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An algebra of chords for a non-degenerate Tonnetz

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For an n-TET tuning system we propose a formalism to study the transformations of k-chords over a generalized non-degenerate Tonnetz generated by a given interval structure. Root and mode are the two components of a directed chord on which the algebra operates, so that chord transformations within one chord cell or towards other cells, and paths or simple circuits over the chord network can be determined without resorting to computational algorithms or geometrical representations. The one-step transformations over the edges of chord network associated with the $k-1$ drift operators generalize the basic operators P, R and L of the Neo-Riemmanian triadic progressions and the maximally smooth cycles of the 12-TET system to any higher dimensional space.

Keywords: chord spaces; generalized Tonnetze; orbifold; voice leading; modes; symmetry group

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1. Introduction

Chords generated from symmetric circular interval series are of great music interest and have been largely studied (e.g., Forte 1973; Chrisman 1977; Lewin 1987; Cohn 1996; Douthett and Steinbach 1998; Tymoczko 2006; Nobile 2013). The basic harmonic structure of trichords[1] in Western tonal music is the starting point of two ways of analyzing music, one, the Schenkerian theory for the deep structure of tonal music, and the other, the Neo-Riemmanian theory for transformations and voice leading over the Tonnetz, which can also be extended to non-tonal music. In the current work, we propose a new and simple formalism to study the structure of chords in an n-TET tuning system and their transformations over the generalized Tonnetz (Tymoczko 2012) and its dual diagram, the chord network, generated by a given interval structure. In extended just intonation a mathematical model for navigation through a complex space of harmonies was proposed by Žabka (2017). Ours is an alternative approach to the Pitch-Class (PC) set theory for atonal music (Forte 1973), based on the notion of *mode* used by musicians such as Messiaen (1944) and Vieru (1980): an ordered k-tuple of PCs from the n-TET scale defines a directed k-chord (a rolled chord), which is composed of a *root* (the starting note) and a *mode* (a partition of the octave). The rotations of a directed chord define an equivalence class, and the shifts[2] of a mode define a mode class. The notes of a directed chord, as a set where order does not matter, form a chord.

Starting by the particular case of trichords, we describe the structure of the chord cell and the *tonal cell*, which are mutual combinatorially dual polytopes. In the 12-TET system, the Tonnetz associated with the mode [3,4,5] generating the major and minor triads is used as a case example. It is worth mentioning theoretical works using alternative Tonnetze, such as Lewin (1998) with the mode [0, 1, 3] and Clough (2002) with the modes [1,4,7] and [2,3,7]. For an n-TET system, we introduce and formalize an algebra of chords for a *non-degenerate* Tonnetz, meaning that the generating mode has *non-two equal* mode intervals, so that transformations can be done in higher dimensions regardless of geometrical representations.

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[1]We use the term trichord as similar to triad, meaning a set of three notes, not necessarily in a particular segment of a scale. This meaning is extended to tetrachords, hexachords, etc.

[2]Mode shifts mean rotations of the mode intervals. Shifts will refer to modes and rotations to directed chords.

Transformations distinguish between operations on the root, such as inversion and translation, and operations on the mode, such as mode inversion, retrogradation, and shifts. Some operations are well defined only for directed chords and others for chords, which necessarily are composition of operations on root and mode.

Translations by mode intervals transform chords within the same mode class, either within the same chord cell or towards another cell. They are equivalent to a combination of mode shifts and chord rotations. However, changing the mode class, although for trichords an inversion suffices, for higher k-dimensional chords must be done through the $k-1$ transpositions of the mode intervals. Among these, a number $k-2$ is associated with one-step *drifts* along the edges within one chord cell, while the remaining drift sends the chord towards a non-congruent chord cell, which generalizes the leading tone exchange of the Neo-Riemmanian triadic progressions (Lewin 1987; Cohn 1996, 1997; Douthett and Steinbach 1998). In this way it is possible to identify in any dimension simple circuits (closed paths where no vertex is repeated) over the chord cell and over the chord network, such as those that generalize the maximal smooth cycles (Cohn 1996).

The paper is organized as follows. In Section 2, notation, definitions and basic concepts are introduced. Section 3 deals with the geometric aspects of trichords and its generalization to higher dimensions. Section 4 formalizes the algebra of chords, namely, operations on the root, on the mode, and on the whole directed chord, respectively. In Section 5 the algebra of chords is applied to the chord network, by analyzing specific transformations and pointing out the main properties. The honeycomb of trichords is used to exemplify how the algebra of chords operates. Finally, in the conclusions, we remark the main results and suggest some points to extend and develop for future works.

2. Preliminaries and definitions

2.1 *Mode*

The octave $\mathcal{O}_n$ containing the whole set of n notes of an equal temperament scale is represented as points evenly distributed along a circle of n units length, oriented clockwise, and beginning with 0. Numerically, a note of an n-TET system is a number in the set $\mathbb{Z}_n$ of integers modulo n.

A k-*mode* is defined by choosing a sequence of k positive integers, $[A_0,\ldots,A_{k-1}]$, so that $A_0+\cdots+A_{k-1}=n$. Therefore, each positive interval has, at most, $n-k+1$ units length and k is not greater than n. In addition, since an interval of n steps leaves invariant any note in the octave (it is equivalent to 0 steps), the whole octave $\mathcal{O}_n$ will correspond to the full mode $[0]=[*]$, where the asterisk means the amount left to fill a full octave. We shall assume that the indices of the mode intervals take values in $\mathbb{Z}_k$. The set of k-modes is notated as $M(n,k)$, where n is the *rank* and k the *dimension*.

We may need to work with modes of rank m, lower[3] than the rank n, i.e., with a part of a mode of rank n. In order to make such a distinction, a mode[4] $\mu=[A_0,\ldots,A_{k-1}]\in M(n,k)$ whose intervals sum 0 in $\mathbb{Z}_n$ will be called *full mode*. We can simplify the notation as $\mu=[A_0,\ldots,A_{k-2},*]$ since in this case one of the intervals is redundant. On the other hand, a mode $\lambda\in M(m,k)$, with $m<n$, will be called *partial mode*. We necessarily write the partial mode without asterisk, as $\lambda=[B_0,\ldots,B_{k-1}]$ with $B_0+\cdots+B_{k-1}=m<n$.

In addition, in an n-TET system, the notion of mode can also be generalized to a series of interval values giving several turns to the circle of the octave, i.e., with total size larger than n, although, in order to determine the notes of a chord, one must always consider $\bmod n$. Therefore, the dimension k of a *generalized mode* may be larger, equal, or lesser than n. Thus, any sequence with components taking values in $\mathbb{Z}_n$ can be interpreted as a set of intervals or as a set of PCs, that is, as a chord. This dual point of view was developed by Vieru (1980) and its mathematical aspects have been further studied, e.g., by Andreatta, Vuza, and Agon (2004). Our aim is not to use this duality but to focus in the interval structure of the chords. In such a case, any mode interval equal to 0 will be suppressed, since it does not provide any information about the chord.

[3]Full rank modes are also called successive interval-arrays (Chrisman 1971) or circular interval series (Chrisman 1977), while partial modes are called interval-arrays or interval series.

[4]Greek characters will be used to represent modes to easily distinguish them from chords and notes.

2.2 *Directed chord and chord*

The result of applying a full mode $\mu=[A_0,\ldots,A_{k-1}]\in M(n,k)$ to the *root* note $a_0\in\{0,\ldots,n-1\}$ is a *directed k-chord*. A directed chord of rank n and dimension k is expressed as

$$a=(a_0,\ldots,a_{k-1})=a_0|\mu \tag{1}$$

with[5] $a_1=a_0+A_0 \bmod n,\ldots,a_{k-1}=a_{k-2}+A_{k-2} \bmod n$; satisfying, in addition, the cyclic property $a_0=a_{k-1}+A_{k-1} \bmod n$. This construction grants the order clockwise direction along the circle $\mathcal{O}_n$ of the directed chord elements. Root and mode are the two components of a directed chord[6]. Hence, given a root a_0 and two full modes $\mu,\nu\in M(n,k)$, $a_0|\mu=a_0|\nu$ if and only if $\mu=\nu$.

All the directed chords containing the same notes as the directed chord a, i.e., which are similar to a except in the cyclic order of the notes, yield the same chord. Thus, a k-chord is defined as this equivalence class of directed k-chords, namely

$$\{a\}=\{a_0,\ldots,a_{k-1}\}=\{a_0|\mu\} \tag{2}$$

The set of directed chords belonging to the same chord class are the *rotations* (we also consider the identity or zero rotation). There are k possible rotations[7] of a directed k-chord a, that are written with a superindex a^i, $i=0,\ldots,k-1$. The directed k-chords of rank n are noted as $K(n,k)$, and its classes, the k-chords, are noted as $\mathcal{K}(n,k)$, where we operate, as usual in set theory, by defining subsets, unions, and intersections.

Any generalized mode $\mu\in M(m,k)$, $m\neq n$, can be *reduced* to a full mode $R(\mu)\in M(n,k)$ so that $\{0|\mu\}=\{0|R(\mu)\}$. Then, for any root a_0, $\{a_0|\mu\}=\{a_0|R(\mu)\}$ and, in addition,

LEMMA 2.1 *If μ,ν are generalized modes such that $\{a_0|\mu\}=\{a_0|\nu\}$, then $R(\mu)=R(\nu)$.*

2.3 *Submodes and supermodes*

Any full mode $\mu=[A_0,\ldots,A_{k-1}]\in M(n,k)$ can be decomposed into partial modes of lower rank and dimension by defining a *concatenation* or *composition* of modes, such as,

$$\mu=\mu_1\cdot\mu_2;\quad \mu_1=[A_0,\ldots,A_{i-1}];\quad \mu_2=[A_i,\ldots,A_{k-1}];\quad 1\le i\le k-1 \tag{3}$$

so that $\text{rank}(\mu_1)+\text{rank}(\mu_2)=n$, and $\dim(\mu_1)+\dim(\mu_2)=k$. Both partial modes μ_1 and μ_2 are then *complementary* in $M(n,k)$.

Any full mode μ can be totally decomposed into $k-1$ concatenations of single modes, $\mu=[A_0]\ldots[A_{k-1}]$, whose intervals equal n (the dot for the composition will be usually omitted). In general, if $\mu_1=[B_0,\ldots,B_{k_1-1}]$ and $\mu_2=[C_0,\ldots,C_{k_2-1}]$ are generalized modes, the composition

$$\mu_1\cdot\mu_2=[B_0,\ldots,B_{k_1-1}][C_0,\ldots,C_{k_2-1}] \tag{4}$$

is defined as the mode that produces the partition of the octave resulting from the successive application of the intervals of μ_1 and μ_2. The composition of generalized modes has a monoid structure, i.e., is associative and has an identity element, which is [0].

A *non-commutative sum* of generalized modes is defined as follows,

$$\mu_1+\mu_2=[B_0,\ldots,B_{k_1-1}+C_0,\ldots,C_{k_2-1}] \tag{5}$$

which has also a monoid structure[8] with identity element [0].

LEMMA 2.2 *The* submodes *of a full mode $\mu=[A_0]\ldots[A_{k-1}]\in M(n,k)$ are obtained as $\lambda=[A_0]\circ\ldots\circ[A_{k-1}]$, where each symbol $\circ$ means either a composition or a sum.*

In this way we will obtain a number of full modes of the same rank but of different dimension, which may vary from 1 to k. When both modes μ and λ are applied to the same root a_0, the chord $\{a_0|\lambda\}$ gives

[5] The expression $r=p+q \bmod n$, although written without parentheses, will mean that r is the reminder in the Euclidean division of $p+q$ by n, otherwise we would write $r=p+(q \bmod n)$.

[6] In general, directed chords will be notated with lowercase Latin letters, without subindices. Subindices will refer to their notes.

[7] While dealing with directed chords, chords, and modes of dimension k we assume that the *indices* for notes a_i, interval modes A_i, and chord rotations a^i are defined in the group $(\mathbb{Z}_k,+)$.

