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The Tonal Graph of a Musical Chord: Arithmetic Relationships Describing Harmonicity and Harmonic Symmetry

Rafael Cubarsi

Departament de Matemàtiques, Universitat Politècnica de Catalunya, 08034 Barcelona, Spain; rafael.cubarsi@upc.edu

Abstract

The harmonic structure of a chord C composed of rational frequency ratios is studied from its tonal graph. The graph describes harmoncity/periodicity properties of the chord allowing to characterize the chord from several parameters, which are easily visualized by the chord spheroid. Relevant harmonic structures are analyzed, such as self-symmetric graphs associated with harmonically symmetric chords. Recurrence relationships for the $\mathrm{lcm}(C)$ in terms of the gcd's of the powerset of C, and for the $\mathrm{gcd}(C)$ in terms of the lcm's, are derived to unveil how the harmonic quotient $\mathrm{lcm}(C)/\mathrm{gcd}(C)$ depends on the common undertones and overtones. In particular, for any number of tones, the harmonic quotient can be univocally expressed from the chord's common undertones. Therefore, although undertones and overtones have not been explicitly taken into account in the current model, the harmonic quotient actually incorporates information about them.

Keywords: least common multiple; greatest common divisor; periodicity; harmonic space; self-symmetric graph; upper-jigsaw graph; lower-jigsaw graph; chord spheroid

MSC: 11A05, 11A51, 11B37, 00A65, 05C12, 54E35

1. Introduction

This paper provides a mathematical approach to study the harmonic structure of chords (combinations of two or more pure tones played together) tuned in just intonation, i.e., as rational frequency proportions. It assumes an ideal model where the fundamental frequencies of the tones composing a chord are different and their amplitudes are similar. The article develops and generalizes [1,2] by describing and interpreting the tonal graph of a chord C in terms of periodicity. The tonal graph is a bidimensional directed acyclic graph representing the tones of the chord together with the common undertones up to $\mathrm{gcd}(C)$ (the common acoustic fundamental) and the common overtones up to $\mathrm{lcm}(C)$ (the first common overtone). The metric used is the harmonic distance [3].

Special attention is payed to reflection symmetric graphs, where the harmonic structure of overtones and undertones is similar. In such a case, the average harmonic distances from the tones of the chord to $\mathrm{gcd}(C)$ (harmonic length) and to $\mathrm{lcm}(C)$ (harmonic co-length) match. This is the case of harmonically symmetric chords. The sum of both lengths is the harmonic distance of the chord. Relevant harmonic structures of the tonal graph are studied, including graphs of chords with a single common undertone or a single common overtone. One would expect the corresponding chords to have some perceptually distinguishable sound qualities.

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According to the forced harmonic oscillator model, the harmonic length determines the average degree of sympathetic vibration between the tones of the chord and the common fundamental, without considering the influence of the mutual undertones and overtones. Sympathetic vibration means consonance (in Latin, consonare means sounding together). However, the overall sympathetic vibration is modulated by the harmonic distance of the chord, i.e., the distance from $\gcd(C)$ to $\mathrm{lcm}(C)$, which condenses the harmonic relationships between the common undertones and between the common overtones.

Over centuries the opposition between consonance and dissonance has attracted a lot of interest and has generated numerous studies and auditory tests to explain what is referred to as consonance, harmony perception, roughness, fusion, etc. [4–14]. Musicians also use other musical terms for referring to chords, such as they sound stable, final, and resolved, while others sound unstable, tense, and unresolved. All of them are quite subjective. The concept of consonance is generally interpreted as the subjective attractiveness of tone combinations. From a physical viewpoint, this is related to the sympathetic vibration phenomenon. It depends on the ratios between the periods of the interfering waves and has an impact on the resonance between the overtones, undertones, and difference tones [14]. Instead, dissonance is generally spoken of as opposite to consonance. However, these terms are ambiguous, as they summarize in a single word perceptions that can be intrinsically different. For this reason, we think that it would be useful to have several parameters to objectively describe a chord.

A chord is usually written as n rational frequency ratios $\nu_1 < \cdots < \nu_n$, referred to an arbitrary fundamental frequency 1, e.g., $(\frac{6}{5}, \frac{4}{3}, \frac{3}{2})$. The lowest ratio is the bass of the chord. What is significant is the relationship between them. For that reason, many times the ratios are written assuming the bass as 1, e.g., $(1, \frac{10}{9}, \frac{5}{4})$. Sometimes there is a distinguished tone to which the other tones are referred, namely playing the role of fundamental, e.g., writing $(\frac{4}{5}, \frac{8}{9}, 1)$ would explicitly refer the ratios to the third tone. If the tone assumed as fundamental matches the bass, it is said that the chord is in root position. If the tones are shifted one position to the left and the old bass is set as the highest tone (by doubling the frequency as many times as necessary in order to maintain the pitch class) musicians say it is the first chord inversion (which is not the same as the inverted chord). By shifting two positions, we get the second chord inversion, etc. These are referred to as voices of the chord. For instance, the chord $(1, \frac{5}{4}, \frac{3}{2})$ is the major triad in root position and $(\frac{5}{4}, \frac{3}{2}, 2)$ is its 1st inversion, which can be written as $(1, \frac{6}{5}, \frac{8}{5})$, with the bass 1, where the old fundamental is now the last tone.

As an example, in some auditory tests the chord $(1, \frac{6}{5}, \frac{8}{5})$ (major triad, 1st inv.) is perceived as more consonant than $(1, \frac{6}{5}, \frac{3}{2})$ (minor triad, root position), although as less consonant in other tests [13]. For these chords, the predicted value for consonance is similar [11], although their perception is clearly different. Therefore, such a difference should be able to be parameterized in some way, regardless subjective considerations. Another example is the famous Tristan chord (the opening harmony of Richard Wagner's opera Tristan und Isolde) $(\frac{5}{7}, 1, \frac{5}{4}, \frac{5}{3})$. The nature of this sonority has been debated for decades, since although considered quite dissonant, musicians say that it has an attractiveness that seems more difficult to explain in terms of the chord itself than in terms of the function and way in which it resolves. Furthermore, chords are usually expressed in lowest terms, such as $(1, \frac{6}{5}, \frac{8}{5}) \sim (5{:}6{:}8)$, $(1, \frac{6}{5}, \frac{3}{2}) \sim (10{:}12{:}15)$ and $(\frac{5}{7}, 1, \frac{5}{4}, \frac{5}{3}) \sim (60{:}84{:}105{:}140)$.

In order to expand the scope of auditory tests, we analyze several objective parameters associated with mathematical and physical properties to describe the chord's harmonicity, which basically means periodicity. These parameters can be derived from arithmetical relationships depending on the gcd's and lcm's of some families of the frequency ratios involved in the chord. The last half of the paper is devoted to investigate what information

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about the undertones and overtones is contained in the harmonic distance of the chord. To this purpose, some recurrence relationships are derived allowing to express the product and quotient of $\gcd(C)$ and $\mathrm{lcm}(C)$ in terms of either the undertones $\gcd(C')$ or the overtones $\mathrm{lcm}(C')$, for all possible subsets $C' \subset C$.

2. Preliminaries

2.1. Periodicity

Harmonic frequency relations, a higher-order sound attribute closely related to pitch perception, has been proposed to account for consonance [9,15–17]. The component frequencies of the notes of some chords combine to produce an aggregate spectrum that is typically harmonic, resembling the spectrum of a single sound with a lower pitch. In contrast, other chords produce an inharmonic spectrum [7]. Chord's consonance is then associated with the spectral content of its tones. With harmonic tone spectra, peak consonance is observed when the fundamental frequencies are related by simple frequency ratios. Simple integer ratios, approximated by prototypically consonant musical chords, tend to produce overtones that either completely coincide or are widely spaced, hence minimizing interference. Consequently, with harmonic tone spectra, the harmonic series provides a pattern for the consonance and consonance is primarily determined by periodicity/harmonicity, although it is also potentially moderated by musical background [8–10,12].

In 1862 Helmholtz invented the double siren, allowing to create phase effects between them in order to study sound interferences. He also added small brass resonators intended to reduce the partials and produce more pure tones. In his foundational work on music acoustics and sound perception, Helmholtz listed the ratios of the main harmonics [4] (p. 187) and their intensity of influence given by $K(p,q) = \frac{100}{pq}$. This parameter, is here referred to as harmonic consonance. According to Helmholtz, the lower the product pq, the greater the degree of beatings in mistuning the interval, i.e., the mistuning is more noticeable, which was interpreted as both harmonics were more consonant before mistuning.

To interpret Helmholtz formula we must recall the forced harmonic oscillator model in the case of periodic solutions associated with commensurable frequencies (Appendix A). If the fundamental tone of the free oscillation has frequency $\nu_1 = 1$ and the forced oscillation has frequency $\nu = \frac{p}{q}$, then $\frac{\nu}{\nu_1} = \frac{p}{q}$ and the total period is $T = qT_{\nu_1} = pT_\nu$. If $q = 1$, the total period T matches T_{ν_1}. As q increases, the subharmonic oscillations of order q are associated with a decreasing resonance. Similarly, if $p = 1$, the total period T matches T_ν. As p increases, the subharmonic oscillations of order p are associated with a decreasing resonance. Then, the lower the sympathetic factors p and q, the greater the sympathetic vibration of the free and forced oscillations. Therefore, Helmholtz's harmonic consonance is an estimate of the sympathetic vibration between two tones based on periodicity.

2.2. Harmonic Distance

Tenney's harmonic distance between two harmonics [3], $\Delta_H(p,q) = \log_2 pq$ for p,q coprime, also accounts for this measure. The product pq in the frequency space, and its logarithm in the interval space, measure the opposite to their harmonic consonance.

For two arbitrary irreducible frequency ratios α, β, the harmonic distance is also obtained as $\Delta_H(\alpha,\beta) = \log_2 \frac{\mathrm{lcm}(\alpha,\beta)}{\gcd(\alpha,\beta)}$. Since, in general, $\Delta_H(x\,\alpha, x\,\beta) = \Delta_H(\alpha,\beta)$, $\forall x > 0$, then, $\Delta_H(\alpha,\beta) = \Delta_H(\frac{\alpha}{\beta},1)$, so that it also evaluates the harmonic distance from the ratio $\frac{\alpha}{\beta}$ to the fundamental $\nu_0 = 1$.

We denote the inverse function of the harmonic distance as $Q = 2^{\Delta_H}$, which we will call harmonic quotient. In the frequency space, instead of the interval space, it is equivalent to the harmonic distance. According to Equation (A3), for two oscillations of frequency ratios α and β (periods T_α and T_β), this value matches the ratio between the periods of the common fundamental and the first common harmonic, like Helmholtz's product pq,

$$Q(\alpha, \beta) = \frac{\mathrm{lcm}(\alpha, \beta)}{\mathrm{gcd}(\alpha, \beta)} = \frac{\mathrm{lcm}(T_\alpha, T_\beta)}{\mathrm{gcd}(T_\alpha, T_\beta)} \tag{1}$$

In the case of two single waves such a value has two equivalent interpretations (which are not equivalent in a more general superposition): on the one hand, Q measures the degree of antiresonance (as opposite to resonance and sympathetic vibration) between the tones composing the chord and the common fundamental; on the other hand, Q also estimates the degree of antiresonance between the lowest common harmonic and the common fundamental.

According to [1], the harmonic distance (and the harmonic quotient) can be generalized for a finite set of frequency ratios as follows (we maintain the name harmonic distance despite referring to several tones, since the name harmonic dissonance used in [1] can lead to confusion). Then, if $\Gamma = \{\gamma_i = \frac{p_i}{q_i},\ i \in I\}$ with $p_i, q_i \in \mathbb{N} \setminus \{0\}$ and $\mathrm{gcd}(p_i, q_i) = 1$, the harmonic distance is obtained by considering separately the numerators and denominators of these fractions as $\Pi = \{p_i,\ i \in I\}$; $\Theta = \{q_i,\ i \in I\}$,

$$\Delta_H(\Gamma) \equiv \log_2 \frac{\mathrm{lcm}(\Gamma)}{\mathrm{gcd}(\Gamma)} = \Delta_H(\Pi) + \Delta_H(\Theta) \tag{2}$$

Notice that numerators and denominators cannot be permuted since the fractions must be irreducible. Similarly, it is hold $\Delta_H(x\,\Gamma) = \Delta_H(\Gamma)$, $\forall x > 0$. Hence, the frequency ratios in Γ can be simplified by dividing by their highest common factor, and they can be expressed in their simplest form.

For a chord with three or more tones, the harmonic distance, although it estimates the degree of antiresonance between the lowest common harmonic and the common fundamental, it does not evaluate the degree of antiresonance of the tones composing the chord relative to the common fundamental.

2.3. Harmonic Space

Tenney [3] (pp. 280–304), in his 1983 *John Cage and the Theory of Harmony*, introduced the concept of harmonic space by generalizing Euler's lattice model. The harmonic space maps coprime harmonics of a fundamental frequency in a multidimensional lattice according to their prime factorization, similarly to the Tonnetz, although it is not periodic in any direction. The harmonic distance is the metric used in the harmonic space to relate two nodes, which basically is Minkowski's L_1 distance, by assuming the directions are orthogonal (also known as city-block metric). Such a representation is also valid for rational frequency ratios [1,18].

According to Tenney, two tones represented by proximate points in the harmonic space tend to be heard as being in a consonant relation to each other, while tones represented by more widely separated points are heard as mutually dissonant. The explanation given by [19] is that in the vicinity of each tuneable ratio there is a region of tolerance within which variations of tuning produce audible beating and phasing. Thus, the harmonic distance between two nodes indicates an important correlation between consonance and dissonance. This corroborates the remark made by Euler in 1766: "The sense of hearing is accustomed to identify with a single ratio, all the ratios which are only slightly different from it, so that the difference between them be almost imperceptible" e.g., [20] (Section 20). Possibly, the transposition invariance, i.e., the ability to identify a melody when its pitches are transposed by the same interval, has the same origin that the interval similarity, i.e., the ability to identify intervals of single frequency ratios, although it is unclear whether it is the intervals or the scale degrees which provide the measurements [21].

Although the current approach does not strictly apply to tones of equal temperament, conditions justifying the approximation of intervals of an equal temperament system as rational proportions were given in [1,11], basically in terms of the convergents and semiconvergents of continued fraction expansions of equal temperament tones.

Thus, the harmonicity of chords can be represented in the harmonic space so that the tone frequencies are associated with nodes and the arithmetic relationships between the prime numbers composing these tones (by products) are associated with directions. This is Tenney's multidimensional harmonic space, which we will call prime tonal graph. For example, for the tones $(4,5,6)$, the prime tonal graph is depicted in Figure 1(left). However, to study their harmonic relations it is possible to build a simpler, planar tonal graph such as that on the right of Figure 1, where one arrow may represent a product of prime numbers. This tonal graph is a subset of the latices composing the prime tonal graph, where the arrows of the former connect nodes through diagonals of the latter.

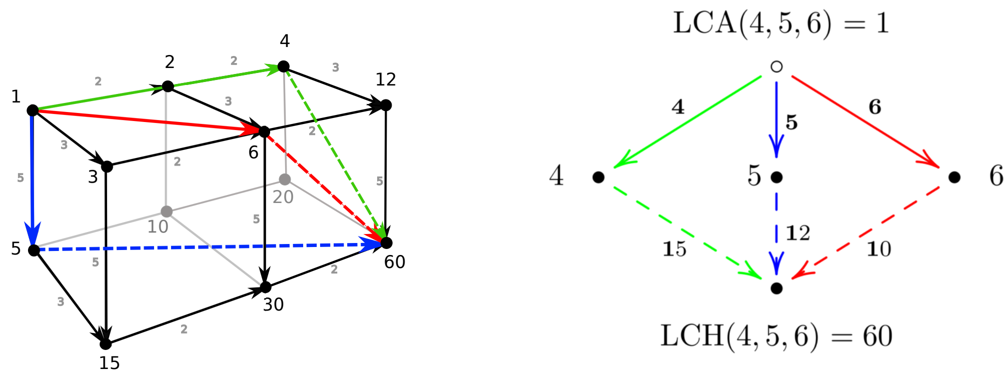

Figure 1. (**Left**) Part of the prime tonal graph generated by prime factors $(2,3,5)$ involving products from 1 to 60. (**Right**) Planar tonal graph for ratios $(4,5,6)$, whose colored arrows are also displayed in the prime tonal graph.

For a chord C, the common fundamental is the lowest common ancestor in the graph, $\mathrm{LCA}(C)$, and the lowest common harmonic is written as $\mathrm{LCH}(C)$ (i.e., lowest common harmonic refers to frequencies and lowest common ancestor refers to the harmonic space). Therefore, in the top-down direction of this directed acyclic graph, these extreme nodes are the closest common ancestor and the closest common descendant. For representing frequencies in the tonal graph we shall use both sets of operators $(\mathrm{LCA}, \mathrm{LCH})$ and $(\gcd, \mathrm{lcm})$, but for periods we only use the latter. Therefore, according to Equation (2) the harmonic distance of a chord C is the distance in the graph from $\mathrm{LCA}(C)$ to $\mathrm{LCH}(C)$, measured as an interval.

