be more precise about the two sizes of a generic step-interval. Consider two tones $\nu_k = \vartheta_j$ and $\nu_{k'} = \vartheta_{j'}$ satisfying $0 < \Delta_j \equiv j - j' < n$. The subtractions of the equations (35) and (37) respectively for both tones have the following pairs of solutions,

$$\Delta_k \equiv k - k' = \Delta_j \, m \bmod n > 0,\ \Delta_k < n\ ; \qquad \Delta'_k = \Delta_k - n < 0$$
$$\Delta_{[\![k]\!]} \equiv [\![k]\!] - [\![k']\!] = \Delta_j \, [\![m]\!] \bmod N > 0,\ \Delta_{[\![k]\!]} < N\ ; \qquad \Delta'_{[\![k]\!]} = \Delta_{[\![k]\!]} - N < 0$$

Then, according to equation (27), subtracted for both tones and bearing in mind the equations (7), we have the respective pairs of solutions for the quantities $\Delta_p \equiv p - p' \geq 0$ and $\Delta_q \equiv q - q' \geq 0$,

$$\Delta_p = ([\![M]\!] + 1)\Delta_k - M\Delta_{[\![k]\!]}\ ; \qquad \Delta'_p = \Delta_p - 1$$
$$\Delta_q = [\![m]\!]\Delta_k - m\Delta_{[\![k]\!]}\ ; \qquad \Delta'_q = \Delta_q + 1$$

Therefore, we meet Myhill's property,

THEOREM 3.3 *An interval of $\Delta_j > 0$ steps is composed of the elementary factors either $U^{\Delta_p} D^{\Delta_q}$ when the difference of iterates between endpoints increases, or $U^{\Delta_p - 1} D^{\Delta_q + 1}$ when the difference of iterates decreases.* ■

4. Generalized diatones

The 12-tone Pythagorean scale, known as chromatic scale, contains the 7-tone Pythagorean scale, known as diatonic scale, whose tones, the *diatones*, are basic to the western pitch notation. The previous cyclic scale to E^3_{12}, with indices of the extreme tones $m = 7$ and $M = 5$ is E^3_7, which indices of the extreme tones are $m = 2$ and $M = 5$. E^3_7 is the former scale that has been subdivided by adding *accidentals* to obtain the next finer cyclic scale E^3_{12}. For $n = 7$, according to the above results, the factors D come alone. The elementary factors follow the scheme

$$\begin{matrix} \nu_0 \xrightarrow{} \nu_2 \xrightarrow{} \nu_4 \xrightarrow{} \nu_6 \xrightarrow{} \nu_1 \xrightarrow{} \nu_3 \xrightarrow{} \nu_5 \xrightarrow{} 2\nu_0 \\ \vartheta_0 \xrightarrow{U} \vartheta_1 \xrightarrow{U} \vartheta_2 \xrightarrow{U} \vartheta_3 \xrightarrow{D} \vartheta_4 \xrightarrow{U} \vartheta_5 \xrightarrow{U} \vartheta_6 \xrightarrow{D} \vartheta_7 \end{matrix}$$

satisfying $U^5 D^2 = 2$. The factor $U = \frac{3^2}{2^3}$ is greater than $D = \frac{2^8}{3^5}$, hence it can be splitted as $U = U'D$, which adds an intermediate tone between U' and D, and gives rise to the scale E^3_{12}, satisfying $(U')^5 D^7 = 2$. In such a case, a tone coming after a factor D is a diatone, i.e., a tone already existing in E^3_7. The complementary set to the diatones is a pentatonic scale, with no two consecutive notes, which, in the chromatic scale, are interpreted as accidentals of the diatones. Likewise, the factor D, which is common to both scales, is the diatonic semitone, while the new factor U' in E^3_{12} is the chromatic semitone.

