Mon. Not. R. Astron. Soc. **380,** 848–864 (2007) doi:10.1111/j.1365-2966.2007.12141.x

The *n*th-order stellar hydrodynamic equation: transfer of comoving moments and pressures

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Accepted 2007 June 19. Received 2007 June 18; in original form 2007 March 30

ABSTRACT

The exact mathematical expression for an arbitrary *n*th-order stellar hydrodynamic equation is explicitly obtained depending on the central moments of the velocity distribution. In such a form the equations are physically meaningful, since they can be compared with the ordinary hydrodynamic equations of compressible, viscous fluids. The equations are deduced without any particular assumptions about symmetries, steadiness or particular kinematic behaviours, so that they can be used in their complete form, and for any order, in future works with improved observational data. Also, in order to work with a finite number of equations and unknowns, which would provide a dynamic model for the stellar system, the *n*th-order equation is needed to investigate in a more general way the closure conditions, which may be expressed in terms of velocity distribution statistics. A case example for a Schwarzschild distribution shows how the infinite hierarchy of hydrodynamic equations is reduced to the equations of orders $n = 0, 1, 2, 3$, owing to the recurrent form of the central moments and to the equations of order $n = 2$ and 3, which become closure conditions for higher even- and odd-order equations, respectively. The closure example is generalized to a quadratic function in the peculiar velocities, so that the equivalence between moment equations and the system of equations that Chandrasekhar had obtained working from the collisionless Boltzmann equation is borne out.

Key words: hydrodynamics – methods: analytical – stars: kinematics – galaxies: kinematics and dynamics – galaxies: statistics.

1 INTRODUCTION

The stellar hydrodynamic equations have been used in a number of works on galactic dynamics to study the stellar mass and velocity distributions, either from an analytical viewpoint (e.g. Vandervoort 1975; Hunter 1979; Evans & Lynden-Bell 1989; Evans, Carollo & de Zeeuw 2000; van de Ven et al. 2003) or as a model for numerical simulations to investigate the shape of the velocity distribution, or to reproduce the spiral structure of galactic discs as an alternative way to the *N*-body approach (e.g. Korchagin et al. 2000; Orlova, Korchagin & Theis 2002; Vorobyov & Theis 2006). However, only equations of mass, momentum and, in few cases, energy transfer are generally handled, and, in most cases, axial symmetry, steady-state stellar system, and other hypotheses are assumed. There are few works that, in a mathematical aspect, have gone beyond such a basic assumption. Sala, Torra & Cortes (1985) proposed a general expression for the *n*th-order equation, without steadiness and axisymmetry, although it was written depending on the absolute, non-comoving moments of the stellar velocity distribution, where, by substitution of the moments as a series of the pressures, they obtained a general but non-explicit expression of the equations. The explicit equations were, in the end, specifically written for orders $n = 0, 1, 2, 3$. However, it is well known that stellar hydrodynamic equations are physically meaningful when they can be compared with the ordinary hydrodynamic equations of a compressible, viscous fluid, and this is only possible when they are written in terms of the tensors of comoving moments or pressures, in the reference frame associated with the local centroid. Often, these expansions or computational procedures are provided instead of their explicit expression, and they are later used to simplify and to close the system of equations, for example, to study a cool, pure rotating disc (Aoki 1985; Amendt & Cuddeford 1991). On the other hand, the work by Cuddeford & Amendt (1991) had also a general and more interesting mathematical scope, although it was restricted to steady-state systems, amid other hypotheses. They studied higher order stellar hydrodynamic equations, by using central velocity moments up to eighth order, and they investigated some quite general conditions over the velocity distribution in order to close the infinite hierarchy of the moment equations.

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However, actual kinematic data, for example, from *Hipparcos* catalogue (ESA 1997), do not support any more the hypotheses of axisymmetry, steadiness or pure galactic rotation (e.g. Cubarsi & Alcobé 2006). In addition, the future *GAIA* mission (Katz et al. 2004; Wilkinson et al. 2005) will represent a major improvement with respect to *Hipparcos*, since the three velocity components will be available for the largest number of stars ever collected, where an unbiased radial velocity component will provide essential information to kinematic and dynamic studies of the Galaxy.

The aim of the present work is to provide the exact and full mathematical expression for an arbitrary nth-order equation depending on the pressures, or alternatively on the comoving moments, without any additional hypotheses. Such a general expression should be taken as a starting point in forthcoming works, either to use improved observational data or to carry out more exhaustive numerical simulations. In addition, the exact nth-order equation is also essential to study more general closure conditions, or under less restrictive assumptions, for building up more accurate dynamical models from a finite number of equations and variables. A case example for Gaussian and ellipsoidal velocity distributions, according to the general form of Chandrasekhar's generalized Schwarzschild functions, illustrates the study of the closure conditions, which make a finite set of moment equations equivalent to the collisionless Boltzmann equation. The analysis shows that the set of hydrodynamic equations of orders $n = 0, 1, 2, 3$ and the equations obtained by Chandrasekhar (1942) for such a type of distributions generate the same dynamical model, since the higher order equations are found to be redundant. This is first proved for Schwarzschild distributions, which are then taken as a basis to expand a generalized ellipsoidal function as a series of them, so that the model is extended in a natural way to the family of quadratic functions in the peculiar velocities.

Let us introduce the notation by reviewing some basic concepts. From a macroscopic approach, a stellar system is described by giving their distribution in the phase space, which consists in couples of three-dimensional vectors $\boldsymbol{r}$ and $\boldsymbol{V}$ representing star position and velocity, measured from an inertial reference system. The stellar distribution is then given through the phase-space density function $f(t, \boldsymbol{r}, \boldsymbol{V})$, which is assumed as continuous and differentiable in nearly every point, providing, at time t, the number of stars with position within the range $\boldsymbol{r}$ and $\boldsymbol{r} + \mathrm{d}\boldsymbol{r}$, and velocity between $\boldsymbol{V}$ and $\boldsymbol{V} + \mathrm{d}\boldsymbol{V}$.

It is generally assumed that the Galaxy is at present in a state in which each star can be idealized as a conservative dynamical system to a very high degree of accuracy. Thus, the forces acting in the system can be associated with a gravitational potential function per unit mass $\mathcal{U}(t, \boldsymbol{r})$, so that the motion of a star is described in a Cartesian coordinate system by the equations

$$\dot{\boldsymbol{r}} = \boldsymbol{V}, \quad \dot{\boldsymbol{V}} = -\mathcal{U}(t, \boldsymbol{r}). \tag{1}$$

This is a Hamiltonian system, where Liouville theorem is satisfied. That is, the density of any element of phase space remains constant during its motion. Therefore f satisfies, by using the Stokes operator $\mathrm{D}/\mathrm{D}t$, the following equation:

$$\frac{\mathrm{D}f}{\mathrm{D}t} \equiv \frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla_r f + \dot{\boldsymbol{V}} \cdot \nabla_V f = 0. \tag{2}$$

The foregoing relationship is usually referred as collisionless Boltzmann equation (Hénon 1982), or fundamental equation of stellar dynamics. It is well known that the right-hand side of equation (2) is in general non-null, but it contains a term giving account of the collisions, $(\partial f/\partial t)_{\mathrm{col}}$, which would provide the complete form of Liouville equation. However, the collisional relaxation time is long in large stellar systems. The time of relaxation for stellar encounters in the solar neighbourhood is greater than 10^{13} years (Binney & Tremaine 1987), while the galactic rotation period is about 10^8 yr. Hence, the encounters are entirely unimportant. The collisions cannot be neglected in a globular cluster which contain 10^5 stars, but for a galaxy of 10^{11} stars, the relaxation time turns out to be much larger than the age of the universe, and the encounters can be neglected.

The collisionless Boltzmann equation may be regarded from two different viewpoints. By substitution of $\dot{\boldsymbol{V}}$ from equation (1) into equation (2), the equation stands for

$$\frac{\partial f}{\partial t} + \boldsymbol{V} \cdot \nabla_r f - \nabla_r \mathcal{U} \cdot \nabla_V f = 0 \tag{3}$$

so that it is either a linear and homogeneous partial differential equation for f, for a given potential $\mathcal{U}$, or a linear non-homogeneous partial differential equation for $\mathcal{U}$, where the density function f is already known. Both approaches have been largely studied since Eddington (1921) and Oort (1928), and among other works, those of Vandervoort (1979), de Zeeuw & Lynden-Bell (1985), Bienaymé (1999) and Famaey, Van Caelenberg & Dejonghe (2002) may be pointed out.

Obviously, neither the phase-space density functions nor the potentials are observable quantities, while we do have enough large data sets of the full space motions in the solar neighbourhood for different types of stars, like those derived from the *Hipparcos* catalogue, or those hopefully forthcoming from the *GAIA* mission, in order to compute the kinematic statistics of the distribution. Then, in order to isolate information about the spatial properties of the stellar system, the collisionless Boltzmann equation can be integrated over the velocity space, or in a more general way, it may be multiplied through by any powers of the velocities before integrating, and each choice of powers leads to a different equation which involves the kinematic statistics describing the stellar system for fixed time and position, which are the mean velocity and the moments of the velocity distribution. Such a strategy, which is usually referred as moment or fluid approach, provides us with an infinite hierarchy of stellar hydrodynamic equations, which could be used as a dynamical model to study the stellar system, on condition that some closure relationships were available in order to work with a finite number of equations and unknowns.

Thus, and since as far as I know the exact and complete nth-order equation, explicitly depending on the comoving moments, has not been yet published, it will be obtained according to the following steps. After reviewing in the next section the statistics describing the stellar system, in Section 3 the collisionless Boltzmann equation will be integrated in the space of peculiar velocities. The resulting equation will be obtained depending on some auxiliary tensors $\boldsymbol{Q}_n$, which give account of the fraction of the transferred pressures $\boldsymbol{P}_n$ as a linear function of the velocity gradient and of the force due to relative stress variations. In Section 4, the relationship between $\boldsymbol{Q}_n$ and $\boldsymbol{P}_n$ will be established, so that the stellar hydrodynamic equations can be expressed in terms of the pressures. In Section 5, the nth-order stellar hydrodynamic equation will be explicitly written depending on the comoving moments of the velocity distribution. In Section 6 a case example of a Schwarzschild distribution, and afterwards of a generalized ellipsoidal velocity distribution, will show how the obtained general expression can be used to study the closure conditions from which the hydrodynamic equations and the collisionless Boltzmann equation are equivalent. Finally, in Section 7, some concluding remarks will be presented.