3. Planar *n*-Tone Graph

An n-tone graph $\mathfrak{G}(\nu_1,\ldots,\nu_n)$ is a planar graph representing the harmonic space associated with a set of n irreducible frequency ratios satisfying $\nu_1 < \ldots < \nu_n$, to avoid duplicity. It is a projection of the harmonic space, useful to study easily their mutual relationships, although not totally equivalent.

3.1. Tonal Graph for $n = 2$

For two tones, the dual graphs of Figure 2 represent the proportions between frequencies and periods. They are similar to the tonal graphs describing the harmonic distances between frequencies and their inverse frequencies.

Let us assume that $\frac{T_{\nu_2}}{T_{\nu_1}} = \frac{p_1}{p_2}$, irreducible. Then, for a positive $x \in \mathbb{R}$, we can write $T_{\nu_2} = p_1\,x$, $T_{\nu_1} = p_2\,x$, $\nu_2 = \frac{y}{p_1}$, and $\nu_1 = \frac{y}{p_2}$, with $y = \frac{1}{x}$. Hence, in the top of the graphs, the common subharmonic of the whole oscillation has the frequency and period given by

$$\nu_{\text{LCA}} = \gcd(\nu_1, \nu_2) = y\,\gcd(\tfrac{1}{p_1}, \tfrac{1}{p_2}) = \tfrac{y}{\text{lcm}(p_1, p_2)}; \quad T_{\text{LCA}} = \text{lcm}(T_{\nu_1}, T_{\nu_2}) = x\,\text{lcm}(p_1, p_2)$$

Similarly, in the bottom, for the common superharmonic, the frequency and period are

$$\nu_{\text{LCH}} = \text{lcm}(\nu_1, \nu_2) = y\,\text{lcm}(\tfrac{1}{p_1}, \tfrac{1}{p_2}) = \tfrac{y}{\gcd(p_1, p_2)}; \quad T_{\text{LCH}} = \gcd(T_{\nu_1}, T_{\nu_2}) = x\,\gcd(p_1, p_2)$$

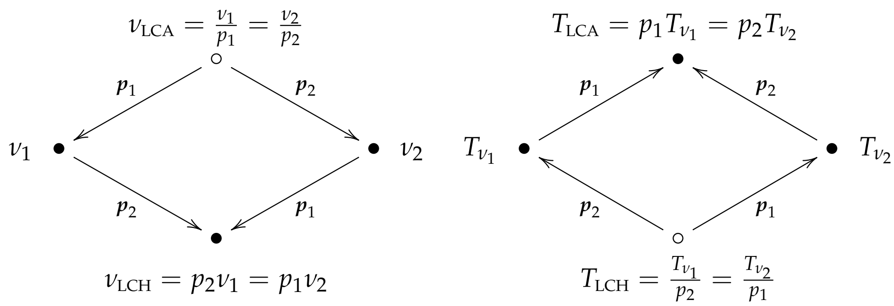

Figure 2. (**Left**) Tonal graph for frequencies (ν_1, ν_2) and (**right**) tonal graph for their periods.

The dual graphs for frequencies and periods have two particular features. One is that, with regard to the arrows, there is reflection symmetry, that is, by inverting the values of the nodes, i.e., for the inverted chord, the graph on the left, from top to bottom, and the graph on the right, from bottom to top, are equivalent. This is also true for graphs with more than two tones, as shown in the examples of the next section. Hence, if $C = (\nu_1, \ldots, \nu_n)$ and $C^{-1} = (\frac{1}{\nu_1}, \ldots, \frac{1}{\nu_n})$, the graph $\mathfrak{G}(C)$ and its inverse $\mathfrak{G}^{-1}(C) \equiv \mathfrak{G}(C^{-1})$ are reflection symmetric.

The other feature, which in general is only true for 2-tone graphs, is that the factors on the arrows of the half upper part are equal to those of the half lower part. In such a case, we will say the graph is self-symmetric (for more than two tones, it will be studied in Section 5).

In a n-tone graph with $n > 2$, any subgraph involving two consecutive tones is also self-symmetric. That is, the structure of Figure 2 is maintained, i.e., given two consecutive primary nodes (a, b) (those of the central row of tones) the node above, $\text{LCA}(a, b)$, has value $\gcd(a, b)$, the node below, $\text{LCH}(a, b)$, has value $\text{lcm}(a, b)$, and it is satisfied

$$\text{LCA}(a, b)\,\text{LCH}(a, b) = ab \tag{3}$$

One thing is important to retain: the degree of resonance/antiresonance of a chord is mainly dictated by the sympathetic factors, i.e., the ratios between the total period T_{LCA} and the partial periods T_{ν_i}, which are the factors involved in the upper half of $\mathfrak{G}(\nu_1, \nu_2, \ldots, \nu_n)$.

3.2. Tonal Graph for $n \geq 3$

We use an example to explain the information associated with the dual graphs for a triad. The frequency ratios associated with the major triad are $(1, \frac{5}{4}, \frac{3}{2})$, which multiplied by a frequency ν_0 produce the major triad rooted in ν_0. Although the factors involved in the graphs are the same as expressing the triad in lowest terms, (4:5:6), the graphs associated with the former (Figure 3) provide some information about the wave superposition which is not provided by the latter (Figure 1, right).

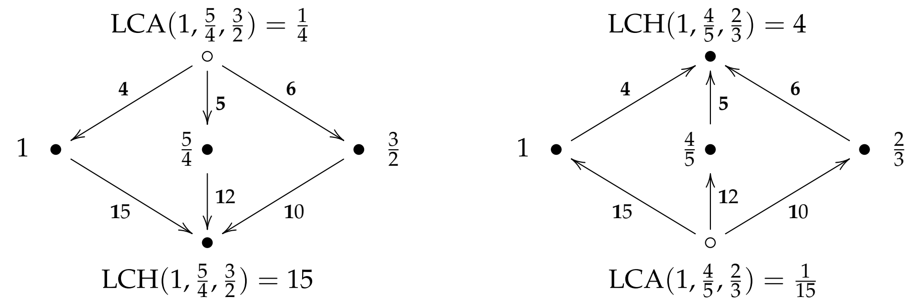

Figure 3. (**Left**) Tonal graph for $C = (1, \frac{5}{4}, \frac{3}{2})$ and (**right**) tonal graph, upside down, for C^{-1}, also indicating the relationships between periods in $\mathfrak{G}(C)$.

On the left panel of Figure 3 the tonal graph $\mathfrak{G}(C)$ describes frequencies and factors between them. The right panel, its dual graph, although read upside down it is valid for the inverse frequencies, read from top to bottom it refers to the periods of $\mathfrak{G}(C)$, with the same factors between them as the frequencies. In the graph on the right, the period of the common subharmonic (period of the whole oscillation) in $\mathfrak{G}(C)$ is $\mathrm{lcm}(1, \frac{4}{5}, \frac{2}{3}) = 4$.

In general, the symmetry existing between both graphs for frequencies and periods implies a similar symmetry between sets of overtones and undertones. Therefore, the harmonic distance is the same for two mutually inverse chords,

$$\mathrm{lcm}(C) = \frac{1}{\mathrm{gcd}(C^{-1})}; \quad \frac{\mathrm{lcm}(C)}{\mathrm{gcd}(C)} = \frac{\mathrm{lcm}(C^{-1})}{\mathrm{gcd}(C^{-1})}; \quad \Delta_H(C) = \Delta_H(C^{-1}) \tag{4}$$

3.3. Interlocking Graphs

A tonal graph for three or more tones can be built from tonal graphs involving adjacent pairs of the composing tones. For instance, the graph on the right of Figure 1 is composed of the two graphs on the left of Figure 4. However, the vertical arrows through the node 5 cannot be fully identified since the arrows labeled as 4 and 6 belong to different directions in the prime tonal graph on the left of Figure 1 (the colored arrows match those of the prime tonal graph). In the prime tonal graph, 5 is not $\mathrm{LCA}(20, 30)$. The two graphs on the left of Figure 4 can be assembled to obtain the one on the right of Figure 4, corresponding to the graph on the right of Figure 1. The added node below is the lcm of the pair above, $\mathrm{lcm}(20, 30) = 60$. Since $\mathrm{lcm}(4, 5) \neq \mathrm{lcm}(5, 6)$, the decomposition of the tonal graph $\mathfrak{G}(4, 5, 6)$ in order to include the graphs $\mathfrak{G}(4, 5)$ and $\mathfrak{G}(5, 6)$ has only involved the lower half graph. Instead, since $\mathrm{gcd}(4, 5) = \mathrm{gcd}(5, 6)$, the upper half graph has not been modified, otherwise we would apply a similar procedure, by adding nodes above from the gcd of the pair below. Notice that, in the graph on the right of Figure 4, $\mathrm{LCA}(20, 30) = 5$, but $\mathrm{gcd}(20, 30) \neq 5$.

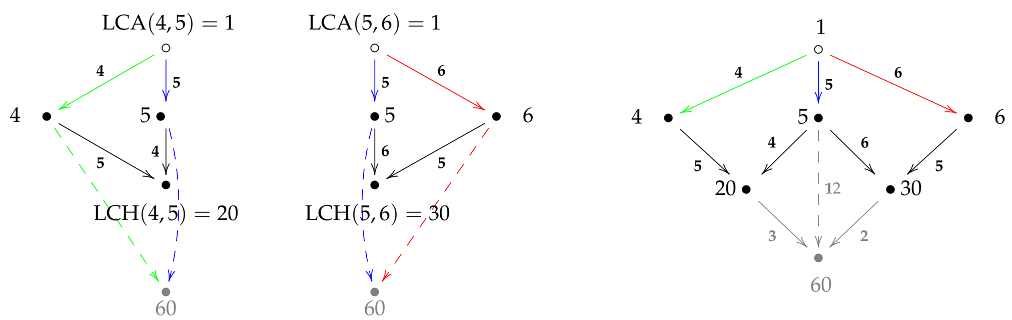

Figure 4. Decomposition of the tonal graph for the major chord $(4, 5, 6)$ into two graphs.

For irreducible frequency ratios $a < b < c$ it is straightforward to see that $\gcd(a,b,c) = \gcd(\gcd(a,b), \gcd(a,c), \gcd(b,c))$ and $\mathrm{lcm}(a,b,c) = \mathrm{lcm}(\mathrm{lcm}(a,b), \mathrm{lcm}(a,c), \mathrm{lcm}(b,c))$. Then, the superposition of three waves can be viewed as the result of a superposition by pairs. Nevertheless, to build the graph it is not necessary to use all the possible pairs; it suffices to use adjacent pairs, so that, in the upper half graph, it is hold

$$\mathrm{LCA}(a,b,c) = \gcd(a,b,c) = \gcd(\gcd(a,b), \gcd(b,c)) = \mathrm{LCA}(\mathrm{LCA}(a,b), \mathrm{LCA}(b,c))$$

while, in the lower half graph,

$$\mathrm{LCH}(a,b,c) = \mathrm{lcm}(a,b,c) = \mathrm{lcm}(\mathrm{lcm}(a,b), \mathrm{lcm}(b,c)) = \mathrm{LCH}(\mathrm{LCH}(a,b), \mathrm{LCH}(b,c))$$

Therefore, the tonal graphs can be nested in multiple layers up and down, similarly to Russian Matryoshka nesting dolls, as shown in Figure 5. However, in general,

$$b \neq \gcd(\mathrm{LCH}(a,b), \mathrm{LCH}(b,c)); \quad b \neq \mathrm{lcm}(\mathrm{LCA}(a,b), \mathrm{LCA}(b,c)) \tag{5}$$

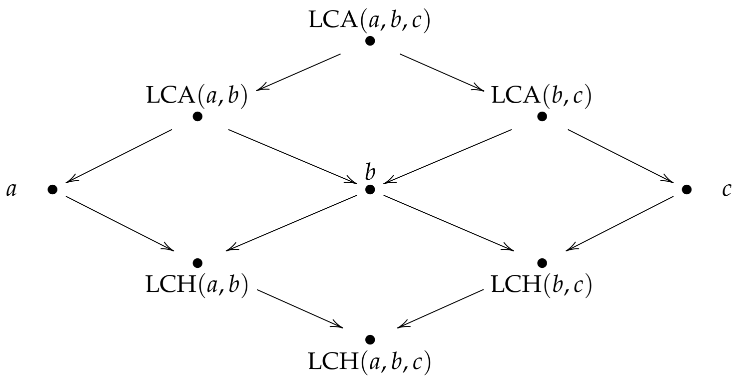

Figure 5. A graph for three tones can be decomposed into several subgraphs for pairs of tones.

That is, for an arbitrary number of tones, the tonal graph involves, from the central row upwards, the undertones by adjacent pairs, which are computed according to the gcd as recursive LCA's, and from the central row downwards, the overtones by adjacent pairs, which are computed according to the lcm as successive LCH's. That is, only in the upper half graph the LCA of a pair of consecutive nodes necessarily matches their gcd, and only in the lower half graph their LCH necessarily matches their lcm. In the central row, the subgraphs involving two consecutive nodes are self-symmetric, i.e., Equation (3) is satisfied, but in the other rows they are not necessarily self-symmetric.

3.4. Upper-Jigsaw Graph

The tones of the chord $C = (\nu_1, \nu_2, \ldots, \nu_n)$, with $\nu_1 < \nu_2 < \ldots < \nu_n$, are the primary nodes of $\mathfrak{G}(C)$. We call the other nodes secondary nodes, either in the upper half or in the lower half of the graph.

For the upper half of $\mathfrak{G}(C)$, including the primary nodes, we consider all possible three consecutive nodes (a,b,c) in one row, and the two nodes $(\gcd(a,b), \gcd(b,c))$ in the row above.

Definition 1. *We say $\mathfrak{G}(C)$ is an upper-jigsaw graph if and only if*

$$b = \mathrm{lcm}(\gcd(a,b), \gcd(b,c)) \tag{6}$$

Lemma 1. $\mathfrak{G}(C)$ *is an upper-jigsaw graph* $\iff \mathrm{lcm}(a,c) = \mathrm{lcm}(a,b,c)$

Proof. Since $\mathrm{lcm}(\gcd(a,b),\gcd(b,c)) = \gcd(b,\mathrm{lcm}(a,c)) = \frac{b\,\mathrm{lcm}(a,c)}{\mathrm{lcm}(a,b,c)}$, by substitution in Equation (6) we prove the lemma. □

Corollary 1. *The 2-subgraphs involving consecutive nodes in a row of the upper half of an upper-jigsaw graph are self-symmetric.*

Thus, an upper-jigsaw graph indicates a greater degree of consistency between the periods of the chord components, so that couples of tones does not generate new harmonics with periods that mix up with those of the components. In other words, it can be associated with a more definite spectrum.

For example, the tonal graph for the Tristan chord $\mathfrak{G}(60{:}84{:}105{:}140)$ is an upper-jigsaw graph. In addition, this chord has another interesting property. All pairs of tones have the same lowest common harmonic (420) (Figure 6, right panel). Therefore, all the secondary nodes of the bottom half graph have the same value, which matches the LCH.

Definition 2. *A chord* $C = (\nu_1, \nu_2, \ldots, \nu_n)$, *where* $\mathrm{lcm}(\nu_i, \nu_j) = \mathrm{lcm}(\nu_1, \nu_2, \ldots, \nu_n)$ *for all* $i < j$, *is a single-overtone chord.*

E.g., $\mathfrak{G}(15{:}20{:}24)$ is an upper-jigsaw graph, but the chord is not single-overtone.

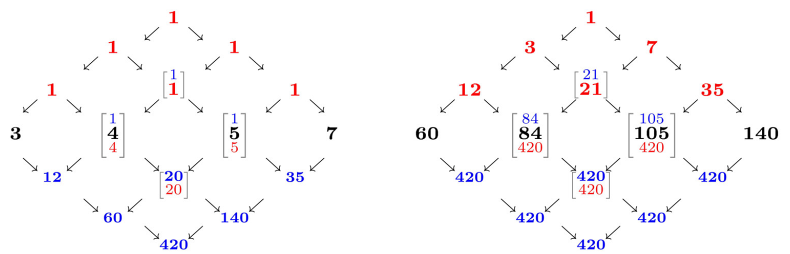

Figure 6. Undertones and overtones for the tonal graphs of $C = (3{:}4{:}5{:}7)$ (**left**) and the Tristan chord $C^{-1} = (60{:}84{:}105{:}140)$ (**right**). The lcm of each pair (above, blue) and the gcd of each pair (below, red) are indicated. The former is a single-undertone chord producing a lower-jigsaw graph and the latter is a single-overtone chord producing an upper-jigsaw graph.