The study of (hyper)diatonic sets of d tones within a chromatic set of c tones has been largely studied (e.g., Clough 1979; Agmon 1989; Clough and Douthett 1991) leading to two diatonic models A ($c = 2d - 1$) and B ($c = 2d - 2$ with d odd), the latter having higher relevance to the study of harmonic tonality (Noll 2015). It is characterized in several ways in Clough and Douthett (1991) and is defined as a particular maximally even set, with precisely one tritone for $d \neq 2$, among other properties. In generalized Pythagorean tuning, diatonic sets are scarce. For instance, if $h = 3$ a diatonic set of type B exists only for $d = 7, c = 12$; if $h = 5$ then the only diatonic set is for $d = 3, c = 4$. Diatonicity has been extended to alternative concepts, such as diatonic/pentatonic systems (Gould 2000), pseudo-diatonic scales (Noll 2006), and generalized diatonic scales (Jedrzejewski 2009, 2008).

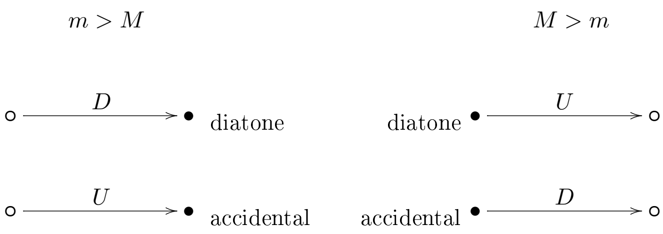

Figure 1. Diatones and accidentals in relation to the scale factors.

In the context of generalized Pythagorean tuning, we shall use the concept of diatone in a generic sense, as a tone already belonging to the previous scale in the chain of cyclic scales. Thus, any cyclic scale E_n^h will contain the *generic diatones*, namely the tones of the previous cyclic scale $E_{n^-}^h \subset E_n^h$, in addition to the new *non-adjacent* tones, which we consider as the *accidentals*. Let us assume that the scale E_n^h has indices of the extreme tones m, M and consider the scale tones in cyclic order as $\{\vartheta_0, \vartheta_1, \ldots\}$, where each one of these tones is the result of increasing the previous one by an elementary factor U or D. According to the cases in equation (3) we have:

Theorem 4.1

(*i*) *If $m > M$, the tones composing the diatonic scale $E_{n^-}^h = E_m^h$ are those of E_n^h that result from increasing the* previous *tone by a* diatonic elementary factor D, *which is common to both scales.*

(*ii*) *If $M > m$, the tones composing the diatonic scale $E_{n^-}^h = E_M^h$ are those of E_n^h whose* next *tone is attained increasing by a* diatonic elementary factor U, *which is common to both scales.*

Proof.

(*i*) Consider the scale tones $\vartheta_j = \Phi(p, q)$ whose next ones are $\vartheta_{j+1} = \Phi(p, q+1)$, which increase by one factor D. According to equation (42), with $i = 1$, this is only possible for the iterates ν_k satisfying $M \le k < n$, i.e., with indices $M, \ldots, n-1$. Thus, according to the second case of equation (38), their respective next tones are the iterates ν_{k-M}, i.e. $\{\nu_0, \nu_1, \ldots, \nu_{m-1}\}$.

(*ii*) Consider the scale tones $\vartheta_j = \Phi(p, q)$ whose next ones are $\vartheta_{j+1} = \Phi(p+1, q)$, increasing by one factor U. According to equation (40) with $i = 1$, this is only possible for the iterates ν_k satisfying $0 \le k < M$, i.e. $\{\nu_0, \nu_1, \ldots, \nu_{M-1}\}$.

■

Consequently, owing to Corollary 3.2, if $m > M$ the *accidentals* are the tones obtained by increasing the previous one by a factor U, and come alone (see Figure 1). If $M > m$, the accidentals are the tones obtained by increasing the previous one by a factor D, and also come alone. Thus, if $M > m$ the diatones are associated with the first case of equation (38) and, if $m > M$ with the second case of equation (39).

If $M > m$, alterations of a tone exclusively in a number of U elementary factors only modify the index of its iterate in packs of m steps. On the other hand, if $m > M$, alterations of a tone exclusively in a number of D elementary factors only modify the index of its iterate in packs of M steps. Therefore, consider a scale tone $\nu_k \in E_n^h$,

Corollary 4.2

(1) For $m > M$, if $k < m$, ν_k is a diatone and $(k + M) \bmod m$ is the index of the previous diatone, otherwise ν_k is an accidental and $k \bmod m$ is the index of the previous diatone. In both cases, $(k - M) \bmod m$ is the index of the next diatone.