[8] In general, if μ_1 and μ_2 are generalized modes, $\mu_1\cdot\mu_2\neq R(\mu_1)\cdot R(\mu_2)$, $R(\mu_1\cdot\mu_2)\neq R(R(\mu_1)\cdot R(\mu_2))$, $\mu_1+\mu_2\neq R(\mu_1)+R(\mu_2)$, $R(\mu_1+\mu_2)\neq R(R(\mu_1)+R(\mu_2))$; but full modes with the operations $\mu_1\odot\mu_2\equiv R(\mu_1\cdot\mu_2)$, $\mu_1\oplus\mu_2\equiv R(\mu_1+\mu_2)$ are a commutative group.

a partition of the octave less fine than the chord $\{a_0|\mu\}$. Then, we write $\lambda \prec \mu$. Conversely, we say that μ is a *supermode* of λ, and write $\mu \succ \lambda$.

PROPOSITION 2.3 *The composition and the sum of* full *modes satisfy* $R(\mu_1 \cdot \mu_2) = R(\mu_1 + \mu_2)$.

Proof. When applying the generalized modes $\mu_1 \cdot \mu_2$ and $\mu_1 + \mu_2$ to the root 0, they describe the following similar chords,

$$\{0|\mu_1 \cdot \mu_2\} = \{0|[A_0,A_1,\dots,A_{i-2},A_{i-1}] \cdot [B_0,B_1,\dots,B_{j-2},B_{j-1}]\} = \{0,A_0,A_0+A_1,\dots,A_0+\dots+A_{i-2},0,B_0,\dots,B_0+\dots+B_{j-2},0\}$$
$$\{0|\mu_1 + \mu_2\} = \{0|[A_0,A_1,\dots,A_{i-2},A_{i-1}+B_0,B_1,\dots,B_{j-2},B_{j-1}]\} = \{0,A_0,A_0+A_1,\dots,A_0+\dots+A_{i-2},B_0,\dots,B_0+\dots+B_{j-2},0\}$$

Hence, according to Lemma 2.1, the respective modes have the same reduced form. ■

COROLLARY 2.4 *If* μ_1,μ_2 *are full modes,* $\{a_0|\mu_1 \cdot \mu_2\} = \{a_0|\mu_1 + \mu_2\} = \{a_0|\mu_1\} \cup \{a_0|\mu_2\}$.

2.4 *Complementary and completed modes*

For any rank $m < n$, a partial mode $\mu = [A_0,\dots,A_{p-2}] \in M(m,p-1)$ can be *completed* on the right-hand side up to a full mode in $M(n,p)$ as $[\mu,*] = [A_0,\dots,A_{p-2},*] = [A_0,\dots,A_{p-2},n-m]$. We express this mode as above or, alternatively, as a concatenation of partial modes, $[\mu,*] = \mu \cdot [n-m]$. Similarly, the partial mode can be completed on the left-hand side as $[*,\mu] = [n-m] \cdot \mu$.

Any directed chord $a = a_0 \,|\, [A_0,\dots,A_{p-2}]$ obtained from a partial mode can obviously be written as $a = a_0 \,|\, [A_0,\dots,A_{p-2},*]$, in terms of the completed mode to the right. Therefore, the directed chord a may be written in either of the forms $a = a_0|\mu = a_0|[\mu,*]$.

Any full mode can be written as the concatenation of two *complementary* partial modes, $\mu \cdot \kappa \in M(n,p)$, with $\mu \in M(m,q)$, $n > m$ and $p > q$, and $\kappa \in M(n-m,p-q)$. Hence, the composition of partial modes can be expressed in terms of their completed modes as

$$\mu \cdot \kappa = [\mu,*] \cdot [*,\kappa] \tag{6}$$

According to Proposition 2.3, the sum of the completed modes satisfies

$$[\mu,*] + [*,\kappa] = [\mu,*] \cdot [*,\kappa] \tag{7}$$

On the other hand, we may assume that a full mode λ is equivalent to the completed modes $[\lambda,*]$ and $[*,\lambda]$. This allows to interpret Proposition 2.3 as a particular case of Eq. 7. In other words, Eq. 7 is valid for *complementary* partial modes, as well as for full modes.

For a chord $\{a_0|\mu \cdot \kappa\} \in \mathcal{K}(n,p)$ generated by the composition of two complementary modes, by taking into account Eq. 6, we obtain a particular case of Corollary 2.4, $\{a_0 \,|\, \mu \cdot \kappa\} = \{a_0 \,|\, [\mu,*]\} \cup \{a_0 \,|\, [*,\kappa]\}$. Obviously, the modes $[\mu,*]$ and $[*,\kappa]$ are submodes of $\mu \cdot \kappa$, and the chords $\{a_0|[\mu,*]\}$ and $\{a_0|[*,\kappa]\}$ are subchords of $\{a_0|\mu \cdot \kappa\}$.

2.5 *Mode shifts and directed chord rotations*

The equivalence relation given by the rotations of the directed chords is now transferred to the modes. Thus, we define the *mode class* with regard to the shifts of a mode. For a full mode $\mu^0 \equiv \mu = [A_0,\dots,A_{k-1}] \in M(n,k)$, a single *shift* is a one-step cyclic permutation[9] of the mode intervals, such as $\mu^1 = [A_1,\dots,A_{k-1},A_0]$. An m-shift (shift of m steps) of μ, with $m \in \mathbb{Z}_k$, is defined as

$$\mu^m = [A_m,A_{m+1},\dots,A_{m-1}] \tag{8}$$

For $m = k$ the mode becomes the original mode or the 0-shift of μ. All the shifts of a mode form a mode class. Then, we are able to write the k equivalent *rotations* of a directed chord $a = (a_0,\dots,a_{k-1})$ by applying the elements of a mode class to a number of k different roots, as follows

$$a^m = a_m|\mu^m = (a_m,\dots,a_{m-1}) \tag{9}$$

[9]A single shift is obtained by the permutation $\sigma = (k,1,\dots,k-1)$ of the symmetric group $\mathcal{S}_k$ with $k!$ elements. Since $\sigma^k = 1$, a shift generates a cyclic subgroup $\langle\sigma\rangle$ of $\mathcal{S}_k$ of order k, which provides a number $(k-1)!$ of cosets in the quotient group $\mathcal{S}_k/\langle\sigma\rangle$. The group $\mathcal{S}_k$ may be obtained from generators in several ways, e.g., from the product of transpositions in the form $(1,i)$, $1 < i \leq k$, or from products of σ and the transposition $(1,2)$.

with indices in $\mathbb{Z}_k$. For a directed chord a^m within the class of the chord $\{a\}$, we notate $a^m \sim \{a\}$ or $\{a\} \sim a^m$ to mean that we take the specific directed chord as a representative of the class.

Depending on the interval values, it is possible to get less than k different shifted modes. For example, if $A_0 = \cdots = A_{k-1}$, all the shifts produce the same mode, although there are still k rotations of any directed chord. The number of rotations is associated with the k different roots where the shifted modes are applied, and not with the number of elements of the mode class.

The set of mode classes of $M(n,k)$ with regard to the shifts[10], for all the possible interval values satisfying $A_0 + A_1 + \cdots + A_{k-1} = n$, is notated as $\mathcal{M}(n,k)$. The set of mode classes for the intervals of a particular mode $\mu \in M(n,k)$ is notated as $\mathcal{C}(\mu)$. The mode class of μ, is written as $\mu^S \in \mathcal{C}(\mu)$ and is composed of the non-equal shifts μ^m for $m \in \{0,\dots,k-1\}$, i.e., is a *fixed necklace* of length k. The number of elements composing a mode class μ^S is notated as $s(\mu^S)$, which is equivalent to the number of different shifts $s(\mu)$ of any mode of the class. If the mode intervals are non-two equal, the notes composing a directed chord $a_0|[A_0,\dots,A_{k-1}]$ are univocally determined. In addition, the mode classes μ^S do not match, i.e., $s(\mu) = k$, and $\mathcal{C}(\mu) = (k-1)!$. If the mode intervals are coprime, the number of different chords in the chord network is $n(k-1)!$ and contains all the notes of the n-TET scale.

2.6 *Uniqueness of chords*

A chord can be defined from several directed chords generated from any mode of one mode class applied to a particular root. If the directed chords are written clockwise direction along the circle of the octave, by starting at any note of the chord, the difference between two directed chords defining the same chord is only a rigid rotation of notes. This is given through the following equivalent conditions.

LEMMA 2.5 *Two directed chords $a_0|\mu$ and $b_0|\nu \in K(n,k)$ produce the same chord $\{a_0|\mu\} = \{b_0|\nu\}$ if and only if* $\exists m \in \mathbb{Z}_k$ so that $b_i = a_{m+i}, \forall i \in \mathbb{Z}_k$.

LEMMA 2.6 *Two directed chords $a_0|\mu$ and $b_0|\nu \in K(n,k)$ produce the same chord $\{a_0|\mu\} = \{b_0|\nu\}$ if and only if* $\exists m \in \mathbb{Z}_k$ so that $b_0 = a_m$ and $\nu = \mu^m$.

2.7 *Inverted modes*

While any full mode $\mu = [A_0, A_1, \dots, A_{k-1}] \in M(n,k)$, with positive intervals, is clockwise directed, a *negative mode* $-\mu = [-A_0, -A_1, \dots, -A_{k-1}]$, also referred to as *inverted mode* of μ, defines a symmetric partition of the octave *anticlockwise direction* when it is applied to the same root.

The negative mode $-[A_0,A_1,\dots,A_{k-1}]$ defines the same partition of the octave that the composition of positive modes $[n - A_0][n - A_1]\cdots[n - A_{k-1}]$. But $[n - A_0, n - A_1, \dots, n - A_{k-1}]$ is not a full mode, since $(n-A_0)+(n-A_1)+\dots+(n-A_{k-1}) = (k-1)n$, instead of equal n. It is a generalized mode. Anticlockwise direction, however, the intervals of the negative mode $-\mu$ do equal $-n$. Therefore, we may speak of a *negative full mode* in the set $M(-n,k)$ when it satisfies such a condition. In this case, when it is applied to a root, a negative mode determines an anticlockwise directed chord. The opposite k-chords have similar properties than the directed chords with regard to the chords they generate. On the other hand, the full mode

$$\overline{\mu} = [A_{k-1},\dots,A_1,A_0,] \tag{10}$$

describes a chord with similar intervals than $-\mu$ when it is applied to the same root, although clockwise direction. Hence,

PROPOSITION 2.7 *The mode $\overline{\mu}$ is the reduced form of $-\mu$, i.e., $\overline{\mu} = R(-\mu)$.*

Proof. To determine the relative partition induced in the octave by the mode $-\mu$ (except rigid rotations) we

[10] In a general, particular families of music objects can be defined according to their behavior with regard to some transformations. For example, a k-chord is defined as a subset of k notes, regardless their order. Therefore, a k-chord is any family of k-tuples of notes which is invariant or closed with regard to permutations. If a k-chord is defined from directed chords, i.e., a k-tuple of notes arranged clockwise direction on the octave, then a chord is a family invariant under rotations, either direct or inverse. Similarly, a mode class is defined as a family of modes which is invariant with regard to shifts. Then there is a group G that acts on a set C by transforming their elements, such as symmetry transformations, and particular families of elements in C are invariant with regard to specific transformations of G. Redfield-Pólya's theorem allows to enumerate in a general way the classes resulting from the action of the group G on C. In Fripertinger and Lackner (2015) these techniques are thoroughly studied.

apply the generalized mode to the root 0. Bearing in mind that the addition of the positive mode intervals equals n and that any integer number of full octaves can be ignored, we write the notes determining the partition, which, in the end, define a directed chord with the notes arranged clockwise,

$$\begin{aligned}
&0|[n-A_0,n-A_1,\dots,n-A_{k-1}]=(0,n-A_0,2n-A_0-A_1,\dots,kn-A_0-A_1-\dots-A_{k-2})=\\
&=(0,n-A_0,n-A_0-A_1,\dots,n-A_0-A_1-\dots-A_{k-2})=\\
&=(0,\cancel{A_0}+A_1+\dots+A_{k-1},\cancel{A_0}+\cancel{A_1}+A_2,\dots+A_{k-1},\dots,\cancel{A_0}+\cancel{A_1}+\dots+\cancel{A_{k-2}}+A_{k-1})=\\
&=(0,A_{k-1},A_{k-1}+A_{k-2},\dots,A_{k-1}+A_{k-2}+\dots+A_0)=0|[A_{k-1},\dots,A_1,A_0]
\end{aligned}$$

■

COROLLARY 2.8 $\mu=R(-\overline{\mu})$ *and* $\{a_0|-\overline{\mu}\}=\{a_0|\mu\}$.