3.5. Lower-Jigsaw Graph

For the lower half of $\mathfrak{G}(C)$, including the primary nodes, we consider all possible three consecutive nodes (a, b, c) in one row, and the two nodes $(\mathrm{lcm}(a,b), \mathrm{lcm}(b,c))$ in the row below.

Definition 3. *We say* $\mathfrak{G}(C)$ *is a lower-jigsaw graph if and only if*

$$b = \gcd(\mathrm{lcm}(a,b), \mathrm{lcm}(b,c)) \tag{7}$$

Lemma 2. $\mathfrak{G}(C)$ *is a lower-jigsaw graph* $\iff \gcd(a,c) = \gcd(a,b,c)$

Proof. Since $\gcd(\mathrm{lcm}(a,b),\mathrm{lcm}(b,c)) = \mathrm{lcm}(b,\gcd(a,c)) = \frac{b\gcd(a,c)}{\gcd(a,b,c)}$, by substitution in Equation (7), the lemma is proven. □

Corollary 2. *The 2-subgraphs involving consecutive nodes in a row of the lower half of a lower-jigsaw graph are self-symmetric.*

Corollary 3. *If* $\mathfrak{G}(C)$ *is a lower-jigsaw graph, then the inverse graph* $\mathfrak{G}^{-1}(C)$ *is an upper-jigsaw graph, and vice versa.*

For example, the tonal graph for the inverse of the Tristan chord $\mathfrak{G}(3{:}4{:}5{:}7)$ is a lower-jigsaw graph (Figure 6, left panel). In addition, its tones are coprime by pairs and have the same common fundamental 1. Then, all the secondary nodes of the top half graph have the same value.

Definition 4. *A chord $C = (\nu_1, \nu_2, \ldots, \nu_n)$ satisfying $\gcd(\nu_i, \nu_j) = \gcd(\nu_1, \nu_2, \ldots, \nu_n)$ for all $i < j$, is a single-undertone chord.*

In particular, if this chord is expressed in lowest terms, $(a_1, a_2, \ldots, a_n)$, it satisfies $\gcd(a_i, a_j) = 1$ for all $i < j$.

E.g., $\mathfrak{G}(5{:}6{:}8)$ is a lower-jigsaw graph and the chord is not single-undertone.

Theorem 1. *If C is a single-undertone chord, then $\mathfrak{G}(C)$ is a lower-jigsaw graph.*

Proof. Let us consider two adjacent secondary nodes $\mathrm{lcm}(a_i, \ldots, a_{i+k})$ and $\mathrm{lcm}(a_{i+1}, \ldots, a_{i+k+1})$ in the kth row of the half bottom graph, for $1 \le i \le n-1$ and $0 \le k \le n-i-1$. The node above in the middle of them is $\mathrm{lcm}(a_{i+1}, \ldots, a_{i+k})$ and the node below in the middle of them is $\mathrm{lcm}(a_i, \ldots, a_{i+k+1})$. We have to prove that $\mathrm{lcm}(a_{i+1}, \ldots, a_{i+k}) = \gcd(\mathrm{lcm}(a_i, \ldots, a_{i+k}), \mathrm{lcm}(a_{i+1}, \ldots, a_{i+k+1}))$.

By writing $A = \mathrm{lcm}(a_{i+1}, \ldots, a_{i+k})$, the previous expression is equivalent to $\mathrm{lcm}(A) = \gcd(\mathrm{lcm}(a_i, A), \mathrm{lcm}(A, a_{i+k+1}))$. Since $\gcd(\mathrm{lcm}(a_i, A), \mathrm{lcm}(A, a_{i+k+1})) = \mathrm{lcm}(A, \gcd(a_i, a_{i+k+1}))$. Then, since $\gcd(a_i, a_{i+k+1}) = 1$, the theorem is proven. □

Corollary 4. *If C is single-undertone chord, then C^{-1} is a single-overtone chord and $\mathfrak{G}(C^{-1})$ is an upper-jigsaw graph.*

For instance, $C = (3{:}4{:}5)$ is a single-undertone chord and $\mathfrak{G}(C)$ is a lower-jigsaw graph. $C^{-1} = (12{:}15{:}20)$ is single-overtone and $\mathfrak{G}(C^{-1})$ is upper-jigsaw.

3.6. Full-Jigsaw Graph

Definition 5. *If $\mathfrak{G}(C)$ is both lower-jigsaw and upper-jigsaw graph, then it is a full-jigsaw graph.*

Then, for all possible three consecutive nodes (a, b, c) in one row, both Equation (6) in the row above and Equation (7) in the row below are fulfilled. All 2-subgraphs involving consecutive nodes in any row are self-symmetric. For example, $\mathfrak{G}(15{:}20{:}28{:}56)$ and its inverse $\mathfrak{G}(15{:}30{:}42{:}56)$ are full-jigsaw graphs. For the chord $(1, \frac{5}{4}, \frac{25}{16}) \sim (16{:}20{:}25)$, $\mathfrak{G}(16{:}20{:}25)$ is a full-jigsaw graph and, in addition, it is self-symmetric.

Figure 7 provides three examples of full-jigsaw graphs. The pair on the left correspond to two mutually inverse chords. The graph on the right is self-symmetric, i.e., by inverting the chord, after expressing it in lowest terms, the chord remains invariant.

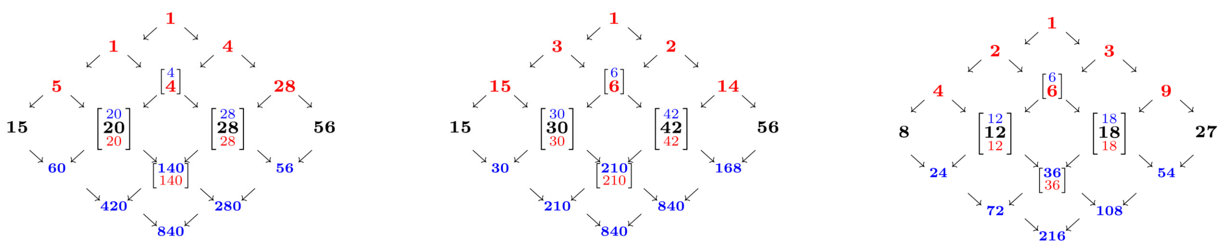

Figure 7. Undertones and overtones for three full-jigsaw tonal graphs. The lcm of each pair (above, blue) and the gcd of each pair (below, red) are indicated. The pair on the left are for two mutually inverse chords. The one on the right is for a harmonically symmetric chord.

4. Characteristic Parameters

4.1. Parameters Describing $\mathfrak{G}(C)$

The graph $\mathfrak{G}(C)$ for a chord $C = (\nu_1, \ldots, \nu_n)$ can be described from a few general parameters. Each one has two versions: one in the same frequency space, likewise the tonal graph, and another one in the interval ($\log_2$) space. The parameters referring to the interval space will be denoted with a subindex H.

The first parameters have already been introduced: the harmonic quotient, whose logarithm is the harmonic distance,

$$Q(C) = \frac{\operatorname{lcm}(C)}{\gcd(C)}; \quad \Delta_H(C) = \log_2 Q(C)$$

In terms of the periods, $Q(C) = \frac{\operatorname{lcm}(T_1,\ldots,T_n)}{\gcd(T_1,\ldots,T_n)} = \frac{T_{\mathrm{LCA}}}{T_{\mathrm{LCH}}}$.

In the upper half graph, the sympathetic factors are $p_i = \frac{\nu_i}{\gcd(C)}$. The geometric mean of the sympathetic factors is the harmonic radius [18], accounting for the average degree of antiresonance. Expressed either in frequencies or in lowest terms it is

$$R(C) = \left(\prod_{1=1}^{n} \frac{\nu_i}{\gcd(C)}\right)^{\frac{1}{n}} = \left(\prod_{1=1}^{n} p_i\right)^{\frac{1}{n}} \tag{8}$$

As a function of the periods, $T_i = \frac{1}{\nu_i}$, $T_{\mathrm{LCA}} = \frac{1}{\gcd(\nu_1,\ldots,\nu_n)} = \operatorname{lcm}(T_1, \ldots, T_n)$, the harmonic radius also provides the average of the periodicities, $R(C) = \left(\prod_{i=1}^{n} \frac{T_{\mathrm{LCA}}}{T_{\nu_i}}\right)^{\frac{1}{n}}$.

Its inverse, the harmonic sympathy,

$$S(C) = R^{-1}(C) = \left(\prod_{1=1}^{n} \frac{\gcd(C)}{\nu_i}\right)^{\frac{1}{n}} = \left(\prod_{1=1}^{n} \frac{1}{p_i}\right)^{\frac{1}{n}} \tag{9}$$

measures the average degree of resonance. Furthermore, according to [2], it also estimates the degree of coincidence among the harmonic series involved in the chord between the LCA and the LCH.

In the interval space, each sympathetic factor is the relative harmonic length $\delta_H(\nu_i|C) = \log_2 \frac{\nu_i}{\gcd(C)}$, whose arithmetic mean is the harmonic length

$$\delta_H(C) = \frac{1}{n}\sum_{i=1}^{n} \delta_H(\nu_i|C) = \frac{1}{n}\sum_{i=1}^{n} \log_2 p_i \,; \quad \delta_H(C) = \log_2 R(C)$$

The sum of the relative harmonic lengths is referred to as the cumulative harmonic length.

Now, we define the counterpart parameters involving the factors $q_i = \frac{\operatorname{lcm}(C)}{\nu_i}$ in the lower half graph. The harmonic co-radius is

$$R'(C) = \left(\prod_{1=1}^{n} \frac{\operatorname{lcm}(C)}{\nu_i}\right)^{\frac{1}{n}} = \left(\prod_{1=1}^{n} q_i\right)^{\frac{1}{n}} \tag{10}$$

In the interval space, the factors are the relative harmonic co-lengths $\delta'_H(\nu_i|C) = \log_2 \frac{\operatorname{lcm}(C)}{\nu_i}$, whose arithmetic mean is the harmonic co-length

$$\delta'_H(C) = \frac{1}{n}\sum_{i=1}^{n} \delta'_H(\nu_i|C) = \frac{1}{n}\sum_{i=1}^{n} \log_2 q_i \,; \quad \delta'_H(C) = \log_2 R'(C)$$

Then, the following relationships are hold,

$$Q(C) = R(C)\,R'(C); \quad \Delta_H(C) = \delta_H(C) + \delta'_H(C) \tag{11}$$

Similarly, by denoting $P(C) = \prod_{1=1}^{n} \nu_i$, we get

$$\frac{\mathrm{lcm}(C)}{P(C)^{\frac{1}{n}}}\,\frac{\mathrm{gcd}(C)}{P(C)^{\frac{1}{n}}} = \frac{R'(C)}{R(C)} \tag{12}$$

It is worth noticing that the parameters Q, R, R', S (therefore, Δ_H, δ_H, δ'_H) are invariant if the tones of the chord C are multiplied by a same factor $x > 0$. Then, by taking $x = \frac{1}{\mathrm{gcd}(C)}$, it is simpler to work with chords expressed in lowest terms, i.e., with $\mathrm{gcd}(C) = 1$ and $Q(C) = \mathrm{lcm}(C)$.

4.2. Parameters Describing $\mathfrak{G}(C^{-1})$

According to Equation (4),

$$Q(C) = Q(C^{-1}); \quad \Delta_H(C) = \Delta_H(C^{-1}) \tag{13}$$

Due to the reflection symmetry of $\mathfrak{G}(C)$ and $\mathfrak{G}(C^{-1})$, it is straightforward to prove that

$$R(C^{-1}) = R'(C); \quad \delta_H(C^{-1}) = \delta'_H(C)$$

Definition 6. *A chord C is harmonically symmetric when $R(C) = R'(C)$, $\delta_H(C) = \delta'_H(C)$.*

Lemma 3. *The harmonic quotient is bounded according to*

$$R(C)^{\frac{n}{n-1}} \le Q(C) \le R(C)^n$$

That is, $\left(\frac{P(C)}{\mathrm{gcd}(C)^n}\right)^{\frac{1}{n-1}} \le \frac{\mathrm{lcm}(C)}{\mathrm{gcd}(C)} \le \frac{P(C)}{\mathrm{gcd}(C)^n}$.

Proof. The result is easily proven if we express the chord C in lowest terms. Then $Q(C) = \mathrm{lcm}(C) \le P(C) = R(C)^n$. Consequently, for the graph of the chord C^{-1}, it is $Q(C^{-1}) \le R(C^{-1})^n$. Since $Q(C^{-1}) = Q(C)$ and $R(C^{-1}) = R'(C) = \frac{Q(C)}{R(C)}$, we get $Q(C) \le \left(\frac{Q(C)}{R(C)}\right)^n$, that is $R(C)^n \le Q(C)^{n-1}$, which proves the lemma. □

Corollary 5. *For a chord $C = (\nu_1, \ldots, \nu_n)$,*

If $\mathrm{gcd}(\nu_i, \nu_j) = \mathrm{gcd}(C)$ for all $i \ne j$ (single-undertone chord), then $Q(C) = R(C)^n$.

If $\mathrm{lcm}(\nu_i, \nu_j) = \mathrm{lcm}(C)$ for all $i \ne j$ (single-overtone chord), then $R(C)^{\frac{n}{n-1}} = Q(C)$.

Proof. By working in lowest terms, the first statement is obvious. For the second one, it is enough to consider the inverted chord $C^{-1} = (\mu_1, \ldots, \mu_n)$ with $\mu_i = \frac{\mathrm{lcm}(C)}{\nu_i}$, for which $\mathrm{gcd}(\mu_i, \mu_j) = 1$. Then, the graph $\mathfrak{G}(C^{-1})$ is under the hypothesis of the first statement. According to the proof of the lemma, the second statement is fulfilled. □

By taking logarithms, we get the equivalent relationship in the interval space, The harmonic distance is bounded according to

$$\tfrac{n}{n-1}\,\delta_H(C) \le \Delta_H(C) \le n\,\delta_H(C) \tag{14}$$

4.3. Chord Spheroid

As previously explained, in the half upper part of the graph $\mathfrak{G}(C)$, the harmonic radius $R(C)$ accounts for the degree of antiresonance associated with the tones composing the chord. However, the common undertones and overtones do also contribute to the total chord's harmonicity and, in particular, to the perception of consonance.

Two recent approaches provide consistent predictions of consonance perception for chords. On the one hand, Stolzenburg's logarithmic periodicity [11] evaluates consonance in terms of perceived periodicity [17,22]. On the other hand, the perceived consonance of chords can be approximately predicted by their relative similarity to voiced speech sounds estimated from a metric called harmonic similarity [13]. The latter authors provide a considerable amount of perceptual data for chords with two, three, and four tones. By using their data, in [2] it has been proven that perception of consonance is inversely correlated with the sum $\delta_H(C) + \Delta_H(C)$, by providing correlations that improve those of harmonic similarity and logarithmic periodicity. In the frequency space, this is measured by the harmonic volume,

$$V(C) = R(C)\,Q(C) \tag{15}$$

The harmonic volume of a chord can be symbolized as an spheroid [2], allowing to grasp at a glance all the parameters described above, i.e., the harmonic radius, co-radius, and the harmonic quotient. Its normalized volume (referred to the volume of a unit sphere $V_0 = \frac{4\pi}{3}$) can be expressed as $V(C) = R(C)^2 R'(C)$. Two of the principal axes, x, y, are associated with the harmonic radius $a = R(C)$ and co-radius $b = R'(C)$. The principal section $z = 0$, involving the radius and the co-radius, has normalized area $Q(C) = R(C)\,R'(C)$ (referred to the area of a unit circle $Q_0 = \pi$), which accounts for the harmonic quotient and incorporates the contribution of the common undertones and overtones to the harmonic volume. The other axis, also associated with the harmonic radius $a = R(C)$, accounts exclusively for the average antiresonance of the tones composing the chord. Depending on the relative values of $R(C)$ and $R'(C)$, the spheroid is prolate ($a < b$), oblate ($a > b$), or an sphere ($a = b$), informing us about the harmonic symmetry of the chord. Figure 8 displays the spheroids of two to mutually inverse chords with the same harmonic quotient (principal section $z = 0$) but different volume. Hence, according to Equation (12), the product $\gcd(C)\,\mathrm{lcm}(C)$ –relative to the quantity $P^{\frac{2}{n}}(C)$– determines the flatness of the spheroid.

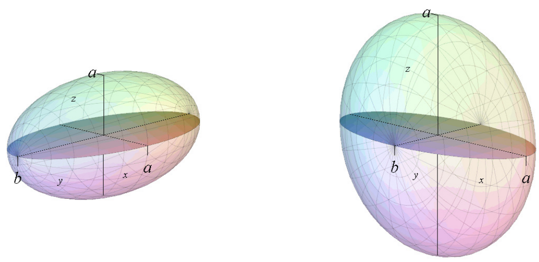

Figure 8. Prolate and oblate spheroids of mutually inverse chords with the same principal section $z = 0$.