(2) For $M > m$, if $k < M$, ν_k is a diatone and $(k + m) \bmod M$ is the index of the next diatone,

otherwise ν_k is an accidental and $k \bmod M$ is the index of the next diatone. In both cases, $(k-m) \bmod M$ is the index of the previous diatone. ■

In order infer such features for the next scale $E^h_{n^+}$, we recall the two possible cases for E^h_n,

$$\begin{array}{ll} \text{(a)} & \delta = 0 \iff \gamma_n > 1 \iff m^+ = m + M,\ M^+ = M \iff 1 < D < U \\ \text{(b)} & \delta = 1 \iff \gamma_n < 1 \iff m^+ = m,\ M^+ = m + M \iff 1 < U < D \end{array} \tag{44}$$

In case (a), for $E^h_{n^+}$ we have $m^+ = m + M$ and $M^+ = M$. Then, $D^+ = D$ and $U^+ < U$, so that the diatones are the tones following a factor D^+. However, whether $D^+ < U^+$ or $D^+ > U^+$ will depend on the value δ^+. Only if $\delta^+ = 1$ we get $U^+ < D^+$ and the scale $E^h_{n^+}$ is optimal.

According to Cubarsi (2020), an optimal scale can be identified from a changing value of the scale digit δ and, therefore, consecutive optimal scales have alternating values of this digit.

In case (b), for $E^h_{n^+}$ we have $m^+ = m$ and $M^+ = m + M$. Then, $D^+ < D$ and $U^+ = U$, so that the diatones are the tones previous to a factor U^+. Only if $\delta^+ = 0$ we get $D^+ < U^+$ and the scale $E^h_{n^+}$ is optimal. Therefore,

COROLLARY 4.3 *Consecutive optimal scales E^h_n and $E^h_{n^+}$ have alternate short and long elementary factors, i.e., either $D < U$ and $U^+ < D^+$, or $U < D$ and $D^+ < U^+$.* ■

However, the role of these factors with regard to the diatones does not alternate, neither does the relation between the indices of the extreme tones.

5. Examples of bidimensional representation

5.1. *12-tone Pythagorean scale*

For $n = 12$ and $h = 3$ the scale is optimal. In such a case, the relevant indices are

$$m = 7, \quad M = 5, \quad [\![m]\!] = 11, \quad [\![M]\!] = 7, \quad [\![n]\!] = 19, \quad N = 19, \quad \mu = 7$$

Since $m > M$, the factor D is the one associated with the diatones, according to the genuine name of the 7-tone Pythagorean scale. The condition of equation (41) is only satisfied for the value $i = 1$. According to equation (40), this increment is only possible for fifths satisfying $0 \le k < 5$. This is exactly the first case of equation (38). Therefore, these five notes, corresponding to fifths $k = 0, \ldots, 4$ (and by equation (33), to the cardinals $j = 0, 7, 2, 9, 4$) will pass from $\Phi(p, q)$ to $\Phi(p+1, q)$ increasing by a factor U, and can no longer be increased further in the first argument. Thus, the cardinals $j = 1, 8, 3, 10, 5$ correspond to accidentals. The remaining fifths will increase in the second argument.

The condition of equation (43) is fulfilled for the values $i = 1, 2$. For $i = 1$, according to equation (42) we get the range $5 \le k < 12$, corresponding to the second case of equation (38). Therefore, these seven notes, for fifths $k = 5, \ldots, 11$ and cardinals $j = 11, 6, 1, 8, 3, 10, 5$, will pass from $\Phi(p, q)$ to $\Phi(p, q+1)$ increasing by a factor D. Their following cardinals $j = 0, 7, 2, 9, 4, 11, 6$ are the diatones and form the 7-tone diatonic scale. In addition, some of these notes may be increased twice in the second argument. The value $i = 2$ allows this possibility for the fifths satisfying $10 \le k < 12$. Hence, two notes, those of fifth $k = 10, 11$ and cardinal $j = 10, 5$, will follow a sequence as $\Phi(p, q), \Phi(p, q+1), \Phi(p, q+2)$.