The mode class of an inverted mode is the one corresponding to its reduced form.

2.8 *Relative inverted chords*

As a consequence of the above result, the retrograde mode of $\overline{\mu}$ is $-\mu$, so that the inverted mode of μ has two expressions: the positive mode $\overline{\mu}$ and the negative mode $-\mu$. If they are applied to the root a_0, we get two directed chords: the *relative inverted*, $a_0|\overline{\mu}$, read clockwise; and the *relative mirror* $a_0|-\mu$, read anticlockwise. Therefore, the full modes $\overline{\mu}$ and $-\mu$ applied to the root a_0 define the same chord,

$$\{a_0|\overline{\mu}\}=\{a_0|-\mu\} \tag{11}$$

Nevertheless, if we write the directed chord $a_0|\mu$ from another root to generate the same chord $\{a_0|\mu\}$, such as the rotation $a_i|\mu^i$, its relative mirror chord $\{a_i|-\mu^i\}$ is different from $\{a_0|-\mu\}$. In other words, the relative mirror and inverted chords depend on the directed chord we are using to express the chord; they are only defined for directed chords.

2.9 *Inverted chords*

The *negative inverted (or mirror) directed chord* of $a_0 \,|\, \mu$ is expressed from the negative mode as $-a_0|-\mu=-(a_0|\mu)$, whose notes are the complementary PCs of $a_0|\mu$. On the other hand, the *(positive) inverted directed chord* is written as $-a_0|\overline{\mu}$, from the positive inversion of the mode.

PROPOSITION 2.9 *The negative inversion of two directed chords, namely* $a_0|\mu$ *and* $b_0|\nu$*, which define the same chord* $\{a_0|\mu\}=\{b_0|\nu\}$*, produce two similar chords* $\{-a_0|-\mu\}=\{-b_0|-\nu\}$.

Proof. It can be easily deduced from the condition given in Lemma 2.5. We know that it exists a value $m\in\mathbb{Z}_k$ such that $(a_m,\dots,a_{m-1})=(b_0,\dots,b_{k-1})$, i.e., both directed chords match. Then, by taking complementary mode classes in both sides and arranging them clockwise along the octave, we get $(-a_{m-1},\dots,-a_m)=(-b_{k-1},\dots,-b_0)$. Hence both mirror directed chords define the same chord. ■

Then, according to Eq. 11, we may speak of *inversion of a chord* as the result of applying the inversion or mirroring to any directed chord defining the same chord, that is, $-\{a_0|\mu\}=\{-a_0|-\mu\}=\{-a_0|\overline{\mu}\}$.

2.10 *Symmetric modes*

By expressing a directed chord $a=a_0|\mu$ as a mode applied to a root we are able to describe easily several well known families of chords. The simplest example is the family of chords obtained by maintaining the mode μ and by translating the root a_0 to any note of the scale. These directed chords have the same interval structure, although they differ in PCs. Another example is the family of directed chords obtained by maintaining the root a_0 and applying different shifts μ^i of the mode. In general, for a mode $\mu=[A_0,\dots,A_{p-1}]\in M(n,p)$, we notate $\mathcal{S}(\mu)$ the family of its *symmetric modes*, that is, the set of modes containing all the permutations of the intervals of μ, containing the classes with regard to shifts, $\mathcal{C}(\mu)$. The cardinality of $\mathcal{S}(\mu)$ depends on the number of equal intervals in μ. If they are all different, $\#\mathcal{S}(\mu)=p!$. If μ has $q\le p$ intervals $A_i, 0\le i\le q-1$, each one repeated n_i times, $\sum_{i=0}^{q-1}n_i=p$, then

$$\#\mathcal{S}(\alpha)=\frac{p!}{n_0!\cdots n_{q-1}!} \tag{12}$$

On the other hand, it is possible to compute the cardinality of $\mathcal{S}(\mu)$ from the mode classes $\mathcal{C}(\mu)$ it contains. Then, since $\mathcal{S}(\alpha)=\mu_1^S\cup\dots\cup\mu_m^S$ for $\mu_i^S\in\mathcal{C}(\mu)$, by taking into account the number of shifts in each mode class, we have $\#\mathcal{S}(\mu)=\sum_{i=1}^{m}s(\mu_i^S)$.

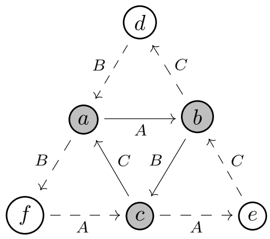

Figure 1. Chord extension $[A,B,C]$ formed by the chords sharing two notes with the initial chord $a|[A,B,C]$ (notes in gray).

3. Geometry of chords

3.1 *Trichords*

Tymoczko (2006, 2012) provide comprehensive studies on the chord geometry. Some of these geometrical concepts appear in the current work to be be studied from an algebraic approach. Therefore, we briefly review the simpler case of the trichords in an n-TET system.

The geometric relationship between one trichord and the three different chords sharing two notes with it is displayed in the graph of Fig. 1. The central triangle of the extension contains the initial directed chord[11] $(a,b,c) \in K(n,3)$, generated by the full mode $[A,B,C] \in M(n,3)$, that we assume with non-two equal intervals (particular cases such as $[A,B,A]$ and $[A,2A,A]$ are easily derived and lead to degenerate Tonnetze to be studied separately). It is surrounded by three triangles sharing an edge with it, corresponding to the chords generated by the shifts of the class $[A,C,B]$.

One chord together with their adjacent chords will be referred to as *chord extension*. This structure may be repeated around each new triangle to form a *tonal network* or Tonnetz originated in the chord $\{a,b,c\} \sim a|[A,B,C]$ with alternance of the two mode classes of the modes $[A,B,C]$ and $[A,C,B]$. The modes of the first class are clockwise directed in the triangles shaped as ∇, while the modes of the second class are anticlockwise directed in the triangles shaped as Δ. For trimodes, we shall say that the former are positive and the latter negative, although this cannot be generalized to higher-order modes. For instance, the symmetric modes of a tetrachord have up to 6 mode classes.

Since $A+B+C=n$, there are actually two degrees of freedom in constructing such a network, allowing us to draw those triangles on a surface. One simple representation consists of, starting with the note a, to take increments by A mod n units along the horizontal direction rightward, and by B mod n units vertically downward, obtaining thus an unbounded bidimensional array of nodes with coordinates (iA,jB) mod $n; i,j \in \mathbb{Z}$, with the center $(0,0)$ in a, that are associated with the notes

$$m_{ij} = a + iA + jB \bmod n; \quad i,j \in \mathbb{Z} \tag{13}$$

When closing one triangle by adding an interval of C units, the octave becomes completed.

The tonal network is periodic in both main directions, and diagonally as well, although the full set of notes of the n-TET system arise in any direction only when the corresponding interval is coprime with n. This structure is equivalent to a Cartesian product of two circles, one for the notes iA mod n and the other one for the notes jB mod n, so that the tonal network can be drawn on a torus. More precisely, the whole notes $k=0,\ldots,n-1$ of the tonal system are represented in the torus if, and only if, $k=a+xA+yB+zn$; for $x,y,z \in \mathbb{Z}$. We may clearly assume the origin at $a=0$. Since the three intervals equal n, the above relationship can be written as

$$k = pA + qB + rC; \quad p,q,r \in \mathbb{Z}; k=0,\ldots,n-1 \tag{14}$$

Therefore, according to Bézout's identity for three integer, in order to express all the notes from any of the intervals it is required that $\gcd(A,B,C)=1$. If $\gcd(A,B,C)=d$, then d is the smallest positive integer that can be expressed in the combination of Eq. 14, which provides a number of $v=\frac{n}{d}$ different notes and vertices for the Tonnetz. Since each mode class generates a unique chord for every note of the n-TET system, if the Tonnetz has v vertices, it is then composed of $f=2v$ different trichords, which are the

[11] To simplify the notation, the chord and mode components of the cases shown in graphs and figures will be without subindices.

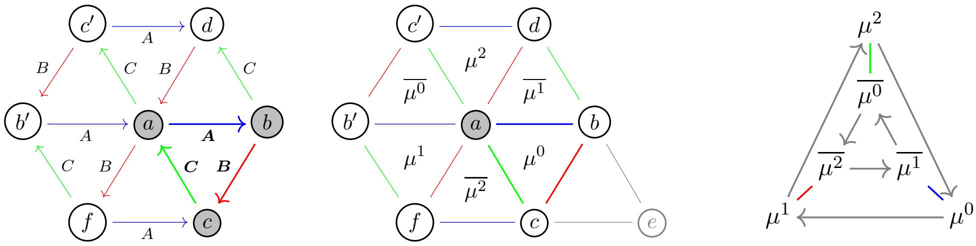

Figure 2. Two ways to represent the hexagonal tonal cell of the trichords originated by the directed chord $a|[A,B,C]$ (in boldface). On the right, Cayley graph of $\mathcal{S}_3$. The triangular diagrams connect the modes of one mode class by a shift. In the vertices, the modes are connected with the color of the shared first interval, with the other intervals are transposed.

faces of the regular tessellation of the torus with triangles. It is worth noticing that three vertices define a face of the triangle only when they are connected by three edges corresponding to the three consecutive mode intervals (ranging one octave), all of them with either positive orientation or negative orientation. Positive and negative alternated intervals do not form a loop. The number of edges in the tessellation is $e=\frac{3}{2}f=3v$, since each triangle has three edges and each edge is shared by two triangles. Thus, we obtain the Euler characteristic of a torus, $\chi=v-e+f=0$.

3.2 *Tonal cell [A,B,C]*

The Tonnetz builds up a mosaic of hexagonal cells composed of six chords in the shape of triangles with a common vertex. These, chords are generated by the symmetric modes of the initial chord $a|[A,B,C]$. Each node of the network is the center of a cell, which contains the root, which is common note of the six chords. In a general case, given a note a_0 and a mode $\mu\in M(n,k)$, the notes of the Tonnetz that are vertices of the chords $\{a_0|\nu\}$, with $\nu\in\mathcal{S}(\mu)$, will be referred to as the *tonal cell* around a_0.

In addition, these chords share another note (and the corresponding interval) with their neighbors. Therefore, the symmetric modes of the original mode $[A,B,C]$ determine a family of neighbor chords that, in musical terms, are harmonically related. For the chord $a|[A,B,C]$, the hexagonal tonal cell shown on the first diagram of Fig. 2 is composed of six different chords and symmetric modes. The symmetric modes are, one class, $\mu=[A,B,C]$, $\mu^1=[B,C,A]$, $\mu^2=[C,A,B]$; and, the other class, $\overline{\mu}=[C,B,A]$, $\overline{\mu^1}=[A,C,B]$, $\overline{\mu^2}=[B,A,C]$. Zero rotations may be notated as $\mu^0\equiv\mu$. With this notation we get the second diagram of Fig. 2.

