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Harmonic complexity of chords

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The harmonic relationships between the frequency ratios composing a chord are analyzed to provide an easy way to evaluate its harmonicity properties. The harmonic space is used to describe a chord, where the harmonic distance and the harmonic length and co-length inform about the degree of the chord's resonance and its deviation from harmonic symmetry. The distribution of the harmonics relatively to the common fundamental is quantified by the harmonic dispersion and is visualized from the chord spheroid, which also accounts for the harmonic asymmetry. Its opposite, the harmonic likeness, is correlated with the perception of consonance.

Keywords: just intonation; tonal graph; sympathetic vibration; consonance/dissonance; harmonic length; harmonic distance; harmonic dispersion; chord spheroid

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1. Introduction

This paper extends Cubarsi (2019) by studying the mutual harmonic relationships of the tones of a chord tuned in just intonation[1], i.e., as rational frequency proportions, finding improved expositions to describe harmonic properties of chords (sets of tones, in general) that can be of interest to music theorists and composers. It does not apply to tones of equal temperament (TET), although mathematical conditions allowing to approximate intervals of an n-TET system as rational proportions are given in the aforementioned paper and in Stolzenburg (2015).

Musicians sometimes use quite subjective musical terms, such as some chords sound stable, final, and resolved, while others sound unstable, tense, and unresolved, but for a scientist it makes more sense to have descriptions associated with physical and mathematical properties, not all of them necessarily related to the perception of consonance, but also in relation to the chord structure.

For instance, the ratio between the periods of each sound component and the total oscillation is an indicator of their sympathetic vibration. For two tones, this ratio is measured by the harmonic distance (Tenney 2015), which can be generalized to an arbitrary number of tones (Cubarsi 2019). It measures the melodic distance between their common acoustic fundamental (the closest common undertone) and their first common overtone, corresponding to the logarithm of the ratio between their periods.

Some properties can be perceived by the human ear. For example, the harmonic distance of each tone of a chord to their common fundamental is much more relevant to perception than the distance to their lowest common partial. It is possible that the common under-

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[1] In just intonation performance, prime factors $2, 3, 5, 7, 11, 13, 17, 19$ and 23 are used and considered for notational representation (Sabat and von Schweinitz 2004). Primary natural intervals with primes higher than 23 are very difficult to tune.

tones and difference tones, that in some cases match, have periods close to the period of the common fundamental, so that they can easily create so much multitonal confusion that the ear cannot cope with analyzing such atonal complexity of the compound sound. This can result in a decrease in consonance perception (Smoorenburg 1972; Parncutt 1989; Tramo et al. 2001). Instead, the overtones always reinforce the harmony by saturating the harmonic spectrum over the common fundamental, with a lower impact in the sensation of consonance.

A quality, not necessarily (or not proved to be) related to perception is the harmonic symmetry (chord invariance by inversion of ratios). Nevertheless, some chords are perceived with a certain degree of consonance while their harmonic inversions are perceived as less consonant. This is the case of the major and minor triads, the former enhancing the presence of the virtual common fundamental more than the latter. Harry Partch's otonality and utonality concepts (Partch 1974) described this feature, which is generalized here by associating the chord with an spheroid: a prolate spheroid will represent a chord that is harmonically closer to its common fundamental than a chord with an oblate spheroid.

The purpose of the current work is to characterize in several ways the harmonic complexity of a chord. In particular, the harmonic dispersion will synthesize some of the chord attributes by evaluating an average of the harmonic distances between the family of harmonics, the common fundamental, and the first common overtone. Its inversion in the frequency space, the harmonic likeness, which is inversely proportional to the volume of the chord spheroid, is proven to be related to the perception of consonance. Instead, the harmonic structure of the common overtones has a lesser impact on the perception of consonance and is rather related to color qualities of the sound.

2. Preliminaries

2.1. *Harmonicity*

The concept of consonance is interpreted as "the subjective attractiveness of tone combinations" (Bowling, Purves, and Gill 2018) or as "a salient perceptual phenomenon that arises from simultaneously sounding musical tones" (Harrison and Pearce 2020), while dissonance is generally spoken of as the opposite concept to consonance in the same continuous scale, and has been widely believed to be the product of beating, i.e., interference between frequency components in the cochlea, more pronounced in dissonant than consonant sounds. Helmholtz (1863) is usually credited with the idea that dissonant chords are unpleasant because they contain the sensation of roughness (Plomp and Levelt 1965; Hutchinson and Knopoff 1978; Sethares 1998). On the other hand, harmonic frequency relations, a higher-order sound attribute closely related to pitch perception, has also been proposed to account for consonance (DeWitt and Crowder 1987; Huron 1991; Ebeling 2008; Cousineau, McDermott, and Peretz 2012). The component frequencies of the notes of consonant chords combine to produce an aggregate spectrum that is typically harmonic, resembling the spectrum of a single sound with a lower pitch. In contrast, dissonant chords produce an inharmonic spectrum. At first, it would seem that, as a dual process (Johnson-Laird, Kang, and Leong 2012), harmonicity would sum in favor of consonance and roughness in favor of dissonance.

From an experiment with amusic listeners (with deficit in melody processing) Cousineau, McDermott, and Peretz (2012) proved that that consonance preferences do not derive from an aversion to beating, but it is an aesthetic dimension distinct from that of dissonance, concluding that harmonicity is more closely related to consonance than is beating (McDermott, Lehr, and Oxenham 2010; McDermott et al. 2016) and that there is a clear dissociation between both properties.

In view of such studies arguing that consonance perception is driven not by interference but by periodicity/harmonicity, Harrison and Pearce (2020) have reassessed this claim by reviewing a wide range of historic literature, modeling perceptual data from four previous empirical studies, and conducting corpus analyses spanning a thousand years of Western music composition (accompanied by many references, which we will not reproduce here). Three blocks summarize these theories of consonance: (1) periodicity/harmonicity, (2) interference between partials, and (3) culture. They consider that vocal similarity and autocorrelation models of consonance are part of (1), while interferences by pure and complex dyads, both in terms of beating and masking, are included in (2). Models with contradictory results, such as fusion, evenness, combination tones, etc., remain excluded. Their conclusion is that "consonance perception in Western listeners seems to be jointly determined by periodicity/harmonicity perception, interference, and learned familiarity with particular musical sonorities". The commonly accepted facts can be summarized as follows. Chord's consonance depends on the spectral content of its tones. With harmonic tone spectra, peak consonance is observed when the fundamental frequencies are related by simple frequency ratios. Simple integer ratios, approximated by prototypically consonant musical chords, tend to produce partials that either completely coincide or are widely spaced, hence minimizing interference. Hence, with harmonic tone spectra, the harmonic series provides a pattern for the consonance. However, although for harmonic tone spectra consonance is primarily determined by periodicity/harmonicity potentially moderated by musical background (McDermott, Lehr, and Oxenham 2010; Cousineau, McDermott, and Peretz 2012; McLachlan et al. 2013; McDermott et al. 2016) it is unclear what proportion of consonance may be explained by periodicity/harmonicity.

Nevertheless, as Langner (2015, 180) says, "our harmonic sense for pitch relations has mathematical reasons and is not, or at least not primarily, due to the adaptation of our auditory system to the physical conditions of our environment. As a result, our brain, or at least our hearing system, reacts almost like a musical instrument and therefore obeys the same mathematical laws as the physical environment". Thus, the present work wants to contribute to the study of the harmonic complexity of chords by providing some simple and mathematically quantifiable properties.

2.2. *Harmonic distance*

The harmonic distance is a measure related to the periodicity/harmonicity of the chord. According to Tenney (2015, 240-279), for two tones, the harmonic distance also quantifies the degree of consonance or dissonance of its interval in the harmonic space. This makes sense since the minima of sensory dissonance of harmonic spectra match those values where their frequencies are in rational proportion (Plomp and Levelt 1965; Sethares 1998). Hence, rational proportions are associated with a relative greater sensation of consonance and, for two tones, it is correlated with the harmonic distance (Cubarsi 2019).

By relating loss of consonance to roughness, for the ratios p/q of the main harmonics Helmholtz (1863, 189) gave the *intensity of influence*, $C_H(p,q) = \frac{100}{pq}$, as a measurement of their consonance. According to Helmholtz, this is the relative strength of the beats resulting from the mistuning of the corresponding interval. The lower the product pq, the greater the degree of beatings in mistuning the interval, i.e., the mistuning is more noticeable. In the logarithmic space, this product is the harmonic distance between p and q, $\Delta_H(p,q) = \log_2(pq)$. Therefore, for two tones, i.e., for an interval, the harmonic distance is inversely correlated with the intensity of influence. However, for a chord with three or more tones, the harmonic distance estimates the degree of sympathetic vibration between the common fundamental and the lowest common harmonic, but not for the tones composing the chord. Therefore, as (Cook and Fujisawa 2006) say, harmony is a three-tone phenomenon. For each single tone of the chord, the degree of sympathetic vibration depends on the ratio between its period and the period of the whole oscillation (e.g., Mickens 1981), that is, on the harmonic distance between each tone and the common acoustic fundamental, and does not depend on the lowest common harmonic. However, when considering the whole chord, the common undertones and overtones do also modulate the resonance.

For instance, the average distance from each tone to the first common undertone (harmonic length) of the major chord $(1, \frac{5}{4}, \frac{3}{2})$ (in lowest terms, 4:5:6) compared to that of the minor chord $(\frac{2}{3}, \frac{4}{5}, 1) = \frac{2}{3} \cdot (1, \frac{6}{5}, \frac{3}{2})$ (i.e., 10:12:15) is much lower, although the harmonic distance of both chords match. Similarly, the harmonic complexity due to the common undertones and difference tones of the Tristan chord[2] $(\frac{5}{7}, 1, \frac{5}{4}, \frac{5}{3})$ (i.e., 60:84:105:140) is associated with a much greater harmonic length than its harmonic inversion $(\frac{3}{5}, \frac{4}{5}, 1, \frac{7}{5})$ (i.e., 3:4:5:7), which follows the harmonic series closely, although their harmonic distance match. In addition, for the Tristan chord, the common fundamental of tones 84 and 105 matches their difference tone 21, and of tones 105 and 140 matches their difference tone 35. Therefore, the different harmonic complexity of the two relative inverted chords is partially related to their harmonic asymmetry.

The above examples are chords that closely follow the overtone series, together with their harmonic inversions, closely following the undertone series, i.e., otonalities and utonalities (Partch 1974). In general, the utonal inversions sound less consonant than their otonal counterparts. In §5.1 we will discuss and generalize these concepts.

On the other hand, the chord $(1, \frac{5}{4}, \frac{3}{2}, \frac{15}{8})$ (i.e., 8:10:12:15) is harmonically symmetric, since it is equivalent to its harmonic inversion $(\frac{8}{15}, \frac{2}{3}, \frac{4}{5}, 1) = \frac{8}{15} \cdot (1, \frac{5}{4}, \frac{3}{2}, \frac{15}{8})$. In this case, the undertones become balanced with the overtones, which means that the harmonic length equals the harmonic co-length (average distance from each tone to the first common overtone).