The elementary factors are $U = \frac{3^7}{2^{11}}$, $D = \frac{2^8}{3^5}$, satisfying $D < U$ and associated with the value $\delta = 0$. The respective intervals measure 113.69 cents for each one of the 5 factors U, corresponding to the chromatic semitone, and 90.22 cents for each one of the 7 factors D, corresponding to the diatonic semitone. Then, the factors composing the 12-tone Pythagorean scale follow the order described below, that can be read as generated by fifths (in the top row) or as increasing cardinals

(in the bottom row)

$$\begin{matrix} \nu_0 & \to & \nu_7 & \to & \nu_2 & \to & \nu_9 & \to & \nu_4 & \to & \nu_{11} & \to & \nu_6 & \to & \nu_1 & \to & \nu_8 & \to & \nu_3 & \to & \nu_{10} & \to & \nu_5 & \to & 2\nu_0 \\ \vartheta_0 & \xrightarrow{U} & \vartheta_1 & \xrightarrow{D} & \vartheta_2 & \xrightarrow{U} & \vartheta_3 & \xrightarrow{D} & \vartheta_4 & \xrightarrow{U} & \vartheta_5 & \xrightarrow{D} & \vartheta_6 & \xrightarrow{D} & \vartheta_7 & \xrightarrow{U} & \vartheta_8 & \xrightarrow{D} & \vartheta_9 & \xrightarrow{U} & \vartheta_{10} & \xrightarrow{D} & \vartheta_{11} & \xrightarrow{D} & \vartheta_{12} \end{matrix}$$

For the whole octave, we get $U^5 D^7 = 2$, according to equation (5) with $j = 1$, $k = m$.

A fifth corresponds to a factor $U^3 D^4$ of the octave, except for the false fifth closing the octave, i.e., between ν_{11} and $2\nu_0$, which corresponds to a factor $U^2 D^5$.

5.2. *53-tone Pythagorean scale*

For $n = 53$ and $h = 3$ the scale is optimal. In such a case, the relevant indices are

$$m = 12, \quad M = 41, \quad [\![m]\!] = 19, \quad [\![M]\!] = 64, \quad [\![n]\!] = 84, \quad N = 84, \quad \mu = 31$$

Depending on whether the increment is in the first or second argument of equation (38), consecutive tones will increase by factors $U = \frac{3^{12}}{2^{19}}$, $D = \frac{2^{65}}{3^{41}}$, satisfying $D < U$ and associated with the value $\delta = 0$. Now, since $M > m$, the factor U is the one associated with the diatones, i.e. the tones of the 41-tone Pythagorean scale that precede this factor. The corresponding intervals are 23.46 cents for the 41 factors U and 19.85 cents for the 12 factors D, associated with the accidentals that precede it. It is fulfilled $U^{41} D^{12} = 2$.

Let us begin by the second argument corresponding to the factor D. The condition of equation (43) only holds for the value $i = 1$ and is fulfilled for $41 \leq k < 53$. These 12 notes, which according to equation (33) have cardinals $j = 52, 30, 8, 39, 17, 48, 26, 4, 35, 13, 44, 22$, no longer increase further in the second argument. These are the accidentals. All other tones of the scale will increase in the first argument.

For the h-iterations allowing to increase in the first argument, i.e. by the factor U, the condition of equation (41) holds for indices $i = 1, 2, 3, 4$. All theses cases will lead to diatones. For $i = 4$, the fifths satisfying equation (41) fulfill $0 \leq k < 5$ and can be increased 4 times. These fifths correspond to the cardinals $j = 0, 31, 9, 40, 18$. Therefore, the notes immediately subsequent, of cardinal $1, 32, 10, 41, 19$, can be increased 3 times, those of cardinal $2, 33, 11, 42, 20$ can be increased 2 times, and those of cardinal $3, 34, 12, 43, 21$ can be increased 1 time.

For $i = 3$, the fifths that will increase 3 times are those that meet $0 \leq k < 17$ and correspond to the cardinals of the above case $i = 4$ and $j = 0, 31, 9, 40, 18$, in addition to the cardinals $1, 32, 10, 41, 19$, as we had discussed, and also the new tones of cardinal $49, 27, 5, 36, 14, 45, 23$.