Out of the tonal cell centered in a, the only chord sharing two notes with the initial chord (marked in boldface) is $\{b,c,e\}$, which is placed in the lower-right vertex of the extension. Therefore, this is the closest chord to the initial chord in the path towards another cell not containing the root a. From this viewpoint, that direction indicates the smoothest voice leading path (Douthett and Steinbach 1998; Tymoczko 2006; Callender, Quinn, and Tymoczko 2008) connecting chords of different tonalities. The note e, that does not belong to the tonal cell of the root, corresponds to the leading tone of the major diatonic scales and to the subtonic of the minor scales.

3.3 *Chord cell*

The dual network of Fig. 2 is Fig. 3, where connections between chords are shown instead of connections between tones. More precisely, each triangle representing a chord in the former graph is now a vertex of the latter, and the edges of the triangles in the first graph become vertices in the second one. On the other hand, one note in Fig. 2 becomes the center of the six vertices of one hexagon in Fig. 3, which are the chords sharing this note. In general, given a note a_0 and a mode $\mu\in M(n,k)$, the set of chords $\{a_0|\nu\}$, with $\nu\in\mathcal{S}(\mu)$ will be referred to as the *chord cell* (or cell of chords), by labeling it with the central note a_0.

In Fig. 3 all the chords are expressed in terms of a shift or an inversion of the same mode μ applied to the same root. The whole chord cell with root a can be transported to the same or another cell by applying a *translation*[12] of the root by a single interval A, B, and C, or its opposite, according to the directions of the arrows in the graph. Since each vertex belongs to three different chord cells, the translation of a chord may

[12]Musicians call transpositions to translations, however this may be confusing since in mathematics a transposition is a permutation which exchanges two elements and keeps all others fixed.

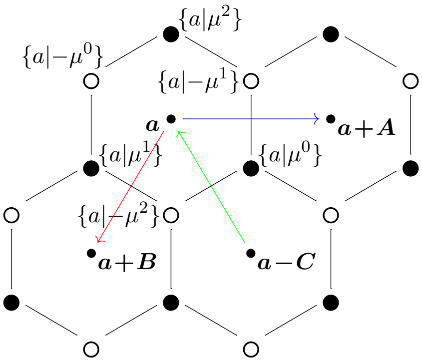

Figure 3. Chord cells composed of six chords, half obtained from directed chords with positive modes (black vertices), and half with negative modes (white circles) in opposite vertices of the cell.

lead either to contiguous cells or to the same cell.

In general, the translations along chord cells of the chord $\{x|\lambda\}$ are defined by successive transformations of the root, according to

$$\tau_u x = x + u \bmod n; \quad u = \pm A, \pm B, \pm C \bmod n \tag{15}$$

that preserves the mode. In addition, the new chord maintains the same relative position in the new cells. These transformations acting on the root have the structure of an abelian group with $\tau_{u+v} = \tau_u \tau_v = \tau_v \tau_u$ and identity element $\epsilon = \tau_0$. Thus, for trichords, all the chords in the network originated by the directed chord $a|\mu$ can be expressed from two kinds of transformations over the chord elements: (1) translations τ_u of the root in any of the two independent directions, by preserving the mode; and (2) inversion of the mode, by preserving the root of the chord.

3.4 *Major and minor chords in a 12-TET scale*

Let us see a well known example of Tonnetz, the one associated with the chord $\{0,4,7\} \sim 0|[4,3,5]$, the major triad. This tonal network is composed of all the major chords of the 12-TET system generated by the mode $[4,3,5]$, as well as all the minor chords generated by the mode $[3,4,5]$. It is build by beginning at the note 0 and, since $\gcd(4,3,5) = 1$, all the notes of the scale appear in this Tonnetz, so that it could be built by beginning at any note. The notes composing the 12-TET scale are obtained separated by intervals of major and minor thirds of 4 and 3 units (semitones).

In Fig. 4 the Tonnetz containing all the major and minor chords of the 12-TET system is represented by explicitly writing the name of the notes. This is a double diagram that displays both the tonal network, with notes in gray, and the chord cells, with chords in black, uppercase letters for the major trichords and lowercase for the minor ones. Blue, red, and green arrows indicate translations of 4 (a major third), 3 (a minor third), and 5 (a fourth) units, respectively. The inverse green arrows correspond to a translation of 7 (a fifth) notes. Each chord is the center of the triangle formed by the notes it contains, and each note is the center of the hexagonal cell of chords containing it.

The connected trichords share two notes. The non-shared notes of two connected trichords differ in 2 semitones for chords vertically connected and in 1 semitone for all the other chords. A closed path in the Tonnetz along non vertically connected chords is known as a *maximally smooth cycle* (Cohn 1996). It is straightforward seeing from Fig. 4 which notes are involved in each maximally smooth cycles for major and minor chords, and which are the common notes among these cycles. Other properties may be easily derived from the graph. For instance, by starting a continuous path from any chord in the Tonnetz, we are able to see which chords can be reached by changing only one note in each step. If the number of steps or edges between two chords is taken as the length of a path over the Tonnetz, then the length of a loop, by beginning and ending in the same chord without repeating any chord, is 6. There are four different maximally smooth cycles of length 6, along the non-vertical polygonal lines. The minimum length between two chords defines a distance on the Tonnetz. In this example, any two chords are connected by a path of at most of 5 steps. For example, the chord more distant to the F major chord is the d$^\sharp$ minor chord, at 5 steps distance. For higher-dimensional chords, these and other properties of the generalized Tonnetz, such as paths connecting

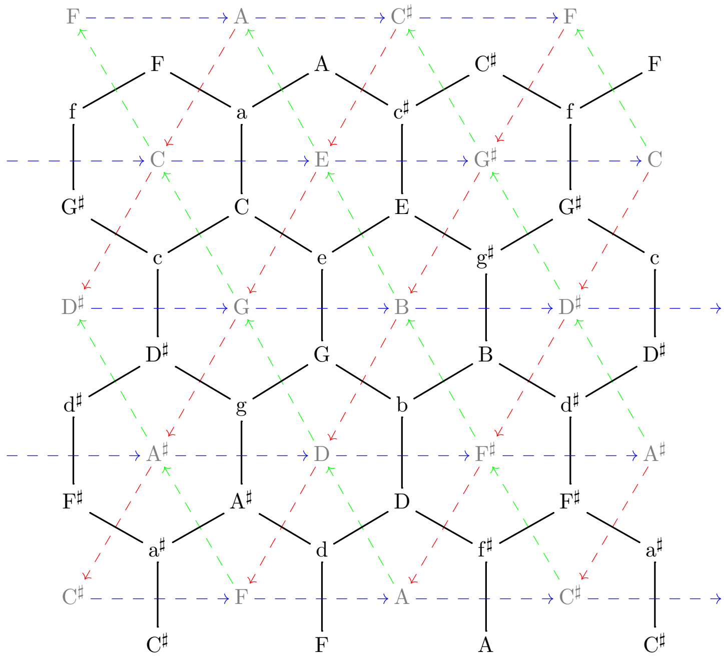

Figure 4. Chord cells and Tonnetz from the chord $\{0|[4,3,5]\}$ in a 12-TET system.

all the chords without repeating any one, can be further studied with the help of an algebra of chords.

3.5 *Higher-dimensional tonal network*

The Tonnetz associated with a mode μ can be generalized to chords and modes of any dimension $k \in \mathbb{N}$. The symmetric modes $\mathcal{S}(\mu)$ contain shifts of several mode classes obtained from transpositions of the mode intervals. By generalizing Eq. 14, the tonal network associated with a k-mode $\mu = [A_0,\ldots,A_{k-1}] \in M(n,k)$, starting at the node a_0, is composed of all the notes in the form $\tau_{c\cdot\mu}\, a_0$, according to the dot product $c\cdot\mu = \sum_{i=0}^{k-2} c_i\, A_i \bmod n;\quad c = (c_0,\ldots,c_{k-2}) \in \mathbb{Z}^k$, where the sum is extended to a subset of $k-1$ mode intervals, since one of them is complementary to the others in $\mathbb{Z}_n$. At any node b_0 of the Tonnetz, all the chords having the note b_0 can be generated as a directed chord $b_0\,|\nu$, with $\nu \in \mathcal{S}(\mu)$. Therefore, the *tonal cell* centered in b_0 is composed of the notes forming the $k!$ directed chords generated by the symmetric modes of μ applied to b_0. The chords around b_0 form *chord cell* associated with this root in the dual network of the Tonnetz.

As in the case described by Eq. 13, the Tonnetz associated with a mode with non-two equal intervals can be formally interpreted as a set of points scattered on a k-dimensional torus. At first, the notes may be represented as an unbounded $(k-1)$-dimensional array of points according to the coordinates given by the intervals of the submode $\mu' = [A_0,\ldots,A_{k-2}] \in M(n,k-1)$, since the interval $A_{k-1} = -(A_0+\ldots+A_{k-2}) \bmod n$ is uninformative. The intervals are then associated with independent *main directions* of the array, so that we may speak of *vector intervals*. We shall call the direction associated with the last mode interval of a directed chord the *returning direction*. Notice that the returning direction A_{k-1} is linearly independent from any single main direction A_i, $i \neq k-1$. Similarly, it is independent of any subset of different intervals A_i, A_j; $i,j \neq k-1$; up to a subset formed by $k-2$ non-equal intervals without including the returning direction.

By placing a note a_0 at the origin of the array, any point of the array with coordinates $(p_0 A_0,\ldots,p_{k-2}A_{k-2}) \bmod n$, with $p_0,\ldots,p_{k-2} \in \mathbb{Z}_n$, represents a note of the n-TET system, of value

$$m = a_0 + p_0 A_0 + \cdots + p_{k-2} A_{k-2} \bmod n \tag{16}$$

Since the structure is periodic in all directions, the array can be shaped as points over a k-dimensional torus[13], which is as a $(k-1)$-dimensional hypersurface within $\mathbb{R}^k$. In degenerate cases, where the generating

[13] According to Tymoczko (2012) the Tonnetz may be organized in other ways other than a generalized torus, depending on the properties to study.

mode has similar intervals, by identifying the similar notes on the torus, the Tonnetz may be converted into a band, a Möbius band, a circle, etc. Similarly as in §3.1, all the notes of the n-TET scale appear on the Tonnetz when $\gcd(A_0,\ldots,A_{k-1})=1$. In general, if $\gcd(A_0,\ldots,A_{k-1})=d$, then d is the smallest positive integer that can be expressed as the combination of Eq. 16, which provides a number of $v=\frac{n}{d}$ different notes and vertices of the Tonnetz.

A subset of vertices of the Tonnetz represents a directed k-chord only when they are connected according to a permutation of the $k-1$ intervals of the submode μ'. In addition, over the Tonnetz we may connect the vertices by using all the mode intervals of μ, either in the positive or the negative direction, which is tantamount to reading the mode clockwise or anticlockwise over the octave, although we cannot mix positive and negative intervals. The last and first vertices of every directed chord must also be connected in order to close the octave. This forms a simplex, which is the convex hull of its vertices. Similarly, its subchords are obtained by connecting the vertices along the corresponding returning directions to form the faces. These new edges are linear combination of the independent directions associated with the intervals of μ'.

3.6 *Generalized tonal cell*

The notes of the Tonnetz that are vertices of the chords $\{a_0|\nu\}$, with $\nu\in\mathcal{S}(\mu)$ and $\mu=[A_0,\ldots,A_{k-1}]$, were referred in §3.2 as the tonal cell around a_0. The number of vertices it contains may be computed as follows. We shall use only positive mode intervals when forming all possible subchords of the network with root a_0.

At one step away from the root, i.e., at the end of one edge, we may reach any vertex expressed as the second vertex of the directed 2-chords in the shape $a_0|[A_i,*]$, i.e., the vertices a_0+A_i. It can be done along the main directions, by choosing i among the indices $0,\ldots,k-2$; and also in the returning direction, with $i=k-1$. Remember that the returning direction is independent from any single main direction, although is a linear combination of all of them. This makes $\binom{k}{1}$ new vertices, in addition to the one corresponding to the root.