2.3. *Harmonic space*

Tenney (2015, 280-304), in his 1983 *John Cage and the Theory of Harmony*, introduced the concept of *harmonic space* by generalizing Euler's lattice model. The harmonic space maps coprime harmonics of a fundamental frequency in a multidimensional lattice according to their prime factorization, similarly to the Tonnetz, although it is not periodic in any direction. The *harmonic distance* is the metric used in the harmonic space to relate two nodes, which basically is Minkowski's L_1 distance, by assuming the directions are orthogonal (also known as city-block metric). Such a representation is also valid for rational frequency proportions, which are usually expressed in lowest terms (Cubarsi 2019; Sabat 2023). According to Tenney, two tones represented by proximate points in the harmonic space tend to be heard as being in a consonant relation to each other, while tones represented by more widely separated points are heard as mutually dissonant, even beyond the degree of beatings. Thus, the harmonic distance between two nodes indicates an important correlation between consonance and dissonance.

The explanation given by Sabat and von Schweinitz (2004) is that in the vicinity of each *tuneable* ratio there is a region of tolerance within which variations of tuning produce audible beating and phasing. These phenomena make it difficult or impossible to tune other more complex ratios falling within this region. Instead, our sense of hearing interprets them as mistuning or detuning of the predominating simpler ratio. The existence of such regions, each dominated by a single frequency ratio, implies that there is a finite number of intervals which can be tuned by ear.

In Cubarsi (2019), Tenney's definition of harmonic distance was extended to an arbitrary

[2] The Tristan chord, in the 12-TET system is formed from the intervals TT-M3-P4, which in just intonation means that the second tone is the frequency ratio $\frac{7}{5}$ of the root, the third tone is $\frac{5}{4}$ of the second tone, and the fourth is $\frac{4}{3}$ of the third. Therefore, expressed in terms of the frequency ratios, the chord has a very simple form $(1, \frac{7}{5}, \frac{7}{\cancel{5}}\frac{\cancel{5}}{4}, \frac{7}{\cancel{5}}\frac{\cancel{5}}{\cancel{4}}\frac{\cancel{4}}{3})$.

set of frequency ratios. It was based on the following fact. In the harmonic space generated by two harmonics (p, q), the harmonic distance involves the following specific concepts. One is the frequency where the overtones p and q cohere at their *lowest common harmonic* $\mathrm{LCH}(p, q) = \mathrm{lcm}(p, q)$. The other is how far should we go back to find their common acoustic fundamental, i.e., their *lowest common ancestor* [3] $\mathrm{LCA}(p, q) = \gcd(p, q)$.

Then, for two harmonics $p, q \in \mathbb{N} \setminus \{0\}$, the harmonic distance is obtained as

$$\Delta_H(p, q) = \log_2 \frac{\mathrm{LCH}(p,q)}{\mathrm{LCA}(p,q)} \tag{1}$$

Therefore, the harmonic distance can be interpreted as the melodic distance where two sinewaves cohere starting from their common fundamental. So it makes sense to generalize it to more sinewaves. For a finite set of frequency ratios, the harmonic distance was named *harmonic dissonance*, since the the name "distance" usually involves a dyad. However, the correlation existing between the harmonic distance of a couple of tones and their sensation of dissonance is partially lost when three or more tones are involved. Thus, in exchanging "distance" by "dissonance" with the intention of using a more rigorous terminology for the former word, it was introduced some imprecision for the latter. Although harmonic dissonance is not the same as sensory dissonance, the term can lead to confusion. Hence, the original term of harmonic distance will continue to be used, in the well-understood that it is the distance in the harmonic space between the LCA of a finite set of frequency ratios and its LCH.

In general, $\Delta_H(x\,p, x\,q) = \Delta_H(p, q)$, $\forall x > 0$, and $\Delta_H(p, q) = \Delta_H(\frac{p}{q}, 1)$. Hence, it also evaluates the harmonic distance from the ratio $\frac{p}{q}$ to the fundamental $\nu_0 = 1$. That is, the harmonic distance of two tones only depends on their relative ratio.

For an arbitrary set of frequency ratios, the harmonic distance can be written from irreducible fractions as follows (Cubarsi 2019). If $\Gamma = \{\gamma_i = \frac{p_i}{q_i},\ i \in I\}$ with $p_i, q_i \in \mathbb{N} \setminus \{0\}$ and $\gcd(p_i, q_i) = 1$, the harmonic distance is obtained by considering separately the numerators and denominators of these fractions[4] as $\Pi = \{p_i,\ i \in I\}$; $\Theta = \{q_i,\ i \in I\}$,

$$\Delta_H(\Gamma) = \log_2 \frac{\mathrm{LCH}(\Gamma)}{\mathrm{LCA}(\Gamma)} = \Delta_H(\Pi) + \Delta_H(\Theta) \tag{2}$$

Similarly, $\Delta_H(x\,\Gamma) = \Delta_H(\Gamma)$, $\forall x > 0$. Hence, the frequency ratios in Γ can be simplified by dividing by their highest common factor and can be expressed in their simplest form.

3. Resonance

3.1. *Sympathetic factors*

The harmonic relations between sounds are rooted on the physical model of the vibrating string that results in the wave equation. Under the appropriate initial and boundary conditions (e.g., Kinsler et al. 2000), its solutions are eigenfunctions (known as normal modes of vibration) associated with the eigenfrequencies $f_n = \frac{nc}{2L}$, $n > 1$ (where c is the propagation velocity and L the length of the string). For $n = 1$, $f_1 = \frac{c}{2L}$ is the fundamental frequency or first harmonic, which generally sounds with more intensity. For $n > 1$, the other harmonics are the overtones or partials, which generally sound with decreasing intensity. These tones are, by definition, consonant[5] (the Latin meaning of con-sonare is

[3] Lowest harmonic refers to frequencies and lowest ancestor refers to the harmonic space. In the top-down direction of a directed acyclic graph, they are the closest common descendant and the closest common ancestor.

[4] Notice that numerators and denominators cannot be permuted since the fractions must be irreducible, e.g., $\Delta_H(\frac{6}{5}, \frac{3}{2}) = \log_2(12 \cdot 15)$, $\Delta_H(\frac{3}{5}, \frac{6}{2}) = \Delta_H(\frac{3}{5}, 3) = \log_2(3 \cdot 15)$.

[5] It does not necessarily mean more pleasant. Small detunings among unison piano strings may introduce slight changes on timbre. This enhance the sustain and the sound can be heard as warmer and more pleasant (Kirk 1959).

sound together). This is a physical property easily demonstrated with a tuning fork. In the presence of another sound wave, it is important to know when the resulting sound is periodic, since this sound will last over time (a quality that instrument builders are looking for). This information can be obtained from the forced harmonic oscillator model (e.g., Mickens 1981, 123). A vibrating string subject to an oscillating external perturbation, with frequencies f_1 for the free oscillation of the string, and f for the external perturbation, admits periodic solutions when the frequencies are commensurable[6], that is, their ratio is a rational number, $\frac{f}{f_1} = \frac{p}{q}$, with $p, q \in \mathbb{N}$ coprime. Two particular cases must be distinguished. If $q = 1$ and $f = p\, f_1$, with $p \geq 1$, the respective periods satisfy $p\, T_f = T_{f_1}$. The total period of the solution is T_{f_1}, the same as the free oscillation. The free oscillation coupled with the perturbation is called *subharmonic* (undertone) oscillation of order p. If $p = 1$ and $q\, f = f_1$, with $q \geq 1$, the periods satisfy $T_f = q\, T_{f_1}$. The total period of the solution is T_f, that of the perturbation. The oscillation is known as *superharmonic* (overtone) oscillation of order q. In the general case, if $\frac{f}{f_1} = \frac{p}{q}$, the periods satisfy $p\, T_f = q\, T_{f_1} \equiv T$, and the minimum period of the whole oscillation is $T = \mathrm{lcm}(T_f, T_{f_1})$. These oscillations maintaining a rational proportion are referred to as *supersubharmonics* or harmonics (not with the same meaning as partial). When the frequencies are commensurable, we say that there is sympathetic vibration. The frequencies of the matching harmonics produce the phenomenon of acoustic resonance, which is perceived as a reinforcement of the normal modes of vibration.

In the simple case where only the fundamental tone f_1 of the free oscillation is considered, the roles of the sounds with frequencies f_1 and f can be swapped. The lower the factor $p = \frac{T}{T_f}$, the closer is the period T_f to the total period T and higher is its contribution to the resonance. Similarly for the factor $q = \frac{T}{T_{f_1}}$. These factors indicate how close/far is each partial oscillation from the common fundamental. We call them *sympathetic factors*. Hence, the tone corresponding to the lowest sympathetic factor is the more reinforced by the whole oscillation. From a physical viewpoint, this explains why the bass of a chord is the most salient sound, mostly perceived in the range of frequencies 300-2000 Hz (e.g., Parncutt 1989). This factor is the main responsible for the perception of consonance, although the other factors may modulate this perception.

3.2. *Harmonic sympathy*

For two frequency ratios ν_1, ν_2, with respective periods T_{ν_1}, T_{ν_2} and total period T, an estimate of the antiresonance (as opposite to resonance) is the geometric mean of the sympathetic factors, i.e., *the average harmonic antiresonance by pairs*,

$$a_H(\nu_1, \nu_2) = \left(\frac{T}{T_{\nu_1}} \frac{T}{T_{\nu_2}}\right)^{\frac{1}{2}} = \left(\frac{\mathrm{lcm}(T_{\nu_1}, T_{\nu_2})^2}{T_{\nu_1} T_{\nu_2}}\right)^{\frac{1}{2}} \tag{3}$$

Taking into account that

$$T_{\nu_1} T_{\nu_2} = \mathrm{lcm}(T_{\nu_1}, T_{\nu_2})\, \mathrm{gcd}(T_{\nu_1}, T_{\nu_2}); \quad \mathrm{lcm}(T_{\nu_1}, T_{\nu_2}) = \frac{1}{\mathrm{gcd}(\nu_1, \nu_2)}; \quad \mathrm{gcd}(T_{\nu_1}, T_{\nu_2}) = \frac{1}{\mathrm{lcm}(\nu_1, \nu_2)}$$

together with Eq. 1, we get

$$a_H(\nu_1, \nu_2) = \left(\frac{\mathrm{lcm}(\nu_1, \nu_2)}{\mathrm{gcd}(\nu_1, \nu_2)}\right)^{\frac{1}{2}} = \exp_2\left(\tfrac{1}{2} \Delta_H(\nu_1, \nu_2)\right) \tag{4}$$

[6] This necessary and sufficient condition comes from the fact that the solutions we have are continuous functions of time, otherwise commensurability is no more a necessary condition.