For $i = 2$, the fifths allowed to increase 2 times are those that meet $0 \leq k < 29$ and correspond to the above cardinals, in addition to $50, 28, 6, 37, 15, 46, 24, 2, 33, 11, 42, 20$. Finally, for $i = 1$, the fifths fulfilling $0 \leq k < 41$ can increase 1 time. These are, in addition to all the previous ones, those of cardinals $51, 29, 7, 38, 16, 47, 25, 3, 34, 12, 43, 21$.

5.3. *Scale Keyboards*

The above procedure is better understood and easier evaluating with the help of the following tables. The tones, ordered by fifths, are written in smaller font size and follow increasing ordering from left to right in the same row. The note cardinals increase from top to bottom in each column. This scheme is a *scale keyboard*, either defined by fifths or by cardinals. The way the scheme is filled in is according to the reverse order: starting from the last row and by filling in each row from right to left according to the iterates, with m columns. These m tones are written in gray and will increase a by factor D to attain the *next* tone, while the remaining tones will increase by a factor U. Rows are indexed by number of fifth k and, just below, the cardinal j is calculated according to $j = k\mu \bmod n$ (equation (33)). Columns are indexed by cardinal j and, just above, the fifth is calculated according to $k = jm \bmod n$ (equation (35)). The value U or D on the right-hand side column is the factor between two consecutive tones of the scale in the same column, the second in the row immediately *below*.

$n{=}53, m{=}12, M{=}41$

$k{\in}[0,n{-}4m)$								0	1	2	3	4	U
$j{=}k\mu \bmod n$								0	31	9	40	18	↓
$k{\in}[53{-}4m,53{-}3m)$	5	6	7	8	9	10	11	12	13	14	15	16	U
j	49	27	5	36	14	45	23	1	32	10	41	19	↓
$k{\in}[n{-}3m,n{-}2m)$	17	18	19	20	21	22	23	24	25	26	27	28	U
j	50	28	6	37	15	46	24	2	33	11	42	20	↓
$k{\in}[n{-}2m,M)$	29	30	31	32	33	34	35	36	37	38	39	40	U
j	51	29	7	38	16	47	25	3	34	12	43	21	↓
$k{\in}[M,n{-}1)$	41	42	43	44	45	46	47	48	49	50	51	52	D
j	52	30	8	39	17	48	26	4	35	13	44	22	↓

Table 1. 53-tone Pythagorean keyboard with $m = 12$ columns. The horizontal order is by fifths k and the vertical by tone cardinals j. The values U (diatonic factor) and D are the factors between consecutive cardinals. Diatones in black and accidentals in gray.

For the 53-tone Pythagorean keyboard listed in table 1, when the bottom of a column is reached, the following tone is always on the top of another column, although for the 12-tone keyboard of table 2 it is not always so. In such a case, the notes that follow the cardinals 5 and 10 are in the same row and are simultaneously on the top and bottom of their own column. In this case, the next cardinals to 5 and 10 are also obtained by multiplying by a factor D. In table 1, since $M > m$, the tones preceding a factor D are accidentals, written in gray, while those preceding a factor U are the diatones, i.e. the tones of the previous cyclic scale. Each column contains at most one accidental as a result of the refinement of the 41-tone Pythagorean scale[9]. This approach allows to account for the alterations of the scale notes (Hook 2007; Douthett and Hook 2009) from a different perspective.

The scheme of the 12-tone scale is also filled in the reverse order: starting from the last row and by filling in each row from right to left according to iterates. The tones of the second row will increase by a factor D and those of the first row by a factor U. Since $m > M$, the accidentals are those preceded by a factor U, written in gray, while those preceded by a factor D are the diatones of the 7-tone Pythagorean scale. In this case, the equality $m = \mu$ is fulfilled. That is, the condition of equation (25) is satisfied together with equation (32). This makes the role of fifths and cardinals interchangeable, as seen in table 2, although not with regard to the factors U, D.

$n{=}12, m{=}7, M{=}5$

$k{\in}[0,M)$			0	1	2	3	4	U
$j{=}k\mu \bmod n$			0	7	2	9	4	↓
$k{\in}[M,n{-}1)$	5	6	7	8	9	10	11	← D
j	11	6	1	8	3	10	5	↓

Table 2. 12-tone Pythagorean scale keyboard with $m = 7$ columns. The horizontal order is by fifths k and the vertical by tone cardinals j. The values U and D (diatonic factor) are the factors between consecutive cardinals. Diatones in black and accidentals in gray.