2 VELOCITY DISTRIBUTION

For fixed values of time t and position $\boldsymbol{r}$ the macroscopic properties of a stellar system can be described from the following statistics, which provide indirect information of the phase-space density function. The stellar density is given by

$$N(t,\boldsymbol{r}) = \int_V f(t,\boldsymbol{r},\boldsymbol{V})\,\mathrm{d}\boldsymbol{V} \tag{4}$$

and the stellar mean velocity, or velocity of the centroid, is

$$\boldsymbol{v}(t,\boldsymbol{r}) = \frac{1}{N(t,\boldsymbol{r})}\int_V \boldsymbol{V}\, f(t,\boldsymbol{r},\boldsymbol{V})\,\mathrm{d}\boldsymbol{V}. \tag{5}$$

If the peculiar velocity of a star is denoted by

$$\boldsymbol{u} = \boldsymbol{V} - \boldsymbol{v}(t,\boldsymbol{r}), \tag{6}$$

then the symmetric tensor of the nth-order central moments is obtained from the following expected value:

$$\boldsymbol{M}_n(t,\boldsymbol{r}) = E[(\boldsymbol{u})^n] = \frac{1}{N(t,\boldsymbol{r})}\int_V (\boldsymbol{V} - \boldsymbol{v}(t,\boldsymbol{r}))^n\, f(t,\boldsymbol{r},\boldsymbol{V})\,\mathrm{d}\boldsymbol{V}, \quad n \geqslant 0, \tag{7}$$

where $(\cdot)^n$ stands for the nth-tensor power. The tensor $\boldsymbol{M}_n$ has $\binom{n+2}{2}$ different elements according to the expression

$$\mu_{\alpha_1\alpha_2\cdots\alpha_n} = E[u_{\alpha_1}u_{\alpha_2}\cdots u_{\alpha_n}] \tag{8}$$

so that the indices belong to the set $\{1, 2, 3\}$, depending on the velocity component. Obviously, $\boldsymbol{M}_0 = 1$, $\boldsymbol{M}_1 = \boldsymbol{0}$.

The tensor of the central moments is equivalent to the tensor of temperatures from the kinetic theory of gases, while the tensor of pressures is given by

$$\boldsymbol{P}_n = N\,\boldsymbol{M}_n. \tag{9}$$

Thus, in order to introduce the kinematic statistics into the collisionless Boltzmann equation, equation (2) is multiplied by the nth-tensor power of the star velocity and then integrated over the whole velocity space,

$$\int_V (\boldsymbol{V})^n \frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}\boldsymbol{V} = (\boldsymbol{0})^n, \quad n \geqslant 0, \tag{10}$$

where, in the integration process, since there are not stars with infinite velocities, the following boundary conditions are assumed,

$$\lim_{|\boldsymbol{V}|\to\infty} (\boldsymbol{V})^n f(t,\boldsymbol{r},\boldsymbol{V}) = (\boldsymbol{0})^n,\ n \geqslant 0. \tag{11}$$

For each value of n, the tensor equation (10) leads to the nth-order stellar hydrodynamic equation, which provides us with a conservation law *along the centroid trajectory*. The most basic cases are the continuity equation, for $n = 0$, which stands for mass conservation, and the momentum conservation equation, for $n = 1$.

However, the methodology of most books on galactic dynamics which devote any chapter to obtain or discuss the stellar hydrodynamic equations (e.g. Chandrasekhar 1942; Kurth 1957; Ogorodnikov 1965; Mihalas 1968; Binney & Tremaine 1987), is to integrate equation (10) – for $n = 0$ and $n = 1$ – over the absolute, non-peculiar velocities, leading to equations involving the absolute moments of the velocity distribution, and afterwards, to give a physical interpretation of each equation, the total moments are explicitly written in function of the central moments. Such a procedure is appropriate for the lowest order equations, but it is not adequate for an arbitrary nth-order equation. The consequence is that, to my knowledge, the general expression for such an arbitrary order equation is nowhere published.

3 HYDRODYNAMIC EQUATIONS

Let us write the collisionless Boltzmann equation, equation (2), in terms of the stellar mean velocity, equation (5), and of the peculiar velocities, equation (6), by expressing the phase-space density function f in the form

$$\phi(t,\boldsymbol{r},\boldsymbol{u}) = f(t,\boldsymbol{r},\boldsymbol{u}+\boldsymbol{v}(t,\boldsymbol{r})), \tag{12}$$

where $t, \boldsymbol{r}$ and $\boldsymbol{u}$ are independent variables. Hence, the derivatives with respect to these variables will be

$$\begin{aligned}\frac{\partial f}{\partial t} &= \frac{\partial \phi}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial t}\cdot\nabla_{\boldsymbol{u}}\phi = \frac{\partial \phi}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla_{\boldsymbol{u}}\phi,\\ \nabla_{\boldsymbol{r}} f &= \nabla_{\boldsymbol{r}}\phi + \nabla_{\boldsymbol{r}}\boldsymbol{u}\cdot\nabla_{\boldsymbol{u}}\phi = \nabla_{\boldsymbol{r}}\phi - \nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi,\\ \nabla_{\boldsymbol{V}} f &= \nabla_{\boldsymbol{V}}\boldsymbol{u}\cdot\nabla_{\boldsymbol{u}}\phi = \nabla_{\boldsymbol{u}}\phi.\end{aligned} \tag{13}$$

Then equation (2) becomes

$$\frac{\partial \phi}{\partial t} - \frac{\partial \boldsymbol{v}}{\partial t}\cdot\nabla_{\boldsymbol{u}}\phi + (\boldsymbol{u}+\boldsymbol{v})\cdot(\nabla_{\boldsymbol{r}}\phi - \nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi) - \nabla_{\boldsymbol{r}}\mathcal{U}\cdot\nabla_{\boldsymbol{u}}\phi = 0 \tag{14}$$

so that, by reorganizing terms, it yields

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\phi - \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v}\right)\cdot\nabla_{\boldsymbol{u}}\phi + \boldsymbol{u}\cdot(\nabla_{\boldsymbol{r}}\phi - \nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi) - \nabla_{\boldsymbol{r}}\mathcal{U}\cdot\nabla_{\boldsymbol{u}}\phi = 0. \tag{15}$$

Now, to simplify the notation, we shall use the material derivative associated with the centroid motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\right)(\cdot). \tag{16}$$

Since $\boldsymbol{r}$ and $\boldsymbol{u}$ are independent variables, we shall use the identity

$$\boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\phi = \nabla_{\boldsymbol{r}}\cdot(\boldsymbol{u}\,\phi) \tag{17}$$

and we shall also consider the following equality[1]:

$$\boldsymbol{u}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{u}}\phi = \nabla_{\boldsymbol{r}}\boldsymbol{v} : (\boldsymbol{u}\otimes\nabla_{\boldsymbol{u}}\phi), \tag{18}$$

where each dot represents an inner product, and $\otimes$ a tensor product.[2] Note that the colon stands for the dot products $\nabla_{\boldsymbol{r}}$ with $\boldsymbol{u}$, and $\boldsymbol{v}$ with $\nabla_{\boldsymbol{u}}$, respectively.

Hence, equation (15) may be written as follows:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} - \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{r}\mathcal{U}\right)\cdot\nabla_{\boldsymbol{u}}\phi + \nabla_{\boldsymbol{r}}\cdot(\boldsymbol{u}\phi) - \nabla_{\boldsymbol{r}}\boldsymbol{v} : (\boldsymbol{u}\otimes\nabla_{\boldsymbol{u}}\phi) = 0. \tag{19}$$

We take now the tensor product of the foregoing equation with the nth-tensor power of the peculiar velocity $(\boldsymbol{u})^n$,

$$(\boldsymbol{u})^n\frac{\mathrm{d}\phi}{\mathrm{d}t} - (\boldsymbol{u})^n\otimes\left[\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{r}\mathcal{U}\right)\cdot\nabla_{\boldsymbol{u}}\phi\right] + \nabla_{\boldsymbol{r}}\cdot[(\boldsymbol{u})^{n+1}\phi] - \nabla_{\boldsymbol{r}}\boldsymbol{v} : \left[(\boldsymbol{u})^{n+1}\otimes\nabla_{\boldsymbol{u}}\phi\right] = (\boldsymbol{0})^n \tag{20}$$

and the resulting equation is then integrated over the peculiar velocities, where the factors depending only on $\boldsymbol{r}$ and t are left out of the integrals. Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\boldsymbol{u}}(\boldsymbol{u})^n\,\phi\,\mathrm{d}\boldsymbol{u} - \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{r}\mathcal{U}\right)\cdot\int_{\boldsymbol{u}}(\boldsymbol{u})^n\otimes\nabla_{\boldsymbol{u}}\phi\,\mathrm{d}\boldsymbol{u} + \nabla_{\boldsymbol{r}}\cdot\int_{\boldsymbol{u}}(\boldsymbol{u})^{n+1}\,\phi\,\mathrm{d}\boldsymbol{u} - \nabla_{\boldsymbol{r}}\boldsymbol{v} : \int_{\boldsymbol{u}}(\boldsymbol{u})^{n+1}\otimes\nabla_{\boldsymbol{u}}\phi\,\mathrm{d}\boldsymbol{u} = (\boldsymbol{0})^n. \tag{21}$$

The first and third terms of the above relationship are directly expressed in function of the pressures, according to equations (7) and (9). Instead, for the other terms an auxiliary tensor may be defined as follows:

$$\boldsymbol{Q}_{n+1} = -\int_{\boldsymbol{u}}(\boldsymbol{u})^n\otimes\nabla_{\boldsymbol{u}}\phi\,\mathrm{d}\boldsymbol{u},\ \ n\geqslant 0 \tag{22}$$

so that equation (21) may be rewritten in a more compact notation,

$$\frac{\mathrm{d}\boldsymbol{P}_n}{\mathrm{d}t} + \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{r}\mathcal{U}\right)\cdot\boldsymbol{Q}_{n+1} + \nabla_{\boldsymbol{r}}\cdot\boldsymbol{P}_{n+1} + \nabla_{\boldsymbol{r}}\boldsymbol{v} : \boldsymbol{Q}_{n+2} = (\boldsymbol{0})^n. \tag{23}$$

However, the tensors $\boldsymbol{Q}_n$ are not directly computable in their current form.

4 CONSERVATION OF PRESSURES

The next step is to write the general hydrodynamic equation (23) explicitly depending on the pressures. Hence we shall find out how the tensors $\boldsymbol{Q}_n$ can be expressed in terms of the pressures $\boldsymbol{P}_n$.

Let us note a particular case of equation (22). For $n = 0$, bearing in mind the boundary condition (11), we get

$$\boldsymbol{Q}_1 = -\int_{\boldsymbol{u}}\nabla_{\boldsymbol{u}}\phi\,\mathrm{d}\boldsymbol{u} = \phi|_{\boldsymbol{u}} = \boldsymbol{0}. \tag{24}$$

[1] In component notation the equality can be written as $u_i\frac{\partial v_j}{\partial r_i}\frac{\partial \phi}{\partial u_j} = \frac{\partial v_j}{\partial r_i}(u_i\frac{\partial \phi}{\partial u_j})$, where Einstein's summation criterion for repeated indices is applied.

[2] The notation used for nabla operators is the usual one. If $\boldsymbol{x}$ is a vector variable and $\boldsymbol{F}_n(\boldsymbol{x})$ an n-rank symmetric tensor field, then for $n\geqslant 1$ the divergence $\nabla_{\boldsymbol{x}}\cdot\boldsymbol{F}_n$ is, in components, $\frac{\partial}{\partial x_{i_1}}F_{i_1\cdots i_n}$, while for $n\geqslant 0$, $\nabla_{\boldsymbol{x}}\,\boldsymbol{F}_n$ is used instead of $\nabla_{\boldsymbol{x}}\otimes\boldsymbol{F}_n$ to represent the gradient $\frac{\partial}{\partial x_{i_1}}F_{i_2\cdots i_{n+1}}$.