This value depends on the relative ratios. Therefore, the mean harmonic distance of two tones is also an estimate of the degree of antiresonance in the logarithmic space, $\frac{1}{2}\Delta_H(\nu_1,\nu_2) = \log_2 a_H(\nu_1,\nu_2)$. However, the above relationship is not valid for three or more frequency ratios since, in general, $\mathrm{lcm}(\nu_1,\nu_2,\ldots,\nu_n)\,\mathrm{gcd}(\nu_1,\nu_2,\ldots,\nu_n) \neq \nu_1\nu_2\cdots\nu_n$. In such a case, for an arbitrary set of frequency ratios $(\nu_1,\nu_2,\ldots,\nu_n)$, one could think that the geometric mean of the average harmonic antiresonance by pairs would also do the trick. Since no two tones are the same, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs. Then, we write it as

$$a_H(\nu_1,\ldots,\nu_n) = \left(\prod_{i<j} a_H(\nu_i,\nu_j)\right)^{\frac{2}{n(n-1)}} \tag{5}$$

However, the factors $a_H(\nu_i,\nu_j)$ with $\mathrm{gcd}(\nu_i,\nu_j) \neq 1$ would measure the relative ratios of the periods T_{ν_i}, T_{ν_j} with regard to their common period $\mathrm{lcm}(T_{\nu_i}, T_{\nu_j})$, instead of the total period T. Therefore, Eq. 5 would not be appropriate for a general case. This can be solved from the following approach.

Sabat (2023) studied and compared several measures to estimate how far, on average, a set of harmonics lies from its common fundamental. Some of them are Benedetti's distance, Euler's Gradus Suavitas, and Barlow's indigestibility. He proposes the generalization to n tones of Eq. 3, referred to as *harmonic radius*. It is defined as

$$R_H(\nu_1,\ldots,\nu_n) = \left(\frac{T}{T_{\nu_1}}\cdots\frac{T}{T_{\nu_n}}\right)^{\frac{1}{n}} \tag{6}$$

The harmonic radius, as a measure of antiresonance, is proportional to the period of the common fundamental T, but, since not all the chords with the same common fundamental elicit the same resonance, the contribution of the other components is considered in average, as inversely proportional to each period, i.e, proportional to each sympathetic factor. The harmonic radius written explicitly in terms of the frequencies is

$$R_H(\nu_1,\ldots,\nu_n) = \frac{(\nu_1\cdots\nu_n)^{\frac{1}{n}}}{\mathrm{gcd}(\nu_1,\ldots,\nu_n)}$$

For $x > 0$, the harmonic radius also satisfies $R_H(x\nu_1,\ldots,x\nu_n) = R_H(\nu_1,\ldots,\nu_n)$, so that it is simpler to express the frequency ratios $(\nu_1,\ldots,\nu_n)$ in their simplest form, as positive integers $(p_1,\ldots,p_n)$ with $\mathrm{gcd}(p_1,\ldots,p_n) = 1$. Then,

$$R_H(p_1,\ldots,p_n) = (p_1\cdots p_n)^{\frac{1}{n}} \tag{7}$$

The harmonic radius is related to the harmonic antiresonance by pairs.

Theorem 3.1 *In terms of the average harmonic antiresonance by pairs, the harmonic radius satisfies*

$$R_H(p_1,\ldots,p_n) = \left(\prod_{i<j} \mathrm{gcd}(p_i,p_j)\, a_H(p_i,p_j)\right)^{\frac{2}{n(n-1)}}$$

Proof. We write the product $\prod_{i=1}^{n} p_i$ in Eq. 7 as a product of $\frac{n(n-1)}{2}$ factors, by pairs, so that each product will appear $n-1$ times. Thus,

$$R_H(p_1,\ldots,p_n)^n = \left(\prod_{i<j} p_i p_j\right)^{\frac{1}{n-1}} = \left(\prod_{i<j} \mathrm{gcd}(p_i,p_j)\ \mathrm{lcm}(p_i,p_j)\right)^{\frac{1}{n-1}} =$$

$$\left(\prod_{i<j} \mathrm{gcd}(p_i,p_j)^2 \frac{\mathrm{lcm}(p_i,p_j)}{\mathrm{gcd}(p_i,p_j)}\right)^{\frac{1}{n-1}} = \left(\prod_{i<j} \mathrm{gcd}(p_i,p_j)\, a_H(p_i,p_j)\right)^{\frac{2}{n-1}}$$

By isolating R_H, the theorem is proven. ■

Corollary 3.2 *If* $C = (p_1, \ldots, p_n)$*, in the last expression of the proof it holds* $\gcd(p_i, p_j) \geq 1$*. Then,* $a_H(C) \leq R_H(C)$*.*

Corollary 3.3 $\gcd(p_i, p_j) = 1 \; \forall i, j \iff a_H(C) = R_H(C)$*.*

In this case, since $\gcd(p_i, p_j) = 1 \; \forall i, j \iff \gcd(C) = 1$ and $\mathrm{lcm}(C) = \prod_{i=1}^{n} p_i$, then $\Delta_H(C) = \log_2 \mathrm{lcm}(C) = n \log_2 (\prod_{i=1}^{n} p_i)^{\frac{1}{n}}$. Therefore,

Corollary 3.4 $\gcd(p_i, p_j) = 1 \; \forall i, j \iff \frac{1}{n}\Delta_H(C) = \log_2 a_H(C)$*.*

Lemma 3.5 *The harmonic distance of a chord* C *is bounded above as*

$$\Delta_H(C) \leq n \; \log_2 R_H(C) \tag{8}$$

Proof. We know that $\Delta_H(C) = \log_2 \frac{\mathrm{lcm}(C)}{\gcd(C)}$. Since $\gcd(C) = 1$, then $\mathrm{lcm}(C) \leq \prod_{i=1} p_i$. According to Eq. 7, it is hold $\mathrm{lcm}(C) \leq R_H(C)^n$. By taking logarithms and simplifying, the result is proven[7]. ■

A chord $C = (\nu_1, \nu_2, \ldots, \nu_n)$ satisfying $\gcd(\nu_i, \nu_j) = \gcd(\nu_1, \nu_2, \ldots, \nu_n)$ for all $i < j$ will be referred to as a *single-undertone* chord. In particular, if this chord is expressed in lowest terms, $(p_1, p_2, \ldots, p_n)$, it satisfies $\gcd(p_i, p_j) = 1$ for all $i < j$. If $\mathrm{lcm}(\nu_i, \nu_j) = \mathrm{lcm}(\nu_1, \nu_2, \ldots, \nu_n)$ for all $i < j$, then it will be referred to as a *single-overtone* chord.

Just as R_H accounts for antiresonance, an estimate of the harmonic resonance is given by its inverse, the *harmonic sympathy*. For a chord in lowest terms, it is defined as

$$S_H(p_1, \ldots, p_n) = R_H^{-1}(p_1, \ldots, p_n) = (\textstyle\prod_{i=1}^{n} \frac{1}{p_i})^{\frac{1}{n}} \tag{9}$$

Theorem 3.6 *For* $C = (p_1, \ldots, p_n)$*,* $p_i \geq 2$*,*

$$S_H(C) = (\textstyle\prod_{i=1}^{n} \frac{1}{p_i})^{\frac{1}{n}} \approx s(C) \equiv \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i} \tag{10}$$

Proof. According to Jensen's inequality, if x is a random variable and f is a concave function, the mean satisfies $\mathrm{E}[f(x)] \leq f(\mathrm{E}[x])$. If f is a straight line, both sides are equal. The log function is concave, hence $\frac{1}{n}\sum_{i=1}^{n} \log x_i \leq \log(\frac{1}{n}\sum_{i=1}^{n} x_i)$. Since $\frac{1}{n}\sum_{i=1}^{n} \log x_i = \log(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$, then, by applying antilogarithms, $(\prod_{i=1}^{n} x_i)^{\frac{1}{n}} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$. If $x_i = \frac{1}{p_i}$, then $0 < x_i \leq \frac{1}{2}$ so that, around the mean $\mathrm{E}[x]$, a straight line would provide a good approximation of $\log(x)$. That is, if $x = \frac{1}{p}$, $p \geq 2$, variations of $\log(\frac{1}{p})$ satisfy $|\Delta \log \frac{1}{p}| \approx |\frac{\Delta p}{p^3}|$. Hence, if $\frac{1}{p_0} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i}$ and $\frac{1}{p_0 + \Delta p_0} = (\prod_{i=1}^{n} \frac{1}{p_i})^{\frac{1}{n}}$, for $p_0 > 2$ and $|\Delta p_0| < 2p_0$, the difference of their logarithms is very small and the arithmetic and geometric means nearly match. ■

3.3. *Harmonic similarity*

Bowling, Purves, and Gill (2018) have shown that the perceived consonance of chords can be approximately predicted by their relative similarity to voiced speech sounds estimated from a metric called harmonic similarity. In its evaluation, the first step is to determine the single series that contain all the harmonics in the chord. The fundamental frequency

[7] Note that we are taking advantage of writing $C = (p_1, \ldots, p_n)$ with $\gcd(C) = 1$. If we were writing $C = (\nu_1, \ldots, \nu_n)$ as a set of irreducible frequency ratios, the inequality $\mathrm{lcm}(C) \leq \prod_{i=1} \nu_i$ would not be satisfied (e.g., $\mathrm{lcm}(\frac{1}{1}, \frac{5}{4}, \frac{3}{2}) = \frac{\mathrm{lcm}(1,5,3)}{\gcd(1,4,2)} = 15$ and $1 \cdot \frac{5}{4} \cdot \frac{3}{2} = \frac{15}{8}$).

of this series is calculated as the gcd of the tones in the chord. The second step is to calculate the percentage of harmonics in the single harmonic series, all of them multiple of the fundamental, that are also present in the chord. This percentage is referred to as harmonic similarity score, H_S. The highest frequency considered in the calculation of the harmonic similarity score is the lcm of the tones in the chord, after which the pattern of harmonics repeats. Instead, to choose another common partial would truncate the harmonic pattern in an arbitrary way.

This estimate is easy to calculate. For a dyad expressed in lowest terms $C = (p_1, p_2)$, it is $H_S(C) = \frac{A}{L}$, with $L = \text{lcm}(C)$ and $A = \frac{L}{p_1} + \frac{L}{p_2} - 1$. For a triad $C = (p_1, p_2, p_3)$, $A = \left(\frac{L}{p_1} - 1\right) + \left(\frac{L}{p_2} - 1\right) + \left(\frac{L}{p_3} - 1\right) - \left(\frac{L}{\text{lcm}(p_1,p_2)} - 1\right) - \left(\frac{L}{\text{lcm}(p_1,p_3)} - 1\right) - \left(\frac{L}{\text{lcm}(p_2,p_3)} - 1\right) + 1 = \frac{L}{p_1} + \frac{L}{p_2} + \frac{L}{p_3} - \frac{L}{\text{lcm}(p_1,p_2)} - \frac{L}{\text{lcm}(p_1,p_3)} - \frac{L}{\text{lcm}(p_2,p_3)} + 1$. The main contribution to H_S is due to the first three terms $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$, proportional to Eq. 10. The other terms make a correction. However, if one harmonic were contained in several single series of the chord, it would have to be interpreted as more consonant than if it only was contained in one single series. Therefore, by subtracting repeated matches in order to get a percentage, the consonance estimate, rather than being corrected, is underestimating the consonance of specific cases where subsets of harmonics reinforce each other[8].