Partch [23] introduced the terms otonality and utonality to describe chords formed by tones that are the overtones or undertones of a given fixed tone. An otonality is a set of frequency ratios expressed by fractions with a common denominator (called numerary nexus) and consecutive numerators, $C = (\frac{p_1}{q}, \ldots, \frac{p_n}{q})$, with $p_{i+1} = p_1 + i$, $\gcd(p_1, \ldots, p_n) = 1$, i.e., a sequence within the harmonic series. The common fundamental is

$$\mathrm{LCA}(C) = \frac{\gcd(p_1,\ldots,p_n)}{\mathrm{lcm}(q,\ldots,q)} = \frac{1}{q}$$

and the maximum consonance is given by the lowest sympathetic factor, $\frac{p_1}{q}:\frac{1}{q} = p_1$.

The associated utonality is the set of inverse frequency ratios expressed by fractions with a common numerator and consecutive denominators, $C^{-1} = (\frac{q}{p_n}, \ldots, \frac{q}{p_1})$. In this case, the common fundamental is

$$\mathrm{LCA}(C^{-1}) = \frac{\gcd(q,\ldots,q)}{\mathrm{lcm}(p_1,\ldots,p_n)} = \frac{q}{\mathrm{lcm}(p_1,\ldots,p_n)}$$

and the maximum consonance corresponds to the lowest sympathetic factor,

$$\frac{q}{p_n} : \frac{q}{\mathrm{lcm}(p_1,\ldots,p_n)} = \frac{\mathrm{lcm}(p_1,\ldots,p_n)}{p_n}$$

Then, since two consecutive integers are always coprime, it is hold

$$\frac{\mathrm{lcm}(p_1,\ldots,p_n)}{p_n} \geq \frac{\mathrm{lcm}(p_{n-1},p_n)}{p_n} = \frac{p_{n-1}p_n}{p_n} = p_{n-1} > p_1$$

Therefore, the otonality C is closer to its common fundamental than the utonality C^{-1}, so that, for chords, the otonality, in particular the bass, is more reinforced by the whole wave than the utonality, by enhancing the presence of the virtual common fundamental. According to Partch, these concepts generalize the major and minor tonalities in conventional music theory.

However, most chords are neither otonalities nor utonalities, hence these concepts were extended to just intonation chords as otonal and utonal [24], not requiring numerators or denominators to be consecutive. Since a prolate spheroid corresponds to a chord that is harmonically closer to its common fundamental than a chord with an oblate spheroid, the flatness of the chord spheroid unambiguously generalizes the concepts of otonal and utonal.

Let us see some examples. We first compare the minor triad $(10, 12, 15)$, the minor chord with the bass doubled $(10, 12, 15, 20)$, and the first inversion of the minor triad $(12, 15, 20)$. These chords sound quite similar, although with some differences that are difficult to describe. Figure 9 (top) shows the respective chord spheroids and the corresponding values for the logarithms of R, R', Q, V. While, Q is maintained, R and V increase, the latter indicating a decreasing degree of the overall sympathetic vibration. The same comparison is made for the major triad $(4, 5, 6)$, the major chord with the bass doubled $(4, 5, 6, 8)$, and the first inversion of the major triad $(5, 6, 8)$. These chords sound a little more differentiated than the minors ones. In this case Q is not always maintained and R as well as V increase. There is significant difference in the key parameters between the major triad and its first inversion. This is also true if we compare the minor triad $(10, 12, 15)$ with the first inversion of the major triad $(5, 6, 8)$, despite having a similar harmonic volume. We can know what makes these chords different, regardless how listeners may interpret such differences in their perception.

Along the current approach we are always evaluating parameters opposite to consonance, resonance, and sympathy. Afterwards, by inverting, we interpret them in terms of those. Then, the estimation of the chord's total consonance is provided by the inverse of the harmonic volume, $V(C)^{-1}$. According to Equation (9), it can be expressed as separable into the two factors $S(C)\,Q(C)^{-1}$, the first one is the harmonic sympathy of the tones composing the chord, which predicts their average periodicity and marks the general trend. The other one is inversely proportional to the harmonic quotient, which modifies the general trend according to the ratio between the periods T_{LCH} and T_{LCA}.

Since those quantities are used to compare chords, if necessary they can be scaled. We shall see that in the interval space it is convenient to define the parameter equivalent to the harmonic volume as depending on a constant k. We define the harmonic dispersion as

$$D_H(C) = k\,(\delta_H(C) + \Delta_H(C)) \tag{16}$$

In Section 6 we will be able to see the usefulness of this constant, by developing [2]. Finally, in the frequency space, as opposite to the harmonic dispersion, the harmonic likeness

$$L(C) = 2^{-D_H(C)}$$

provides the prediction of how consonant a chord is. By assuming $k = 1$, it can be expressed as $L(C) = V(C)^{-1} = S(C)\,Q(C)^{-1}$. According to [2], the perception of consonance is proportional to the degree of coincidence among the chord's components within the interval between the common fundamental and the first common overtone, and inversely proportional to the harmonic quotient.

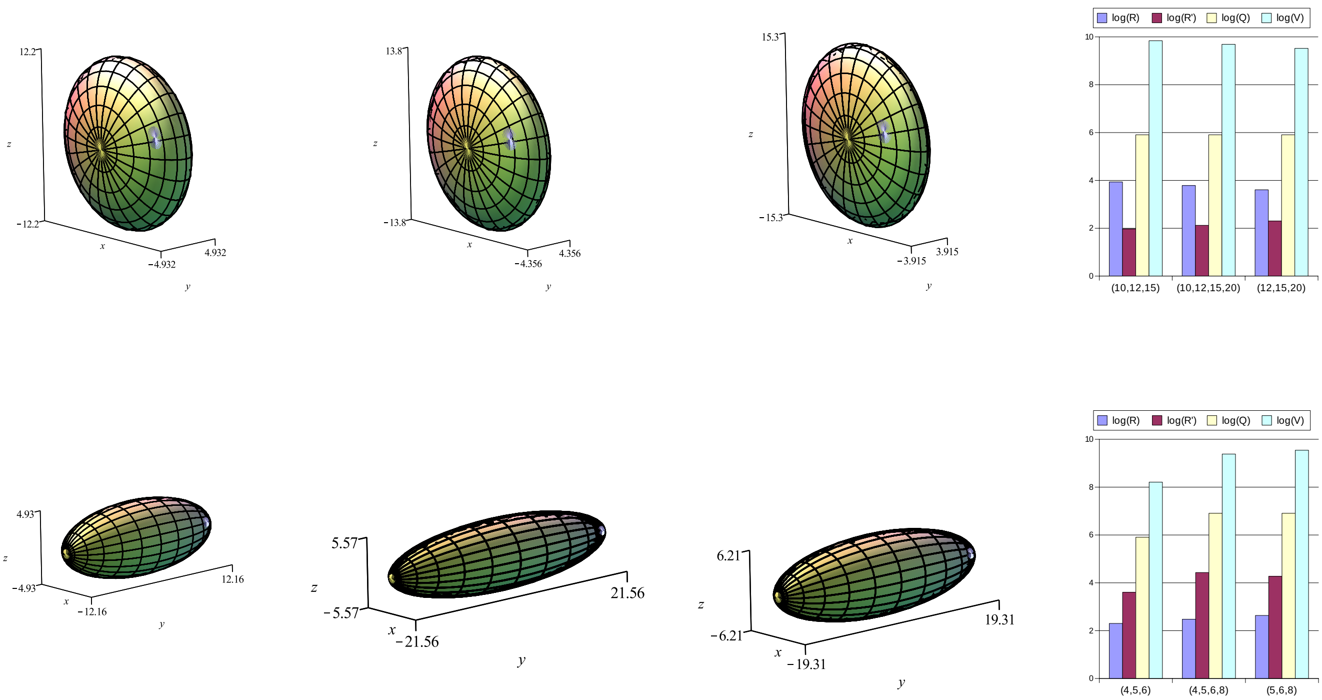

Figure 9. Chord spheroids for (**top**) $(10, 12, 15)$, $(10, 12, 15, 20)$, $(12, 15, 20)$, and (**bottom**) $(4, 5, 6)$, $(4, 5, 6, 8)$, $(5, 6, 8)$, and parameter values.

4.4. What Is Not Explained Here

As Ben Johnston [25] (p. 53) explains, "Western European music is characterized by counterpoint and harmony and by its constant stylistic evolution. Counterpoint refers to the simultaneous unfolding of several melodic lines. The combinations of tones that result are harmonies, classified as consonant (bearing harmonic repose) and dissonant (bearing harmonic tension) and arranged since the seventeenth century in hierarchical families called tonalities (relationships between tones)".

Melody and harmony are usually referred to as the horizontal and vertical dimensions of the music staff. From a listener viewpoint, in terms of time this could be interpreted as harmony being the present (what we actually hear), while melody is the past and future.

"The harmonic structure of tones as generated by musical instruments is not a product of careful manufacturing. Instead, it arises from resonances where the delays due to travel times in tubes or on strings are compensated by the periods of the harmonics (depending on boundary conditions, integer multiples of the delays or the periods may have the same effect). Similarly, the integer relationships of horizontal harmony arise—in the spirit of Pythagoras—as inevitable mathematical consequences of the neuronal correlation analysis. Our coincidence neuron would prefer certain harmonically related tones even if such relationships were not also observable in our acoustic environment" [26] (p. 180).

Nonetheless, in music theory, counterpoint has its rules based on aesthetics and stylistic practice, which have been in continuous change along centuries. In these rules, for tonal music the leading tone plays an important role. For a major diatonic scale (any translation of the scale with seven tones CDEFGAB) the leading tone is the seventh degree, namely the note just one semitone below the fundamental (the root of a chord). For a minor diatonic scale (any translation of the scale with seven tones ABCDEFG), the seventh degree is called subtonic and is one tone below the fundamental. Thus, there is a close relationship between voice leading, chord geometry, and chord transformations [27–30].

Consequently, the chord transformations are driven by the melodic voices (either in favor or against the counterpoint rules, depending on the composer's intention). A chord which bass is strongly reinforced by the common fundamental (such as an otonal chord) is sometimes referred to as stable or resolved since it tends to retain the melody. That is, it is necessary a major change in the melody lines to get out of its tonality. Instead, a chord where the reinforcement on the bass is weak or it falls on another tone (such as an utonal chord) is sometimes referred to as unstable, tense, or unresolved since it offers less resistance to melodic evolution. In other words, it easily allows to pass the leading role from one tonality to another. A physics analogy could be used: the melody would be kinetic energy and each chord would represent a potential well associated with tonality from which it would be more or less easy to get out or stay in.

5. Self-Symmetric Graph

We denote the factors in the upper half graph, the sympathetic factors, as $p_i = \frac{\nu_i}{\gcd(C)}$, and the factors in the lower half graph as $q_i = \frac{\mathrm{lcm}(C)}{\nu_i}$.

Definition 7. *A tonal graph $\mathfrak{G}(C)$ is self-symmetric if it is isomorphic to $\mathfrak{G}(C^{-1})$.*

This means that they have the same factors, although the nodes can be proportional. If the factors in the lower half graph are redefined as $p'_i = q_{n-i+1}$, then they satisfy $p_1 = p'_1, \ldots, p_n = p'_n$ (Figure 10). Hence, according to Equations (8) and (10), $R(C) = R'(C)$. Furthermore, owing to Equation (11), it is straightforward to see that

Lemma 4. *If $\mathfrak{G}(C)$ is self-symmetric, then $R(C) = R'(C) = Q(C)^{\frac{1}{2}}$. This also applies to $\mathfrak{G}(C^{-1})$.*

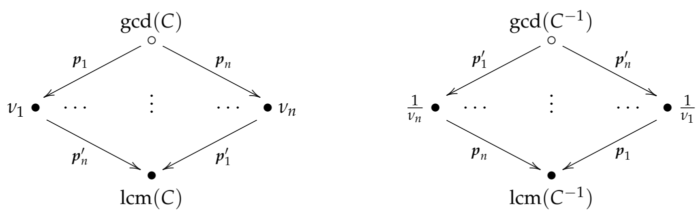

Figure 10. Tonal graphs $\mathfrak{G}(C)$ and $\mathfrak{G}(C^{-1})$ for chords $C = (\nu_1, \ldots, \nu_n)$ and $C^{-1} = (\frac{1}{\nu_n}, \ldots, \frac{1}{\nu_1})$.

Therefore, a chord C is harmonically symmetric if and only if $\mathfrak{G}(C)$ is self-symmetric.

Notice that the harmonic symmetry is a property concerning the intervals between the ordered tones. If by inverting the tones the intervals are maintained, then the chord is harmonically symmetric.

Corollary 6. *If C is harmonically symmetric, then C is proportional to C^{-1}. In their simplest form, they are the same chord.*

5.1. Case n = 3

Without loss of generality, we may express the chord $C = (p_1, p_2, p_3)$ in lowest terms, with $\gcd(p_1, p_2, p_3) = 1$. The graph is self-symmetric when the factors (according to Figure 10) satisfy $p_1 = p_1'$, $p_2 = p_2'$, and $p_3 = p_3'$, so that $\mathrm{lcm}(C) = p_2^2 = p_1 p_3$.

Theorem 2. *Consider $p_1 < p_2 < p_3$ and $\gcd(p_1, p_2, p_3) = 1$.*
$\mathfrak{G}(p_1, p_2, p_3)$ is self-symmetric $\iff$ $\gcd(p_1, p_3) = 1$ and $\mathrm{lcm}(p_1, p_2, p_3) = \mathrm{lcm}(p_1, p_3)$.

Proof. Let us assume $\gcd(p_1, p_3) = g \geq 1$ and write $p_1 = gk$, $k \geq 1$ and $p_3 = gl$, $l \geq 1$, with $\gcd(k, l) = 1$. Since $p_1 p_3 = p_2^2$, then the product $p_1 p_3 = g^2 kl$ must be a perfect square, but since k and l have not any common factor, they must be perfect squares, $k = u^2$, $l = v^2$. Thus, $p_1 = gu^2$, $p_2 = guv$, $p_3 = gv^2$, with $\gcd(u, v) = 1$. Since $\gcd(p_1, p_2, p_3) = \gcd(gu^2, guv, gv^2) = g$, then it must be $g = 1$. Therefore, $\gcd(p_1, p_3) = 1$, which implies $\mathrm{lcm}(p_1, p_3) = p_1 p_3 = \mathrm{lcm}(p_1, p_2, p_3)$. $\square$

The following equalities are satisfied. On the one hand, $\gcd(p_1, p_2) = u$, $\gcd(p_2, p_3) = v$, and $\mathrm{lcm}(uv) = p_2$. On the other hand, $\mathrm{lcm}(p_1, p_2) = u^2 v$, $\mathrm{lcm}(p_2, p_3) = uv^2$, and $\gcd(u^2 v, uv^2) = p_2$. Then, owing to Equations (6) and (7),

Corollary 7. *If $\mathfrak{G}(p_1, p_2, p_3)$ is self-symmetric, then it is a full-jigsaw graph.*

There are several equivalent expressions for writing a harmonically symmetric triad $C = (p_1, p_2, p_3)$, which allow us identify easily such a property. Since $p_1 p_3 = p_2^2$, then $\frac{p_1}{p_2} = \frac{p_2}{p_3}$ and $\frac{p_3}{p_1} = \frac{p_2^2}{p_1^2}$. By dividing C by p_2, an equivalent form is $(\frac{p_1}{p_2}, 1, \frac{p_3}{p_2}) = (\frac{p_1}{p_2}, 1, \frac{p_2}{p_1})$. However, these fractions need to be simplified such as $C' = (\frac{u}{v}, 1, \frac{v}{u})$.

Another equivalent form is obtained by dividing C by p_1. Then we get $(1, \frac{p_2}{p_1}, \frac{p_3}{p_1}) = (1, \frac{p_2}{p_1}, \frac{p_2^2}{p_1^2})$. Once simplified, this expression can be written as $(1, \frac{v}{u}, \frac{v^2}{u^2})$. Still by multiplying the above expression by u^2, we get the equivalent expression (u^2, uv, v^2).