For $n = 1$, the tensor product $(\boldsymbol{u})^n \otimes \nabla_{\boldsymbol{u}}\, \phi$ within the integral of equation (22) verifies, in components,

$$u_i \, \frac{\partial \phi}{\partial u_j} = \frac{\partial (u_i \phi)}{\partial u_j} - \delta_{ij}\, \phi, \tag{25}$$

where δ_{ij} is the Kronecker delta, and for $n \geqslant 2$,

$$u_{i_1} \cdots u_{i_n} \, \frac{\partial \phi}{\partial u_{i_{n+1}}} = \frac{\partial (u_{i_1} \cdots u_{i_n} \phi)}{\partial u_{i_{n+1}}} - \left(\delta_{i_1 i_{n+1}}\, u_{i_2} \cdots u_{i_n} + \cdots + \delta_{i_j i_{n+1}}\, u_{i_1} \cdots \widehat{u_{i_j}} \cdots u_{i_n} + \cdots + \delta_{i_n i_{n+1}}\, u_{i_1} \cdots u_{i_{n-1}} \right) \phi, \tag{26}$$

where the hat remarks the omitted factors.

Then, the tensor $\boldsymbol{Q}_{n+1}$ can be evaluated by integrating equations (25) and (26). The conditions of equation (11) are once more applied over the integration boundary, so that the first term on the right-hand side of equation (26), when integrating over $u_{i_{n+1}}$, yields

$$\int_{u_{i_{n+1}}} \frac{\partial (u_{i_1} \cdots u_{i_n} \phi)}{\partial u_{i_{n+1}}} \, \mathrm{d}u_{i_{n+1}} = u_{i_1} \cdots u_{i_n}\, \phi|_{u_{i_{n+1}}} = 0. \tag{27}$$

Hence, the tensor $\boldsymbol{Q}_{n+1}$ is obtained by integrating only the remaining terms, and by taking into account equations (8) and (9).

Thus, for $n = 1$ we are led to

$$(\boldsymbol{Q}_2)_{ij} = \delta_{ij}\, \boldsymbol{P}_0 \tag{28}$$

and for $n \geqslant 2$, we get the following expression depending on the pressures,

$$(\boldsymbol{Q}_{n+1})_{i_1 \cdots i_{n+1}} = \delta_{i_1 i_{n+1}}\, P_{i_2 \cdots i_n} + \cdots + \delta_{i_j i_{n+1}}\, P_{i_1 \cdots \widehat{i_j} \cdots i_n} + \cdots + \delta_{i_n i_{n+1}}\, P_{i_1 \cdots i_{n-1}}. \tag{29}$$

The foregoing relationships will be used to write both terms in equation (23), which involve the tensor $\boldsymbol{Q}_{n+1}$. One of the terms contains a single dot product of this tensor with a vector, namely, $\boldsymbol{a} \cdot \boldsymbol{Q}_{n+1}$. Hence, by applying equation (29), we get

$$\begin{aligned}(\boldsymbol{a} \cdot \boldsymbol{Q}_{n+1})_{i_1 \cdots i_n} &= a_{i_{n+1}} \left(\delta_{i_1 i_{n+1}}\, P_{i_2 \cdots i_n} + \cdots + \delta_{i_j i_{n+1}}\, P_{i_1 \cdots \widehat{i_j} \cdots i_n} + \cdots + \delta_{i_n i_{n+1}}\, P_{i_1 \cdots i_{n-1}} \right) \\ &= a_{i_1}\, P_{i_2 \cdots i_n} + \cdots + a_{i_j}\, P_{i_1 \cdots \widehat{i_j} \cdots i_n} + \cdots + a_{i_n}\, P_{i_1 \cdots i_{n-1}},\end{aligned} \tag{30}$$

where Einstein's summation convention is used. The result is the symmetrized tensor product in regard to permutations of indices, $\mathcal{S}(\boldsymbol{a} \otimes \boldsymbol{P}_{n-1})$, which will be represented according to the following notation[3] (Cubarsi 1992):

$$\mathcal{S}(\boldsymbol{a} \otimes \boldsymbol{P}_{n-1}) = n\, \boldsymbol{a} \star \boldsymbol{P}_{n-1}, \;\; n \geqslant 1 \tag{31}$$

so that the number of summation terms, which are needed in order to symmetrize the tensor product, is explicitly written. Note that the only non-standard notation used in the current work is such a star product, which is defined in the footnote, since it worthy simplifies the forthcoming formulas.

Hence, equation (30) now stands for

$$\boldsymbol{a} \cdot \boldsymbol{Q}_{n+1} = n\, \boldsymbol{a} \star \boldsymbol{P}_{n-1}, \;\; n \geqslant 1 \tag{32}$$

and, therefore,

$$\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{\boldsymbol{r}} \mathcal{U} \right) \cdot \boldsymbol{Q}_{n+1} = n \, \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \nabla_{\boldsymbol{r}} \mathcal{U} \right) \star \boldsymbol{P}_{n-1}, \;\; n \geqslant 1. \tag{33}$$

For the particular case $n = 0$, by taking into account equation (A25), the relation $(\mathrm{d}\boldsymbol{v}/\mathrm{d}t + \nabla_{\boldsymbol{r}} \mathcal{U}) \cdot \boldsymbol{Q}_1 = 0$ is fulfilled, which means, from an algebraic viewpoint, that equation (33) is also valid for $n \geqslant 0$, since the factor n appearing in equation (33) would make null the equality, even though $\boldsymbol{P}_{-1}$ is not defined.

In a similar way, equation (23) will be calculated for the double product $\nabla_{\boldsymbol{r}}\, \boldsymbol{v} : \boldsymbol{Q}_{n+2}$. Indeed, for $n = 0$ equation (28) simply leads to

$$\nabla_{\boldsymbol{r}} \boldsymbol{v} : \boldsymbol{Q}_2 = (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\, \boldsymbol{P}_0 \tag{34}$$

while for $n \geqslant 1$, according to equation (29), it can be written in components as follows:

$$\begin{aligned}\left(\nabla_{\boldsymbol{r}} \boldsymbol{v} : \boldsymbol{Q}_{n+2} \right)_{i_1 \cdots i_n} &= \frac{\partial v_{i_{n+2}}}{\partial r_{i_{n+1}}} \left(\delta_{i_1 i_{n+2}}\, P_{i_2 \cdots i_{n+1}} + \cdots + \delta_{i_j i_{n+2}}\, P_{i_1 \cdots \widehat{i_j} \cdots i_{n+1}} + \cdots + \delta_{i_n i_{n+2}}\, P_{i_1 \cdots \widehat{i_n} i_{n+1}} + \delta_{i_{n+1} i_{n+2}}\, P_{i_1 \cdots i_n} \right) \\ &= \frac{\partial v_{i_1}}{\partial r_{i_{n+1}}}\, P_{i_2 \cdots i_{n+1}} + \cdots + \frac{\partial v_{i_j}}{\partial r_{i_{n+1}}}\, P_{i_1 \cdots \widehat{i_j} \cdots i_{n+1}} + \cdots + \frac{\partial v_{i_n}}{\partial r_{i_{n+1}}}\, P_{i_1 \cdots \widehat{i_n} i_{n+1}} + \frac{\partial v_{i_{n+1}}}{\partial r_{i_{n+1}}}\, P_{i_1 \cdots i_n}.\end{aligned} \tag{35}$$

[3] In general, if $\boldsymbol{A}_m$ and $\boldsymbol{B}_n$ are two m- and n-rank symmetric tensors, we can define the tensor $\boldsymbol{A}_m \star \boldsymbol{B}_n$ as the obtained by symmetrizing the tensor product $\boldsymbol{A}_m \otimes \boldsymbol{B}_n$, and by normalizing then with respect to the number of summation terms, T. The result is a $(m + n)$-rank symmetric tensor, whose components are

$$(\boldsymbol{A}_m \star \boldsymbol{B}_n)_{i_1 i_2 \cdots i_{m+n}} = \frac{1}{T}\, \mathcal{S}\, (\boldsymbol{A}_m \otimes \boldsymbol{B}_n)_{i_1 i_2 \cdots i_{m+n}} = \frac{1}{T} \sum_{\substack{\alpha i_1 < \cdots < \alpha i_m \\ \alpha i_{m+1} < \cdots < \alpha i_{m+n}}} A_{\alpha i_1 \cdots \alpha i_m}\, B_{\alpha i_{m+1} \cdots \alpha i_{m+n}},$$

where α belongs to the symmetric group S $(m + n)$. If both tensors are different ones, then $T = \frac{(m+n)!}{n!m!}$. Note that, in particular, if $\boldsymbol{A}_m = \boldsymbol{B}_n$ the number of summation terms is $T = \frac{(2n)!}{2!n!n!}$, and for the symmetric tensor product $\mathcal{S}(\bigotimes^k \boldsymbol{A}_n)$ the number of terms is $T = \frac{(kn)!}{k!(n!)^k}$.

The last term of the expression above is equivalent to $(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\, \boldsymbol{P}_n$, while the first n terms are the components of the symmetrized tensor product $n\,(\boldsymbol{P}_n \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v}$. Thus, for $n \geqslant 1$, the foregoing equation can be written as

$$\nabla_{\boldsymbol{r}} \boldsymbol{v} : \boldsymbol{Q}_{n+2} = n\,(\boldsymbol{P}_n \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} + (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\, \boldsymbol{P}_n. \tag{36}$$

Once more such a relation can formally be used also for $n = 0$, since, even though the dot product $(\boldsymbol{P}_0 \cdot \nabla_{\boldsymbol{r}})$ which would appear in the first term of the right-hand side is not defined, it would become null being multiplied by n.

Finally, by substitution of equation (33) and (36) into equation (23), and taking into account the definition of the material derivative, the general expression for an arbitrary nth-order hydrodynamic equation becomes

$$\frac{\partial \boldsymbol{P}_n}{\partial t} + \boldsymbol{v} \cdot \nabla_r\, \boldsymbol{P}_n + n\, \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{v} + \nabla_r \mathcal{U} \right) \star \boldsymbol{P}_{n-1} + \nabla_r \cdot \boldsymbol{P}_{n+1} + n\, (\boldsymbol{P}_n \cdot \nabla_r) \star \boldsymbol{v} + (\nabla_r \cdot \boldsymbol{v})\, \boldsymbol{P}_n = (\boldsymbol{0})^n. \tag{37}$$

Therefore, for each n, the foregoing equation, which is written in terms of the generalized tensor of pressures $\boldsymbol{P}_n$, is explicitly providing its conservation law.

5 MOMENT EQUATIONS

The lowest order hydrodynamic equations are generally used, together with some additional hypotheses like axisymmetry, steadiness, incompressible flow, etc., and together with some closure assumptions related to diffusion (e.g. by neglecting off-diagonal second moments), conductivity (e.g. by neglecting third moments and higher odd-order moments), etc., in order to estimate either kinematic parameters of the local stellar populations, or the local stellar density, similarly to the earliest works by Jeans (1922) and Oort (1932), or like more recent works by Bahcall (1984a,b), Jarvis & Freeman (1985), van der Marel (1991), Famaey & Dejonghe (2003), most of them by using also the Poisson equation for self-gravitating systems or Stäckel models in order to close the system of equations. For the lowest orders, it is easy to give a physical interpretation of the stellar hydrodynamic equations by comparing them with the ones of fluid dynamics. Thus, for $n = 0$, bearing in mind that $\boldsymbol{P}_0 = N$ and $\boldsymbol{P}_1 = \boldsymbol{0}$, the continuity equation can be written in its transfer form, and by using the material derivative equation (16), as

$$\frac{\mathrm{d} \ln N}{\mathrm{d}t} = -\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}. \tag{38}$$

Hence, since $\frac{\mathrm{d} \ln N}{\mathrm{d}t} = -\frac{\mathrm{d} \ln N^{-1}}{\mathrm{d}t}$, the divergence of the mean velocity yields the fractional time rate of change of the density N, as well as of the specific volume N^{-1}.