4. Tonal graph

An n-tone graph $\mathcal{G}(\nu_1, \ldots, \nu_n)$ is a bidimensional (planar) graph representing the harmonic space associated with a set of n irreducible frequency ratios satisfying $\nu_1 < \ldots < \nu_n$. It is a projection of the harmonic space, useful to study easily their mutual relationships, although not totally equivalent.

For two tones, the dual graphs of Fig. 1 represent the proportions between frequencies and periods described in §3.1. They are similar to the tonal graphs describing the harmonic distances between frequencies and their inverse (Cubarsi 2019).

Let us assume that $\frac{T_{\nu_2}}{T_{\nu_1}} = \frac{p_1}{p_2}$. Then, for a positive $x \in \mathbb{R}$, we can write $T_{\nu_2} = p_1\, x$, $T_{\nu_1} = p_2\, x$, $\nu_2 = \frac{y}{p_1}$, and $\nu_1 = \frac{y}{p_2}$, with $y = \frac{1}{x}$. Hence, in the top of the graphs, the common subharmonic has the frequency of the whole oscillation, with total period[9] $T = T_{\text{LCA}}$,

$$\nu_{\text{LCA}} = \gcd(\nu_1, \nu_2) = y \gcd(\tfrac{1}{p_1}, \tfrac{1}{p_2}) = \frac{y}{\text{lcm}(p_1,p_2)}; \quad T_{\text{LCA}} = \text{lcm}(T_{\nu_1}, T_{\nu_2}) = x\ \text{lcm}(p_1, p_2)$$

Similarly, in the bottom, for the common superharmonic with period T_{LCH},

$$\nu_{\text{LCH}} = \text{lcm}(\nu_1, \nu_2) = y\ \text{lcm}(\tfrac{1}{p_1}, \tfrac{1}{p_2}) = \frac{y}{\gcd(p_1,p_2)}; \quad T_{\text{LCH}} = \gcd(T_{\nu_1}, T_{\nu_2}) = x \gcd(p_1, p_2)$$

The dual graphs for frequencies and periods have two particular features. One is that, with regard to the arrows, there is reflection symmetry, that is, by inverting the values of the nodes, the graph on the left, from top to bottom, and the graph on the right, from bottom to top, are equivalent. This is also true for graphs with more than two tones, as shown in the examples of the next section. The graphs $\mathcal{G}(\nu_1, \nu_2, \ldots, \nu_n)$ and its inverse $\mathcal{G}^{-1}(\nu_1, \nu_2, \ldots, \nu_n) \equiv \mathcal{G}(\frac{1}{\nu_n}, \ldots, \frac{1}{\nu_2}, \frac{1}{\nu_1})$ are *reflection symmetric*.

[8] For instance, $H_S(2, 3, 4) = 8$, although by pairs in the series of 2 and 3 there are 2 matches, in the series of 2 and 4 there are 3 matches, and in the series of 3 and 4 there is 1 match. In contrast, $H_S(2, 3, 12) = 8$, although by pairs in the series of 2 and 3 there are 2 matches, in the series of 2 and 12 there is 1 match, and in the series of 3 and 12 there is 1 match. Both chords have the same similarity score, but by pairs, the series of 2 and 4 of the former chord have a greater mutual reinforcement than in the latter chord.

[9] For frequencies we use both sets of operators, (LCA, LCH) and (gcd, lcm). For periods we use the latter. When referring to a graph, sometimes it is clearer to use both denominations, e.g., the gcd(a, b) is the LCA of a.

The other feature, which in general is only true for 2-tone graphs, is that the factors on the arrows of the half upper part are equal to those of the half lower part. Then, we will say the graph is *self-symmetric*. In a 2-tone self-symmetric graph, *even if it is a subgraph of a n-tone graph with $n > 2$*, the structure of Fig. 1 is maintained, i.e., given two consecutive nodes (a, b) in any row, the node above, $\mathrm{LCA}(a, b)$, has value $\gcd(a, b)$, the node below, $\mathrm{LCH}(a, b)$, has value $\mathrm{lcm}(a, b)$, and it is satisfied $\gcd(a, b)\,\mathrm{lcm}(a, b) = ab$.

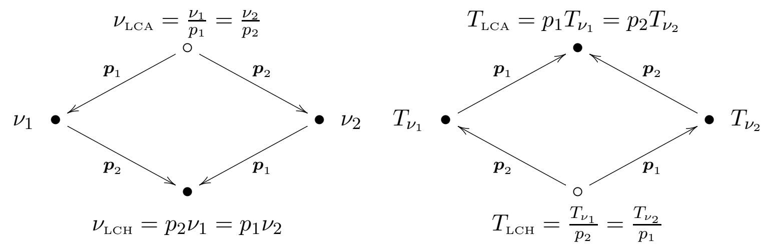

Figure 1. (Left) Tonal graph for frequencies (ν_1, ν_2) and (right) tonal graph for their periods.

As explained in §3.1, according to equations 3 and 6, the important thing to retain is that the degree of resonance/antiresonance of a chord depends on the sympathetic factors, i.e., the ratios between the overall period of the superposition T_{LCA} and the partial periods T_{ν_i}, that is, *on the factors involved in the upper half of* $\mathcal{G}(\nu_1, \nu_2, \ldots, \nu_n)$. For $n = 2$ are the same than those of the lower half, but for $n \geq 3$ are different, unless the graph is self-symmetric. The upper half of the tonal graph involves the undertones, in particular, pairs of common fundamentals, which in many cases match the difference tones. Notice that, given two frequencies a and b such that $a > b$, the common divisors of a and b are the same as the common divisors of $a - b$ and b. Hence, the difference tone $a - b$ shares the common fundamental of a and b, and sometimes it is also their common undertone.

We use an example to review the information associated with the dual graphs for a triad. The frequency ratios associated with a major triad are $(1, \frac{5}{4}, \frac{3}{2})$, which multiplied by a frequency ν_0 produce the major triad rooted in ν_0. Although the factors involved in the graphs are the same as expressing the triad in lowest terms, $(4, 5, 6)$, the graphs associated with the former (Fig. 2) provide some information about the wave superposition which is not provided by the latter[10].

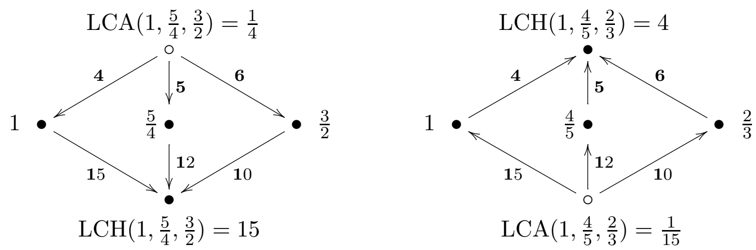

Figure 2. (Left) Tonal graph for frequencies $(1, \frac{5}{4}, \frac{3}{2})$ and (right) tonal graph, upside down, for their inverse.

On the left panel of Fig. 2 the tonal graph describes frequencies and factors between them, while the right panel, its dual graph, although it is valid for the inverse frequencies read upside down, read from top to bottom refers to the periods with the same factors between them as the frequencies. In the graph on the right, the period of the lowest common subharmonic (of the whole oscillation) is $T = \mathrm{lcm}(1, \frac{4}{5}, \frac{2}{3}) = 4$.

[10]Notice that for the major chord $(4, 5, 6)$, their common fundamental 1 matches the difference tones $6-5$ and $5-4$. For the minor chord $(10, 12, 15)$, the common fundamental of 10 and 12 is its difference tone 2, and the common fundamental of 12 and 15 is its difference tone 3.

In general, the symmetry existing between both graphs for frequencies and periods implies a similar symmetry between sets of overtones and undertones, since, if $C = (\nu_1, \ldots, \nu_n)$ and $C^{-1} = (\frac{1}{\nu_n}, \ldots, \frac{1}{\nu_1})$, then

$$\mathrm{LCH}(C) = \frac{1}{\mathrm{LCA}(C^{-1})} \iff \Delta_H(C) = \frac{\mathrm{LCH}(C)}{\mathrm{LCA}(C)} = \Delta_H(C^{-1}) = \frac{\mathrm{LCH}(C^{-1})}{\mathrm{LCA}(C^{-1})} \tag{11}$$

4.1. *Harmonic length*

Let us consider a chord $C = (\nu_1, \ldots, \nu_n)$ with common fundamental $\gcd(C)$.

Definition 4.1 The *relative harmonic length* (or δ-distance) of $\nu_i \in C$ is its harmonic distance to $\gcd(C)$,

$$\delta(\nu_i|C) = \Delta_H(\nu_i, \gcd(C)) = \log_2 \frac{\nu_i}{\gcd(C)} \tag{12}$$

The relative harmonic length of $\nu_i \in C$ is the logarithm of its sympathetic factor.

Definition 4.2 The *harmonic length* of C is the average

$$\delta_H(C) = \frac{1}{n}\sum_{i=1}^{n} \delta(\nu_i|C) = \log_2 \frac{(\nu_1 \cdots \nu_n)^{\frac{1}{n}}}{\gcd(C)} = \log_2 R_H(C) \tag{13}$$

The sum of the relative harmonic lengths is referred to as *cumulative harmonic length.* According to Eq. 9, the harmonic sympathy of the chord is

$$S_H(C) = 2^{-\delta_H(C)} \tag{14}$$

Theorem 4.3 *The harmonic distance of a chord C satisfies*

$$\Delta_H(C) = \delta_H(C) + \delta_H(C^{-1}) \tag{15}$$

Proof. If we consider the graphs associated with the chords C and C^{-1} (Fig. 3), for $\mathcal{G}(C)$ we have $\frac{1}{\gcd(C)}\left(\prod_{i=1}^{n}\nu_i\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n}p_i\right)^{\frac{1}{n}} = R_H(C)$, and for $\mathcal{G}(C^{-1})$, $\frac{1}{\gcd(C^{-1})}\left(\prod_{i=1}^{n}\frac{1}{\nu_i}\right)^{\frac{1}{n}} = \mathrm{lcm}(C)\left(\prod_{i=1}^{n}\frac{1}{\nu_i}\right)^{\frac{1}{n}} = \left(\prod_{i=1}^{n}p'_i\right)^{\frac{1}{n}} = R_H(C^{-1})$, with $p'_i = \frac{\mathrm{lcm}(C)}{\nu_i}$. By taking logarithms and adding, according to Eq. 1 and to Definition 4.2, we get the result. ■

Owing to Eq. 13, Eq. 8 provides the upper bound for the harmonic distance, $\Delta_H(C) \leq n\,\delta_H(C)$. For $n = 2$, $\Delta_H(C) = 2\,\delta_H(C)$. We determine its lower bound.