5.2. General Case $n \geq 3$, Odd

For an odd number $n \geq 3$, a harmonically symmetric chord will have the following form. We write, in lowest terms, $C = (p_1, \ldots, p_n) = (a_m, \ldots, a_1, a_0, b_1, \ldots, b_m)$, with $n = 2m + 1$, so that $\gcd(a_m, \ldots, a_1, a_0, b_1, \ldots, b_m) = 1$. As in the previous case, $\mathrm{lcm}(C) = a_0^2 = a_1 b_1 = a_2 b_2 = \cdots = a_m b_m$. Then, $b_i = \frac{a_0^2}{a_i}$, and

$$C = \left(a_m, \ldots, a_1, a_0, \frac{a_0^2}{a_1}, \ldots, \frac{a_0^2}{a_m} \right)$$

By dividing C by a_0, the chord can be written as inverse ratios around the central 1, such as $C' = \left(\frac{a_m}{a_0}, \ldots, \frac{a_1}{a_0}, 1, \frac{a_0}{a_1}, \ldots, \frac{a_0}{a_m} \right)$. Notice that these fractions need to be simplified and $\gcd(C') = \frac{1}{a_0}$, $\mathrm{lcm}(C') = a_0$.

As for $n = 3$, for $i = 1, \ldots, m$ if $\gcd(a_i, b_i) = g_i$ then $\gcd(a_i, a_0, b_i) = g_i$. In general, $g_i \neq 1$, although $\gcd(g_1, \ldots, g_m) = 1$. Separately, for three centered principal nodes, each tonal graph $\mathfrak{G}(a_i, a_0, b_i)$ (after being expressed in their simplest form) is a full-jigsaw graph.

In particular, it is the central one. But for any three consecutive, non-centered principal nodes, $\mathfrak{G}(p_{i-1}, p_i, p_{i+1})$ is not a full-jigsaw graph.

5.3. General Case Case $n \geq 2$, Even

For an even number $n \geq 2$, a harmonically symmetric chord will have the following form. We write $C = (p_1, \ldots, p_n) = (a_m, \ldots, a_1, b_1, \ldots, b_m)$, with $n = 2m$. We assume it is expressed in lowest terms, so that $\gcd(a_m, \ldots, a_1, b_1, \ldots, b_m) = 1$. These nodes satisfy, $\lambda \equiv \mathrm{lcm}(C) = a_1 b_1 = a_2 b_2 = \cdots = a_m b_m$. Then, $b_i = \frac{\lambda}{a_i}$, and

$$C = \left(a_m, \ldots, a_1, \tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right)$$

In this case, λ is not necessarily a perfect square, so that the chord maybe could not be written as inverse ratios symmetrically around the center.

Theorem 3. *Let us consider a set of positive integers $A = (a_m, \ldots, a_1)$, with $a_m < \ldots < a_1$ and $l = \mathrm{lcm}(A), g = \gcd(A)$. For any positive integer α such that $\gcd(g, \alpha) = 1$, if $\lambda = \alpha l$, then it is fulfilled*

$$\mathrm{lcm}\left(a_m, \ldots, a_1, \tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right) = \lambda; \qquad \gcd\left(a_m, \ldots, a_1, \tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right) = 1$$

Proof. On the one hand,

$$\begin{aligned}\mathrm{lcm}\left(a_m, \ldots, a_1, \tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right) &= \mathrm{lcm}\left(\mathrm{lcm}(a_m, \ldots, a_1), \mathrm{lcm}(\tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m})\right) = \\ = \mathrm{lcm}\left(l, \tfrac{\lambda}{\gcd(a_1, \ldots, a_m)}\right) = \mathrm{lcm}\left(l, \tfrac{\lambda}{g}\right) &= \mathrm{lcm}\left(l, \tfrac{l\alpha}{g}\right) = l\ \mathrm{lcm}(1, \tfrac{\alpha}{g}) = l\ \tfrac{\mathrm{lcm}(1,\alpha)}{\gcd(1,g)} = l\alpha = \lambda\end{aligned}$$

On the other hand,

$$\begin{aligned}\gcd\left(a_m, \ldots, a_1, \tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right) &= \gcd\left(\gcd(a_m, \ldots, a_1), \gcd\left(\tfrac{\lambda}{a_1}, \ldots, \tfrac{\lambda}{a_m}\right)\right) = \\ = \gcd\left(g, \tfrac{\lambda}{\mathrm{lcm}(a_1, \ldots, a_m)}\right) &= \gcd\left(g, \tfrac{\lambda}{l}\right) = \gcd(g, \alpha) = 1\end{aligned}$$

Notice that the components of $\left(a_m, \ldots, a_1, \frac{\lambda}{a_1}, \ldots, \frac{\lambda}{a_m}\right)$ have no common factors, owing to the condition $\gcd(g, \alpha) = 1$. □

Corollary 8. *If $\alpha > \dfrac{a_1^2}{l}$, then $C = \left(a_m, \ldots, a_1, \frac{\lambda}{a_1}, \ldots, \frac{\lambda}{a_m}\right)$ satisfies the condition $a_m < \ldots < a_1 < \frac{\lambda}{a_1} < \ldots < \frac{\lambda}{a_m}$, allowing to express the chord without duplicity, with $\mathrm{lcm}(C) = \lambda$.*

In general, harmonically symmetric chords for $n \geq 4$ do not generate full-jigsaw graphs. However, for a ratio r, the symmetric chord $C = (1, r, r^2, \ldots, r^{n-1})$, regardless n is odd or even, generates a full-jigsaw graph. For instance, for $\left(1, \frac{3}{2}, (\frac{3}{2})^2, (\frac{3}{2})^3\right)$, which is equivalent to the simplest form (8:12:18:27).

6. Harmonic Dispersion Referred to a Symmetric Chord

In Equation (15) we have defined the harmonic volume of a chord C. For the inverse chord C^{-1}, since $Q(C) = Q(C^{-1})$, then $V(C^{-1}) = R(C^{-1})\,Q(C)$. Hence, if $R(C) < R(C^{-1})$ then $V(C) < V(C^{-1})$. In such a case, the spheroid associated with $V(C)$ is prolate and the one of $V(C^{-1})$ is oblate. This means that the chord C is more consonant than C^{-1}.

If C is symmetric, i.e., $C = C^{-1}$, then $R(C) = R'(C)$ and, according to Equation (11), $Q(C) = R(C)^2$. The chord spheroid is a sphere of volume $V(C) = R(C)^3$. Therefore, for chords with a similar harmonic radius, the symmetric chord is less consonant than those with a prolate spheroid, but more consonant than those with an oblate spheroid.

In order to evaluate whether a chord C is more or less consonant than a virtual equivalent symmetric chord C^s with the same harmonic radius $R(C^s) = R(C)$, we may refer the chord C, with harmonic volume $V(C) = R(C)^2 R'(C)$, to the symmetric chord with volume $V(C^s) = R(C)^3$. Its deviation will inform us about the difference of consonance due to asymmetry.

Equivalently, such a comparison can be made in the interval space in terms of the harmonic dispersion,

$$D_H(C) = k(2\,\delta_H(C) + \delta'_H(C)); \quad D_H(C^s) = k\,3\,\delta_H(C)$$

Therefore, if we assume $k = \frac{1}{3}$, the comparison consists of referring $D_H(C)$ to $D_H(C^s) = \delta_H(C)$. That is, the mean harmonic dispersion is given by the symmetric chord with $D_H(C^s) = \delta_H(C)$, in such a way that the harmonic distance $\Delta_H(C)$ will increase or decrease the harmonic dispersion $D_H(C)$ relatively to that of the symmetric chord. In other words, a symmetric chord with harmonic length $\delta_H(C^s) = \delta_H(C)$ provides a mean harmonic dispersion so that if $\delta'_H(C) > \delta_H(C)$, the chord C has greater harmonic dispersion than the symmetric chord, and if $\delta'_H(C) < \delta_H(C)$, the chord C has less harmonic dispersion than the symmetric chord. This is shown in Figure 11 (top panel) for the most common triads (including the major, minor, suspended, and diminished triads –with their voices–, and the augmented triad).

We can also investigate what features of the tonal graph are related to such deviation of the harmonic distance with regard to that of a symmetric chord. To this purpose, it is necessary to recall Equation (14). Owing to Corollary 5, the lower bound takes place for a single-overtone chord and, by Corollary 4, also according to an upper-jigsaw graph. The upper bound corresponds to a single-undertone chord and, by Theorem 1, also according to a lower-jigsaw graph.

Depending on the number of tones, these bounds are

$$\cdots \leq \tfrac{3}{2}\,\delta_H(C) \leq \tfrac{2}{1}\,\delta_H(C) \leq \Delta_H(C) \leq 2\,\delta_H(C) \leq 3\,\delta_H(C) \leq \cdots$$

The symmetric chord always satisfies $\Delta_H(C) = 2\,\delta_H(C)$, as corresponds to any 2-tone symmetric graph. Hence, the weighted average of both extremes,

$$\alpha\,\tfrac{n}{n-1}\,\delta_H(C) + (1-\alpha)\,n\,\delta_H(C) = 2\,\delta_H(C); \quad \alpha = \tfrac{n-1}{n}$$

determines the harmonic distance of the symmetric chord. Figure 11 (bottom panel) depicts the harmonic distance oscillating between both extremes for the most common chords.

Similarly, for the most common chords, Figure 12 compares the harmonic likeness to the harmonic sympathy. The latter is the harmonic likeness of the equivalent virtual symmetric chord. It is clearly seen that, for low harmonic radii, the harmonic quotient introduces an important correction to the harmonic sympathy, while for high harmonic radii the harmonic sympathy marks the asymptotic behavior.

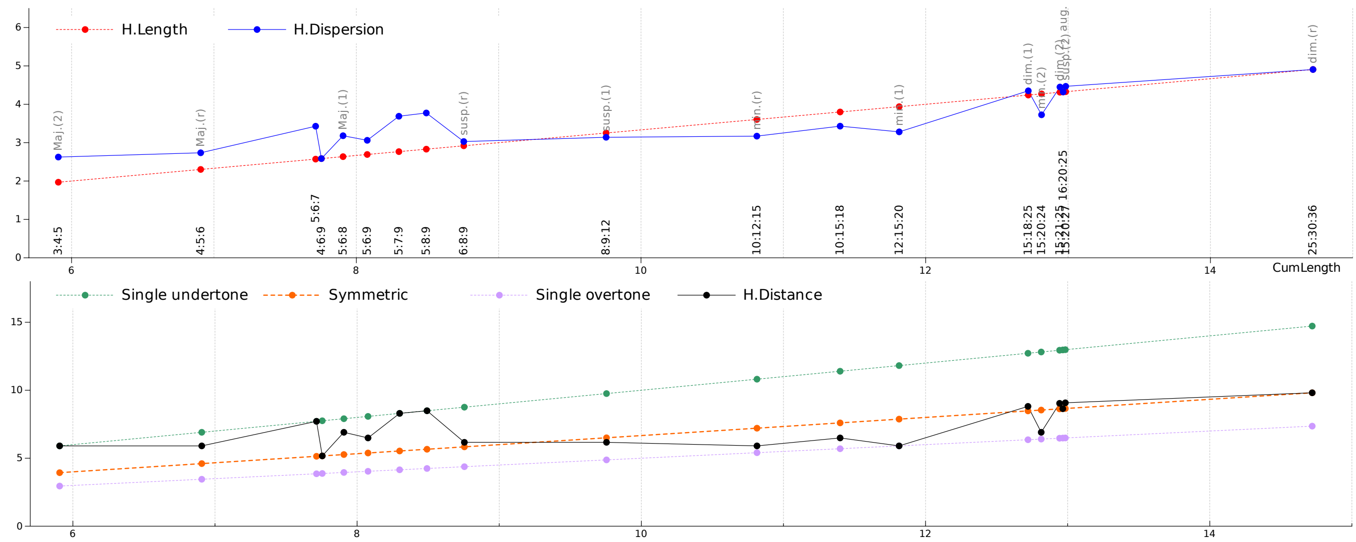

Figure 11. (**Top**) Harmonic dispersion of common chords (blue) and symmetric chords (red) with the same harmonic length. (**Bottom**) Harmonic distance of common chords (black) and symmetric chords (orange) varying between the lower bound of a single-overtone chord and the upper bound of a single-undertone chord.

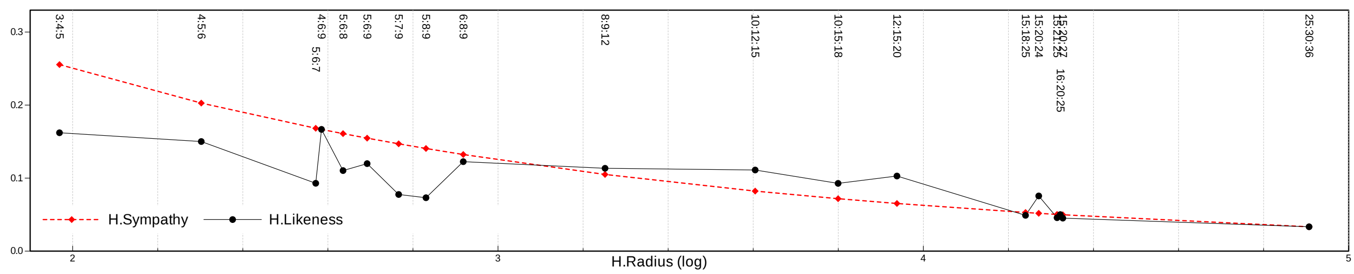

Figure 12. Harmonic likeness and harmonic sympathy of most common chords.

7. Recurrence Relationships for $\gcd(C)$ and $\mathrm{lcm}(C)$

It is important to recall that the common fundamental and the difference tones can actually be perceived in acoustical instruments tuned in just intonation [31–33]. In the treaty by [4], he describes the distortion produced by the snail (cochlea) of the inner ear, responsible for hearing combined tones, mainly of frequencies $f - f'$ and $f + f'$, and, in general, $f - n(f - f')$ and $f + n(f - f')$, $n \in \mathbb{N}$, with decreasing intensity. In the book [34] (p. 380) it is mentioned that Riccati, in 1767, foresaw the essential role of the bundles of hair cells, that he associated with the auditory nerve. There are several explanations based on nonlinear oscillator models [35–37] of how combined tones arise. In general, the undertones and difference tones can easily create so much multitonal confusion that the ear cannot cope with analyzing such atonal complexity of the compound sound [31–33]. Instead, the overtones always reinforce the harmony by saturating the harmonic spectrum over the common fundamental.

The upper half of the tonal graph involves the undertones of consecutive tones, i.e., backward common fundamentals by pairs, although a multidimensional prime graph would account for all the undertones. In many cases, these undertones match the difference tones, since given two frequencies a and b such that $a > b$, the common divisors of a and b are the same as the common divisors of $a - b$ and b. Hence, the difference tone $a - b$ shares the common fundamental of a and b, and sometimes it is also their common undertone. For instance, in Figure 6 (right panel) the common fundamental of tones 84 and 105 matches their difference tone 21, and of tones 105 and 140 matches their difference tone 35. All this complexity should be taken into account to characterize the chord harmonicity.

Here we will pay attention to the subsets of common undertones and overtones. We shall see that the periodicity of the undertones and the overtones is constrained. For a chord C we want to determine how the overall $\gcd(C)$ and $\operatorname{lcm}(C)$ depend exclusively on either the common undertones or the overtones.

It has been proven in Corollary 5 that single-undertone chords reach the maximum possible value of the harmonic quotient, whereas single-overtone chords match the minimum bound. We are going to see that along the tonal graph $\mathfrak{G}(C)$, the relationship the undertones maintain is in some way balanced with that of the overtones, so that they contribute in an equivalent way to determine the harmonic quotient.

We are going to use the simplified notation $(a,b) = \gcd(a,b)$ and $[a,b] = \operatorname{lcm}(a,b)$. In such a case the chord will be noted as $X_n = \{x_1, \dots, x_n\}$. For $n = 2$,

$$[x_1, x_2] = \frac{x_1 x_2}{(x_1, x_2)} \tag{17}$$

For $n = 3$, since $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$, applying Equation (17) together with the distributive property yields

$$[x_1, x_2, x_3] = \frac{\frac{x_1 x_2}{(x_1,x_2)}\, x_3}{([x_1, x_2], x_3)} = \frac{\frac{x_1 x_2 x_3}{(x_1,x_2)}}{[(x_1, x_3), (x_2, x_3)]}$$

Applying Equation (17) again to the denominator, we obtain

$$[x_1, x_2, x_3] = \frac{x_1 x_2 x_3}{(x_1, x_2)} \frac{(x_1, x_2, x_3)}{(x_1, x_3)(x_2, x_3)}$$

Therefore,

$$[x_1, x_2, x_3] = \frac{x_1 x_2 x_3\, (x_1, x_2, x_3)}{(x_1, x_2)(x_1, x_3)(x_2, x_3)} \tag{18}$$

Both the least common multiple and the greatest common divisor are invariant under permutations of their arguments. In this sense, the previous expressions are symmetric. More generally, for $k > 1$, products of $\binom{n}{k}$ factors naturally arise. These are defined through the following symmetric operators, two of them associated with the gcd and lcm,

$$ {}^{n}_{k}\mathcal{G} = \prod_{i_1 < \cdots < i_k \le n} (x_{i_1}, \dots, x_{i_k}) \; ; \quad {}^{n}_{k}\mathcal{L} = \prod_{i_1 < \cdots < i_k \le n} [x_{i_1}, \dots, x_{i_k}] \tag{19}$$

and, for $k > 0$, another one associated with the product of factors,

$$ {}^{n}_{k}\mathcal{P} = \prod_{i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} \tag{20}$$

Thus, ${}^{n}_{n}\mathcal{P} = {}^{n}_{1}\mathcal{P}$.