For $n = 1$ the equation of momentum transfer, which is usually referred as Jeans equation, is

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla_r \mathcal{U} - \frac{1}{N} \nabla_{\boldsymbol{r}} \cdot \boldsymbol{P}_2 \tag{39}$$

and it is equivalent to the Navier–Stokes equation of fluid dynamics. Thus, the acceleration of the centroid is partially due to the force coming from the potential (per unit mass), and partially due to the force coming from relative pressure variations, that is, from the surface forces applied on a volume element. Let us remember that the second-order pressure tensor, which is also known as the comoving stress tensor, gives account, in its diagonal elements, of the normal stresses, and, in its non-diagonal elements, of the tangential stresses, which are associated with viscosity and diffusion effects.

For $n = 2$, equation (37) yields

$$\frac{\mathrm{d}\boldsymbol{P}_2}{\mathrm{d}t} = -(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v})\, \boldsymbol{P}_2 - 2\,(\boldsymbol{P}_2 \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} - \nabla_{\boldsymbol{r}} \cdot \boldsymbol{P}_3. \tag{40}$$

The first and second terms in the right-hand side of equation (40) are related to the rate of strain. The first one contains the divergence of the mean velocity, or volumetric rate of strain, while the second term contains the symmetric tensor $\nabla_{\boldsymbol{r}} \star \boldsymbol{v}$, which is the shear rate of strain. Although the trace of the above tensor equation provides the law for energy transfer, the six scalar equations involved in it give account of the work balance along the different transfer directions. Thus, according to the usual interpretation from fluid dynamics, the variation of internal energy in each direction is partially due to the work coming from the specific volume variation (first right-hand term), to the viscous dissipation through the surface of a volume element (second right-hand term), and from the heat added through conduction (third right-hand term).

In a general way, the transfer of the nth-order pressure, equation (23), which we had obtained in Section 3, can be interpreted by taking into account equation (39), and by writing it in the following form:

$$\frac{\mathrm{d}\boldsymbol{P}_n}{\mathrm{d}t} = -\boldsymbol{Q}_{n+1} \cdot \frac{\nabla_r \cdot \boldsymbol{P}_2}{N} - \boldsymbol{Q}_{n+2} : \nabla_{\boldsymbol{r}} \boldsymbol{v} - \nabla_{\boldsymbol{r}} \cdot \boldsymbol{P}_{n+1}. \tag{41}$$

Thus the changes in pressure $\boldsymbol{P}_n$ are explained from a first term linearly depending on the rate of stress variation per unit mass, through the tensor $\boldsymbol{Q}_{n+1}$, from a second term linearly depending on the velocity gradient, through the tensor $\boldsymbol{Q}_{n+2}$, and from a third term giving account of the nearest higher order pressure variation $\boldsymbol{P}_{n+1}$.

However, the general hydrodynamic equation (37) is not explicitly written in terms of data actually available from stellar velocity catalogues, since the pressures obviously depends on the stellar density, according to equation (9), while the central velocity moments can be directly computable from large stellar samples. By working from the velocity moments, together with the hydrodynamic equations and some appropriate closure conditions, it is possible to estimate or to model the stellar density, the velocity distribution or the potential function (e.g. Cuddeford & Amendt 1991). Similarly, numerical approaches and simulations by using the moment equations have not to be restricted either to orders $n < 2$, or to the assumption of vanishing odd-order moments (e.g. Vorobyov & Theis 2006).

For that reason, the general nth-order equation will be expressed in terms of the central moments $\boldsymbol{M}_n$. The continuity equation, equation (38), does not need to be rewritten, while for $n \geqslant 1$, equation (38) can be used together with equation (9) to rewrite the general relation equation (37) in the following form:

$$\frac{\partial \boldsymbol{M}_n}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\, \boldsymbol{M}_n + n\, \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{v} + \nabla_r \mathcal{U}\right) \star \boldsymbol{M}_{n-1} + (\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_{n+1} + n\, (\boldsymbol{M}_n \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^n. \tag{42}$$

Hence, for $n = 1$, the momentum equation can be expressed as follows:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{v} + \nabla_r \mathcal{U} = -(\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_2 \tag{43}$$

and for $n = 2$, since $\boldsymbol{M}_1 = \boldsymbol{0}$, we have

$$\frac{\partial \boldsymbol{M}_2}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\, \boldsymbol{M}_2 + (\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_3 + 2\, (\boldsymbol{M}_2 \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^2. \tag{44}$$

In general, equation (43) may be introduced into the higher order equations to replace the terms depending on the potential function, so that they remain written in terms of the comoving moments. Then, for $n \geqslant 2$ we have

$$\frac{\partial \boldsymbol{M}_n}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\, \boldsymbol{M}_n - n\, \left[(\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_2\right] \star \boldsymbol{M}_{n-1} + (\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_{n+1} + n\, (\boldsymbol{M}_n \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^n \tag{45}$$

which is the general expression,[4] giving the contributing terms to the conservation of the nth-order moment.

Nevertheless, let us remember the typical situation we are led when working with hydrodynamic equations. The equations (38) and (43), for $n = 0$ and 1, contain four different scalar equations, which involve a set of eleven unknown scalar functions, namely, $N, \mathcal{U}, \boldsymbol{v}$ and the symmetric tensor $\boldsymbol{M}_2$. It is well known that, even in the case of taking into account higher order equations the system remains always open, since by picking up the mth-order equation, which contains $\binom{m+2}{2}$ scalar equations, we are also introducing as many as $\binom{m+3}{2}$ new unknowns, which are the different components of the tensor $\boldsymbol{M}_{m+1}$.

In most cases, the lowest order equations are used under some particular assumptions to reduce the number of unknowns. For example, by assuming the epicyclic approach, like in Oort (1965), by assuming axial symmetry (e.g. Vandervoort 1975), or by taking a velocity distribution function depending on specific isolation integrals of the star motion (e.g. Jarvis & Freeman 1985). In such a way, some constraint relationships for the central moments may be reached (e.g. van der Marel 1991). Now, for specific velocity distributions and working from the general moment equation, the closure conditions could be studied in a more general way.

6 CLOSURE EXAMPLE

Although a general study on closure conditions is beyond the scope of the present work, we shall see an example of how to use the nth-order general expression, equation (45), to find out the closure conditions in terms of the velocity distribution statistics.

For an appropriate framework of the problem let us remember that a consequence of the collisionless Boltzmann equation is that if I_1, $I_2, \ldots, I_6$ are any six independent isolating integrals of the equation of motion of a star for a given potential, in regard to equation (1), then the phase-space density function must be of the form

$$f(t, \boldsymbol{r}, \boldsymbol{V}) = f(I_1, I_2, \ldots, I_6), \tag{47}$$

where the quantity on the right-hand side stands for an arbitrary function of the specified arguments, on condition that the mass of the system be finite and that the density in the phase space be non-negative. This property is an alternative form of the Liouville theorem. However, isolating integrals like the energy integral, the integral for the axial component of the angular momentum, and sometimes a third integral, are only found for all orbits under steady-state, axisymmetric potentials, or other particular potentials. In order to avoid these limitations, a functional approach may be adopted, which takes advantage of some kinematic knowledge about the stellar system. Thus, after a transient period, the velocity distribution of some stellar groups tends to be of Maxwell type, Schwarzschild type, or, in a more general way, it is ellipsoidal shaped (e.g. de Zeeuw & Lynden-Bell 1985). The functional approach then focuses in the study of one stellar population alone, which is associated

[4] The component $i_1\ \ i_n$ for $n \geqslant 2$ of the equation stands for

$$\begin{aligned} &\frac{\partial M_{i_1\cdots i_n}}{\partial t} + v_\alpha \frac{\partial M_{i_1\cdots i_n}}{\partial r_\alpha} - \left(\frac{\partial M_{\alpha i_1}}{\partial r_\alpha} M_{i_2\cdots i_n} + \frac{\partial M_{\alpha i_2}}{\partial r_\alpha} M_{i_1\widehat{i_2}\cdots i_n} + \cdots + \frac{\partial M_{\alpha i_n}}{\partial r_\alpha} M_{i_1\cdots i_{n-1}}\right) \\ &+ \frac{\partial \ln N}{\partial r_\alpha}\left(M_{\alpha i_1\cdots i_n} - M_{\alpha i_1} M_{i_2\cdots i_n} - M_{\alpha i_2} M_{i_1\widehat{i_2}\cdots i_n} - \cdots - M_{\alpha i_n} M_{i_1\cdots i_{n-1}}\right) \\ &+ \frac{\partial M_{\alpha i_1\cdots i_n}}{\partial r_\alpha} + \frac{\partial v_{i_1}}{\partial r_\alpha} M_{\alpha i_2\cdots i_n} + \frac{\partial v_{i_2}}{\partial r_\alpha} M_{\alpha i_1\widehat{i_2}\cdots i_n} + \cdots + \frac{\partial v_{i_n}}{\partial r_\alpha} M_{\alpha i_1\cdots i_{n-1}} = 0 \end{aligned} \tag{46}$$

.

with the given velocity distribution. Note that under such a viewpoint there is no need of the collisions term in the Boltzmann equation. On the contrary, it is assumed that there are sufficient collisions to keep the system in statistical equilibrium accordingly to the specific phase-space density function. Such an approach was first explored by Eddington (1921) and Oort (1928), and was formulated in a more general way by Chandrasekhar (1942).

Thus, for example, if the phase-space density function is taken of Schwarzschild type, as Gilmore, King & Kruit (1987) point out, even though it is a quite simple case, the distribution then leads to a solution that predicts many details of the Galactic structure and kinematics, and it is possible that a realistic model could be built up as a superposition of such solutions (e.g. Cubarsi 1989), and it also leads in a natural way to Stäckel potentials and the quadratic third integral that goes with them.

In order to allow some more degrees of freedom to the velocity distribution, as well as to the dynamical model, an arbitrary ellipsoidal function in the peculiar velocities was investigated by Chandrasekhar (1942). Under such an approach the dynamical model can be derived from a finite set of equations by substitution of the phase-space density function into the collisionless Boltzmann equation in its form of equation (15). But we may wonder about how is it related to the moment approach and, in particular, to the infinite hierarchy of hydrodynamic equations we have deduced before. Although we know that all the hydrodynamic equations must be formally fulfilled, we may guess that there is a finite subset of hydrodynamic equations which are strictly equivalent to the collisionless Boltzmann equation. Then, which are the orders of these equations? Why and which are the redundant equations? Can we explicitly write the conditions that make them redundant? In other words, which are the closure conditions?

The answers to these questions are illustrated in the following subsections, where instead of studying at first hand the generalized Schwarzschild distribution, it is better to take advantage of the algebraic simplicity of the Gaussian distribution, and in a further step, to generalize the derived results to an arbitrary ellipsoidal velocity distribution.