Lemma 4.4 *The harmonic distance is bounded as*

$$\frac{n}{n-1}\,\delta_H(C) \leq \Delta_H(C) \leq n\,\delta_H(C) \tag{16}$$

Proof. For C^{-1}, Eq. 8 yields $\Delta_H(C^{-1}) \leq n\,\delta_H(C^{-1})$. According to Definition 4.2 and Eq. 15, for C^{-1} we have $\delta_H(C^{-1}) = \Delta_H(C^{-1}) - \delta_H(C)$. Since $\Delta_H(C) = \Delta_H(C^{-1})$, then $\Delta_H(C) \leq n\;\Delta_H(C^{-1}) - n\;\delta_H(C)$ and $n\,\delta_H(C) \leq (n-1)\;\Delta_H(C)$. Hence, $\frac{n}{n-1}\;\delta_H(C) \leq \Delta_H(C)$ provides the lower bound. ■

Under the conditions of Corollary 3.4,

Corollary 4.5 *If C is a single-undertone chord, its harmonic distance attains the maximum value, $\Delta_H(C) = n\,\delta_H(C)$. Then, C^{-1} is a single-overtone chord and its harmonic distance attains the minimum value, $\Delta_H(C^{-1}) = \frac{n}{n-1}\,\delta_H(C^{-1})$.*

4.2. *Harmonic co-length*

Similar definitions can be referred to the lowest common harmonic.

Definition 4.6 The *relative harmonic co-length* of ν_i is its harmonic distance to $\mathrm{lcm}(C)$,

$$\delta'(\nu_i|C) = \Delta_H(\nu_i, \mathrm{lcm}(C)) = \log_2 \frac{\mathrm{lcm}(C)}{\nu_i} \tag{17}$$

Definition 4.7 The *harmonic co-length* of C is the average

$$\delta'_H(C) = \frac{1}{n}\sum_{i=1}^{n} \delta'(\nu_i|C) = \log_2 \frac{\mathrm{lcm}(C)}{(\nu_1 \cdots \nu_n)^{\frac{1}{n}}} \tag{18}$$

Theorem 4.8 *It is satisfied $\delta'_H(C) = \delta_H(C^{-1})$, so that*

$$\Delta_H(C) = \delta_H(C) + \delta'_H(C)$$

Proof. If $C{=}(\nu_1, \ldots, \nu_n)$ and $C^{-1}{=}(\frac{1}{\nu_n}, \ldots, \frac{1}{\nu_1})$, then $\frac{\mathrm{lcm}(C)}{(\nu_1 \cdots \nu_n)^{\frac{1}{n}}} {=} \frac{(\frac{1}{\nu_n} \cdots \frac{1}{\nu_1})^{\frac{1}{n}}}{\mathrm{gcd}(C^{-1})}$. Hence, according to Definition 4.7, $\delta'_H(C) = \delta_H(C^{-1})$. By substitution in Eq. 15, the theorem is proven. ■

Therefore, the ratio

$$\beta = \frac{\delta'_H(C)}{\delta_H(C)}; \quad \frac{1}{n-1} \leq \beta \leq n-1 \tag{19}$$

measures the deviation from harmonic symmetry of C, attained when $\beta = 1$. For $n = 2$ a chord is always harmonically symmetric. If the chord C has ratio β, then C^{-1} has ratio β^{-1}.

We also define the *harmonic co-radius* as

$$R'_H(C) = \frac{\mathrm{lcm}(C)}{(\nu_1 \cdots \nu_n)^{\frac{1}{n}}} \tag{20}$$

so that $\delta'_H(C) = \log_2 R'_H(C)$; $\quad R_H(C^{-1}) = R'_H(C)$.

When the harmonic length is shorter than the harmonic co-length, the frequency of the LCA is relatively close to the bass and reinforce each other. If the chord is single-undertone, no other undertones will contribute to create confusion.

When the harmonic co-length is shorter, the frequency of the LCH can be relatively close to the highest tone of the chord. In some cases, this may create the sensation that the LCH is a tone belonging to the chord and reinforce the higher frequencies. In particular, if the chord is single-overtone, there are no intermediate overtones before LCH, by strengthening the above sensation. In return, the reference to the common fundamental may fade.

4.3. *Harmonically symmetric chord*

Definition 4.9 A tonal graph $\mathcal{G}(C)$ is self-symmetric if it is isomorphic to $\mathcal{G}(C^{-1})$.

This means that they have the same factors, i.e., $p_1 = p'_1, \ldots, p_n = p'_n$ in Fig. 3, although the nodes can be proportional. According to Eq. 15, it is straightforward to see that

Lemma 4.10 *If $\mathcal{G}(C)$ is self-symmetric, then $\delta_H(C) = \delta'_H(C)$. This also applies to $\mathcal{G}(C^{-1})$.*

Definition 4.11 A chord C is harmonically symmetric if and only if $\mathcal{G}(C)$ is self-symmetric.

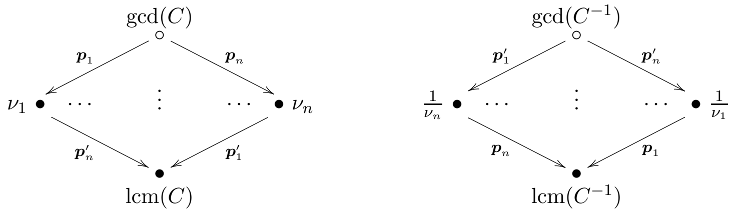

Figure 3. Tonal graphs $\mathcal{G}(C)$ and $\mathcal{G}(C^{-1})$ for chords $C = (\nu_1, \ldots, \nu_n)$ and $C^{-1} = (\frac{1}{\nu_n}, \ldots, \frac{1}{\nu_1})$.

Corollary 4.12 *If C is harmonically symmetric, then C is proportional to C^{-1}. In their simplest form, they are the same chord.*

The augmented triad 16:20:25 and the diminished triad (r) 25:30:36 are symmetric.

5. Harmonic dispersion

In the tonal graph of a chord C, the harmonic distances between the LCA(C), its tones, and the LCH(C) inform about several aspects of the harmonic complexity of the chord that concerns the resonance. As previously explained, the harmonic length $\delta_H(C)$ is indicative of the average degree of resonance of the components, relative to the period of the whole oscillation. However, their harmonics must also be taken into account. Those from the common fundamental to the lowest common harmonic are associated with the harmonic distance $\Delta_H(C)$, and the pattern of harmonics repeats beyond these values.

If to the harmonic length we add the harmonic distance, this amount provides a slight modification of the harmonic sympathy accounting for the harmonics elicited by the chord. Then, a chord with greater resonance would have its tones and the undertones between LCA(C) and C closer to LCA(C), which is associated with a lower value of $\delta_H(C)$, otherwise they would take up a greater extent of the tonal graph, so that they would be more dispersed. Similarly, a chord with greater resonance would have its overtones between C and LCH(C) closer to LCA(C), associated with a lower value of $\Delta_H(C)$.

Definition 5.1 The harmonic dispersion of a chord C is defined for $n \geq 2$ as

$$D_H(C) = \delta_H(C) + \Delta_H(C) \tag{21}$$

The harmonic dispersion increases with both the harmonic length and the harmonic distance. In terms of the mean harmonic length and co-length, it is written as

$$D_H(C) = 2\,\delta_H(C) + \delta'_H(C) \tag{22}$$

The weight of the former is twice the weight of the latter. According to Eq. 16, the harmonic dispersion is bounded as

$$\tfrac{2n-1}{n-1}\,\delta_H(C) \leq D_H(C) \leq (n+1)\,\delta_H(C)$$

Its maximum value is attained for a single-undertone chord. The minimum value is attained for a single-overtone chord. For the chord C^{-1}, $D_H(C^{-1}) = \delta_H(C^{-1}) + \Delta_H(C^{-1}) = \delta'_H(C) + \Delta_H(C)$, so that $D_H(C^{-1}) - D_H(C) = \delta'_H(C) - \delta_H(C)$.

By using the ratio β of Eq. 19, $D_H = (2+\beta)\,\delta_H$. For chords with similar δ_H, the harmonic dispersion increases with β, i.e., as δ'_H becomes greater than δ_H and the LCH moves away in the tonal graph. Also, $D_H = (1 + \frac{1}{1+\beta})\,\Delta_H$. For chords with similar Δ_H, the harmonic dispersion increases as β decreases, i.e., as δ'_H becomes smaller than δ_H and the LCA moves away from the tones of the chord.

5.1. *The chord spheroid*

Partch (1974) introduced the terms otonality and utonality to describe chords formed by tones that are the overtones or undertones of a given fixed tone. An otonality is a set of frequency ratios expressed by fractions with a common denominator (called numerary nexus) and consecutive numerators, $C = (\frac{p_1}{q}, \ldots, \frac{p_n}{q})$, with $p_{i+1}{=}p_1{+}i$, $\gcd(p_1, \ldots, p_n) = 1$, i.e., a sequence within the harmonic series. The common fundamental is $\mathrm{LCA}(C) = \frac{\gcd(p_1,\ldots,p_n)}{\mathrm{lcm}(q,\ldots,q)} = \frac{1}{q}$. The maximum consonance is given by the lowest sympathetic factor $\frac{p_1}{q} : \frac{1}{q} = p_1$. The associated utonality is the set of inverse frequency ratios expressed by fractions with a common numerator and consecutive denominators, $C^{-1} = (\frac{q}{p_n}, \ldots, \frac{q}{p_1})$. According to Partch, these concepts generalize the major and minor tonalities in conventional music theory. The common fundamental is $\mathrm{LCA}(C^{-1}) = \frac{\gcd(q,\ldots,q)}{\mathrm{lcm}(p_1,\ldots,p_n)} = \frac{q}{\mathrm{lcm}(p_1,\ldots,p_n)}$ and the maximum consonance corresponds to the lowest sympathetic factor $\frac{q}{p_n} : \frac{q}{\mathrm{lcm}(p_1,\ldots,p_n)} = \frac{\mathrm{lcm}(p_1,\ldots,p_n)}{p_n}$. Then, since two consecutive integers are always coprime, $\frac{\mathrm{lcm}(p_1,\ldots,p_n)}{p_n} \geq \frac{\mathrm{lcm}(p_{n-1},p_n)}{p_n} = \frac{p_{n-1}p_n}{p_n} = p_{n-1} > p_1$. Therefore, the otonality C is closer to its common fundamental than the utonality C^{-1}, so that, for chords, the otonality is more reinforced by the whole wave than the utonality, by enhancing the presence of the virtual common fundamental.

Since most chords are neither otonalities nor utonalities, these concepts were extended to just intonation chords as otonal and utonal, not requiring numerators or denominators to be consecutive. Any chord can be written either with a numerary nexus in the numerator or with a (different) numerary nexus in the denominator. Several criteria (Wikipedia contributors 2025) based on the odd-limit[11] of their tones can be used to classify a chord as otonal or utonal. However, focusing in the proximity of the bass to the common fundamental, it is easily seen from the tonal graph of Fig 3 that, if $p_1 < p'_1$ then C owns the otonal property and C^{-1} the utonal. Therefore, the result arises from comparing C to C^{-1}. We have seen that this comparison of average distances to the common fundamental is made from the harmonic length and co-length.

A graphical representation through an spheroid allows to encapsulate the overall information, i.e., the average distances to the common fundamental and to the first common harmonic (together determining the harmonic distance), and the harmonic dispersion.