For convenience, we write

$$ {}_{n}\mathcal{P} \equiv {}^{n}_{n}\mathcal{P} \equiv \prod_{i} x_i; \quad {}_{n}\mathcal{G} \equiv {}^{n}_{n}\mathcal{G} = (x_1, \dots, x_n); \quad {}_{n}\mathcal{L} \equiv {}^{n}_{n}\mathcal{L} = [x_1, \dots, x_n]$$

With this notation, Equations (17) and (18) can be written as

$$ {}_{2}\mathcal{L} = \frac{{}^{2}_{1}\mathcal{P}}{{}^{2}_{2}\mathcal{G}} = \frac{{}_{2}\mathcal{P}}{{}_{2}\mathcal{G}}; \quad {}_{3}\mathcal{L} = \frac{{}^{3}_{1}\mathcal{P}\;{}^{3}_{3}\mathcal{G}}{{}^{3}_{2}\mathcal{G}} = \frac{{}_{3}\mathcal{P}\;{}_{3}\mathcal{G}}{{}^{3}_{2}\mathcal{G}} \tag{21}$$

Theorem 4. *For $n \geq 2$,*

$$ {}_n\mathcal{L} = {}^n_1\mathcal{P} \,/\, ({}^n_2\mathcal{G} \,/\, (\cdots / ({}^{\;\,n}_{n\text{-}1}\mathcal{G} \,/\, {}^n_n\mathcal{G}))) = {}^n_1\mathcal{P} \prod_{k=2}^{n} ({}^n_k\mathcal{G})^{(-1)^{k-1}} \tag{22} $$

That is (notice the difference between both cases, either ${}_n\mathcal{L} \propto {}_n\mathcal{G}^{-1}$ or ${}_n\mathcal{L} \propto {}_n\mathcal{G}$),

$$ (a)\ \ {}_n\mathcal{L} = \frac{{}_n\mathcal{P} \cdots {}^{\;\,n}_{n\text{-}1}\mathcal{G}}{{}^n_2\mathcal{G} \cdots {}_n\mathcal{G}},\ \textit{if } n = \dot{2}; \qquad (b)\ \ {}_n\mathcal{L} = \frac{{}_n\mathcal{P} \cdots {}_n\mathcal{G}}{{}^n_2\mathcal{G} \cdots {}^{\;\,n}_{n\text{-}1}\mathcal{G}},\ \textit{if } n = \dot{2} + 1 \tag{23} $$

Proof. We prove the result by induction for even n. Assume that the relation holds for ${}_{n\text{-}1}\mathcal{L} = [x_1, \ldots, x_{n-1}]$. Using Equation (17) and the distributive property, we obtain

$$ {}_n\mathcal{L} = [{}_{n\text{-}1}\mathcal{L}, x_n] = \frac{{}_{n\text{-}1}\mathcal{L}\, x_n}{([x_1, \ldots, x_{n-1}], x_n)} = \frac{{}_{n\text{-}1}\mathcal{L}\, x_n}{[(x_1, x_n), \ldots, (x_{n-1}, x_n)]} \tag{24} $$

Now we apply Equation (23) to numerator and denominator,

$$ {}_{n\text{-}1}\mathcal{L} = \frac{\prod\limits_{i_1 \leq n-1} x_{i_1} \ \cdots \ (x_{i_1}, \ldots, x_{i_{n-1}})}{\prod\limits_{i_1 < i_2 \leq n-1} (x_{i_1}, x_{i_2}) \ \cdots \prod\limits_{i_1 < \cdots < i_{n-2} \leq n-1} (x_{i_1}, \ldots, x_{i_{n-2}})} $$

$$ [(x_1, x_n), \ldots, (x_{n-1}, x_n)] = \frac{\prod\limits_{i_1 \leq n-1} (x_{i_1}, x_n) \ \cdots \ (x_{i_1}, \ldots, x_{i_{n-1}}, x_n)}{\prod\limits_{i_1 < i_2 \leq n-1} (x_{i_1}, x_{i_2}, x_n) \ \cdots \prod\limits_{i_1 < \cdots < i_{n-2} \leq n-1} (x_{i_1}, \ldots, x_{i_{n-2}}, x_n)} $$

Substituting into Equation (24) we obtain

$$ {}_n\mathcal{L} = \frac{x_n \prod\limits_{i_1 \leq n-1} x_{i_1} \ \cdots \ (x_{i_1}, \ldots, x_{i_{n-1}}) \prod\limits_{i_1 < \cdots < i_{n-2} \leq n-1} (x_{i_1}, \ldots, x_{i_{n-2}}, x_n)}{\prod\limits_{i_1 < i_2 \leq n-1} (x_{i_1}, x_{i_2}) \prod\limits_{i_1 \leq n-1} (x_{i_1}, x_n) \ \cdots \ (x_{i_1}, \ldots, x_{i_{n-1}}, x_n)} = $$

$$ = \frac{\prod\limits_{i_1 \leq n} x_{i_1} \ \cdots \prod\limits_{i_1 < \cdots < i_{n-1} \leq n} (x_{i_1}, \ldots, x_{i_{n-1}})}{\prod\limits_{i_1 < i_2 \leq n} (x_{i_1}, x_{i_2}) \ \cdots \ (x_{i_1}, \ldots, x_{i_{n-1}}, x_n)} $$

which, in accordance with the definitions in Equation (19), completes the proof. The case of odd n follows analogously. □

Corollary 9. *By the duality property of the* lcm *and* gcd, *the following holds:*

$$ {}_n\mathcal{G} = {}^n_1\mathcal{P} \,/\, ({}^n_2\mathcal{L} \,/\, (\cdots / ({}^{\;\,n}_{n\text{-}1}\mathcal{L} \,/\, {}^n_n\mathcal{L}))) = {}^n_1\mathcal{P} \prod_{k=2}^{n} ({}^n_k\mathcal{L})^{(-1)^{k-1}} \tag{25} $$

that is,

$$ (a)\ \ {}_n\mathcal{G} = \frac{{}_n\mathcal{P} \cdots {}^{\;\,n}_{n\text{-}1}\mathcal{L}}{{}^n_2\mathcal{L} \cdots {}_n\mathcal{L}},\ \textit{if } n = \dot{2}; \qquad (b)\ \ {}_n\mathcal{G} = \frac{{}_n\mathcal{P} \cdots {}_n\mathcal{L}}{{}^n_2\mathcal{L} \cdots {}^{\;\,n}_{n\text{-}1}\mathcal{L}},\ \textit{if } n = \dot{2} + 1 \tag{26} $$

Thus, we have obtained how $\gcd(X_n)$ depends on the lcm's of the powerset of X_n, and conversely, the $\mathrm{lcm}(X_n)$ depends on the partial gcd's.

Using a notation similar to that of continued fractions, these nested divisions could be notated as

$$c_1/(c_2/(\cdots/(c_{n-1}/c_n))) = c_1 \times \frac{1}{c_2 \times \cdots \times} \frac{1}{c_{n-1} \times} \frac{1}{c_n}$$

However, they become clearer when taking logarithms. Defining ${}^n_k g = \log_2 {}^n_k\mathcal{G}$, ${}^n_k l = \log_2 {}^n_k\mathcal{L}$, ${}^n_k p = \log_2 {}^n_k\mathcal{P}$, Equations (22) and (25) correspond to the alternating sums ${}^n_n g = {}^n_1 p - \sum_{k=2}^n (-1)^k\ {}^n_k l$; ${}^n_n l = {}^n_1 p - \sum_{k=2}^n (-1)^k\ {}^n_k g$.

Corollary 10. *If X_n is single-undertone, then ${}_n\mathcal{L} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}^{n-1}}$, $\quad Q(X_n) = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}^n}$.*

If X_n is single-overtone, then ${}_n\mathcal{G} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{L}^{n-1}}$, $\quad Q(X_n) = \frac{{}_n\mathcal{L}^n}{{}_n\mathcal{P}}$.

Proof. Single undertone means that $\gcd(x_i, x_i) = {}_n\mathcal{G}, \forall i < j$. Then ${}^n_k\mathcal{G} = {}_n\mathcal{G}^{\binom{n}{k}}$. According to Equation (23), ${}_n\mathcal{L} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}^{\sum_{k=2}^n (-1)^k \binom{n}{k}}}$. Since

$$\textstyle\sum_{k=2}^n (-1)^k \binom{n}{k} = \sum_{k=0}^n (-1)^k \binom{n}{k} - \binom{n}{0} + \binom{n}{1} = (1-1)^n - 1 + n,$$

we get ${}_n\mathcal{L} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}^{n-1}}$.

Single overtone means that $\operatorname{lcm}(x_i, x_i) = {}_n\mathcal{L}, \forall i < j$. Hence ${}^n_k\mathcal{L} = {}_n\mathcal{L}^{\binom{n}{k}}$. According to Equation (26), ${}_n\mathcal{G} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{L}^{\sum_{k=2}^n (-1)^k \binom{n}{k}}}$. Again, $\sum_{k=2}^n (-1)^k \binom{n}{k} = n - 1$. Then we get ${}_n\mathcal{G} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{L}^{n-1}}$. □

If we express X_n in lowest terms, then ${}_n\mathcal{G} = 1$. If X_n is single-undertone, $Q(X_n) = {}_n\mathcal{P}$. If X_n is single-overtone, then $Q(X_n) = {}_n\mathcal{P}^{\frac{1}{n-1}}$. Bearing in mind that the harmonic radius satisfies $R(X_n)^n = {}_n\mathcal{P}$, the above result is equivalent to the bounds of Corollary 5.

Example

We provide a numerical example to easily understand how these operators work. Let us consider $n = 3$ and the major chord $\{4, 5, 6\}$. The operators are the following ones (we still use the notation $(x_1, \ldots, x_n) = \gcd(x_1, \ldots, x_n)$ and $[x_1, \ldots, x_n] = \operatorname{lcm}(x_1, \ldots, x_n)$):

$$\begin{aligned}
&{}^3_1\mathcal{P} = 4 \cdot 5 \cdot 6 = 120; && \\
&{}^3_2\mathcal{G} = (4,5)(4,6)(5,6) = 1 \cdot 2 \cdot 1 = 2; && {}^3_3\mathcal{G} = (4,5,6) = 1; \\
&{}^3_2\mathcal{L} = [4,5][4,6][5,6] = 20 \cdot 12 \cdot 30 = 7200; && {}^3_3\mathcal{L} = [4,5,6] = 60.
\end{aligned}$$

$${}^3_3\mathcal{L} = \frac{{}^3_1\mathcal{P}\ {}^3_3\mathcal{G}}{{}^3_2\mathcal{G}} = \frac{120 \cdot 1}{2} = 60; \quad {}^3_3\mathcal{G} = \frac{{}^3_1\mathcal{P}\ {}^3_3\mathcal{L}}{{}^3_2\mathcal{L}} = \frac{120 \cdot 60}{7200} = 1 \tag{27}$$

Since the chord is expressed in lowest terms, then ${}^3_3\mathcal{G} = 1$ and $Q(4,5,6) = {}^3_3\mathcal{L}$.

In addition, from Equation (27), we get

$$\left(\frac{{}^3_3\mathcal{L}}{{}^3_3\mathcal{G}}\right)^2 = \frac{{}^3_2\mathcal{L}}{{}^3_2\mathcal{G}} \iff Q(4,5,6)^2 = Q(4,5)\,Q(4,6)\,Q(5,6) = 60^2 \tag{28}$$

Such a "propagation" of the harmonic quotient along the tonal graph will be studied in Section 9.

Now we will determine how the above recurrence relationships are transferred to the parameters $R(X_n)$ and $Q(X_n)$.

8. Recurrence Relationships for the Harmonic Radius and Co-Radius

8.1. Symmetric Product of Factors

We first derive a property for the symmetric product of factors. Let us denote the individual factors appearing in Equations (19) and (20) by

$$ {}^{n}_{k}\mathcal{P}_{i_1 \ldots i_k} = x_{i_1} \cdots x_{i_k}; \quad {}^{n}_{k}\mathcal{G}_{i_1 \ldots i_k} = (x_{i_1}, \cdots, x_{i_k}); \quad {}^{n}_{k}\mathcal{L}_{i_1 \ldots i_k} = [x_{i_1}, \cdots, x_{i_k}] $$

Accordingly,

$$ {}^{n}_{k}\mathcal{P} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{P}_{i_1 \ldots i_k}; \quad {}^{n}_{k}\mathcal{G} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{G}_{i_1 \ldots i_k}; \quad {}^{n}_{k}\mathcal{L} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{L}_{i_1 \ldots i_k} $$

Lemma 5. *The symmetric product of factors satisfies*

$$ {}_{n}\mathcal{P}^{\binom{n-1}{k-1}} = {}^{n}_{k}\mathcal{P} \tag{29} $$

Proof. The result is immediate. In the product of the $\binom{n}{k}$ factors ${}^{n}_{k}\mathcal{P}_{i_1 \ldots i_k} = x_{i_1} \cdots x_{i_k}$, each fixed variable x_j appears together with the remaining $n-1$ variables in $\binom{n-1}{k-1}$ factors. □

Corollary 11. *For $k = 2$, ${}_{n}\mathcal{P}^{n-1} = {}^{n}_{2}\mathcal{P}$. Hence, ${}_{n}\mathcal{P}^{n-1} = \prod_{i<j} {}^{n}_{2}\mathcal{P}_{ij}$.*

Therefore, there is the following connection between several geometrical means,

$$ {}_{n}\mathcal{P}^{\frac{1}{n}} = \left(\prod_{i<j} {}^{n}_{2}\mathcal{P}^{\frac{1}{2}}_{ij} \right)^{\binom{n}{2}^{-1}} $$

Since ${}^{n}_{2}\mathcal{P}_{ij} = {}^{n}_{2}\mathcal{G}_{ij}\, {}^{n}_{2}\mathcal{L}_{ij}$, then ${}_{n}\mathcal{P}$ can be expressed in terms of the gcd and lcm by pairs,

$$ {}_{n}\mathcal{P}^{n-1} = \prod_{i<j} {}^{n}_{2}\mathcal{G}_{ij}\, {}^{n}_{2}\mathcal{L}_{ij} = {}^{n}_{2}\mathcal{G}\, {}^{n}_{2}\mathcal{L} $$

8.2. Symmetric Product of Harmonic Radii

Following the same notation, the harmonic radius associated with a subset of $k > 1$ frequencies of the set X_n is written as

$$ {}^{n}_{k}\mathcal{R}_{i_1 \ldots i_k} = \frac{{}^{n}_{k}\mathcal{P}^{\frac{1}{k}}_{i_1 \ldots i_k}}{{}^{n}_{k}\mathcal{G}_{i_1 \ldots i_k}}; \quad {}^{n}_{k}\mathcal{P}_{i_1 \ldots i_k} = {}^{n}_{k}\mathcal{R}^{k}_{i_1 \ldots i_k}\, {}^{n}_{k}\mathcal{G}^{k}_{i_1 \ldots i_k} \tag{30} $$

As before, we define the symmetric operator

$$ {}^{n}_{k}\mathcal{R} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{R}_{i_1 \ldots i_k}; \quad {}_{n}\mathcal{R} \equiv {}^{n}_{n}\mathcal{R} $$

Theorem 5. *The total harmonic radius depends on the partial ones according to*

$$ {}_{n}\mathcal{R} = \left(\prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{R}_{i_1 \ldots i_k} \frac{{}^{n}_{k}\mathcal{G}_{i_1 \ldots i_k}}{{}_{n}\mathcal{G}} \right)^{\binom{n}{k}^{-1}} $$

Proof. We rewrite Equation (29) as ${}_{n}\mathcal{P}^{\binom{n-1}{k-1}} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{P}_{i_1 \ldots i_k}$. Taking Equation (30) into account we get ${}_{n}\mathcal{P}^{\binom{n-1}{k-1}} = \prod_{i_1 < \cdots < i_k \leq n} {}^{n}_{k}\mathcal{R}^{k}_{i_1 \ldots i_k}\, {}^{n}_{k}\mathcal{G}^{k}_{i_1 \ldots i_k}$. The exponent on the left side is identically equal to $\binom{n}{k}\frac{k}{n}$. Taking the $\frac{1}{k}$ power on both sides yields ${}_n\mathcal{P}^{\binom{n}{k}\frac{\cancel{k}}{n}} = \prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}^{\cancel{k}}_{i_1\ldots i_k}\, {}^n_k\mathcal{G}^{\cancel{k}}_{i_1\ldots i_k}$. Dividing by ${}_n\mathcal{G}^{\binom{n}{k}}$, we obtain ${}_n\mathcal{R}^{\binom{n}{k}} = \prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}_{i_1\ldots i_k} \frac{{}^n_k\mathcal{G}_{i_1\ldots i_k}}{{}_n\mathcal{G}}$, which proves the statement. □

Notice that ${}^n_k\mathcal{G}_{i_1\ldots i_k} \ge {}_n\mathcal{G}$. Therefore, ${}_n\mathcal{R} \ge \left(\prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}_{i_1\ldots i_k}\right)^{\binom{n}{k}^{-1}}$.