6.1 Schwarzschild distribution

Let us assume the phase-space density function $f(t, \boldsymbol{r}, \boldsymbol{V})$ of Schwarzschild type. Then the equation (12) takes the form

$$\phi(t, \boldsymbol{r}, \boldsymbol{u}) = \mathrm{e}^{-(1/2)(Q+\sigma)}, \quad Q = \boldsymbol{u}^{\mathrm{T}} \cdot \boldsymbol{A}_2 \cdot \boldsymbol{u}, \tag{48}$$

where Q is a quadratic, positive definite form, with $\boldsymbol{A}_2(t, \boldsymbol{r})$ a second-rank symmetric tensor and $\sigma(t, \boldsymbol{r})$ a scalar function, which are continuous and differentiable in both arguments. Hence, the distribution is of Gaussian type in the peculiar velocities, although it can be multiplied by an arbitrary function of time and position. In such a way the quadratic form Q can give account of the three aforesaid isolating integrals of star motions, so that, in general, it is allowing some friction phenomena which are quantified by the off-diagonal second central moments of the distribution.

As it is known, the second moments of the Schwarzschild distribution satisfy

$$\boldsymbol{M}_2 = \boldsymbol{A}_2^{-1} \tag{49}$$

and all the odd-order central moments are obviously null. Let us remark that the velocity moments do not depend on the function σ appearing in equation (48). Only the stellar density, which is obtained from equation (4), depends on it according to

$$N(t, \boldsymbol{r}) = (2\pi)^{3/2}\, |A|^{-1/2} \mathrm{e}^{-(1/2)\sigma}, \quad |A| = \det \boldsymbol{A}_2. \tag{50}$$

The more general way to characterize such a trivariate distribution is from its cumulants, which, in addition and opposite to the central moments, have unbiased sample estimators. In general, the relationship between moments with arbitrary mean $\boldsymbol{M}_n$ and cumulants $\boldsymbol{K}_n$ (Stuart & Ord 1987, sections 13.11–15) is given by

$$\begin{aligned}
\boldsymbol{M}_1 &= \boldsymbol{K}_1 \\
\boldsymbol{M}_2 &= \boldsymbol{K}_2 + \mathcal{S}(\boldsymbol{K}_1 \otimes \boldsymbol{K}_1) \\
\boldsymbol{M}_3 &= \boldsymbol{K}_3 + \mathcal{S}(\boldsymbol{K}_2 \otimes \boldsymbol{K}_1) + \mathcal{S}(\boldsymbol{K}_1 \otimes \boldsymbol{K}_1 \otimes \boldsymbol{K}_1) \\
\boldsymbol{M}_4 &= \boldsymbol{K}_4 + \mathcal{S}(\boldsymbol{K}_3 \otimes \boldsymbol{K}_1) + \mathcal{S}(\boldsymbol{K}_2 \otimes \boldsymbol{K}_2) + \mathcal{S}(\boldsymbol{K}_2 \otimes \boldsymbol{K}_1 \otimes \boldsymbol{K}_1) + \mathcal{S}(\boldsymbol{K}_1 \otimes \boldsymbol{K}_1 \otimes \boldsymbol{K}_1 \otimes \boldsymbol{K}_1) \\
&\vdots
\end{aligned} \tag{51}$$

where the notation for symmetrized tensors defined in Section 4 is applied. If centred variables are used, then the odd-order cumulants vanish and the Gaussian distribution remains characterized only from its second cumulants $\boldsymbol{K}_2 = \boldsymbol{M}_2$. In other words, the symmetry properties of the Gaussian distribution do also provide vanishing even-order cumulants $\boldsymbol{K}_n = 0$ for $n \geqslant 4$ (Stuart & Ord 1987, section 15.3). Then, under those premises, the relationships of equation (51) are reduced to the following ones, which are written by using the star product notation also defined in Section 4,

$$
\begin{aligned}
&\boldsymbol{M}_4 = \mathcal{S}(\boldsymbol{M}_2 \otimes \boldsymbol{M}_2) = 3\,\boldsymbol{M}_2 \star \boldsymbol{M}_2\\
&\boldsymbol{M}_6 = \mathcal{S}(\boldsymbol{M}_2 \otimes \boldsymbol{M}_2 \otimes \boldsymbol{M}_2) = 15\,\boldsymbol{M}_2 \star \boldsymbol{M}_2 \star \boldsymbol{M}_2\\
&\vdots\\
&\boldsymbol{M}_{2n} = \mathcal{S}\left(\bigotimes^n \boldsymbol{M}_2\right) = C_n\,\overbrace{\boldsymbol{M}_2 \star \cdots \star \boldsymbol{M}_2}^{n}, \quad C_n = \frac{(2n)!}{2^n n!}\\
&\vdots
\end{aligned}
\tag{52}
$$

Therefore, we can easily obtain the relationship between two consecutive even-order moments. The previous coefficient C_n satisfies the recurrence relation

$$
C_{n+1} = \frac{(2n+2)(2n+1)!(2n)!}{2^n 2(n+1)n!} = (2n+1)\,C_n \tag{53}
$$

which allow us to write equation (52) as

$$
\boldsymbol{M}_{2n+2} = \frac{C_{n+1}}{C_n}\,\mathcal{S}\left(\bigotimes^n \boldsymbol{M}_2\right) \star \boldsymbol{M}_2 = (2n+1)\,\boldsymbol{M}_{2n} \star \boldsymbol{M}_2. \tag{54}
$$

The relation (54) of moment recurrence will be used to simplify and to reduce higher order moment equations to lower order ones, so that such a relationship will provide the key to the closure problem.

6.2 Even-order equations, $n \geqslant 2$

For even-order equations, $n = 2k$ and $k \geqslant 1$, bearing in mind that the odd moments are null, equation (45) becomes

$$
\frac{\partial \boldsymbol{M}_{2k}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\,\boldsymbol{M}_{2k} + 2k\,(\boldsymbol{M}_{2k} \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^{2k} \tag{55}
$$

which, by substitution of the moment expression equation (52), is transformed into

$$
\frac{\partial}{\partial t}\,\mathcal{S}\left(\bigotimes^k \boldsymbol{M}_2\right) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \mathcal{S}\left(\bigotimes^k \boldsymbol{M}_2\right) + 2k\left(\mathcal{S}\left(\bigotimes^n \boldsymbol{M}_2\right) \cdot \nabla_{\boldsymbol{r}}\right) \star \boldsymbol{v} = (\boldsymbol{0})^{2k}. \tag{56}
$$

After some algebra, we have

$$
\frac{C_k}{C_{k-1}}\,k\,\mathcal{S}\left(\bigotimes^{k-1} \boldsymbol{M}_2\right) \star \frac{\partial \boldsymbol{M}_2}{\partial t} + \frac{C_k}{C_{k-1}}\,k\,\mathcal{S}\left(\bigotimes^{k-1} \boldsymbol{M}_2\right) \star (\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{M}_2) + \frac{C_k}{C_{k-1}}\,2k\,\mathcal{S}\left(\bigotimes^{k-1} \boldsymbol{M}_2\right) \star (\boldsymbol{M}_2 \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^{2k}. \tag{57}
$$

And, by taking into account equation (54) we can write

$$
\frac{C_k}{C_{k-1}}\,\boldsymbol{M}_{2k-2} \star \left[\frac{\partial \boldsymbol{M}_2}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\,\boldsymbol{M}_2 + 2\,(\boldsymbol{M}_2 \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v}\right] = (\boldsymbol{0})^{2k}. \tag{58}
$$

Since $\boldsymbol{M}_{2k-2}$ never vanishes, all the even-order equations, $n \geqslant 2$, are then reduced to the moment equation of second order,

$$
\frac{\partial \boldsymbol{M}_2}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\,\boldsymbol{M}_2 + 2\,(\boldsymbol{M}_2 \cdot \nabla_{\boldsymbol{r}}) \star \boldsymbol{v} = (\boldsymbol{0})^{2}. \tag{59}
$$

Therefore, such a relationship, along with the moment recurrence given by equation (54), provides a closure condition for the even-order hydrodynamic equations.

6.3 Odd-order equations, $n \geqslant 3$

In a similar way, for odd-order equations, $n = 2k + 1, k \geqslant 1$, and provided that the odd-order moments are null, from equation (45) we can write

$$
-(2k+1)\;[(\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_2)] \star \boldsymbol{M}_{2k} + (\nabla_{\boldsymbol{r}} \ln N + \nabla_{\boldsymbol{r}}) \cdot \boldsymbol{M}_{2k+2} = (\boldsymbol{0})^{2k+1}. \tag{60}
$$

Then, by substitution of the recurrence law for the moments, equation (54), into the foregoing equation we have

$$
-(2k+1)\;(\nabla_{\boldsymbol{r}} \ln N \cdot \boldsymbol{M}_2) \star \boldsymbol{M}_{2k} - (2k+1)\;(\nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2) \star \boldsymbol{M}_{2k} + (2k+1)\,\nabla_{\boldsymbol{r}} \ln N \cdot (\boldsymbol{M}_2 \star \boldsymbol{M}_{2k}) + (2k+1)\,\nabla_{\boldsymbol{r}} \cdot (\boldsymbol{M}_2 \star \boldsymbol{M}_{2k}) = (\boldsymbol{0})^{2k+1}. \tag{61}
$$

In order to simplify the previous equation, by regarding the dependence of $\boldsymbol{M}_{2k}$ in terms of $\boldsymbol{M}_2$ given by equation (52), and since $\nabla_{\boldsymbol{r}} \ln N$ is a vector, we use the equivalence[5]

$$
(\nabla_{\boldsymbol{r}} \ln N \cdot \boldsymbol{M}_2) \star \boldsymbol{M}_{2k} = \nabla_{\boldsymbol{r}} \ln N \cdot (\boldsymbol{M}_2 \star \boldsymbol{M}_{2k}) \tag{62}
$$

[5] In general, for a vector $\boldsymbol{a}$ and for any symmetric tensors $\boldsymbol{A}_m$ and $\boldsymbol{B}_n$, the following equivalence is satisfied,
$\boldsymbol{a} \cdot \mathcal{S}(\boldsymbol{A}_m \otimes \boldsymbol{B}_n) = \mathcal{S}((\boldsymbol{a} \cdot \boldsymbol{A}_m) \otimes \boldsymbol{B}_n) + \mathcal{S}(\boldsymbol{A}_m \otimes (\boldsymbol{a} \cdot \boldsymbol{B}_n))$.

so that equation (61) yields

$$-(2k+1)\ (\nabla_{\boldsymbol{r}}\cdot\boldsymbol{M}_2)\star\boldsymbol{M}_{2k}+(2k+1)\,\nabla_{\boldsymbol{r}}\cdot(\boldsymbol{M}_2\star\boldsymbol{M}_{2k})=(\boldsymbol{0})^{2k+1} \tag{63}$$

And now, to further simplify equation (63), and once more by taking into account equation (52), we use the following identity[6]:

$$\nabla_{\boldsymbol{r}}\cdot(\boldsymbol{M}_2\star\boldsymbol{M}_{2k})=(\nabla_{\boldsymbol{r}}\cdot\boldsymbol{M}_2)\star\boldsymbol{M}_{2k}+k\,(\boldsymbol{M}_2\cdot\nabla_{\boldsymbol{r}})\star\boldsymbol{M}_{2k} \tag{64}$$

so that equation (63) takes the form

$$(\boldsymbol{M}_2\cdot\nabla_{\boldsymbol{r}})\star\boldsymbol{M}_{2k}=(\boldsymbol{0})^{2k+1}. \tag{65}$$

Finally, since $\boldsymbol{M}_{2k}=(2k-1)\,\boldsymbol{M}_2\star\boldsymbol{M}_{2k-2}$, and $\boldsymbol{M}_{2k-2}$ is always non-null for $k\geqslant 1$, equation (65) is reduced to the third-order equation

$$(\boldsymbol{M}_2\cdot\nabla_{\boldsymbol{r}})\star\boldsymbol{M}_2=(\boldsymbol{0})^3. \tag{66}$$

Thus, the foregoing expression stands for all the odd-order equations, $n=2k+1\geqslant 3$, and, in virtue of the moment recurrence equation (54), such a relation provides a closure condition for odd-order hydrodynamic equations in terms of the second central moments $\boldsymbol{M}_2$.