[9] The cyclic scales with $n = 17, 29, 41, 53$ have the same index m and factor U, therefore, they are also represented in this keyboard if the respective rows are eliminated, beginning by the bottom. Then, the last factor should accumulate the ones removed, namely $D' = UD$ for $n = 41$, $D' = UUD$ for $n = 29$, and $D' = UUUD$ for $n = 17$.

5.4. *Matrix form and reversible keyboards*

The keyboards of tables 1 and 2 can also be defined in matrix form. Each element of the matrix represents a key that can be labeled either as iterate or as cardinal, i.e., we use only one of the indices in the table. The key of the fundamental is 0 for iterates as well as for cardinals and occupies the position $(1, 1)$. By labeling the keys as iterates, the value k of the iterate in the row $\xi \geq 1$ and column $\eta \geq 1$ is

$$k(\xi, \eta) = [(\eta - 1) + (\xi - 1)\, m] \bmod n \tag{45}$$

By labeling them as cardinals, the value j of the cardinal in the row ξ and column η is

$$j(\xi, \eta) = [(\xi - 1) + (\eta - 1)\, \mu] \bmod n \tag{46}$$

A keyboard is *reversible* if $k(\xi, \eta) = j(\eta, \xi)$ for all ξ and η. Since $0 < m < n$ and $0 < \mu < n$, this equality is fulfilled only when

$$m = \mu \tag{47}$$

In such a case, for the respective matrices and keyboards, iterates and cardinals are transposable for each other with regard to the indices. The condition of equation (47) is the same than the involution condition of equation (25) in Section 2.5, bearing in mind equation (32).

For any value of h, the trivial case $n = 2$, i.e. the scale with tones ν_0 and ν_1, always provides an optimal scale and reversible keyboards. Among optimal and non-optimal Pythagorean scales ($h = 3$) there are no more reversible keyboards than for the optimal 12-tone scale and the non-optimal 3-tone scale. If we analyze the cyclic scales generated by the median ($h = 5$), the scale with $n = 3$ is the only optimal scale with reversible keyboards. The tones of such a scale are three major thirds arranged consecutively, i.e., $1, \frac{5}{4}, \frac{25}{16}$, with closure $\gamma_3 = \frac{125}{128}$. Non-optimal scales with $n = 4$ and 87 have also reversible keyboards.

The matrices in table 3 make evident the reversibility of keyboards for the 12-tone scale.

k	→ **fifths** →				(n=12)
↓ (cardinals) $k\mu \bmod n$	⓪	①	②	③	④	⋯
	[7]	[8]	[9]	[10]	[11]	⋯
	②	③	④	[5]	[6]	⋯
	[9]	[10]	[11]	⓪	①	⋯
	④	[5]	[6]	[7]	[8]	⋯
	⋮	⋮	⋮	⋮	⋮	

j	$jm \bmod n$ (fifths) →				(n=12)
↓ **cardinals** ↓	⓪	⑦	②	⑨	④	⋯
	[1]	[8]	[3]	[10]	[5]	⋯
	②	⑨	④	[11]	[6]	⋯
	[3]	[10]	[5]	⓪	⑦	⋯
	④	[11]	[6]	[1]	[8]	⋯
	⋮	⋮	⋮	⋮	⋮	

Table 3. Matrix keyboards for the 12-tone Pythagorean scale. On the left, the keyboard is labeled as fifths, k, and on the right as tone cardinals j.

On the left hand side of table 3 the keyboard is labeled by the number of fifth k corresponding to each tone. The fifths are arranged consecutively from left to right, while from top to bottom the sequences correspond to consecutive tones of the scale calculated from k as indicated in equation (33). On the right, the keyboard is labeled by the cardinal j, following an order from top to bottom. The fifths then remain arranged consecutively from left to right, calculated from

j as indicated in equation (35). Passing from one note to the next cardinal is done by applying a factor U if the first tone is circled and D if it is boxed. Consecutive fifths are separated by a factor U^3D^4, except the one marked in gray, which precedes a factor U^2D^5.