At two steps away from a_0 we may reach some new vertices expressed as the third vertex of the directed 3-chords such as $a_0\,|\,[A_i,A_j,*]$, with $i\neq j$ among the indices $0,\ldots,k-1$ that are associated with either the main directions or the returning direction. Once again, the path along these intervals follows two independent directions of the network. So, we reach $\binom{k}{2}$ new vertices at two steps away from the root.

This procedure may continue up to the kth vertex $a_0+A_{i_1}+\ldots+A_{i_{k-1}}$ of the directed k-chords, for a number of $k-1$ different indices among the values $0,\ldots,k-1$ by providing $\binom{k}{k-1}$ new notes. If one of the intervals is A_{k-1}, we bear in mind that this returning direction is still independent of the other $k-2$ main directions involved in the k-chord. This makes a tonal cell with a number $N_k=\sum_{i=0}^{k-1}\binom{k}{i}=2^k-1$ of notes, including the root.

3.7 *Generalized Tonnetz*

According to the preceding section, the geometric structure of a k-chord $\{a\}$ in the Tonnetz is a $(k-1)$-simplex, that is, any note of the chord can be connected with the others in the way described to form a subchord, which is a simplex of lower dimension. The k vertices and notes of the k-chord are the 0-faces of the $(k-1)$-simplex, the $\binom{k}{2}=\frac{k(k-1)}{2}$ edges are the 1-faces forming 2-chords, etc., and the $(k-1)$-face is the whole chord. Hence, each m-face $(m<k)$ is an m-simplex and is an $(m+1)$-chord, subchord of $\{a\}$.

By focusing on the vertices of the chord, the following aspects may be pointed out. The $(k-2)$-faces or *facets* of the chord $\{a\}$ are subchords containing $k-1$ notes. If one note not included in $\{a\}$ is added to one facet, we obtain a new k-chord, which is adjacent to $\{a\}$, i.e., shares the $k-1$ vertices of the facet. Therefore, the number of adjacent chords to $\{a\}$ is the same than the number of facets, that is $\binom{k}{k-1}=k$. This set of chords was called the *chord extension* in §3.1. In that case, the vertex to be added to the $(k-1)$-subchord is not arbitrary, since these chords belong to the Tonnetz generated by the intervals of μ. As explained in §2.10, only two different chords may have a particular subset of $k-1$ vertices in common. They are obtained by writing one chord as the directed chord $a=a_0|[A_0,\ldots,A_{k-3}][A_{k-2},A_{k-1}]$, so that the directed chord $a'=a_0|[A_0,\ldots,A_{k-3}][A_{k-1},A_{k-2}]$ is the only one that shares the $k-1$ first vertices of a. By rotations on the original directed chord we find the k chords sharing the other $k-1$ subsets of vertices.

The chord extension, i.e., the chord $\{a\}$ and its k adjacent chords, may be extended to all the chords over the Tonnetz. The $(k-1)$-simplices associated with the chords form a $(k-1)$-dimensional tessellation of the Tonnetz. The conglomerate of $k!$ simplices, which are the chords sharing one root, is the tonal cell.

If the Tonnetz is non-degenerate, these vertices are associated with different notes.

3.8 *Generalized chord network*

The topography of the chords in the Tonnetz can also be represented with the dual network diagram, the *chord network*, where the chords are associated with vertices instead of the notes. Under this viewpoint, the chords are vertices grouped according to the notes they share, that is, each note is in the center of a chord cell formed by the chords containing it. Each k-chord can then be rooted in anyone of its notes, so that it belongs to k chord cells. In other words, we may think of the k notes composing a k-chord as the center of k adjacent chord cells.

We now focus in chords containing the note a_0, i.e., the cell of chords labeled with the root a_0 in its center, so that it is possible to write all these chords from directed chords that have a_0 as root. Since the directed chords having the same root can be expressed from the symmetric modes $\mathcal{S}(\mu)$ as $a_0|[A_{\sigma(0)},\dots,A_{\sigma(k-1)}]$, with $\sigma\in\mathcal{S}_k$, there is a number of $k!$ chords in this cell. Among them, the number of different chords is given by Eq. 12. In general, the chord cell is a *permutahedron* of order k (e.g., De Loera, Rambau, and Santos 2010), i.e., a $(k-1)$-dimensional polytope embedded in a k-dimensional space, whose vertices and edges are isomorphic to the Cayley graph of the symmetric group $\mathcal{S}_k$.

Each couple of chords $\{x\}$ and $\{x'\}$ that have $k-1$ notes[14] in common and differ in one note (they share a facet in the tonal network) are adjacent chords. These chords are shared by $k-1$ congruent chord cells. A new chord $\{x''\}$, different from $\{x\}$ and $\{x'\}$, that shares a facet with $\{x'\}$ in the tonal network (hence differs in one note which does not belong to $\{x\}$, otherwise it would be $\{x''\}=\{x\}$) shares, at least, $k-2$ notes with $\{x\}$ in the chord network. Then, we can connect two chords that have a common facet in the tonal network with an edge in the chord network. Each edge is shared by $k-1$ chord cells and the cells on the opposite sides are labeled according to the note they differ.

Thus, from the tessellation of the tonal network with vertices as notes and $(k-1)$-simplices as k-chords, we obtain a dual tessellation, the chord network, with chords as vertices and notes as polytopes associated with the chord cells. In the former, the $(k-1)$-faces are k-chords, the $(k-2)$-faces are $(k-1)$-subchords, etc., and the 0-faces are notes. In the latter, the 0-faces are k-chords, the 1-faces are $(k-1)$-subchords, etc., and the $(k-1)$-faces are the notes labeling the chord cells. These structures are respectively co-dimensional to $k-1$.

Since the tonal cell around one root in the tonal network becomes the chord cell labeled as the root in the chord network, then the number $f_k=2^k-2$ of vertices around (excluding) the root of the tonal cell becomes the number of facets of the chord cell centered in the root.

3.9 *Chord cell facets*

In the chord cell a_0 containing the directed chord $a_0|[A_0,\dots,A_{k-1}]$, $k>1$, we pay attention to the chords that share the notes a_0 and a_0+A_0. These chords are generated from directed chords in the form $a_0|[A_0][A_{\sigma(1)},\dots,A_{\sigma(k-1)}]$ with $\sigma\in\mathcal{S}_{k-1}$. There are $(k-1)!$ of them along different directions of the chord network. They share a facet of the chord cell a_0, which is also a facet of the cell a_0+A_0. Since these notes differ in the interval A_0, we shall say that the facet is *orthogonal* to this main direction. Similarly, the chords that share the notes a_0 and a_0+A_1 are generated according to a mode composition $a_0|[A_1][A_{\sigma(0)},\cancel{A_1},\dots,A_{\sigma(k-1)}]$ with $\sigma\in\mathcal{S}_{k-1}$, where the component that is left out in the second partial mode is marked. There are $(k-1)!$ chords that belong to the facet shared by the cells a_0 and a_0+A_1, orthogonal to the main direction A_1. This may be done for each mode component A_i, until reaching the last facet, which is orthogonal to the returning direction A_{k-1}, shared by the cells a_0 and a_0+A_{k-1}. We call these k disjoint facets *front facets* of the chord cell a_0. By collecting the chords of the k front facets we get the full chord cell with $k!=k(k-1)!$ chords.

For $k>2$, we also look for the facets containing the chords that share the two notes a_0 and $a_0+\Delta$, with $\Delta=A_0+A_1$. In that case, these chords are generated as either $a_0|[A_0,A_1][\cancel{A_0},\cancel{A_1},A_{\sigma(2)},\dots,A_{\sigma(k-1)}]$ or $a_0|[A_1,A_0][\cancel{A_0},\cancel{A_1},A_{\sigma(2)},\dots,A_{\sigma(k-1)}]$ with $\sigma\in\mathcal{S}_{k-2}$, so that there are $2!(k-2)!$ of them. This can be done by selecting $\binom{k}{2}$ different couples of intervals for Δ, by determining a similar number of disjoint facets, so that collecting them we get the full set of chords in the cell $k!=\binom{k}{2}2!(k-2)!$.

Similarly, for $k>m$, the facet containing the chords that share the notes a_0 and $a_0+\Delta$, with

[14] We use the term note for a vertex of the tonal network, while the term vertex alone refers to the chord network.

$\Delta = A_0 + ... + A_{m-1}$ is composed of the chords $a_0|[A_{\rho(0)},...,A_{\rho(m-1)}][\cancel{A_0},...,\cancel{A_{m-1}},A_{\sigma(m)},...,A_{\sigma(k-1)}]$ with $\rho \in \mathcal{S}_m$ and $\sigma \in \mathcal{S}_{k-m}$, so that there are $m!(k-m)!$ of them in each facet.

In general, we may select $\binom{k}{m}$ subsets of m mode intervals for Δ, which will produce a similar number of disjoint facets. By collecting them, we always get the full set of $k! = \binom{k}{m} m!(k-m)!$ chords in the cell. In particular, the case $m = k-1$ leads to facets with a similar number of $(k-1)!$ chords as the front facets. Their last mode interval is a main direction or the returning direction. That is, the notes a_0 and $a_0 + \Delta$ differ in an interval $-A_i$. Hence these facets are also orthogonal to these directions, although they are in the negative part of the axes, by assuming the center of the cell a_0 in the origin. We call them *back facets*. Front and back facets are *principal facets* of the cell of chords. The other facets are the *mid facets*, existing only for $k \geq 4$. The way of moving along the chord network between two chords by maintaining the maximum number of common notes regardless tonality criteria is perpendicular to one facet. Such a movement implies changes between mode classes. Therefore, once again we obtain the cell of chords as composed of a number of facets

$$f_k = \sum_{m=1}^{k-1} \binom{k}{m} = \binom{k}{1} + \binom{k}{2} + ... + \binom{k}{k-2} + \binom{k}{k-1} = 2^k - 2$$

where the facets corresponding to $m = i$ and $m = k - i$ (indicated with arrows in the above equation) have the same number $(k-i)!i!$ of vertices. In particular, the number of principal facets in the chord cell is $2k$, corresponding to the first and last terms of the series. Then, if k is odd, there is a number $\frac{k-1}{2}$ of different chord configurations in the facets, i.e., the amount of different facets with regard to their number of vertices. If k is even, there are $\frac{k}{2}$ different types facets, with a different number of chords. Each chord of the cell is contained in each type of facet.

4. Algebra of chords

4.1 *Operations on the root*

4.1.1 Translations and inversions on directed chords. It is possible to transform chords over a two- or three-dimensional Tonnetz by using geometrical or graphical means (e.g., Gollin 1998; Jedrzejewski 2006; McCartin 2015, 2016), however, for higher dimensional networks it is necessary to define algebraically such a transformations. It is a mandatory reference to the work by Lewin (1987), one of the pioneers in applying group theory to chord transformations. Necessarily with some common notation, the chord operations of the current paper will use a simpler approach, by taking the advantage that we are always moving on the chord network generated by a fixed k-mode.

First, we center our attention in two operations on the root of a directed chord of an n-TET system. They will be noted by Greek letters. The first operation is the *inversion of the root* of a directed chord $a = a_0|\mu$, defined in operator form as $\iota(a_0|\mu) = -a_0|\mu$. Let us recall the equivalence $-a_0 \equiv n - a_0$, since $a_0 \in \mathbb{Z}_n$. If the identity operation on the root is expressed as ϵ, then $\iota^2 = \epsilon$. Hence, the group generated by ι is isomorphic to $\mathbb{Z}_2$. This operation is not defined for chords, since it results in different chords depending on the directed chord we apply it.

The other operation is the *translation of the root* of a directed chord, which induces translations of chords along the tonal network. For $u,v \in \mathbb{Z}_n$, according to the notation $\tau_u v = v + u \bmod n$, a translation by an interval $u \in \mathbb{Z}_n$ is the transformation $\tau_u(a_0|\mu) = \tau_u a_0|\mu$. In particular, when u matches an interval of the mode μ, we are able to represent geometrically the translation of the directed chord on the Tonnetz.