We write the harmonic dispersion of the chord C as $D_H(C){=}\log_2 R_H^2(C)R'_H(C)$. First, we represent the harmonic distance from $\mathrm{LCA}(C)$ to $\mathrm{LCH}(C)$ with an ellipse of semiaxes $a = R_H(C)$, in the direction x, and $b = R'_H(C)$, in the direction y. The eccentricity of the ellipse is determined by the relative proportions of the harmonic radius and coradius. Its area is $A = \pi ab$, so that when $a = b = 1$ we obtain the area of the unit circle $A_0 = \pi$. Then, the harmonic distance is the logarithm of the normalized area of the ellipse $A_H = \frac{A}{A_0}$, $\Delta_H(C) = \log_2 ab = \log_2 A_H$. If C is harmonically symmetric (i.e., $C = C^{-1}$, in lowest terms), then $a = b$ and the ellipse degenerates into a circle. In general, for two mutually inverse chords, the major and minor semiaxes swap, although the area of the ellipse and the harmonic distance are the same, as shown in Fig. 4 (ellipse in gray).

To the above ellipse we add a third semiaxis, in the direction z, accounting exclusively for the degree of harmonic antiresonance. Thus, two semiaxes have the same value, $c = a = R_H(C)$, and the ellipse transforms into an ellipsoid of revolution, i.e., an spheroid.

The principal section $z = 0$ is the previous ellipse (similar to the principal section $x = 0$). If $c = a = b$, we get a sphere, which correspond to a harmonically symmetric

[11] According to Harry Partch's notation, for a positive odd number q, the q-odd-limit contains all rational numbers such that the largest odd number that divides either the numerator or denominator is not greater than q.

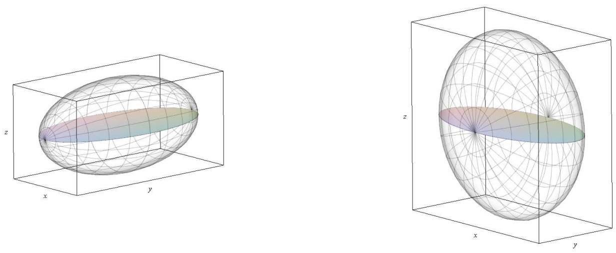

Figure 4. Chord spheroids for non harmonically symmetric chords C and C^{-1}. Their volume represents their harmonic dispersion. The area of the ellipse of the principal section $z = 0$ represents their equal harmonic distance.

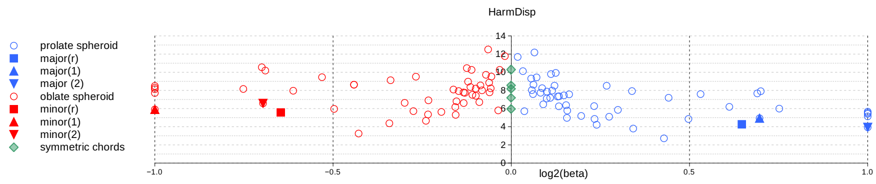

Figure 5. Distribution of harmonic dispersion D_H in terms of $\log_2 \beta$.

chord. If $b < a$, the spheroid is oblate (flattened like a lentil). If $b > a$, the spheroid is prolate (elongate like a rugby ball). The volume of the spheroid is $V = \frac{4\pi}{3} a^2 b$. When $a = b = c = 1$ we obtain the volume of the unit sphere $V_0 = \frac{4\pi}{3}$. Then, the harmonic dispersion of the chord is the logarithm of the normalized *harmonic volume* $V_H = \frac{V}{V_0}$, so that $D_H(C) = \log_2 a^2 b = \log_2 V_H$.

The shape of the spheroid informs us about the deviation from symmetry. For two non-symmetric chords C and C^{-1}, since their spheroids have the semiaxes b and a swapped (Fig. 4), they yield different harmonic volume and harmonic dispersion, although they have the same harmonic distance. For a similar harmonic distance (principal section $z = 0$), a harmonic length (axis x) lower than the co-length (axis y) indicates a prolate spheroid, which indicates more reinforcement between the acoustic fundamental and the bass. Therefore, the flattening of the spheroid, either prolate or oblate, generalizes the concepts otonal and utonal, while the volume indicates the chord dispersion.

If the triads[12] analyzed by Bowling, Purves, and Gill (2018), not exceeding one octave, are completed with their inversions, the same number of prolate and oblate chords will be present. Fig. 5 shows the distribution of the harmonic dispersion D_H in terms of the logarithm of the ratio β. In the first quartile ($D_H \leq 5.8$) there are much more prolate chords, although harmonic symmetry is not directly correlated with harmonic dispersion. Each chord with $\log_2 \beta > 0$ (prolate) has less harmonic dispersion than its inverted chord with $\log_2 \beta < 0$ (oblate). The chord with less harmonic dispersion, with frequency ratios $(2, 3, 4)$, is prolate and the following one, $(3, 4, 6)$, is oblate. Just to compare, in Fig. 5 five green dots have been added. They are the just intonation symmetric chords, from lower to greater dispersion, $1{:}\frac{4}{3}{:}\frac{16}{9}$, $1{:}\frac{5}{4}{:}\frac{25}{16}$, $1{:}\frac{6}{5}{:}\frac{36}{25}$, $1{:}\frac{7}{5}{:}\frac{49}{25}$, and $1{:}\frac{9}{8}{:}\frac{81}{64}$.

The spheroids for the major and minor triads as well as their inversions are displayed in Fig. 6. They are ordered by harmonic dispersion of the chord C (top). The resulting order for C^{-1} (bottom) is not necessarily the same. The major triad and its inversions are prolate and the minor triad and its inversions are oblate. Also prolate are the augmented

[12] These authors use simple frequency ratios but they notate them by number of semitones of the closest pitch class in the 12-TET system. This is not the optimal notation, since ratios producing significant variations of harmonic similarity may correspond to the same number of semitones. For instance, in general, the triads $(1, \frac{7}{5}, \frac{9}{5})$ and $(1, \frac{11}{8}, \frac{11}{6})$, also expressed as ratios $1{:}\frac{7}{5}{:}\frac{9}{5}$ and $1{:}\frac{11}{8}{:}\frac{11}{6}$, would share the notation $(0\,6\,10)$ in number of semitones. For these authors, $(0\,6\,10)$ is equivalent exclusively to $1{:}\frac{7}{5}{:}\frac{9}{5}$.

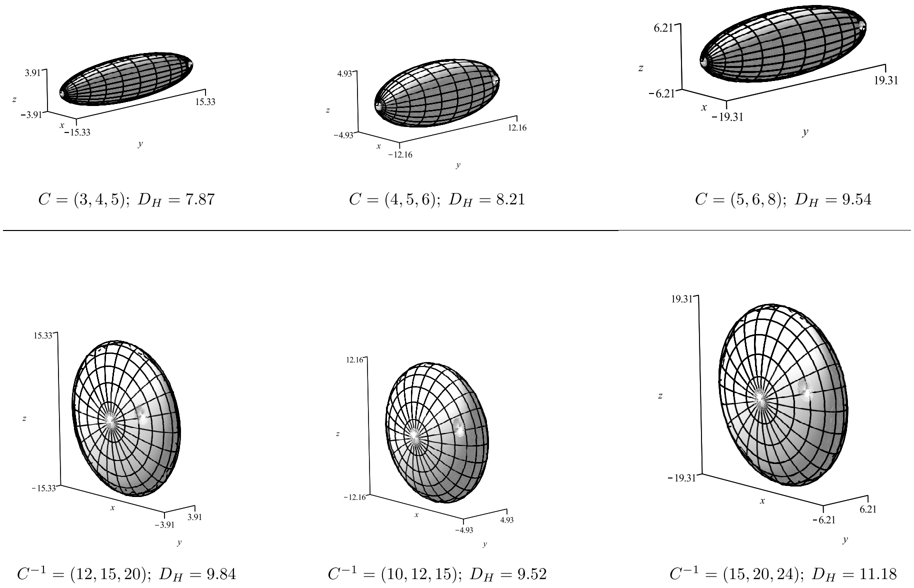

Figure 6. (Top) Prolate spheroids for the major triads (2nd inv., root, 1st inv.) and (bottom) oblate spheroids of their inverted chords, the minor triads (1st inv., root, 2nd inv.), with their harmonic dispersion.

triad[13], diminished triads, and their inversions.

5.2. *Harmonic likeness*

For two tones, Helmholtz's intensity of influence was a measure of the harmonic consonance. It was inversely correlated with the harmonic distance, $C_H(p,q) \propto 2^{-\Delta_H(p,q)}$. From a similar relationship, although in terms of the harmonic dispersion, it can be generalized for more than two tones to estimate harmonicity, as opposite to harmonic dispersion.

Definition 5.2 The *harmonic likeness* of a chord C is given by[14] $L_H(C) = 2^{-\,D_H(C)}$.

The harmonic likeness L_H is the inverse of the harmonic volume V_H.

By taking into account equations 9, 14, and 21, it can be expressed as proportional to the harmonic sympathy with a correction factor, as

$$L_H(C) = \tfrac{\mathrm{gcd}(C)}{\mathrm{lcm}(C)}\, S_H(C) \tag{23}$$

Also, bearing in mind equations 6 and 10, the harmonic likeness can be written in terms of the periods of the tones of the chord and their LCA and LCH,

$$L_H(C) = \tfrac{T_{\mathrm{LCH}}}{T_{\mathrm{LCA}}} \left(\prod_{i=1}^{n} \tfrac{T_i}{T_{\mathrm{LCA}}}\right)^{\frac{1}{n}} \approx \tfrac{T_{\mathrm{LCH}}}{T_{\mathrm{LCA}}}\, \tfrac{1}{n} \sum_{i=1}^{n} \tfrac{T_i}{T_{\mathrm{LCA}}} \tag{24}$$

[13]The augmented triad $(1, \frac{5}{4}, \frac{25}{16})$ (16:20:26, in its simplest form) is symmetric, although the assumed augmented triad (0 4 8)=$(1, \frac{5}{4}, \frac{8}{5})$ according to Bowling, Purves, and Gill (2018) (20:25:32, in its simplest form) is slightly prolate and with greater harmonic dispersion (one tone differs by 41¢, which is not negligible). Similarly, the diminished triad (r) $(1, \frac{6}{5}, \frac{36}{25})$ (25:30:36, in its simplest form) is symmetric, although the assumed diminished triad (r) (0 3 6)=$(1, \frac{6}{5}, \frac{7}{5})$ (5:6:7, in its simplest form) is strongly prolate and with lower harmonic dispersion (one tone differs by 49¢).

[14]This is a relative measure. A symmetric chord satisfies $D_H(C)$=3 $\delta_H(C)$. Then, by defining $L_H(C) = 2^{-\frac{1}{3}\,D_H(C)}$ we would get the harmonic likeness referred to a virtual symmetric chord with harmonic length $\delta_H(C)$.