In conclusion, for a velocity distribution of Schwarzschild type, if the closure conditions given by equations (54), (59) and (66) are satisfied, then all the moment equations are reduced to the four equations of orders $n=0$, 1, 2, 3, which are a set of 20 scalar equations.

6.4 Equations for A_2 and σ

In the previous sections we have been left only with four independent hydrodynamic equations involving the statistics N, $\boldsymbol{v}$ and $\boldsymbol{M}_2$. The equations for conservation of mass and momentum, $n=0,1$, do not provide the same model as the collisionless Boltzmann equation, but, since the higher even-order moments can be expressed in the recurrence form of equation (54), and, in virtue of the equations of orders $n=2$ and 3 that are acting as closure conditions, then the four hydrodynamic equations are completely equivalent to the Boltzmann equation.

Hereafter two implications of such a result are proved. First, the referred set of equations is totally equivalent to the system of equations obtained by Chandrasekhar (1942) for generalized Schwarzschild distributions and, secondly, our result for Gaussian velocity distributions is also valid for generalized ellipsoidal velocity distributions. The reason of proceeding in two steps is for mathematical simplicity. It is thus formally proved not only that Chandrasekhar's equations are equivalent to a subset of hydrodynamic equations (de Orús 1952; Juan-Zornoza 1995) but also that, because of the closure conditions, they are equivalent to the infinite hierarchy of hydrodynamic equations. Indeed, de Orús (1952) had proved that if Chandrasekhar's equations were fulfilled, then the continuity equation and Jean's equation were also satisfied. On the other hand, working from velocity moments up to fourth order, Juan-Zornoza (1995) showed that Chandrasekhar's equations could be derived from the first four hydrodynamic equations. Now, from a new and more general approach, the whole set of hydrodynamic equations and velocity moments have been taken into account.

Therefore, since Chandrasekhar's equations provide the functional dependence of $\boldsymbol{A}_2$ and σ, let us transform the moment equations in terms of those quantities. In Section A1 we find the algebraic details showing that the equation (66), which corresponds to the equation of order $n=3$, and only involves the tensor of moments $\boldsymbol{M}_2$, is equivalent to the following condition on the tensor $\boldsymbol{A}_2$,

$$3\,\nabla_{\boldsymbol{r}}\star\boldsymbol{A}_2=(\boldsymbol{0})^3. \tag{67}$$

From equation (66), in Section A2 it is also derived an auxiliary property which will be used later. The equation (67) represents a set of 10 scalar, first-order linear equations in partial derivatives for the elements of the symmetric tensor $\boldsymbol{A}_2$.

Working from the hydrodynamic equation of order $n=2$, equation (59), along with equation (67), in Section A3 the following relationship is obtained,

$$\frac{\partial\boldsymbol{A}_2}{\partial t}-2\,\nabla_{\boldsymbol{r}}\star(\boldsymbol{A}_2\cdot\boldsymbol{v})=(\boldsymbol{0})^2 \tag{68}$$

which stands for a set of six scalar first-order linear partial differential equations for $\boldsymbol{A}_2$ and $\boldsymbol{v}$. Also, as an auxiliary property, the divergence of the mean velocity is determined in Section A4.

By using both aforesaid auxiliary properties, the relationship equivalent to the hydrodynamic equation of order $n=1$, equation (43), is obtained in Section A5:

$$\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\boldsymbol{v}+\nabla_{\boldsymbol{r}}\mathcal{U}=-\frac{1}{2}\boldsymbol{A}_2^{-1}\cdot\nabla_{\boldsymbol{r}}\sigma \tag{69}$$

whose components are three scalar equations, the only ones which involve the potential function.

[6] In general, for any symmetric tensors $\boldsymbol{A}_m$ and $\boldsymbol{B}_n$, the following equality is satisfied,

$$\nabla\cdot\mathcal{S}(\boldsymbol{A}_m\otimes\boldsymbol{B}_n)=\mathcal{S}((\nabla\cdot\boldsymbol{A}_m)\otimes\boldsymbol{B}_n)+\mathcal{S}((\boldsymbol{A}_m\cdot\nabla)\otimes\boldsymbol{B}_n)+\mathcal{S}((\nabla\otimes\boldsymbol{A}_m)\cdot\boldsymbol{B}_n)+\mathcal{S}(\boldsymbol{A}_m\otimes(\nabla\cdot\boldsymbol{B}_n))$$

in particular, if $\boldsymbol{A}_m=\boldsymbol{B}_n$, we have

$$\nabla\cdot\mathcal{S}(\boldsymbol{A}_m\otimes\boldsymbol{A}_m)=\mathcal{S}((\nabla\cdot\boldsymbol{A}_m)\otimes\boldsymbol{A}_m)+\mathcal{S}((\boldsymbol{A}_m\cdot\nabla)\otimes\boldsymbol{A}_m)$$

and, if $\boldsymbol{B}_n=\overbrace{\boldsymbol{A}_m\star\cdots\star\boldsymbol{A}_m}^{k}$, the equality yields

$$\nabla\cdot\mathcal{S}(\boldsymbol{A}_m\otimes\boldsymbol{B}_n)=\mathcal{S}((\nabla\cdot\boldsymbol{A}_m)\otimes\boldsymbol{B}_n)+k\,\mathcal{S}((\boldsymbol{A}_m\cdot\nabla)\otimes\boldsymbol{B}_n).$$

Note that if the phase-space density function is strictly of Gaussian type, with $\sigma(t, \boldsymbol{r}) = 0$, then the centroid motion does not change neither due to pressure nor due to viscosity. In other words, there are no transport phenomena and the centroid moves like a particle under the gravitational potential.

Finally, in Section A6, the continuity equation (38), for order $n = 0$, is proved equivalent to the following condition:

$$\frac{\partial\sigma}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}}\sigma = 0 \tag{70}$$

which is one scalar linear differential equation for σ, giving account of the conservation of such a quantity along the centroid path.

6.5 Generalized Schwarzschild distribution

In this subsection it is shown that if the moment equations are fulfilled for a Schwarzschild distribution $\mathrm{e}^{-(1/2)(Q+\sigma)}$, then they are also satisfied for a generalized ellipsoidal distribution in the form

$$f(t, \boldsymbol{r}, \boldsymbol{V}) = \psi(Q+\sigma), \tag{71}$$

where ψ is an arbitrary function of the specified argument as defined in equation (48).

For such a distribution the even-order cumulants are not null, as they were in the Gaussian case, although all the even-order central moments can also be computed in terms of the second ones. Thus, the odd-order moments obviously vanish, and the even moments can be expressed from symmetrized tensor products of the second moments, similarly to the Gaussian case, but with a new factor that depends on σ (de Orús 1977) through the integral[7]

$$\phi_n(\sigma) = \int_0^\infty Q^{n/2}\psi(Q+\sigma)Q^{1/2}\,\mathrm{d}Q \tag{72}$$

for n even or null. Then, equation (4) provides a stellar density in the form

$$N(t, \boldsymbol{r}) = 2\pi\,|A|^{-1/2}\phi_0(\sigma) \tag{73}$$

and, by using the notation which was introduced in equations (52) and (53), the even-order moments can be written as

$$\boldsymbol{M}_{2n} = \frac{1}{C_{n+1}}\,\frac{\phi_{2n}(\sigma)}{\phi_0(\sigma)}\,C_n\,\overbrace{\boldsymbol{A}_2^{-1}\star\cdots\star\boldsymbol{A}_2^{-1}}^{n} = \frac{1}{C_{n+1}}\,\frac{\phi_{2n}(\sigma)}{\phi_0(\sigma)}\,\mathcal{S}\left(\bigotimes^{n}\boldsymbol{A}_2^{-1}\right). \tag{74}$$

In order to investigate the closure conditions, a similar procedure as for Gaussian distributions could be applied, but in the current case it is quite more long and tedious, since the moments depend on the functions $\phi_k(\sigma)$. Hence I shall use an indirect and shorter approach.

In Appendix B it is shown that the family of Schwarzschild functions

$$\left\{\mathrm{e}^{-(1/2)(Q+\sigma)k}\right\}_k, \quad k\in\mathcal{N}-\{0\} \tag{75}$$

constitutes a non-orthogonal basis of the space of square-integrable functions over the interval $(0, +\infty)$ for the variable Q, so that the integrals of equation (72), which arise when computing the moments, are convergent. Then the velocity distribution can be formally expressed as a linear combination of them, according to

$$\psi(Q+\sigma) = \sum_{k=1}^{\infty}\gamma_{k-1}\mathrm{e}^{-(1/2)(Q+\sigma)k}. \tag{76}$$

Therefore, an arbitrary quadratic distribution $\psi(Q+\sigma)$ can be written as a convergent series of Gaussian functions in the peculiar velocities, $\mathrm{e}^{-(1/2)(Q+\sigma)k}$, for $k \geqslant 1$, all of them having zero mean. Hence, all the centroids of the partial distributions – for each term of the series – have the same mean velocity $\boldsymbol{v}$. Under those premises, not only the collisionless Boltzmann equation, equation (2), is linear for the phase-space density function, but also all the hydrodynamic equations, equation (37), are linear in the pressures, since the total nth-order pressure is simply the sum of the partial nth-order pressures. Note that if the mean velocities were different for each partial distribution, as the linearity on the pressures would not then hold, the hydrodynamic equations might be considered separately for each distribution component.

Then, the tensor involved in the exponent of the kth term in equation (76) is $k\,\boldsymbol{A}_2$, and the accompanying function of t and $\boldsymbol{r}$ is $k\,\sigma$. In addition, we can see that for each Schwarzschild component the moment equations in terms of $\boldsymbol{A}_2$ and σ, equations (67)–(70), remain invariant whether $\boldsymbol{A}_2$ and σ are, respectively, exchanged by $k\,\boldsymbol{A}_2$ and $k\,\sigma$. Therefore, due to the linear condition of the problem, if each Gaussian summation term in equation (75) satisfies the moment equations, then an arbitrary generalized ellipsoidal distribution $\psi(Q+\sigma)$, according to equation (76), do satisfy them too.

7 CONCLUDING REMARKS

The exact and full expression of the stellar hydrodynamic equations for any arbitrary order was deduced from the collisionless Boltzmann equation. It was written without any restrictive assumptions, like those of steady-state system, axial symmetry, galactic plane of symmetry,

[7] The integral is related to the Mellin transform, which, for a function $f(x)$, is defined as $\varphi(s) = \int_0^\infty x^{s-1}f(x)\,\mathrm{d}x$ (e.g. Ditkin & Prudnikov 1965).

pure rotating system, vanishing odd-order moments, etc., so that it can be used, for example, under some of these hypotheses to test analytical dependences of the phase-space density function on the integrals of motion, or in its complete form to carry out numerical simulations about either the velocity distribution or stellar density variations.