The following equivalences hold, $\tau_{u+v} = \tau_u \tau_v$, $\tau_{uv} = \tau_v^u = \tau_v^u$, so that we may write $\tau_u = \tau_1^u$. The composition of translations has the structure of a cyclic group inherited from the sum in $\mathbb{Z}_n$, generated by the element τ_1. The identity element is $\epsilon = \tau_0 = \tau_1^n$, and the composition satisfies the properties, $\tau_u \tau_v = \tau_v \tau_u$, $\tau_u(\tau_v \tau_w) = (\tau_u \tau_v)\tau_w$, $\tau_u \tau_{-u} = \epsilon$. Hence, the inverse translation of τ_u may be written in several ways, e.g., $\tau_{-u} = \tau_u^{-1} = \tau_1^{-u}$. It is straightforward to see that translations and inversions on the root of a directed chord are non-commutative[15], instead, they satisfy

$$\iota\tau_x = \tau_{-x}\iota \tag{17}$$

4.1.2 Translations on chords. For a non-null value $u \in \mathbb{Z}_n$, a translation τ_u on a directed chord $a_0|\mu$ generates, in general, a different chord, $\{\tau_u a_0|\mu\} \neq \{a_0|\mu\}$. When a translation is applied to two directed chords producing the same chord, namely $\{a_0|\mu\} = \{b_0|\nu\}$, it is straightforward to see that the translated chords are the same, i.e., $\{\tau_u a_0|\mu\} = \{\tau_u b_0|\nu\}$. Therefore, we may speak of *translation of a chord* as the result of applying the translation on any rotation of the chord. Hence, we shall consider $\tau_u\{a_0|\mu\} \equiv \{\tau_u a_0|\mu\}$.

4.1.3 Dependent translations. Translations may be expressed without subindices by using the notation $\tau[x](a_0|\mu) \equiv \tau_x a_0|\mu$, or, in general, for $m > 0$, $\tau^m[x](a_0|\mu) \equiv \tau_x^m a_0|\mu$ and $\tau^m[-x](a_0|\mu) = \tau^{-m}[x]a_0|\mu$. However, sometimes the translation is carried out by one or several mode intervals[16]. For example, if the ith interval of the full mode $\mu = [A_0, A_1, \dots, A_{k-1}] \in M(n,k)$ is notated as $[\mu]_i = A_i$, $i \in \mathbb{Z}_k$, then we will express a translation of the directed chord $a_0|\mu$ by this quantity simply as $\tau[\mu]_i(a_0|\mu) \equiv \tau[A_i]a_0|\mu$.

Since the full modes have a cyclic structure, it is possible to refer to an interval anticlockwise direction, with a negative index, by defining $[\mu]_{-i} \equiv [\mu]_{k-i} = A_{k-i}$, $i \in \mathbb{Z}_k$.

A *retrograde* translation by a mode interval is also possible, $\tau[-\mu]_i(a_0|\mu) = \tau[-A_i]a_0|\mu$. We assume the null translation as $\tau[\mu]_0 = \tau_0$ and we use the equivalence $\tau[-\mu]_i = \tau^{-1}[\mu]_i$ to refer to the translation to an interval of the mode μ instead of using its inverted mode. In addition, we can write the intervals of the positive inversion $\overline{\mu}$ by using the intervals of the mode μ, as $[\overline{\mu}]_i = [\mu]_{-i-1}$, so that we get the equivalence $\tau[\overline{\mu}]_i = \tau[\mu]_{-i-1}$.

4.1.4 Prograde translations by mode intervals. Composition of translations according to successive mode intervals are denoted as

$$\tau[\mu]_{i,\dots,i+j} \equiv \tau[\mu]_i \cdots \tau[\mu]_{i+j} = \tau[A_i] \cdots \tau[A_{i+j}] = \tau[A_i + \dots + A_{i+j}] \tag{18}$$

We pay attention to the root of chord rotations as defined in Eq. 9. The root of the ith rotation of a directed chord, namely $a^i = a_i|\mu^i$, satisfies $a_{i+1} = \tau[A_i]a_i = \tau[\mu]_i a_i$. Then, according to Eq. 18, the root of successive rotations is

$$a_{i+j} = \tau[A_i + \dots + A_{i+j-1}]a_i = \tau[\mu]_{i,\dots,i+j-1}a_i \tag{19}$$

bearing in mind that the whole set of intervals of the full mode satisfy $\tau[A_0 + \dots + A_{k-1}] = \tau_0$. In a similar way, translations on the root by a negative interval of the mode satisfy

$$\begin{gathered} a_i = \tau[-A_i]a_{i+1} = \tau[-\mu]_i a_{i+1} \\ a_i = \tau[-(A_i + \dots + A_{i+j-1})]a_{i+j} = \tau[-\mu]_{i,\dots,i+j-1}a_{i+j} \end{gathered} \tag{20}$$

by providing the roots of retrograde rotations of the directed chord. A particular case of Eq. 19 is

$$a_i = \tau[A_0 + \dots + A_{i-1}]a_0 = \tau[\mu]_{1,\dots,i-1}\,a_0 \tag{21}$$

where the successive translations begin in the first interval of the mode.

Nevertheless, a more compact notation can be used by writing

$$\tau[\mu]_p^q \equiv \tau[\mu]_{p,\dots,p+q-1} = \tau[A_p + \dots + A_{p+q-1}] \tag{22}$$

where q *denotes the number of mode intervals involved in the translation*. The foregoing expression is valid for positive and negative interval indices p. In general, since $\tau^m[x] = \tau[mx]$ for any $m \in \mathbb{Z}$, it is hold $\tau^m[\mu]_p^q = \tau^m[\mu]_{p,\dots,p+q-1}$. Then, by notating $\tau[\mu]_p^0 = \tau_0$, it is satisfied,

$$\tau[\mu]_p^q\,\tau[-\mu]_p^q = \tau_0 \tag{23}$$

By writing in a shorter form equations 19, 20, and 21, the *successive notes of a directed chord* are

$$a_i = \tau[\mu]_0^i\,a_0\,, \quad a_{i+j} = \tau[\mu]_i^j\,a_i\,, \quad a_i = \tau[-\mu]_i^j\,a_{i+j} \tag{24}$$

[15] In an n-TET system the translations form a group isomorphic to $\mathbb{Z}_n$ and the inversions are a group isomorphic to $\mathbb{Z}_2$. Then, the operations on the root generated by τ_1 and ι have the structure of a semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ with $2n$ elements $\{\tau_0, \tau_1, \tau_1^2, \dots, \tau_1^{n-1}, \iota, \tau_1\iota, \tau_1^2\iota, \dots, \tau_1^{n-1}\iota\}$, where $\mathbb{Z}_2$ acts on $\mathbb{Z}_n$ by inversion, since the generators of the factor groups satisfy $\iota\tau_1\iota = \tau_1{}^{-1}$. Therefore, the operations on the root of directed chords are isomorphic to the dihedral group D_n (e.g., Dummit and Foote 2004), so called because it is the group of symmetries of a regular polygon.

[16] Instead of referring to the components of a full mode $\mu \in M(n,k)$ from 1 to k, we refer to the mode intervals from 0 to $k-1$ in order to use their indices according to the cyclic structure of $\mathbb{Z}_k$.

4.1.5 ***Retrograde translations by mode intervals.*** We also define successive translations by the mode intervals, read anticlockwise direction,

$$\tau[\mu]_p^{-q} = \tau[\mu]_{p,p-1,\dots,p-q+1} = \tau[A_p + \dots + A_{p-q+1}]; \quad q \le p \tag{25}$$

The subindex p indicates the highest mode interval in the composition and the superindex q is the number of retrograde compositions. Hence, Eq. 23 is also true for negative powers of the translations,

$$\tau[\mu]_p^{-q}\tau[-\mu]_p^{-q} = \tau_0 \tag{26}$$

For example, for $p = -1$ in Eq. 25, we may apply translations by the mode intervals starting from the last one, anticlockwise direction, such as

$$\tau[\mu]_{-1}^{-i} = \tau[\mu]_{-1,\dots,-i} = \tau[A_{k-1} + \dots + A_{k-i}] \tag{27}$$

which, by using the positive inversion of the mode, $\overline{\mu} = [A_{k-1},\dots,A_0]$, may be computed as

$$\tau[\mu]_{-1}^{-i} = \tau[\overline{\mu}]_0^i \tag{28}$$

In this way it is possible to express retrograde translations by intervals of a mode as prograde translations by intervals of the positive inversion of the mode.

4.2 *Operations on the mode*

4.2.1 ***Positive and negative inversions.*** In §2.8 we introduced the positive and negative inverted modes. We reformulate these transformations to express them as operators acting on directed chords, notating them with lowercase Gothic letters. The negative and positive forms of a full mode $\mu = [A_0, A_1, \dots, A_{k-1}] \in M(n,k)$, i.e., $-\mu$ and $\overline{\mu}$, will be respectively written as

$$\mathfrak{n}\mu = [-A_0, -A_1, \dots, -A_{k-1}]; \quad \mathfrak{p}\mu = [A_{k-1}, \dots, A_1, A_0] \tag{29}$$

Both forms were used to determine the relative inverted chord of $\{a_0|\mu\}$ about a_0, so that the directed chords $\mathfrak{n}(a_0|\mu) = a_0|\mathfrak{n}\mu$, read anticlockwise direction, and $\mathfrak{p}(a_0|\mu) = a_0|\mathfrak{p}\mu$, read in the positive orientation, according to Eq. 11 produce the same chord, $\{a_0|\mathfrak{n}\mu\} = \{a_0|\mathfrak{p}\mu\}$.

By expressing the identity transformation on modes as $\mathfrak{e}$, it is obviously satisfied $\mathfrak{n}^2 = \mathfrak{p}^2 = \mathfrak{e}$.

4.2.2 ***Retrogradation and shifts.*** In §2.7 we referred to the retrograde mode as $-\overline{\mu} = -[A_{k-1},\dots,A_1,A_0]$. Applied anticlockwise direction to a_0, it generates the same chord $\{a_0|\mu\} = \{a_0|-\overline{\mu}\}$. We shall write

$$\mathfrak{r}\mu \equiv \mathfrak{n}\mathfrak{p}\mu = -[A_{k-1},\dots,A_1,A_0] \tag{30}$$

so that $\mathfrak{r}(a_0|\mu) = a_0|\mathfrak{r}\mu$. Hence, $\{a_0|\mathfrak{r}\mu\} = \{a_0|\mu\}$. The retrogradation $\mathfrak{r}$ of a directed chord actually does not modify the mode, but chooses the direction the directed chord is read, without changing the root. Thus, retrogradation can be applied to a chord, although it leaves the chord invariant. Therefore, it is an operation well defined for chords. It is straightforward to see that[17]

$$\mathfrak{r}^2 = \mathfrak{e}, \quad \mathfrak{r} = \mathfrak{n}\mathfrak{p} = \mathfrak{p}\mathfrak{n}, \quad \mathfrak{n} = \mathfrak{r}\mathfrak{p} = \mathfrak{p}\mathfrak{r}, \quad \mathfrak{p} = \mathfrak{r}\mathfrak{n} = \mathfrak{n}\mathfrak{r} \tag{31}$$

On the other hand, the shifts on modes defined in Eq. 8 can be extended to directed chords as operations that leave the root unaltered. For a positive mode $\mu^0 \equiv \mu$, a prograde single shift can be written as $\mu^1 \equiv \mathfrak{s}\mu^0 = [A_1,\dots,A_{k-1},A_0]$, while, in general, the jth shift is obtained as

$$\mu^j = \mathfrak{s}\mu^{j-1} = \mathfrak{s}^j\mu^0; \quad j \in \mathbb{Z}_k \tag{32}$$

with $\mathfrak{s}^0 = \mathfrak{e}$. Notice that, for full modes of dimension k, this transformation is k-periodic, so that $\mathfrak{s}^k = \mathfrak{e}$. Therefore, $\langle\mathfrak{s}\rangle$ is a cyclic group of order k, although the shift does not commute with the previous operations. Hence, the exponent indicating the shift number can be assumed as a number in $\mathbb{Z}_k$. Then, a retrograde or anticlockwise shift may also be applied to μ, i.e., $\mu^{-1} = \mathfrak{s}^{-1}\mu^0 = [A_{k-1}, A_0, \dots, A_{k-2}]$. In general, we have $\mu^{-j} = \mathfrak{s}^{-1}\mu^{-j+1} = \mathfrak{s}^{-j}\mu^0$.