6. Discussion

Bowling, Purves, and Gill (2018) provided a considerable amount of perceptual data for chords with two, three, and four tones. They measured the degree of vocal similarity of the chords with two metrics and compared the results with perceived consonance. The first metric was the harmonic similarity score, previously explained. The second metric was a frequency interval analysis that compared the absolute frequency intervals between the tones in each chord to the absolute frequency intervals that occur between harmonics in human vocalizations. The minimum absolute frequency interval between successive harmonics typically encountered in human vocalizations is of about 50 Hz (Risberg 1961). Chords containing intervals smaller than 50 Hz were treated as having lower vocal similarity and were predicted to be heard as less consonant. According to the authors, both analysis correctly predict the chord perceived consonance in most cases, and conclude that the neural response seems to be unitary, responding to harmonic similarity only when harmonics are appropriately spaced. Therefore, only harmonic similarity will be compared to harmonic likeness. Consonance ratings were evaluated from two subject groups: (I) musicians and (II) non-musicians. For triads, the ratings of musicians were significantly closer to the empirical standard than those of non-musicians, proving that the effects of musical training on consonance perception are relatively pronounced. Many of the incorrect predictions involved at least one chord commonly used in Western music in comparison with a chord more rarely encountered. For dyads, our harmonic likeness L_H and their harmonic similarity H_S yield similar results (Fig. 7), likewise the rank orderings of both music theory and psychological experiments (Hutchinson and Knopoff 1978; Plomp and Levelt 1965).

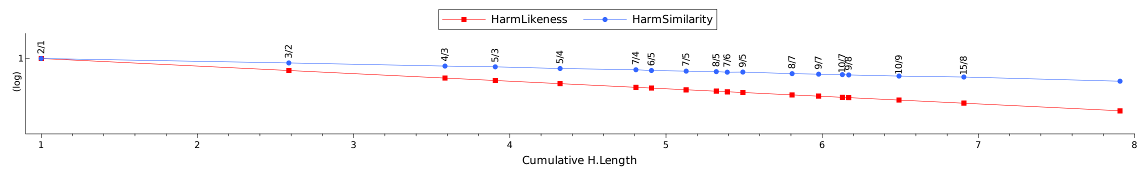

Figure 7. Harmonic likeness L_H and harmonic similarity H_S (in logarithmic scale) for two-tone frequency ratios in terms of the cumulative harmonic length λ.

We compare them for all triads (those that match with their equivalences). By using Pearson's correlation coefficient r, which coincides with Spearman's rank correlation coefficient on rankings provided that there are not too duplicate values (significance is tested from Student's t-distribution and $p < 0.0001$ in all cases), the correlation between the ranks of both groups and those of the harmonic similarity H_S is $r = 0.665$ (adjusted $R^2 = 0.433$) and for the harmonic likeness L_H is $r = 0.732$ (adjusted $R^2 = 0.528$), hence slightly improved. If, instead of ranks, we compare estimates, the correlation coefficients are $r = 0.728$ (adjusted $R^2 = 0.522$) for H_S and $r = 0.731$ (adjusted $R^2 = 0.527$) for L_H.

Figure 8 displays the relation between the mean consonance ratings and the ranks for L_H and H_S, since the relationship between the mean consonance ratings and the rank classification is nearly linear (correlation coefficient $r = 0.984$) and the distribution around the mean less dispersed. Figure 9 plots L_H and H_S, as well as the mean consonance ratings, in terms of the cumulative harmonic length. Lines do not indicate trends but provide a better visualization. For both estimates, higher values of the harmonic length are dominated by the right-hand factor S_H of Eq. 23, i.e., the harmonic sympathy, inverse of the harmonic radius. Instead, for lower harmonic lengths, the harmonic distance, which condenses the relationships between the undertones and the overtones, makes the difference.

For the main incorrect predictions, discussed in Bowling, Purves, and Gill (2018), H_S predicts incorrectly more consonance for 5:8:9$\sim$(0 8 10) than for 6:8:9$\sim$(0 5 7) (suspended fourth), more consonance for 5:6:7$\sim$(0 3 6) (diminished) than for 5:6:8$\sim$(0 3 8) (major 1st

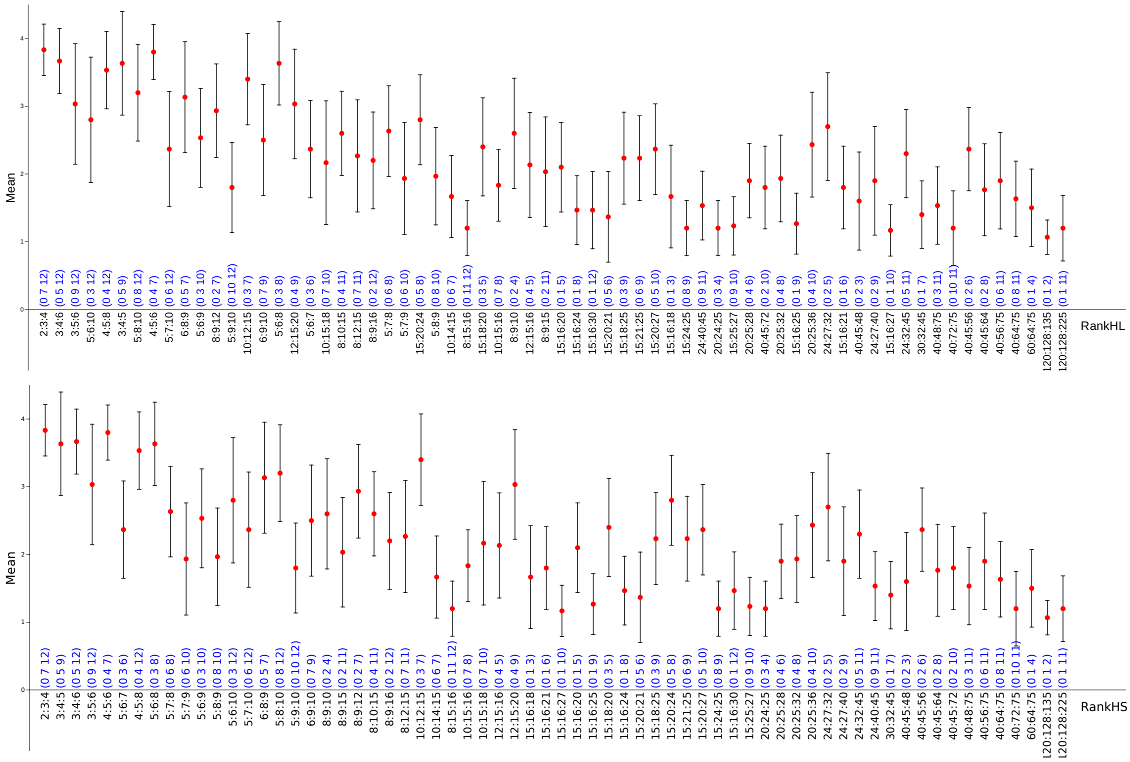

Figure 8. Mean of consonance ratings with standard deviations vs. ranks of (top) L_H and (bottom) H_S. Chords are notated as ratios p:q:r and pitch classes (P,Q,R).

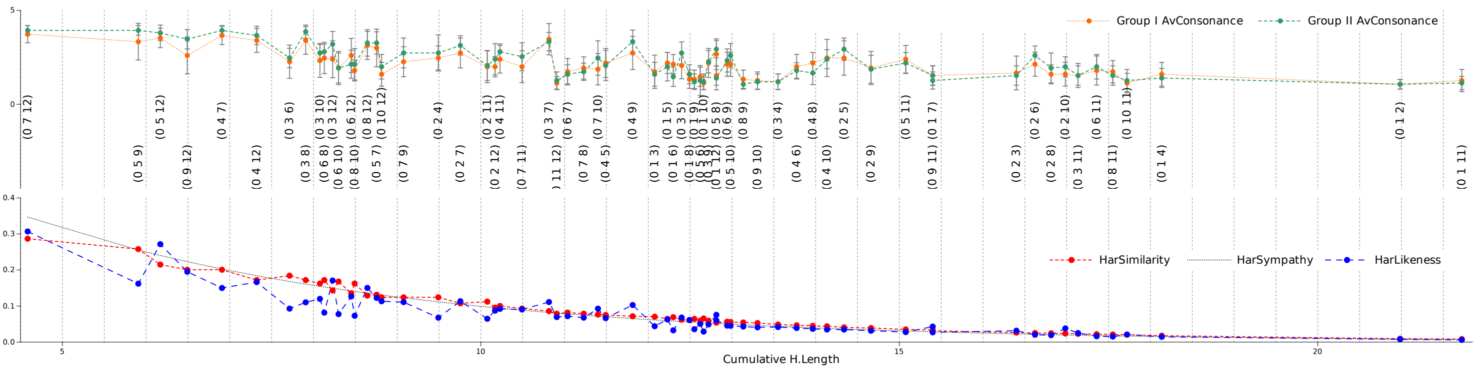

Figure 9. Average consonance ratings of subject groups I and II (with standard deviations), S_H, H_S, and L_H in terms of the cumulative harmonic length.

inv.), more consonance for 5:7:9∼(0 6 10) than for 5:6:9∼(0 3 10), and more consonance for 5:6:7∼(0 3 6) (diminished) than for 10:12:15∼(0 3 7) (minor). All of them have been addressed according to L_H. Figure 10 (blue line for H_S, black line for L_H) provides a more detailed view.

There are two special cases. One is the chord 3:4:5∼(0 5 9) (major 2nd inv.), for which both estimates predict as more consonant than 4:5:6∼(0 4 7) (major) and 10:12:15∼(0 3 7) (minor). However, the average consonance ratings have standard deviations that are still consistent with both estimates. Curiously, Group II of non-musicians feels 3:4:5∼(0 5 9) as equal consonant than 4:5:6∼(0 4 7) and more than 10:12:15∼(0 3 7), as opposite to Group I of musicians. As discussed in Bowling, Purves, and Gill (2018), these incorrect predictions (or perceptions, since in some cases they have considerable standard deviations) can be partially explained by the frequent occurrence in popular music of major and minor chords compared with diminished chords. The other case involves the chords 5:6:8∼(0 3 8) (major

1st inv.) and 5:6:9∼(0 3 10), the former predicted as less consonant than the latter by L_H, contrary to H_S and to the consonance ratings. Group I estimates 5:6:8∼(0 3 8) more consonant than the minor triad, while Group II as less consonant. The harmonic likeness predicts it as similar to the minor triad, but less consonant than 5:6:9∼(0 3 10). As before, it is possible that the reason is that this chord, which is simply formed by a minor third and a fifth, is less frequent. Therefore, for chords in root position we get a consistent rank order of increasing dissonance: major < minor < diminished < augmented, also according to Roberts (1986); Cook (2001); Cook and Fujisawa (2006); Johnson-Laird, Kang, and Leong (2012) (particularly, M_4^6<M<M^6∼m<m^6<m_4^6). Bowling, Purves, and Gill (2018) conclude than they cannot determine whether the consonance ratings collected here apply to all listeners regardless of their musical background. Against to thinking that consonance is primarily a result of exposure to Western music rather than auditory system neurobiology (Lundin 1947; Parncutt 1989; McDermott et al. 2016), they argue that culture is itself a biological phenomenon and the auditory neurobiology is shaped by experience.