Corollary 12. *If all pairs of frequencies have the same* gcd*, then*

$$ {}^n_k\mathcal{G}_{i_1\ldots i_k} = {}_n\mathcal{G} \;\Rightarrow\; {}_n\mathcal{R} = \left(\prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}_{i_1\ldots i_k}\right)^{\binom{n}{k}^{-1}} $$

That is, for a single-undertone chord, the harmonic radius is the geometric mean of the harmonic radii of the powerset of X_n.

8.3. Symmetric Product of Harmonic Co-Radii

The co-harmonic radius of a subset of $k > 1$ frequencies of X_n is defined as

$$ {}^n_k\mathcal{R}'_{i_1\ldots i_k} = \frac{{}^n_k\mathcal{L}_{i_1\ldots i_k}}{{}^n_k\mathcal{P}^{\frac{1}{k}}_{i_1\ldots i_k}} $$

that is, ${}^n_k\mathcal{P}_{i_1\ldots i_k} = \dfrac{{}^n_k\mathcal{G}^k_{i_1\ldots i_k}}{{}^n_k\mathcal{R}'^k_{i_1\ldots i_k}}$. As before, we define the symmetric operator

$$ {}^n_k\mathcal{R}' = \prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}'_{i_1\ldots i_k}; \quad {}_n\mathcal{R}' \equiv {}^n_n\mathcal{R}' $$

Recall that the relationship between the radius and the co-harmonic radius is given by

$$ {}^n_k\mathcal{R}\, {}^n_k\mathcal{R}' = \frac{{}^n_k\mathcal{L}}{{}^n_k\mathcal{G}} $$

It then follows immediately:

Corollary 13. *The total co-harmonic radius depends on the partial ones in the form*

$$ {}_n\mathcal{R}' = \left(\prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}'_{i_1\ldots i_k} \frac{{}_n\mathcal{L}}{{}^n_k\mathcal{L}_{i_1\ldots i_k}}\right)^{\binom{n}{k}^{-1}}. $$

Since ${}_n\mathcal{L} \ge {}^n_k\mathcal{L}_{i_1\ldots i_k}$, then ${}_n\mathcal{R}' \ge \left(\prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}'_{i_1\ldots i_k}\right)^{\binom{n}{k}^{-1}}$.

Corollary 14. *If all pairs of frequencies have the same* lcm*,*

$$ {}^n_k\mathcal{L}_{i_1\ldots i_k} = {}_n\mathcal{L} \Rightarrow {}_n\mathcal{R}' = \left(\prod_{i_1<\cdots<i_k\le n} {}^n_k\mathcal{R}'_{i_1\ldots i_k}\right)^{\binom{n}{k}^{-1}} $$

Therefore, for a single-overtone chord, the harmonic co-radius is the geometric mean of the harmonic co-radii of the powerset of X_n.

9. Product and Quotient of gcd(C) and lcm(C)

The extreme nodes of the tonal graph $\mathfrak{G}(X_n)$ are $\gcd(X_n) = {}_n\mathcal{L}$ and $\mathrm{lcm}(X_n) = {}_n\mathcal{G}$, whose product and quotient are directly related to the harmonic radius $R(X_n) = {}_n\mathcal{R}$ and co-radius $R'(X_n) = {}_n\mathcal{R}'$. The quotient, according to Equation (11) is $\frac{{}_n\mathcal{L}}{{}_n\mathcal{G}} = {}_n\mathcal{R}\,{}_n\mathcal{R}'$ (which logarithm provides the harmonic distance of the chord). The product, according to Equation (12), satisfies $\frac{{}_n\mathcal{L}\,{}_n\mathcal{G}}{{}_n\mathcal{P}^{\frac{2}{n}}} = \frac{{}_n\mathcal{R}'}{{}_n\mathcal{R}}$ (informing us about the chord's symmetry). Our interest is to express $\frac{{}_n\mathcal{L}}{{}_n\mathcal{G}}$ and ${}_n\mathcal{L}\,{}_n\mathcal{G}$ as a function of the same quantities in terms of the powerset of X_n.

For $n = 2$, these relations are well known, since ${}_2\mathcal{G}\,{}_2\mathcal{L} = {}_2\mathcal{P}$ and $\frac{{}_2\mathcal{L}}{{}_2\mathcal{G}} = \frac{{}_2\mathcal{P}}{{}_2\mathcal{G}^2}$. However, for $n > 2$, these relations are no longer immediate.

We shall use the following notation, valid for any symmetric quantity ${}^n_k\mathcal{Z}$ among those previously defined,

$$
\begin{aligned}
&\text{if } n = \dot{2}, \qquad \phi_n(\mathcal{Z}) = \frac{{}^n_1\mathcal{Z}}{{}^n_n\mathcal{Z}}\,\psi_n(\mathcal{Z}); \qquad \begin{cases} \psi_2(\mathcal{Z}) \equiv 1 \\ \text{if } n > 2,\ \psi_n(\mathcal{Z}) = \prod_{k=2}^{n-1} ({}^n_k\mathcal{Z})^{(-1)^{k-1}} \end{cases} \\
&\text{if } n = \dot{2}+1, \quad \phi_n(\mathcal{Z}) = {}^n_1\mathcal{Z}\ {}^n_n\mathcal{Z}\ \psi_n(\mathcal{Z}); \quad \psi_n(\mathcal{Z}) = \prod_{k=2}^{n-1} ({}^n_k\mathcal{Z})^{(-1)^{k-1}}
\end{aligned}
\tag{31}
$$

It is straightforward to verify that

$$
\begin{aligned}
\phi_n^{-1}(\mathcal{Z}) \equiv [\phi_n(\mathcal{Z})]^{-1} = \phi_n(\mathcal{Z}^{-1}); \quad \phi_n(\mathcal{Z})\,\phi_n(\mathcal{Y}) = \phi_n(\mathcal{Z}\,\mathcal{Y}) \\
\psi_n^{-1}(\mathcal{Z}) \equiv [\psi_n(\mathcal{Z})]^{-1} = \psi_n(\mathcal{Z}^{-1}); \quad \psi_n(\mathcal{Z})\,\psi_n(\mathcal{Y}) = \psi_n(\mathcal{Z}\,\mathcal{Y})
\end{aligned}
\tag{32}
$$

Lemma 6. *For any $\alpha \in \mathbb{R}$, the following holds:*

$$
\begin{aligned}
&\text{if } n = \dot{2}, \qquad \phi_n^\alpha(\mathcal{P}) = \phi_n(\mathcal{P}^\alpha) = \psi_n^\alpha(\mathcal{P}) = \psi_n(\mathcal{P}^\alpha) = 1 \\
&\text{if } n = \dot{2}+1, \quad \phi_n^\alpha(\mathcal{P}) = \phi_n(\mathcal{P}^\alpha) = {}_n\mathcal{P}^{2\alpha}\ \psi_n^\alpha(\mathcal{P}) = {}_n\mathcal{P}^{2\alpha}\ \psi_n(\mathcal{P}^\alpha) = 1
\end{aligned}
\tag{33}
$$

Proof. Owing to Equation (29), for any $n \geq 2$,

$$
\phi_n(\mathcal{P}) = \frac{{}^n_1\mathcal{P}\cdots}{{}^n_2\mathcal{P}\cdots} = {}_n\mathcal{P}^{\binom{n-1}{0}-\binom{n-1}{1}+\cdots\pm\binom{n-1}{n-1}} = {}_n\mathcal{P}^{(1-1)^{n-1}} = 1
$$

Hence, $\phi_n^\alpha(\mathcal{P}) = \phi_n(\mathcal{P}^\alpha) = 1$. Then, according to Equation (31),

$$
\begin{aligned}
&\text{if } n = \dot{2}, \qquad \phi_n(\mathcal{P}) = \frac{\cancel{{}_n\mathcal{P}}}{\cancel{{}_n\mathcal{P}}}\psi_n(\mathcal{P}) \Longrightarrow \phi_n^\alpha(\mathcal{P}) = \psi_n^\alpha(\mathcal{P}) = \psi_n(\mathcal{P}^\alpha) \\
&\text{if } n = \dot{2}+1, \quad \phi_n(\mathcal{P}) = {}_n\mathcal{P}^2\ \psi_n(\mathcal{P}) \Longrightarrow \phi_n^\alpha(\mathcal{P}) = {}_n\mathcal{P}^{2\alpha}\ \psi_n^\alpha(\mathcal{P}) = {}_n\mathcal{P}^{2\alpha}\ \psi_n(\mathcal{P}^\alpha)
\end{aligned}
$$

□

Corollary 15. *If $n = \dot{2}$, $\psi_n(\mathcal{P}^\alpha) = 1$. If $n = \dot{2}+1$, $\psi_n(\mathcal{P}^\alpha) = {}_n\mathcal{P}^{-2\alpha}$*

To work with products and quotients of gcd's and lcm's, the following auxiliary variables (normalized $\mathcal{L}$- and $\mathcal{G}$-operators) are appropriate,

$$
{}^n_k\mathcal{M} \equiv \frac{{}^n_k\mathcal{L}}{{}^n_k\mathcal{P}^{\frac{1}{2}}}; \quad {}^n_k\mathcal{M}_{i_1\ldots i_k} \equiv \frac{{}^n_k\mathcal{L}_{i_1\ldots i_k}}{{}^n_k\mathcal{P}^{\frac{1}{2}}_{i_1\ldots i_k}}; \quad {}_n\mathcal{M} \equiv {}^n_n\mathcal{M}
$$

$$
{}^n_k\mathcal{D} \equiv \frac{{}^n_k\mathcal{G}}{{}^n_k\mathcal{P}^{\frac{1}{2}}}; \quad {}^n_k\mathcal{D}_{i_1\ldots i_k} \equiv \frac{{}^n_k\mathcal{G}_{i_1\ldots i_k}}{{}^n_k\mathcal{P}^{\frac{1}{2}}_{i_1\ldots i_k}}; \quad {}_n\mathcal{D} \equiv {}^n_n\mathcal{D}
$$

Theorem 6. *For $n \geq 2$, the following relationships hold:*

$$
(n = \dot{2}) \qquad \psi_n\left(\frac{\mathcal{M}}{\mathcal{D}}\right) = 1; \qquad {}_n\mathcal{M}\,{}_n\mathcal{D} = \psi_n(\mathcal{D}) = \psi_n(\mathcal{M}) \tag{34}
$$

$$
(n = \dot{2}+1) \qquad \psi_n(\mathcal{M}\,\mathcal{D}) = 1; \qquad \frac{{}_n\mathcal{M}}{{}_n\mathcal{D}} = \psi_n(\mathcal{D}) = \psi_n^{-1}(\mathcal{M}) \tag{35}
$$

Proof. Taking into account Equations (32) and (33), let us first assume $n = \dot{2}$. From Equations (23) and (26)a, we have

$$
{}_n\mathcal{L} = \phi_n(\mathcal{G}) = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}}\,\psi_n(\mathcal{G}) = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}}\,\frac{\psi_n(\mathcal{G})}{\psi_n(\mathcal{P}^{\frac{1}{2}})} = \frac{{}_n\mathcal{P}}{{}_n\mathcal{G}}\,\psi_n\left(\frac{\mathcal{G}}{\mathcal{P}^{\frac{1}{2}}}\right) \iff \frac{{}_n\mathcal{L}}{{}_n\mathcal{P}^{\frac{1}{2}}}\,\frac{{}_n\mathcal{G}}{{}_n\mathcal{P}^{\frac{1}{2}}} = \psi_n\left(\frac{\mathcal{G}}{\mathcal{P}^{\frac{1}{2}}}\right)
$$

By the dual property, it also follows that $\frac{{}_n\mathcal{L}}{{}_n\mathcal{P}^{\frac{1}{2}}}\,\frac{{}_n\mathcal{G}}{{}_n\mathcal{P}^{\frac{1}{2}}} = \psi_n\left(\frac{\mathcal{L}}{\mathcal{P}^{\frac{1}{2}}}\right)$.

Expressing these relations in terms of the auxiliary variables, we obtain the following constraint:

$$
{}_n\mathcal{M}\;{}_n\mathcal{D} = \psi_n(\mathcal{M}) = \psi_n(\mathcal{D}) \Longrightarrow \psi_n\left(\frac{\mathcal{M}}{\mathcal{D}}\right) = 1 \tag{36}
$$

Therefore, the quantity that is replicated along the powerset of X_n is

$$
({}_n\mathcal{M}\,{}_n\mathcal{D})^2 = \psi_n(\mathcal{M})\,\psi_n(\mathcal{D}) = \psi_n(\mathcal{M}\,\mathcal{D})
$$

From Equation (36) we may compute the harmonic quotient $\frac{{}_n\mathcal{M}}{{}_n\mathcal{D}} = \frac{{}_n\mathcal{L}}{{}_n\mathcal{G}}$ in terms of the gcd's of the upper half of the graph, or equivalently, in terms of the lcm's of the lower half of the graph,

$$
\frac{{}_n\mathcal{M}}{{}_n\mathcal{D}} = {}_n\mathcal{D}^{-2}\;\psi_n(\mathcal{D}) = {}_n\mathcal{M}^2\;\psi_n^{-1}(\mathcal{M}) \tag{37}
$$

Let us now assume $n = \dot{2}+1$. From Equations (23) and (26)b, we obtain

$$
{}_n\mathcal{L} = \phi_n(\mathcal{G}) = {}_n\mathcal{P}\;{}_n\mathcal{G}\;\psi_n(\mathcal{G}); \quad {}_n\mathcal{G} = \phi_n(\mathcal{L}) = {}_n\mathcal{P}\;{}_n\mathcal{L}\;\psi_n(\mathcal{L})
$$

Due to Corollary 15, with $\alpha = \frac{1}{2}$, we have ${}_n\mathcal{P} = \psi_n^{-1}(\mathcal{P}^{\frac{1}{2}})$. Hence, dividing by ${}_n\mathcal{P}^{\frac{1}{2}}$, we write

$$
\frac{{}_n\mathcal{L}}{{}_n\mathcal{P}^{\frac{1}{2}}} = \frac{{}_n\mathcal{G}}{{}_n\mathcal{P}^{\frac{1}{2}}}\,\psi_n\left(\frac{\mathcal{G}}{\mathcal{P}^{\frac{1}{2}}}\right); \quad \frac{{}_n\mathcal{G}}{{}_n\mathcal{P}^{\frac{1}{2}}} = \frac{{}_n\mathcal{L}}{{}_n\mathcal{P}^{\frac{1}{2}}}\,\psi_n\left(\frac{\mathcal{L}}{\mathcal{P}^{\frac{1}{2}}}\right)
$$

In terms of the auxiliary variables, it be expressed as

$$
{}_n\mathcal{M} = {}_n\mathcal{D}\;\psi_n(\mathcal{D}); \quad {}_n\mathcal{D} = {}_n\mathcal{M}\;\psi_n(\mathcal{M}) \Longrightarrow \psi_n(\mathcal{M})\,\psi_n(\mathcal{D}) = \psi_n(\mathcal{M}\,\mathcal{D}) = 1
$$

Therefore,

$$\frac{{}_{n}\mathcal{M}}{{}_{n}\mathcal{D}} = \psi_n(\mathcal{D}) = \psi_n^{-1}(\mathcal{M}) \tag{38}$$

The quantity that is replicated along the powerset of X_n is

$$\left(\frac{{}_{n}\mathcal{M}}{{}_{n}\mathcal{D}}\right)^2 = \frac{\psi_n(\mathcal{D})}{\psi_n(\mathcal{M})} = \psi_n^{-1}\left(\frac{\mathcal{M}}{\mathcal{D}}\right) \tag{39}$$

□

If we express the chord X_n in its simplest form, then ${}_{n}\mathcal{G} = 1$ (although ${}^{n}_{k}\mathcal{G} \neq 1$, in general) and $Q(X_n) = {}_{n}\mathcal{L}$.