It is worth noting that most of the aforementioned hypotheses are not already valid in the solar neighbourhood, as it can be shown working from *Hipparcos* catalogue. Alcobé & Cubarsi (2005) and Cubarsi & Alcobé (2004, 2006) discussed how the local thin disc was clearly non-axisymmetric, with a non-vanishing vertex deviation, whereas increasing nested subsamples of thick disc stars showed a trend to axisymmetry. Similarly, they found that the local thin disc was not in steady state, which was related to its net radial velocity towards the galactic centre, whereas thick disc stars did showed a trend to steady state. Therefore, the moment equations have to be actually used in their complete form, at least for Galactic disc analysis.

On the other hand, the general expression of moment equations may be also useful to study closure conditions, which are associated with specific velocity distributions, in order to reduce the infinite hierarchy of equations and unknowns to a finite number of them, so that a feasible dynamical model can be available. Obviously, when working with Jeans equation alone, or whatever finite set of hydrodynamic equations, the collisionless Boltzmann equation is not generally fulfilled. It is an interesting mathematical problem to study how the system of equations can be closed, and which are then the conditions over the velocity distribution function in order to exactly fulfil Boltzmann equation. For example, since a Schwarzschild velocity distribution may satisfy the collisionless Boltzmann equation, Cuddeford & Amendt (1991) adopted some closure assumptions involving the moments of the velocity distribution, which did match some known constraints between the moments of the Schwarzschild distribution, like those related to the skewness and the kurtosis of the distribution in specific directions. By this way they made the stellar hydrodynamic equations equivalent to the collisionless Boltzmann equation, as well as they obtained a phase-space density function which was more general than of Schwarzschild type. However, the closure conditions they found, working even up to eighth-order moments, were only valid in a steady-state, cool and axisymmetric stellar system, with vanishing radial mean velocity. Therefore, in regard to actual data, the exact general expression of moment equations is a must to establish more general closure assumptions.

The case example has shown how the general hydrodynamic equations is proved useful to investigate the closure conditions problem. In general, under the functional approach, the statistical properties of the velocity distribution function may be derived and used to reduce the whole set of moment equations to a finite subset. For trivariate Schwarzschild distributions it is easy to find a recurrence relation between even central moments, which allow to reduce the moment equations only to four different orders, the even-order equations for $n \geqslant 2$ are equivalent to the one of order $n = 2$, and the odd-order equations for $n \geqslant 3$ are equivalent to the one of order $n = 3$. Therefore, the equations for mass and momentum transfer do not generate the same model as the collisionless Boltzmann equation, whereas if the moment equations of orders $n = 2$ and $n = 3$ are considered, along with the relation of moment recurrence, which are acting as closure conditions, the model they provide is the same as the one Chandrasekhar had derived by working from Boltzmann equation, as it has been proved in Appendix A. Such a result has been extended in a natural way to generalized Schwarzschild distributions, which can be expanded as a power series of Schwarzschild functions with the same mean velocity. Therefore the closure example shows the advantage of having derived the full form of moment equations, and it may be the starting point to forthcoming works on some more general distribution functions or analytical integral models, and on the closure conditions they provide.

ACKNOWLEDGMENTS

The author wishes to record his gratitude to the late Prof. Juan J. de Orús. His earliest notes on galactic dynamics and his always stimulating comments have been the basis to the present work.

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APPENDIX A: MOMENT EQUATIONS FOR A SCHWARZSCHILD DISTRIBUTION IN TERMS OF $\boldsymbol{A}_2$ AND σ

A1 Equation of order n = 3

In components, equation (49) is equivalent to

$$M_{i\alpha} A_{\alpha j} = \delta_{ij}. \tag{A1}$$

By taking gradient in both sides we get the identity

$$\frac{\partial}{\partial r_k}\left(M_{i\alpha} A_{\alpha j}\right) = \frac{\partial}{r_k}\left(\delta_{ij}\right) = 0 \tag{A2}$$

which is valid for any index k of the derivative variable, either with or without contraction of indices, and it is also valid if the time derivative is taken.

The following relationship is then satisfied

$$\frac{\partial M_{i\alpha}}{\partial r_k} A_{\alpha j} = -M_{i\alpha}\frac{\partial A_{\alpha j}}{\partial r_k}. \tag{A3}$$

If the equation (66) corresponding to order $n = 3$ is contracted three times with the tensor $\boldsymbol{A}_2 \otimes \boldsymbol{A}_2$, which is always non-null, we have

$$A_{ij} A_{kl}\left(M_{i\alpha}\frac{\partial M_{jk}}{\partial r_\alpha} + M_{j\alpha}\frac{\partial M_{ik}}{\partial r_\alpha} + M_{k\alpha}\frac{\partial M_{ij}}{\partial r_\alpha}\right) = 0. \tag{A4}$$

By taking into account equation (A1), we may then write

$$\delta_{j\alpha} A_{kl}\frac{\partial M_{jk}}{\partial r_\alpha} + \delta_{i\alpha} A_{kl}\frac{\partial M_{ik}}{\partial r_\alpha} + \delta_{l\alpha} A_{ij}\frac{\partial M_{ij}}{\partial r_\alpha} = A_{kl}\frac{\partial M_{jk}}{\partial r_j} + A_{kl}\frac{\partial M_{ik}}{\partial r_i} + A_{ij}\frac{\partial M_{ij}}{\partial r_l} = 0 \tag{A5}$$

and by equation (A3), as well as by changing the sign, we get

$$M_{jk}\frac{\partial A_{kl}}{\partial r_j} + M_{ik}\frac{\partial A_{kl}}{\partial r_i} + M_{ij}\frac{\partial A_{ij}}{\partial r_l} = 0. \tag{A6}$$

If some repeated indices are changed, we can write

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j} + \frac{\partial A_{jk}}{\partial r_i} + \frac{\partial A_{ij}}{\partial r_k}\right) = 0. \tag{A7}$$

Therefore, since the double contraction of indices is carried out with the non-null symmetric tensor $\boldsymbol{M}_2$, inverse of $\boldsymbol{A}_2$, which is associated with a positive definite quadratic form, we are led to the equation

$$3\,\nabla_{\boldsymbol{r}} \star \boldsymbol{A}_2 = (\boldsymbol{0})^3. \tag{A8}$$

Such a relation gives then account of the spatial dependence of the tensor $\boldsymbol{A}_2$.

A2 Property

We deduce a consequence of equation (A7), which will be useful in the following sections. By applying the relation equation (A3), since the tensors there involved are symmetric, the left-hand side of equation (A7) can be written as

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j} + \frac{\partial A_{jk}}{\partial r_i} + \frac{\partial A_{ij}}{\partial r_k}\right) = 2M_{ij}\frac{\partial A_{ik}}{\partial r_j} + M_{ij}\frac{\partial A_{ij}}{\partial r_k} = -2A_{ik}\frac{\partial M_{ij}}{\partial r_j} + M_{ij}\frac{\partial A_{ij}}{\partial r_k}. \tag{A9}$$

Now, if $\bar{A}_{ij}$ denotes the cofactor of the element A_{ij}, which is the same one as for its transposed element A_{ji}, then the relation equation (49) obviously implies

$$M_{ij} = \frac{\bar{A}_{ij}}{|A|}. \tag{A10}$$

Hence equation (A9) can be converted into

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j} + \frac{\partial A_{jk}}{\partial r_i} + \frac{\partial A_{ij}}{\partial r_k}\right) = -2A_{ik}\frac{\partial M_{ij}}{\partial r_j} + \frac{\bar{A}_{ij}}{|A|}\frac{\partial A_{ij}}{\partial r_k}. \tag{A11}$$

It is well known that if the tensor $\boldsymbol{A}_2$ depends on a variable ξ, then the relation

$$\frac{\partial |A|}{\partial \xi} = \frac{\partial A_{ij}}{\partial \xi}\bar{A}_{ij} \tag{A12}$$

is satisfied.[8] Hence, by writing $\xi = r_k$ we have

$$M_{ij}\left(\frac{\partial A_{ik}}{\partial r_j} + \frac{\partial A_{jk}}{\partial r_i} + \frac{\partial A_{ij}}{\partial r_k}\right) = -2A_{ik}\frac{\partial M_{ij}}{\partial r_j} + \frac{1}{|A|}\frac{\partial |A|}{\partial r_k}. \tag{A13}$$

Therefore, bearing in mind that $\boldsymbol{M}_2 = \boldsymbol{A}_2^{-1}$, it is fulfilled:

$$3\boldsymbol{M}_2 : (\nabla_{\boldsymbol{r}} \star \boldsymbol{A}_2) = -2\boldsymbol{A}_2 \cdot (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2) + \nabla_{\boldsymbol{r}} \ln |A|. \tag{A14}$$

Nevertheless, in virtue of equation (A7), the left-hand side of the above equation is zero. Hence, the following relationship is satisfied,

$$\boldsymbol{A}_2 \cdot (\nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2) = \nabla_{\boldsymbol{r}} \ln |A|^{1/2} \tag{A15}$$

which, by taking dot product with $\boldsymbol{M}_2$, can also be written as

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2 = \boldsymbol{M}_2 \cdot \nabla_{\boldsymbol{r}} \ln |A|^{1/2}. \tag{A16}$$

A3 Equation of order $n = 2$

For the second-order hydrodynamic equation, if we take the colon product of $\boldsymbol{A}_2 \otimes \boldsymbol{A}_2$ with equation (59), we have

$$A_{ik}A_{jl}\left(\frac{\partial M_{ij}}{\partial t} + v_\alpha\frac{\partial M_{ij}}{\partial r_\alpha} + M_{i\alpha}\frac{\partial v_j}{\partial r_\alpha} + M_{j\alpha}\frac{\partial v_i}{\partial r_\alpha}\right) = 0. \tag{A17}$$

By taking into account equation (A3) we can write

$$-A_{ik}M_{ij}\frac{\partial A_{jl}}{\partial t} - A_{ik}M_{ij}v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha} + A_{ik}M_{i\alpha}A_{jl}\frac{\partial v_j}{\partial r_\alpha} + A_{ik}A_{jl}M_{j\alpha}\frac{\partial v_i}{\partial r_\alpha} = 0 \tag{A18}$$

and now, by equation (A1), we have

$$\begin{aligned} &-\delta_{kj}\frac{\partial A_{jl}}{\partial t} - \delta_{kj}v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha} + \delta_{k\alpha}A_{jl}\frac{\partial v_j}{\partial r_\alpha} + A_{ik}\delta_{j\alpha}\frac{\partial v_i}{\partial r_\alpha} \\ &= -\frac{\partial A_{kl}}{\partial t} - v_\alpha\frac{\partial A_{kl}}{\partial r_\alpha} + A_{jl}\frac{\partial v_j}{\partial r_k} + A_{ik}\frac{\partial v_i}{\partial r_j} = 0. \end{aligned} \tag{A19}$$

On the other hand, in equation (A8), if we take the inner product with v_α, we obtain the identity

$$-v_\alpha\frac{\partial A_{jl}}{\partial r_\alpha} = v_\alpha\frac{\partial A_{j\alpha}}{\partial r_l} + v_\alpha\frac{\partial A_{\alpha l}}{\partial r_j} \tag{A20}$$