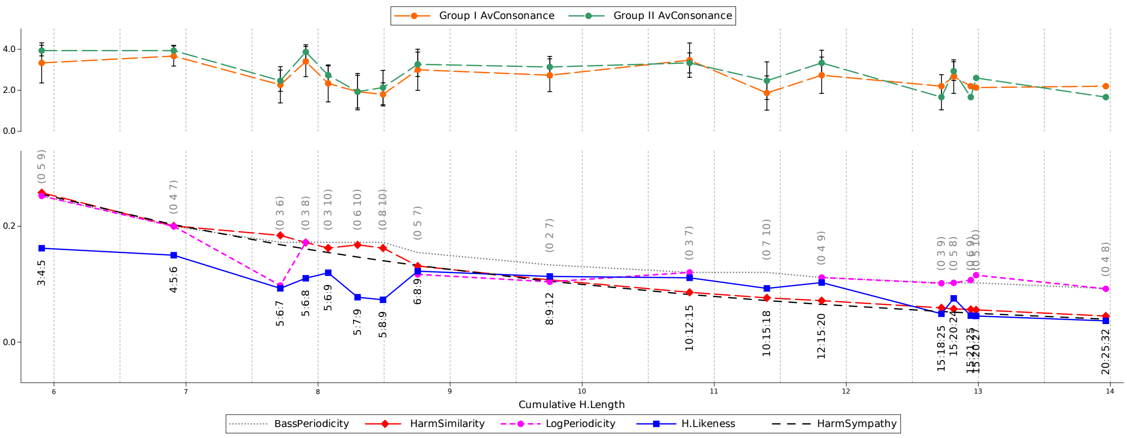

Figure 10. (Right) Average consonance ratings of subject groups I and II (with standard deviations) and comparison of bass periodicity, H_S, $\log P$, S_H, and L_H in terms of the cumulative harmonic length.

We now compare harmonic likeness and harmonic similarity to Stolzenburg's (2015) periodicity. This author provides an interesting approach to evaluate consonance in terms of *perceived* periodicity (Langner 1997; Ebeling 2008). By using the concepts already described, for a triad $C = (\nu_1, \nu_2, \nu_3)$ we summarize his procedure. The frequency ratios are referred to $\nu_1 = 1$. Since the common fundamental (LCA) is the main responsible for the periodicity detected in the brain (Tramo et al. 2001; Roederer 2008), then, according to Stolzenburg (2015), the degree of perceived consonance depends on the ratio between the period of the common fundamental T_{LCA} and the period of the bass of the chord. This ratio is exactly the same for chords that maintain similar proportions, i.e., $C' = (\nu_1', \nu_2', \nu_3') = (\frac{1}{\nu_2}, 1, \frac{\nu_3}{\nu_2})$, with $T_{\text{LCA}}' = \nu_2 T_{\text{LCA}}$, and $C'' = (\nu_1'', \nu_2'', \nu_3'') = (\frac{1}{\nu_3}, \frac{\nu_2}{\nu_3}, 1)$, with $T_{\text{LCA}}'' = \nu_3 T_{\text{LCA}}$. Obviously, $\frac{T_{\text{LCA}}}{T_1} = \frac{1}{3}(\frac{T_{\text{LCA}}}{T_1} + \frac{T_{\text{LCA}}'}{T_1'} + \frac{T_{\text{LCA}}''}{T_1''})$. However, depending on the listener, the above ratios are slightly modified so that, not only each frequency ratio is approximated by the closest simple ratio but their mutual proportions are also approximated. In evaluating the ratio $\frac{\nu_3}{\nu_2}$ of C' the tone ν_3 is virtually exchanged by ν_3'', which provides a simpler ratio, so that the period of the LCA for the modified chord becomes $T_{\text{LCA},3}$. Similarly, for the ratio $\frac{\nu_2}{\nu_3}$ of C'' the tone ν_2 is virtually exchanged by ν_2', also providing a simpler ratio, and the period of the LCA for the new chord becomes $T_{\text{LCA},2}$. Then, the relative periodicity, is given by $P = \frac{1}{3}(\frac{T_{\text{LCA}}}{T_1} + \frac{T_{\text{LCA},3}}{T_1'} + \frac{T_{\text{LCA},2}}{T_1''})$. The way the simple ratios are evaluated follows the pattern of the 12-TET system. A just intonation ratio $\frac{a}{b}$ is associated with its closest pitch class, i.e., a power of 2 with exponent $\frac{k \bmod 12}{12}$, $\frac{-k \bmod 12}{12}$ for its inverse. The *best rational approximation* $\frac{p}{q}$ of the corresponding pitch class of the

12-TET system provides the approximation of the just intonation ratio, which is referred to as *rational intonation* (with the lowest possible value of q allowing a relative error of 1.1%). To easily understand the procedure, we use the same example as Stolzenburg (2015). If $C = (1, \frac{6}{5}, \frac{5}{3})$, with $T_{\text{LCA}} = 15$, then $C' = (\frac{5}{6}, 1, \frac{25}{18}) \approx (\frac{5}{6}, 1, \frac{7}{5})$ and $T'_{\text{LCA},3} = 30$. Similarly, $C'' = (\frac{3}{5}, \frac{18}{25}, 1) \approx (\frac{3}{5}, \frac{7}{10}, 1)$ and $T''_{\text{LCA},2} = 10$. Then, the relative periodicity[15] is $P = \frac{1}{3}(\frac{15}{1} + \frac{30}{6/5} + \frac{10}{5/3})$.

Let us note that the new chord C' is equivalent to $(1, \frac{6}{5}, \frac{42}{25})$ (the difference between $\frac{42}{25}$ and $\frac{5}{3}$ is 14¢) and C'' is equivalent to $(1, \frac{7}{6}, \frac{5}{3})$ (the difference between $\frac{7}{6}$ and $\frac{6}{5}$ is 49¢). Then, there are two main points to discuss. On the one hand, although it is totally plausible that the listener approximates the ratios between the tones composing the chord, the process of building equivalent chords in order to estimate the resonance of each bass implies that, if the bass of the original chord is maintained, the listener would obviate the mistunings just pointed out (which for the second case is not negligible). On the other hand, for listeners with a better musical training than the 12-TET system, say with an auditory resolution capable of distinguishing 17, 29, 31, or even 53 tones, the estimated periodicity would change and would tend to the exact periodicity. Hence, this approach evaluates the subjectivity in the perception of the harmonicity/periodicity of the chord for a specific listener model, rather than objective parameters of the chord's harmonicity.

Now, we will compare the three measurements with the thirteen common triads in Stolzenburg (2015) (major, minor, suspended, and diminished, with their inversions, and augmented) by using the ratings of the previous groups I and II (Fig. 10, red line for $\log_2 P$, black line for L_H). In this case, the correlation coefficients for ranks are $r = 0.709$ for H_S, $r = 0.907$ for $\log_2 P$, and $r = 0.912$ for L_H ($p < 0.0001$). If, instead of ranks, we compare estimates, the correlation coefficients are $r = 0.673$ for H_S, $r = 0.732$ for $\log_2 P$, and $r = 0.899$ for L_H. Therefore, the harmonic likeness, which gives an objective estimation of the chord's harmonicity, is also highly correlated with consonance perception and it can be used as an approximate predictor.

7. Conclusions

Harmonic properties of a chord composed of just intonation ratios, with any number of tones, whether they span more or less than an octave, have been studied from a combination of several parameters. The distribution of the undertones and overtones has been represented in a planar tonal graph, which is a simplification of Tenney's harmonic space. The graph is characterized by three geometrical properties, namely, the harmonic length –accounting for the average harmonic distance from the common fundamental to the tones of the chord–, the harmonic co-length –accounting for the average harmonic distance from the tones of the chord to the first common overtone–, and the harmonic distance, which is the sum of the harmonic length and co-length. When the harmonic length and co-length match, the chord is harmonically symmetric. This property is likely not perceived by the human ear, unless by interval perception.

A greater harmonic complexity of the chord can be associated with a loss of reference to the common fundamental. It can be mainly attributed to the many undertones and difference tones, generally associated with a greater harmonic length and a weaker bass resonance. Lower values of the harmonic length indicate that the tones follow the harmonic series more closely and that there is a greater bass resonance. However, the harmonic complexity of the overtones and their difference tones in the lower-half graph has a lesser impact on the perception of consonance and is rather related to color qualities of the sound.

[15] A better estimation is provided by the logarithmic periodicity, $\log_2 P = \frac{1}{3}(\log_2 \frac{15}{1} + \log_2 \frac{30}{6/5} + \log_2 \frac{10}{5/3})$.

One parameter is clearly inversely correlated with the perception of consonance: the harmonic dispersion. It depends on the harmonic length, which estimates the degree of sympathetic vibration between the common fundamental and the tones of the chord in terms of the sympathetic factors, and on the harmonic distance, which estimates the degree of sympathetic vibration between the common fundamental and the first common overtone. The harmonic distance condenses the harmonic relationships between the overtones and undertones between both extremes into a single value. In the frequency space, the inversion of the harmonic dispersion, the harmonic likeness, remains separable into two factors, one, the harmonic sympathy (inverse of the harmonic length), provides the main pattern of the overall periodicity. The other, proportional to the ratio between the periods of the first common overtone and the common fundamental, provides the correction of the main pattern, which makes the difference for chords of low harmonic length. The harmonic likeness gives an improved estimation of the logarithmic periodicity (Stolzenburg 2015) and the harmonic similarity (Bowling, Purves, and Gill 2018).

Harmonic dispersion, distance, length, and co-length can be visualized by the chord spheroid: the shape of the spheroid informs about the deviation from harmonic symmetry (i.e., difference between length and co-length); the area of the principal section informs about the harmonic distance; and the volume informs about the harmonic dispersion. The spheroids of two non-symmetric chords C and C^{-1} have different volume although the same principal section. This generalizes Harry Partch's concepts of otonality and utonality, which is directly related to consonance when we compare mutually inverted chords.

The present approach, rooted in the forced harmonic oscillator model and combining aspects of roughness and harmonicity theories, even improving the predictions given by the chord periodicity and harmonic similarity, clearly provides a simplified view of the problem. Future research should analyze more specific ways to characterize the complexity of the harmonics involved in a chord. For example, the major triad (1st inv.) $(5, 6, 8)$ is perceived as more consonant than the minor triad (r) $(10, 12, 15)$ by Group II, and less by Group I. The harmonic dispersion predicts a similar value ~ 9.5. Therefore, although with a similar consonance, these chords can be distinguished in terms of other attributes related to the distribution of their undertones, overtones, and difference tones (that in many cases match and reinforce the harmonicity). This would be associated with specific properties of the tonal graph, which in part have already been captured by the chord spheroid.

Furthermore, it would be interesting to study whether some properties related to harmonic symmetry, prolate/oblate chords, single overtone/undertone chords, etc., could be perceived in auditory tests beyond evaluating the attractiveness of the sound. Nevertheless, even if the harmonicity of a chord is well characterized, there will always be the question of subjectivity: will a listener be sensitive to a change in harmoniousness due to a variation of a tone of the chord by one or two Pythagorean commas? Surely, this will depend not only on culture, but also on musical training and practice.

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