Unlike translations, the operations $\mathfrak{n}, \mathfrak{p}$, and $\mathfrak{s}$ on the modes cannot be extended to chords, since the same operation may result in different chords depending on the directed chord rotation it is applied.

It is straightforward to prove the following properties of the mode operations involving shifts, being the

[17] The set of mode operations $\{\mathfrak{e},\mathfrak{n},\mathfrak{p},\mathfrak{r}\}$ is an abelian Klein four-group, which is the direct product $\langle\mathfrak{n}\rangle \times \langle\mathfrak{p}\rangle$ of cyclic groups of order 2. However, if the modes are used strictly clockwise definite, we only have to deal with $\langle\mathfrak{p}\rangle$.

first one commutative and the other two anticommutative:

$$\mathfrak{s}\mathfrak{n}=\mathfrak{n}\mathfrak{s}\,,\quad \mathfrak{s}\mathfrak{p}=\mathfrak{p}\mathfrak{s}^{-1}\,,\quad \mathfrak{s}\mathfrak{r}=\mathfrak{r}\mathfrak{s}^{-1} \tag{33}$$

Therefore, for any $j\in\mathbb{Z}_k$, it is fulfilled

$$\mathfrak{s}^j\mathfrak{n}=\mathfrak{n}\mathfrak{s}^j\,,\quad \mathfrak{s}^j\mathfrak{p}=\mathfrak{p}\mathfrak{s}^{-j}\,,\quad \mathfrak{s}^j\mathfrak{r}=\mathfrak{r}\mathfrak{s}^{-j} \tag{34}$$

Let us recall that the shifts $\mathfrak{s}^j\mu$, $j\in\mathbb{Z}_k$, of the full mode $\mu\in M(n,k)$ determine one mode class μ^S. The inverted mode, e.g., written as a positive mode $\mathfrak{p}\mu$, and their shifts belong also to one mode class $(\mathfrak{p}\mu)^S$, not necessarily the same as μ^S. For the sake of the second relationship in Eq. 33, these shifts satisfy $\mathfrak{p}\mathfrak{s}^j\mu=\mathfrak{s}^{-j}\mathfrak{p}\mu$, so that all the inversions of the modes in μ^S belong to the mode class $(\mathfrak{p}\mu)^S$. Therefore, we may speak of mutually inverted mode classes, which have the same cardinality. In particular, if μ and $\mathfrak{p}\mu$ belong to the same class, i.e., for some $m\in\mathbb{Z}_k$, $\mathfrak{p}\mu=\mathfrak{s}^m\mu$, then $(\mathfrak{p}\mu)^S=\mu^S$, since, for any $j\in\mathbb{Z}_k$, it is fulfilled $\mathfrak{p}\mathfrak{s}^j\mu=\mathfrak{s}^{m-j}\mu$.

4.2.3 ***Neighbor chords.*** The way of moving between chords along the chord network by maintaining the maximum number of common notes is along the edges of congruent cells, where, between adjacent vertices only one note is changed. This is a general case that includes the parsimonious voice leading (Cohn 1997) along the Tonnetz generated by a major triad and its dual network[18]. In such a path, the chords at the ends of the edge belong to each one of the cells that share this edge, but depending on the rotation we use to describe them we can make explicit that they also belong to the opposite non-congruent cells that are separated by this edge. In a non-degenerate Tonnetz, this efficient path *combines chords generated by different mode classes.*

Before describing how to move in the chord network from one chord to its neighbor, we must remark that some of the mode transformations can also be applied to partial modes. In particular, the operators $\mathfrak{p}$ and $\mathfrak{s}$ may act on partial modes by maintaining the total mode as a full mode. For instance, given a full mode written as a composition of partial modes, i.e., $\mu=\nu\cdot\kappa$, with $\nu\in M(n,p),\kappa\in M(n,q)$ and $p+q=k$, we can generate new full modes such as $\mathfrak{p}\nu\cdot\mathfrak{s}\kappa$, $\mathfrak{p}\mathfrak{s}\nu\cdot\kappa$, etc. The only condition is not to mix positive and negative partial modes.

For a mode $\mu=[A_0,A_1][A_2,...,A_{k-1}]$, $k>2$, we notate a single transposition operator as

$$\mathfrak{t}\mu=\mathfrak{p}[A_0,A_1][A_2,...,A_{k-1}] \tag{35}$$

It is the transposition of the first two mode intervals. Obviously, $\mathfrak{t}^2=\mathfrak{e}$.

The ith transposition of two consecutive intervals A_i, A_{i+1} is obtained by combining the foregoing operator with the mode shifts as

$$\mathfrak{t}_i=\mathfrak{s}^{-i}\mathfrak{t}\mathfrak{s}^i\,,\quad i\in\mathbb{Z}_k \tag{36}$$

In this way it is possible to reach all the transpositions of the symmetric group $\mathcal{S}_k$. The $k-1$ generators $\{\mathfrak{t}_0,...,\mathfrak{t}_{k-2}\}$ provide a presentation of $\mathcal{S}_k$ $(k>2)$ with the following constraints[19]

$$\text{(i)}\ \mathfrak{t}_i^2=\mathfrak{e}\,,\quad \text{(ii)}\ (\mathfrak{t}_j\mathfrak{t}_i)^2=\mathfrak{e}\Leftrightarrow|i-j|\neq 1\,,\quad \text{(iii)}\ (\mathfrak{t}_{i+1}\mathfrak{t}_i)^3=\mathfrak{e} \tag{37}$$

These constraints are also valid *for all* the indices in $\mathbb{Z}_k$, by assuming that the second condition $|i-j|\neq 1$ takes place in $\mathbb{Z}_k$, i.e., it that does not apply to the pair $\mathfrak{t}_0$ and $\mathfrak{t}_{k-1}$.

PROPOSITION 4.1 *For $j=2,...,k$, given a subset of $j-1$ consecutive numbers $\{i,i+1,...,i+j-2\}$ in $\mathbb{Z}_k$ and a permutation $\sigma\in\mathcal{S}_{j-1}$, it is satisfied $(\mathfrak{t}_{\sigma(i+j-2)}\cdots\mathfrak{t}_{\sigma(i+1)}\mathfrak{t}_{\sigma(i)})^j=\mathfrak{e}$.*

Proof. Any composition of $j-1$ different transpositions involving a subset of consecutive indices in $\mathbb{Z}_k$ (regardless of the order of composition) is a cycle of length j and is a permutation of order j. ■

In particular, by assuming the indices in $\mathbb{Z}_k$, this applies to a full mode shift expressed from any composition in the form[20] $\mathfrak{s}=\mathfrak{t}_{i+k-2}\cdots\mathfrak{t}_{i+1}\mathfrak{t}_i$.

When the single transposition operator $\mathfrak{t}=\mathfrak{t}_0$ is applied to the directed chord $a_0|\mu$, the resulting directed chord $\mathfrak{t}(a_0|\mu)=a_0|\mathfrak{t}\mu=(a_0,a_0+A_1,a_0+A_0+A_1,a_0+A_0+A_1+A_2,...,a_0+A_0+...,A_{k-1})$ has exchanged

[18] Originally, the concept of parsimonious voice leading was applied when two notes of a trichord were maintained during a transformation and the third note moved by a minor second or a major second. Douthett and Steinbach (1998) gave a more flexible definition by requiring that just one note were maintained, with the other two moving by minor or major second.

[19] Equivalently, $\mathfrak{t}_i^{-1}=\mathfrak{t}_i$, $\mathfrak{t}_j\mathfrak{t}_i=\mathfrak{t}_i\mathfrak{t}_j\Leftrightarrow|i-j|\neq 1$, and $\mathfrak{t}_{i+1}\mathfrak{t}_i\mathfrak{t}_{i+1}=\mathfrak{t}_i\mathfrak{t}_{i+1}\mathfrak{t}_i$. The second condition only applies for $k>3$.

[20] The transposition $\mathfrak{t}_{k-1}$, since $\mathfrak{s}=\mathfrak{t}_{k-2}\cdots\mathfrak{t}_0$, can be expressed as $\mathfrak{t}_{k-1}=\mathfrak{t}_{k-2}\cdots\mathfrak{t}_1\mathfrak{t}_0\mathfrak{t}_1\cdots\mathfrak{t}_{k-2}$.

the note $a_1=a_0+A_0$ of the original chord for a_0+A_1. Both directed chords are still referred to the same cell, although after one rotation, both chords can be written as

$$\begin{aligned}\{a_0|\mu\}&=\{a_0+A_0|\mathfrak{s}\mu\}=\{a_0+A_0,a_0+A_0+A_1,\dots,a_0+A_0+\dots,A_{k-1},a_0\}\\ \{\mathfrak{t}(a_0|\mu)\}&=\{a_0+A_1|\mathfrak{s}\mathfrak{t}\mu\}=\{a_0+A_1,a_0+A_0+A_1,\dots,a_0+A_0+\dots,A_{k-1},a_0\}\end{aligned} \tag{38}$$

now explicitly referred to the different cells centered in the notes they differ. Therefore, the operator $\mathfrak{t}$ has displaced the original chord along the edge connecting the cells $a_1=a_0+A_0$ and a_0+A_1.

In general, if the transposition operator is applied to the ith rotation of the directed chord, $a^i=a_i|\mu^i$, it is immediately to prove that

THEOREM 4.2 *The chords $\{a_i|\mu^i\}$ and $\{\mathfrak{t}(a_i|\mu^i)\}$ only differ in the notes $a_{i+1}=a_i+A_i$ and a_i+A_{i+1}, respectively.*

Proof. The reasoning is similar to that of Eq. 38. The chord $\{a_0\mid\mu\}$ is equivalent to its rotations $\{a_i|\mu^i\}=\{a_i+A_i|\mathfrak{s}\mu^i\}$, while $\{\mathfrak{t}(a_i|\mu^i)\}=\{a_i+A_{i+1}|\mathfrak{s}\mathfrak{t}\mu^i\}$. These roots correspond to opposite cells in the ends of the edge that connects these chords. ■

To recover the original position of the common notes, it suffices to do the ith inverse rotation. For $i=0,\dots,k-2$, the root a_0 appears in the inversions of the directed chords $\mathfrak{t}(a_i\mid\mu^i)$, i.e., $\{\mathfrak{t}(a_0\mid\mu^0)\}=\{a_0\mid[A_1,A_0,\dots,A_{k-1}]\}$ and $\{\mathfrak{t}(a_i\mid\mu^i)\}=\{a_0\mid[A_0,\dots,A_{i+1},A_i,\dots]\}$; $i=1,\dots,k-2$. For $i=k-1$, the root a_0 is exchanged by the note $a_0+A_0-A_{k-1}$, i.e., $\{\mathfrak{t}(a_{k-1}|\mu^{k-1})\}=\{a_0+A_0-A_{k-1}|[A_{k-1},A_1,\dots,A_{k-2},A_0]\}$. Therefore,

COROLLARY 4.3 *For $i=0,\dots,k-2$, the chords $\{\mathfrak{t}(a_i\mid\mu^i)\}$ remain in the chord cell a_0. The chord $\{\mathfrak{t}(a_{k-1}\mid\mu^{k-1})\}$ does not belong to the chord cell a_0. The edge connecting the chord cells a_0 and $a_0+A_0-A_{k-1}$ is the one to follow in order to move to a non-congruent cell with a_0.*

In the end of §4.3.5 we come back to these transformations in more detail.