Corollary 16. *The harmonic quotient has a unique expression in terms of the undertones of the powerset of X_n,*

$$Q(X_n) = {}_{n}\mathcal{P}\,\psi_n(\mathcal{G})$$

Proof. Bearing in mind Equation (33), the Equations (34) and (35) become,

$$\begin{aligned}
(n = \dot{2}) \quad & Q(X_n) = {}_{n}\mathcal{P}\,\psi_n(\mathcal{M}) = {}_{n}\mathcal{P}\,\psi_n(\mathcal{D}) = {}_{n}\mathcal{P}\,\psi_n(\mathcal{L}) = {}_{n}\mathcal{P}\,\psi_n(\mathcal{G}) \\
(n = \dot{2}+1) \quad & Q(X_n) = \psi_n^{-1}(\mathcal{M}) = \psi_n(\mathcal{D}) = ({}_{n}\mathcal{P}\,\psi_n(\mathcal{L}))^{-1} = {}_{n}\mathcal{P}\,\psi_n(\mathcal{G})
\end{aligned}$$

□

Therefore, the harmonic quotient of the tonal graph $\mathfrak{G}(X_n)$ can be determined either from the overtones or from the undertones, independently. That is, the harmonic structure of the former has its counterpart in the latter. In both cases, the harmonic quotient has a unique expression in terms of the common undertones of the powerset of X_n.

9.1. Example

In order to exemplify how these operators work, we continue with the numerical example of Equations (27) and (28) for the major chord $\{4,5,6\}$, with n odd. The equation on the right side of Equation (35) is equivalent to Equation (39). Since $\frac{{}_{3}\mathcal{M}}{{}_{3}\mathcal{D}} = \frac{{}_{3}\mathcal{L}}{{}_{3}\mathcal{G}}$, then

$$\psi_3^{-1}\left(\frac{\mathcal{M}}{\mathcal{D}}\right) = \psi_3^{-1}\left(\frac{\mathcal{L}}{\mathcal{G}}\right) = \frac{\psi_3^{-1}(\mathcal{L})}{\psi_3^{-1}(\mathcal{G})} = \frac{\psi_3(\mathcal{G})}{\psi_3(\mathcal{L})} = \frac{{}^{3}_{2}\mathcal{L}}{{}^{3}_{2}\mathcal{G}}$$

Hence, we get

$$Q(X_n)^2 = \left(\frac{{}_{3}\mathcal{L}}{{}_{3}\mathcal{G}}\right)^2 = \frac{{}^{3}_{2}\mathcal{L}}{{}^{3}_{2}\mathcal{G}} \iff \left(\frac{[4,5,6]}{(4,5,6)}\right)^2 = \frac{[4,5]\,[4,6]\,[5,6]}{(4,5)(4,6)(5,6)} = 60^2$$

which is the propagation relationship obtained in Equation (28).

For the constraint on the left in Equation (35), we use its equivalent expression in Equation (38), which can be written as $Q(X_n) = \psi_3(\mathcal{D}) = \psi_3^{-1}(\mathcal{M})$. Then, $\psi_3\left(\frac{\mathcal{G}}{\mathcal{P}^{\frac{1}{2}}}\right) = \psi_3^{-1}\left(\frac{\mathcal{L}}{\mathcal{P}^{\frac{1}{2}}}\right) = \psi_3\left(\frac{\mathcal{P}^{\frac{1}{2}}}{\mathcal{L}}\right)$. Hence $\frac{\psi_3(\mathcal{G})}{\psi_3(\mathcal{P}^{\frac{1}{2}})} = \frac{\psi_3(\mathcal{P}^{\frac{1}{2}})}{\psi_3(\mathcal{L})}$. By Corollary 15 $\psi_3(\mathcal{P}^{\frac{1}{2}}) = {}_{3}\mathcal{P}^{-1}$, so that $\psi_3(\mathcal{G})\,{}_{3}\mathcal{P} = \frac{1}{\psi_3(\mathcal{L})\,{}_{3}\mathcal{P}}$. Therefore,

$$Q(X_n) = \frac{{}_{3}\mathcal{P}}{{}^{3}_{2}\mathcal{G}} = \frac{{}^{3}_{2}\mathcal{L}}{{}_{3}\mathcal{P}} \iff \frac{4\cdot 5\cdot 6}{(4,5)(4,6)(5,6)} = \frac{[4,5]\,[4,6]\,[5,6]}{4\cdot 5\cdot 6} = 60$$

9.2. Deviation from Harmonic Symmetry

For $n > 2$, in order to determine the chord's deviation from harmonic symmetry we had to compare the harmonic radius and co-radius. The case ${}_n\mathcal{R} < {}_n\mathcal{R}'$ corresponds to otonal chords, i.e., chords with a prolate spheroid, where the bass is more reinforced by common fundamental. In the opposite case, ${}_n\mathcal{R} > {}_n\mathcal{R}'$, corresponding to utonal chords, i.e., chords with an oblate spheroid, the reference to the common fundamental and the reinforcement of the bass weakens. These properties, which ultimately depend on the sympathetic factors, are related to the chords' qualities such as that "sound stable, final, and resolved" and "sound unstable, tense, and unresolved" used by musicians.

Such different qualities are in either sides of the condition for a symmetric chord, $\frac{{}_n\mathcal{R}}{{}_n\mathcal{R}'} = 1$. If the chord X_n is expressed in lowest terms, according to Equation (12) and Corollary 16,

$$\frac{{}_n\mathcal{R}'}{{}_n\mathcal{R}} = \frac{Q(X_n)}{{}_n\mathcal{P}^{\frac{2}{n}}} = {}_n\mathcal{P}^{\frac{n-2}{n}}\,\psi_n(\mathcal{G})$$

Therefore, the shape of the chord spheroid is determined by the harmonic structure of the common undertones according to the following expression.

Corollary 17. *A prolate chord spheroid satisfies*

$$Q(X_n) > {}_n\mathcal{P}^{\frac{2}{n}} \iff \psi_n(\mathcal{G}) > {}_n\mathcal{P}^{\frac{2-n}{n}}$$

An oblate chord spheroid satisfies

$$Q(X_n) < {}_n\mathcal{P}^{\frac{2}{n}} \iff \psi_n(\mathcal{G}) < {}_n\mathcal{P}^{\frac{2-n}{n}}$$

10. Conclusions

The present approach has studied the harmonic structure of just intonation chords on the basis provided by the forced harmonic oscillator model. The approach assumes an ideal model where each single tone of the chord sounds according to its fundamental frequency, with similar amplitudes. The wave superposition produces a whole oscillation which frequency is the greatest common divisor of the tone components, i.e., the common fundamental frequency. Therefore, the harmonic series of the common fundamental contains all the harmonics present in the chord. The decaying partials of each single tone are not considered, except for the harmonic corresponding to the least common multiple of the chord, where all the harmonic series concur, i.e., the first common overtone, after which the pattern of harmonics repeats. Of course, when this simplified arithmetic model is applied to real acoustic instruments, their theoretical harmonicity values will be modulated by the decaying harmonic amplitudes and by masking and critical band interference.

The average harmonic distance δ_H (harmonic length) between the common fundamental and each tone of the chord estimates the degree of sympathetic vibration of the chord. It is inversely correlated with the degree of coincidence among the harmonic series involved in the chord. In the tonal graph, this parameter is complemented by the average distance δ'_H (harmonic co-length) to the first common overtone. Their relative size determines the deviation of the chord from harmonic symmetry, and their sum is the harmonic distance of the chord Δ_H. These are the basic parameters involved in the tonal graph, which are mainly associated with qualities of the chord in relation to timbre/color, although the harmonic dispersion $D_H = \delta_H + \Delta_H$ was proven to be inversely correlated with the average perceptual consonance [2].

In the frequency space, the harmonic volume associated with the chord spheroid allows to visualize the above parameters. The distinction between chords with prolate and oblate chord spheroids, the former closer to the common fundamental and the latter closer to the first common overtone, generalizes the concepts of otonal and utonal chords.

Some parameters characterizing the tonal graph have been referred to a harmonically symmetric chord, for which the harmonic length and co-length coincide. This is the case of the harmonic dispersion in the interval space and the harmonic likeness in the frequency space. Therefore, the properties of symmetric chords have deserved particular interest. For a fixed harmonic length, the chords having a lower co-length are less consonant than those with a higher co-length, hence a symmetric chord represents an intermediate degree of consonance. In such a case, the harmonic dispersion matches the harmonic length, which provides the generic trend (Figure 11, top). Nevertheless, for low harmonic lengths, the harmonic distance can make the difference by introducing an important correction. Similarly, it happens with the harmonic likeness and the harmonic sympathy (Figure 12) in the frequency space.

Single-overtone and single-undertone chords, which generate upper-jigsaw and lower-jigsaw graphs, have also been studied. To this purpose, the tonal graph has been completed with pairs of common undertones (in the upper half) and with pairs of common overtones (in the lower half). It has been proven that, when a chord is single-overtone, the harmonic distance reaches its minimum relatively to a symmetric chord with the same harmonic length. If a chord is single-undertone, it reaches its maximum value.

Then, a question arose: how the harmonic quotient of a chord depends on the set of common undertones and overtones? To answer this question, for a chord X_n we have determined what is the connection between $\mathrm{lcm}(X_n)$, $\gcd(X_n)$, and similar values for the powerset of X_n. It has been found that, for any number of tones, the harmonic quotient can be univocally expressed from their common undertones. In addition, the periodicity of the common undertones and overtones is constrained according to the Equations (37) and (38). Consequently, although the undertones and overtones of the chord have not been explicitly taken into account in our model, the harmonic quotient actually incorporates information about the nested ratios of the common undertones and overtones involved in the chord.

From a strictly mathematical viewpoint, the above general relationships describe how the $\mathrm{lcm}(X_n)$ depends on all possible common factors involved in X_n, and how the $\gcd(X_n)$ depends on all possible common multiples involved in X_n. They can be useful in particular cases, such as when $X_n = \{x_1, \ldots, x_n\}$ is formed from the set $A_n = \{a_1, \ldots, a_n\}$ so that $x_i = \mathrm{lcm}(A_i)$, or $x_i = \gcd(A_i)$, with $\cup_i A_i = A_n$. From a musical viewpoint, such relationships can also be used to analyze particular cases of chords.

It is known that some of these parameters, such as those directly indicating consonance, can be perceived in auditory tests, but other parameters have not been asked for. It is important to have them well characterized in order to psychoacoustics researchers can study whether they are perceptible or not, and to learn how to recognize those who are, since many things depend on training and practice.

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Glossary of Symbols

Special symbols used in this manuscript in order of appearance:

Δ_H	harmonic distance;	Q	harmonic quotient;
$\mathfrak{G}$	tonal graph;	R	harmonic radius;
S	harmonic sympathy;	δ_H	harmonic length;
R'	harmonic co-radius;	δ'_H	harmonic co-length;
V	harmonic volume;	D_H	harmonic dispersion;
L	harmonic likeness;	${}^n_k\mathcal{G}$	symmetric gcd of $\binom{n}{k}$ arguments;
${}^n_k\mathcal{L}$	symmetric lcm of $\binom{n}{k}$ arguments;	${}^n_k\mathcal{P}$	symmetric product of $\binom{n}{k}$ arguments;
${}^n_k\mathcal{R}$	symmetric product of $\binom{n}{k}$ harmonic radii;	${}^n_k\mathcal{R}'$	symmetric product of $\binom{n}{k}$ harmonic co-radii;
${}^n_k\mathcal{M}$	normalized $\mathcal{L}$-operator;	${}^n_k\mathcal{D}$	normalized $\mathcal{G}$-operator.

Appendix A. Periodic Solutions of the Forced Harmonic Oscillator

According to the model of the vibrating string and assuming that nothing dampens the oscillations, at a fixed point x of the string, if the perturbation has angular frequency ω, the motion of the forced harmonic oscillator can be modeled by the equation

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} + \omega_n^2 Y = F\cos\omega t; \quad Y(0) = A,\ \dot{Y}(0) = 0$$

where ω_n is the angular frequency of the free oscillation.

In particular, we may focus in the most significant angular frequency ω_1, corresponding to the fundamental tone, and look for values ω_1 and ω producing a resulting periodic vibration. Then, the general solution is

$$\begin{aligned} Y(t) &= c_1\cos\omega t + c_2\sin\omega t + \tfrac{F}{2\omega}\, t\sin\omega t, \text{ if } \omega = \omega_1 \\ Y(t) &= c_1\cos\omega_1 t + c_2\sin\omega_1 t - \tfrac{F}{\omega^2-\omega_1^2}\cos\omega t, \text{ if } \omega \neq \omega_1 \end{aligned} \tag{A1}$$

The terms with the constants c_1, c_2 correspond to the general solution of the homogeneous equation, while the last term is a particular solution of the complete equation. After imposing the initial conditions, the constants c_1, c_2 remain fixed. If the two frequencies match, in which case we say that there is resonance, the first solution of Equation (A1) becomes

$$Y(t) = A\cos\omega t + \tfrac{F}{2\omega}\, t\sin\omega t$$

leading to a solution with a total amplitude that increases over time. However, in a real case there are always factors that dampen the vibrations. Then, this solution become modulated by a exponentially decreasing amplitude, so that, after reaching a maximum of amplitude, the oscillations tend to disappear. Therefore, if an oscillation of frequency ω is subjected to an oscillating external perturbation of the same frequency, the resulting oscillation intensifies, and when there is damping, simply stays for long. This is what happens with simultaneous sounds with matching harmonics, such as when playing chords.

If $\omega \neq \omega_1$, the second solution of Equation (A1) becomes

$$Y(t) = (A + \tfrac{F}{\omega^2-\omega_1^2})\cos\omega_1 t - \tfrac{F}{\omega^2-\omega_1^2}\cos\omega t \tag{A2}$$

The first term corresponds to the solution of the free oscillation and the second one is due to the perturbation. The amplitude of the waves depends on the frequencies.

Equation (A2) is periodic in the following two situations (see, e.g., [37] (p. 123)):

1. If the amplitude of the first wave vanishes, i.e., $\omega_1^2 - \omega^2 = \frac{F}{A}$, the oscillations have the frequency of the perturbation. The periodic solution is $Y(t) = A\cos\omega t$. They are known as harmonic oscillations.
2. Otherwise frequencies ω_1 and ω must be commensurable (this necessary and sufficient condition comes from the fact that the solutions we have are continuous functions of time, otherwise commensurability is no more a necessary condition), that is, their ratio must be a rational number, $\frac{\omega}{\omega_1} = \frac{p}{q}$, with $p, q \in \mathbb{N}$ coprime.

   Then, the periodic solution is

$$Y(t) = \left(A + \tfrac{Fp^2}{\omega^2(p^2-q^2)}\right)\cos\tfrac{q}{p}\,\omega t - \tfrac{Fp^2}{\omega^2(p^2-q^2)}\cos\omega t$$

   There are two particular cases:

2.1. If $q = 1$ and $\omega = p\,\omega_1$, $p \geq 1$, the respective periods satisfy $p\,T_\omega = T_{\omega_1}$. The total period of the solution is T_{ω_1}, the same as the free oscillation. The free oscillation coupled with the perturbation is called subharmonic oscillation of order p (undertone).

2.2. If $p = 1$ and $q\,\omega = \omega_1$, $q \geq 1$, the periods satisfy $T_\omega = q\,T_{\omega_1}$. The total period of the solution is T_ω, that of the perturbation. The oscillation is known as superharmonic oscillation of order q (sometimes referred to as harmonic, overtone, or $(q-1)$th partial).

In the general case, if $\frac{\omega}{\omega_1} = \frac{p}{q}$, the periods satisfy $p\,T_\omega = q\,T_{\omega_1} \equiv T$. Note that in such a case we may write $T_\omega = q\,t_0$ and $T_{\omega_1} = p\,t_0$ with $t_0 \in \mathbb{R}$, so that the minimum period of the whole oscillation is $T = \mathrm{lcm}(T_\omega, T_{\omega_1}) = p\,q\,t_0$. This is the period of the highest subharmonic common to both oscillations (common fundamental), while $t_0 = \gcd(T_\omega, T_{\omega_1})$ is the period of the lowest superharmonic common to both oscillations. The ratio

$$\frac{T}{t_0} = \frac{\mathrm{lcm}(T_\omega, T_{\omega_1})}{\gcd(T_\omega, T_{\omega_1})} = p\,q \tag{A3}$$

measures the period of the common fundamental in terms of the period of the lowest common harmonic. In general, for a finite set of irreducible frequency ratios C, this value is the harmonic quotient $Q(C)$.

In general, these oscillations are referred to as supersubharmonics (or harmonics, in a general sense). When the frequencies of free and forced oscillations are commensurable we say that there is sympathetic vibration. Therefore, in order to highlight a sound we must look for a constructive superposition of periodic solutions of frequencies in commensurable ratio. The frequencies of the matching supersubharmonics will produce an effect of resonance and amplification of sound, which is interpreted as a higher degree of consonance. In the simple case where only the fundamental tone ω_1 of the free oscillation is considered, the roles of ω_1 and ω of the forced oscillation can be swapped, i.e., there is symmetry between both waves, which is the case assumed in the present paper.

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