Let us remark that these bidimensional representations, in addition to relate the indices of the scale tones, inform about the scale temperament, that is, they specify the elementary intervals existing between the scale tones. Nevertheless, such a feature is not transposable.

6. Conclusions

In order to study the distribution of tones along the octave for any n-tone cyclic scale (Cubarsi 2020), the scale tones are expressed as the product of integer powers of two factors. We determine the necessary and sufficient condition allowing such factor decompositions (FD), which are particular cases of duality (Regener 1973; Carey and Clampitt 1996; Clampitt and Noll 2011). The two main FD's are analyzed. The *first* FD is provided by the generator/co-generator pair. The *second* FD is obtained from the step/co-step factors. It determines the partition of the octave from the two elementary factors $U = \nu_m$ and $D = \frac{2}{\nu_M}$, where m and M are the indices of the extreme tones. These factors are the generalization of the chromatic and diatonic semitones of the Pythagorean scale. The sequence of elementary factors composing the scale matches the definition of a Christoffel word, so that the scale is a Christoffel word of the alphabet $\{U, D\}$ with slope $\frac{m}{M}$ and length n (e.g., Noll 2008).

According to the third particular case of FD's, other decompositions are possible by taking into account the indices of the extreme tones of smaller cyclic scales in the hierarchy. For instance, for $n = 12$ with $U^5D^7 = 2$, in addition to the first and second FD's, it is possible to get other FD's from the pairs $\{\nu_2 = UD, \frac{2}{\nu_1} = U^2D^3\}(n = 3)$, $\{\nu_2 = UD, \frac{2}{\nu_3} = UD^2\}(n = 5)$, $\{\nu_2 = UD, \frac{2}{\nu_5} = D\}(n = 7)$.

For any n-tone cyclic scale we prove that the maximum number of consecutive U factors is the highest integer i_U satisfying $i_U\, m < n$, and the maximum number of consecutive D factors is the highest integer i_D satisfying $i_D\, M \leq n$. Hence, if $m > M$, the U factors come alone and, if $M > m$, the D factors come alone. In relation to Myhill's property, we determine that the factorization of any generic step-interval is composed of the elementary factors either U^aD^b when the difference of iterates between endpoints increases, or $U^{a-1}D^{b+1}$ when the difference of iterates decreases, for specified non-negative integers a, b .

The generalization of the diatonic intervals of the Pythagorean scale is interpreted in the following sense. A cyclic scale E^h_n contains the tones of the previous cyclic scale $E^h_{n^-}$, which are considered as generic diatones, in addition to the new non-adjacent tones, interpreted as accidentals. In any cyclic scale E^h_n, they can be identified as follows. If $m > M$, the diatones are those that result from increasing the previous tone by a factor D, the diatonic factor that is shared by both scales. If $M > m$, the diatones are those whose next tone is attained increasing by a factor U, which is now the diatonic factor common to both scales. Such a criterion allow us to conclude that two consecutive optimal scales (those providing the best scale closure, i.e., the continued fraction convergents of $\log_2 h$) have alternate short and long elementary factors, either U or D.

The previous results provide us with a method to fill in, in a simple way, a bidimensional table, the scale keyboard, to represent the scale tones arranged bidimensionally by iterates and/or by cardinals, *together with their elementary intervals between them*, where the generalized diatones and accidentals become easily identified. The method is valid for any n-tone cyclic scale with an arbitrary generator h, although the examples shown apply to the 12- and 53-tone Pythagorean scales. Therefore, we have shown how the refinement of generalized Pythagorean tuning works, that can also serve to find larger sets of justly intoned concords between their tones, as Mersenne's intention was.

The keyboard is associated with two matrix forms, labeled either by tone iterates or by cardinals. If both matrix representations of the keyboard are mutual transpose, we say the keyboard is reversible. It corresponds to the case where the relationship between both main FD's is given by an involutory matrix, taking place when the cardinal of the first iterate matches the index of the minimum tone, as in the rare case of the 12-tone Pythagorean scale.

The current approach provides tools to estimate, for instance, how close the tones of a cyclic scale are from those of an equally tempered scale, which will be studied in a future work.

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