[8] The determinant of $\boldsymbol{A}_2$ can be expressed as $|A| = \epsilon_{i_1\cdots i_n}A_{1i_1}\cdots A_{ni_n}$, where $\epsilon_{i_1\cdots i_n}$ denotes the Levi–Civitta tensor. Then,

$$\begin{aligned} \frac{\partial |A|}{\partial \xi} &= \epsilon_{i_1\cdots i_n}\left(\frac{\partial A_{1i_1}}{\partial \xi}A_{2i_2}\cdots A_{ni_n} + A_{1i_1}\frac{\partial A_{2i_2}}{\partial \xi}\cdots A_{ni_n} + \cdots + A_{1i_1}\cdots\frac{\partial A_{ni_n}}{\partial \xi}\right) \\ &= \frac{\partial A_{1i_1}}{\partial \xi}\bar{A}_{1i_1} + \frac{\partial A_{2i_2}}{\partial \xi}\bar{A}_{2i_2} + \cdots + \frac{\partial A_{ni_n}}{\partial \xi}\bar{A}_{ni_n} = \frac{\partial A_{ij}}{\partial \xi}\bar{A}_{ij}. \end{aligned}$$

which, by substitution in equation (A19), yields

$$-\frac{\partial A_{kl}}{\partial t} + v_\alpha \frac{\partial A_{j\alpha}}{\partial r_l} + v_\alpha \frac{\partial A_{\alpha l}}{\partial r_j} + A_{jl}\frac{\partial v_j}{\partial r_k} + A_{ik}\frac{\partial v_i}{\partial r_j} = 0. \tag{A21}$$

Hence, by reordering and changing some repeated indices, we have

$$-\frac{\partial A_{kl}}{\partial t} + \frac{\partial (A_{k\alpha} v_\alpha)}{\partial r_l} + \frac{\partial (A_{\alpha l} v_\alpha)}{\partial r_k} = 0 \tag{A22}$$

which can be written in the form

$$\frac{\partial \boldsymbol{A}_2}{\partial t} - 2\,\nabla_{\boldsymbol{r}} \star (\boldsymbol{A}_2 \cdot \boldsymbol{v}) = (\boldsymbol{0})^2. \tag{A23}$$

A4 Property

If we take the colon product of $\boldsymbol{M}_2$ with equation (A23), by equation (A1) and by changing some repeated indices, we can write

$$\begin{aligned} &M_{ij}\frac{\partial A_{ij}}{\partial t} - M_{ij}\frac{\partial A_{j\alpha}}{\partial r_i} v_\alpha - M_{ij} A_{j\alpha}\frac{\partial v_\alpha}{\partial r_i} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j} v_\alpha - M_{ij} A_{i\alpha}\frac{\partial v_\alpha}{\partial r_j} \\ &\quad = M_{ij}\frac{\partial A_{ij}}{\partial t} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j} v_\alpha - \delta_{i\alpha}\frac{\partial v_\alpha}{\partial r_i} - M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j} v_\alpha - \delta_{j\alpha}\frac{\partial v_\alpha}{\partial r_j} \\ &\quad = M_{ij}\frac{\partial A_{ij}}{\partial t} - 2M_{ij}\frac{\partial A_{i\alpha}}{\partial r_j} v_\alpha - 2\frac{\partial v_\alpha}{\partial r_\alpha} = 0. \end{aligned} \tag{A24}$$

Now we apply equations (A11) and (A12) with $\xi = t$ to the first summation term of the last equation, and we also apply the property expressed by equation (A13) to the second summation term, so that we obtain

$$\frac{\partial \ln |A|}{\partial t} + \frac{\partial \ln |A|}{\partial r_\alpha} v_\alpha - 2\frac{\partial v_\alpha}{\partial r_\alpha} = 0. \tag{A25}$$

Thus, the foregoing relation gives account of the divergence of the centroid velocity,

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = \frac{\partial \ln |A|^{1/2}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \ln |A|^{1/2}. \tag{A26}$$

A5 Equation of order $n = 1$

By substitution of the stellar density *N*, equation (50), into the equation corresponding to $n = 1$, equation (43), we write

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{v} + \nabla_{\boldsymbol{r}} \mathcal{U} = -\frac{1}{2}\nabla_{\boldsymbol{r}} \ln |A| \cdot \boldsymbol{M}_2 - \frac{1}{2}\nabla_{\boldsymbol{r}}\sigma \cdot \boldsymbol{M}_2 + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2. \tag{A27}$$

Then, by taking into account equation (A16), we have

$$\nabla_{\boldsymbol{r}} \cdot \boldsymbol{M}_2 - \frac{1}{2}\nabla_{\boldsymbol{r}} \ln |A| \cdot \boldsymbol{M}_2 = (\boldsymbol{0}). \tag{A28}$$

Therefore, equation (A27) reduces to

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \boldsymbol{v} + \nabla_{\boldsymbol{r}} \mathcal{U} = -\frac{1}{2}\boldsymbol{A}_2^{-1} \cdot \nabla_{\boldsymbol{r}}\sigma. \tag{A29}$$

A6 Equation of order $n = 0$

Let us write the continuity equation (38) in the following form

$$\frac{\partial \ln N}{\partial t} + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} N = 0. \tag{A30}$$

Then, by substitution of *N*, equation (50), and by reordering terms, we can write

$$-\frac{1}{2}\left(\frac{\partial \sigma}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\sigma\right) - \left(\frac{\partial \ln |A|^{1/2}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} \ln |A|^{1/2}\right) + \nabla_{\boldsymbol{r}} \cdot \boldsymbol{v} = 0. \tag{A31}$$

Nevertheless, the relationship we obtained for $\nabla_{\boldsymbol{r}} \cdot \boldsymbol{v}$ in equation (A26) makes also null the above first term, which is independent from the tensor $\boldsymbol{A}_2$. Hence we have

$$\frac{\partial \sigma}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{r}}\sigma = 0. \tag{A32}$$

Thus, σ is conserved along the centroid trajectory.

APPENDIX B: GENERALIZED SCHWARZSCHILD DISTRIBUTION EXPRESSED AS A POWER SERIES OF SCHWARZSCHILD DISTRIBUTIONS

Let $\psi(Q+\sigma)$ be a square-integrable function over the interval $I = (0, +\infty)$ in regard to the variable Q, so that it is denoted as $\psi \in \mathcal{L}^2(I)$. Note that ψ cannot be constant.

Let us remember that Q depends on the velocities through the positive definite quadratic form defined in equation (48), σ depends only on time and position, and ψ is the velocity distribution. Then ψ may be written as [9]

$$\psi(Q+\sigma) = F\left(\frac{Q+\sigma}{2}\right) \mathrm{e}^{-(1/2)(Q+\sigma)}, \tag{B1}$$

where F can be expressed as a series of Laguerre polynomials, which are an orthogonal basis of the vector space $\mathcal{L}^2(I)$ with respect to the weight $w(Q) = \mathrm{e}^{-(1/2)(Q+\sigma)}$ on I.

Hence, by defining the variable

$$\tau = \frac{1}{2}(Q+\sigma) \tag{B2}$$

we can write

$$F(\tau) = \sum_{k=0}^{\infty} \alpha_k L_k^0(\tau), \tag{B3}$$

where each Fourier coefficient α_k, which multiplies the Laguerre polynomial L_k^α with $\alpha = 0$, can be evaluated from the inner product defined as

$$\langle f(\tau), g(\tau)\rangle = \int_0^\infty f(\tau) g(\tau) \mathrm{e}^{-\tau} \mathrm{d}\tau. \tag{B4}$$

On the other hand, the foregoing conditions do also allow us to compute the central velocity moments, which are given through the integral equation (72). Thus, bearing in mind the equation (B1), the integral takes the form

$$\phi_n(\sigma) = \int_0^\infty Q^{n/2} F\left(\frac{Q+\sigma}{2}\right) \mathrm{e}^{-(1/2)(Q+\sigma)} Q^{1/2}\, \mathrm{d}Q. \tag{B5}$$

Note that such an integral is convergent because, after the change of variable given in equation (B2), F can be expressed through the associated Laguerre polynomials, now with $\alpha = 1/2$.

The above polynomial form of F can be formally useful in the case we want to estimate it from all the available central moments. Nevertheless, to our current purpose, we need to write F depending on the new variable

$$\eta = \mathrm{e}^{-\tau} = \mathrm{e}^{-(1/2)(Q+\sigma)}. \tag{B6}$$

Then, for a fixed σ it is possible to establish an isomorphism between the respective domains of Q and η, namely, $I = (0, +\infty)$ and $J = (0, \mathrm{e}^{-(1/2)\sigma})$, so that with the notation

$$\tilde{F}(\eta) = F(\tau(\eta)) \tag{B7}$$

the function $\tilde{F}(\eta)$ can be expressed as depending on the new variable from a basis of orthogonal polynomials $\{P_k(\eta)\}_{k\in\mathcal{N}}$ over J, according to an inner product which is equivalent to the previous one defined on I from equation (B4). That is,

$$\langle f(\tau), g(\tau)\rangle_I \equiv \int_0^\infty f(\tau) g(\tau) \mathrm{e}^{-\tau} \mathrm{d}\tau = \int_0^{\mathrm{e}^{-(1/2)\sigma}} f(\tau(\eta)) g(\tau(\eta)) \mathrm{e}^{-\tau(\eta)} \left|\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right|^{-1} \mathrm{d}\eta = \int_0^{\mathrm{e}^{-(1/2)\sigma}} \tilde{f}(\eta)\tilde{g}(\eta)\mathrm{d}\eta \equiv \langle \tilde{f}(\eta), \tilde{g}(\eta)\rangle_J. \tag{B8}$$

Thus, we can write $\tilde{F}(\eta)$ as a series of polynomials $\{P_k(\eta)\}_{k\in\mathcal{N}}$ or, by reorganizing terms, as power series of η in the following form:

$$\tilde{F}(\eta) = \sum_{k=0}^{\infty} \beta_k P_k(\eta) = \sum_{k=0}^{\infty} \gamma_k \eta^k. \tag{B9}$$

Hence, according to equation (B7), by substitution of equation (B9) into equation (B1), and by writing it in terms of the variable Q, we finally have

[9] Let us remember that any square-integrable function $f(x) \in \mathcal{L}^2(I)$ admits an expansion as a series of the associated Laguerre functions,

$$f(x) = \sum_{k=0}^{\infty} L_k^\alpha(x) x^\alpha \mathrm{e}^{-x}$$

with $\alpha > -1$, where $\{L_k^\alpha(x)\}_{k\in\mathcal{N}}$ is the family of the associated Laguerre polynomials, which is an orthogonal basis of the space $\mathcal{L}^2(I)$ with respect to the weight $x^\alpha\ \mathrm{e}^{-x}$ (e.g. Abramowitz & Stegun 1965).

$$\psi(Q+\sigma) = \mathrm{e}^{-(1/2)(Q+\sigma)} \sum_{k=0}^{\infty} \gamma_k \mathrm{e}^{-(1/2)(Q+\sigma)\,k} = \sum_{k=1}^{\infty} \gamma_{k-1} \mathrm{e}^{-(1/2)(Q+\sigma)\,k}. \tag{B10}$$

Thus, any arbitrary quadratic function $\psi(Q+\sigma)$ can be expressed as a convergent series of the Gaussian functions $\mathrm{e}^{-(1/2)(Q+\sigma)k}$, with $k \geqslant 1$, although they are not an orthogonal system.

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