4.2.4 Translations by mode intervals.

PROPOSITION 4.4 *Prograde translations of a mode $\mu\in M(n,k)$ satisfy $\tau[\mu^{i-j}]_0^j=\tau\left[\mathfrak{p}\mu^i\right]_0^j$.*

Proof. According to Eq. 8, since the subindices of the mode intervals belong to $\mathbb{Z}_k$, we have $\mu^i=[A_i,\dots,\overbrace{A_{i-j},\dots,A_{i-1}}^{j}]$ and $\mathfrak{p}\mu^i=[\overbrace{A_{i-1},\dots,A_{i-j}}^{j},\dots,A_i]$. Similarly, $\mu^{i-j}=[\overbrace{A_{i-j},A_{i-j+1},\dots,A_{i-j+j-1}}^{j},\dots,A_{i-j+k-1}]$. Hence, the sums of the j first terms of both foregoing equations match, $\tau\left[\mathfrak{p}\mu^i\right]_0^j=\tau[\mu^{i-j}]_0^j=\tau[A_{i-j}+\dots+A_{i-1}]$, leading to the desired result. ■

In particular, for $i=0$ we get $\tau[\mu^{-j}]_0^j=\tau[\mathfrak{p}\mu]_0^j$ and, for $i=j$, we get $\tau[\mu]_0^j=\tau\left[\mathfrak{p}\mu^j\right]_0^j$.

4.2.5 Relationships involving shifts and translations. We give several properties combining translations by intervals of a shifted mode, which will be used in the following sections. Equations involving translations by mode intervals are also valid for inverse translations.

An interval A_{i+j} of the mode $\mu=[A_0,A_1,\dots,A_{k-1}]$ can be denoted in different ways, depending on the mode shift and the chosen interval, as follows, $[\mu^{i+j}]_0=[\mu^i]_j=[\mu^j]_i; i,j\in\mathbb{Z}_k$. Also, according to Eq. 28,

$$\tau[\mu^j]_{-1}^{-i}=\tau\left[\mathfrak{p}\mu^j\right]_0^i\quad(=\tau[A_{j+k-1}+\dots+A_{j+k-i}]) \tag{39}$$

By taking into account Eq. 22, for $1\le i\le k$ and any integer l, we have

$$\tau[\mu^j]_0^i=\tau[\mu^{j-l}]_l^i\quad(=\tau[A_j+\dots+A_{j+i-1}]) \tag{40}$$

Since the first i intervals of μ^j are the last i intervals of $\mathfrak{p}\mu^j$ in reverse order, their sum match. Thus, $\tau[\mu^j]_0^i=\tau[\mathfrak{p}\mu^j]_{k-i}^i$. Hence, by equations 34 and 40, we get $\tau[\mu^j]_0^i=\tau[\mathfrak{p}\mu^{j+i+l}]_l^i$. On the other hand, since $\tau[\mu^j]_0^k=\tau_0$, then $\tau[\mu^j]_0^i\tau[\mu^j]_i^{k-i}=\tau_0$. Therefore, in $\mathbb{Z}_n$, $\tau[\mu^j]_0^i=\tau[\mathfrak{n}\mu^j]_i^{k-i}$.

Owing to equations 34 and 40, going backwards $i-l$ intervals and compensating with a prograde shift of $i-l$ steps, we get the more general relationship for Eq. 40, $\tau[\mu^j]_0^i = \tau[\mathfrak{n}\mu^{j+i-l}]_l^{k-i}$. Finally, we also give an alternative version of Eq. 24 for the notes of a directed chord $a_0|\mu$,

$$a_{i+j} = \tau[\mu^i]_0^j a_i \tag{41}$$

4.3 *Chord transformations*

4.3.1 Operations on root and mode. As mentioned above, the operations on the mode $\mathfrak{n}$,$\mathfrak{p}$, $\mathfrak{s}$ and $\mathfrak{t}$ are defined only for directed chords. According to the condition given by Lemma 2.6, the operations on two directed chords generating the same chord will produce the same chord only if the operations that modify their modes are balanced with operations on their roots. Operations on directed chords involving root and mode will be noted with uppercase Gothic letters.

As we have already seen, inversion of directed chords, using either the positive or negative mode inversions, combine operations on root and mode. Both forms of inversion applied to a directed chord generate the same chord, although read in opposite orientations. Under operational form, the inverted and mirror directed chords are expressed as

$$\mathfrak{I}(a_0|\mu) = \iota\mathfrak{p}(a_0|\mu) = -a_0|\mathfrak{p}\mu\,, \quad \mathfrak{M}(a_0|\mu) = \iota\mathfrak{n}(a_0|\mu) = -a_0|\mathfrak{n}\mu \tag{42}$$

We notate the identity transformation of a directed chord as $\mathfrak{E} = \epsilon\mathfrak{e}$, which is the commutative composition of both identities acting on the root and the mode, respectively. Then, $\mathfrak{I}^2 = \mathfrak{M}^2 = \mathfrak{E}$. Let us remember that, as directed chords, $\mathfrak{I}(a_0|\mu) \neq \mathfrak{M}(a_0|\mu)$, although, as studied in §2.9, they are well defined operations on chords and generate the same chord.

According to Eq. 42, and bearing in mind Eq. 17, it is straightforward to derive the following anticommutative properties that combine inversions and translations of directed chords,

$$\mathfrak{I}\,\tau_x(a_0|\mu) = \tau_{-x}\mathfrak{I}(a_0|\mu)\,, \quad \mathfrak{M}\,\tau_x(a_0|\mu) = \tau_{-x}\mathfrak{M}(a_0|\mu) \tag{43}$$

4.3.2 Inversion of chords. Thus, the inversion of a chord is well defined regardless the directed chord we use. That is, $\{a_0|\mu\} = \{b_0|\nu\}$ if, and only if, $\{\mathfrak{I}(a_0|\mu)\} = \{\mathfrak{I}(b_0|\nu)\}$. In addition, the anticommutative properties of Eq. 43 are also valid for chords, regardless the rotation of the directed chord that is being transformed,

$$\mathfrak{I}\,\tau_x\{a_0|\mu\} = \tau_{-x}\mathfrak{I}\{a_0|\mu\}\,, \quad \mathfrak{M}\,\tau_x\{a_0|\mu\} = \tau_{-x}\mathfrak{M}\{a_0|\mu\} \tag{44}$$

We still give a more precise result.

PROPOSITION 4.5 *If the directed k-chord $a_0|\mu$ is a rotation of m steps of the directed chord $b_0|\nu$, then the inversion $\mathfrak{I}(a_0|\mu)$ is a rotation of $k-m$ steps of $\mathfrak{I}(b_0|\nu)$*

Proof. According to Lemma 2.6, there is one value $m \in \mathbb{Z}_k$ such that $a_m = b_0$ and $\mu^m = \nu$. Hence,

$$a_m|\mathfrak{s}^m\mu = b_0|\nu \tag{45}$$

Then, for the respective inverted directed chords, $\mathfrak{I}(a_0|\mu) = -a_0|\mathfrak{p}\mu$ and $\mathfrak{I}(b_0|\nu) = -b_0|\mathfrak{p}\nu$, we look for the index i that satisfies

$$-a_m|\mathfrak{s}^i\mathfrak{p}\mu = -b_0|\mathfrak{p}\nu \tag{46}$$

By inverting the directed chords of Eq. 45, according to Eq. 42 we get $-a_m|\mathfrak{p}\mathfrak{s}^m\mu = -b_0|\mathfrak{p}\nu$, which can be expressed, by applying the second relationship of Eq. 33, as

$$-a_m|\mathfrak{s}^{-m}\mathfrak{p}\mu = -b_0|\mathfrak{p}\nu \tag{47}$$

By comparing Eq. 46 and Eq. 47, we obtain the shift $i{=}k-m$, which makes the inverted directed chords fit them together. The result is identical for modes oriented anticlockwise with the operator $\mathfrak{M}$. ■

4.3.3 Inversion and mirror by x. Other operations on the root and mode of a directed chord are defined, such as the inversion by an interval x, in one of the following forms,

$$\begin{aligned}\mathfrak{I}_x(a_0|\mu) \equiv \tau_x\mathfrak{I}(a_0|\mu) = \tau_x\iota\mathfrak{p}(a_0|\mu) = (x-a_0)|\mathfrak{p}\mu\\ \mathfrak{M}_x(a_0|\mu) \equiv \tau_x\mathfrak{M}(a_0|\mu) = \tau_x\iota\mathfrak{n}(a_0|\mu) = (x-a_0)|\mathfrak{n}\mu\end{aligned} \tag{48}$$

depending on the orientation the mode is read. Thus, the inversion by x combines, first, an inversion of the directed chord and, afterwards, a translation by an interval x, leading to a new root $x-a_0$.

By Eq. 17, these operations are not commutative. We write $\mathfrak{I}_0 \equiv \mathfrak{I}$ and $\mathfrak{M}_0 \equiv \mathfrak{M}$. Since inversions by an interval are composition of translations and inversions, they are also well defined operations on chords.

Operations on the root and operations on the mode are mutually independent and commutative. On the other hand, as seen in Eq. 17, translations and inversions on the root do not commute. In addition to the property $\tau_x \tau_y = \tau_{x+y}$, it is easy to borne out the following properties involving translations τ_x and inversions $\mathfrak{I}_y$ (or alternatively $\mathfrak{M}_y$):

$$\mathfrak{I}_x \mathfrak{I}_y = \tau_{x-y} \qquad \mathfrak{M}_x \mathfrak{M}_y = \tau_{x-y} \tag{49}$$

$$\tau_x \mathfrak{I}_y = \mathfrak{I}_{x+y} = \tau_y \mathfrak{I}_x \qquad \tau_x \mathfrak{M}_y = \mathfrak{M}_{x+y} = \tau_y \mathfrak{M}_x \tag{50}$$

$$\mathfrak{I}_x \tau_y = \mathfrak{I}_{x-y} = \tau_{-y} \mathfrak{I}_x \qquad \mathfrak{M}_x \tau_y = \mathfrak{M}_{y-x} = \tau_{-y} \mathfrak{M}_x \tag{51}$$

From Eq. 49 we see that $\mathfrak{I}_x^2 = \mathfrak{E}$, i.e., the identity element[21].

4.3.4 ***Rotations.*** Like inversions, a directed chord rotation, as defined in Eq. 9, is a combination of a translation on the root and a shift on the mode. A one-step prograde rotation of the directed chord $a = a_0|\mu \in K(n,k)$, depending on the first mode interval, can be expressed in either of the following forms,

$$\mathfrak{R}a = a^1 = \tau_{A_0} a_0 | \mathfrak{s}\mu = \tau[\mu]_0 a_0 | \mathfrak{s}\mu \tag{52}$$

Similarly, an i-step prograde rotation applied to the same directed chord, according to Eq. 32 and by taking into account Eq. 40, can be written as

$$\mathfrak{R}^i a = a^i = \tau_{A_0+\ldots+A_{i-1}} a_0 | \mathfrak{s}^i \mu = \tau[\mu]_0^i a_0 | \mathfrak{s}^i \mu \tag{53}$$

Successive rotations can be notated as follows,

$$\mathfrak{R}a^i = a^{i+1} = \tau_{A_i} a_i | \mathfrak{s}\mu^i = \tau[\mu^i]_0 a_i | \mathfrak{s}\mu^i \tag{54}$$

and, in general, the consecutive i- and j-step rotations are expressed as

$$\mathfrak{R}^j a^i = a^{i+j} = \tau_{A_i+\ldots+A_{i+j-1}} a_i | \mathfrak{s}^j \mu^i = \tau[\mu^i]_0^j a_i | \mathfrak{s}^j \mu^i \tag{55}$$

Rotations on directed k-chords clearly satisfy $\mathfrak{R}^i a^j = \mathfrak{R}^j a^i = \mathfrak{R}^{i+j} a^0$, $\mathfrak{R}^k = \mathfrak{R}^0$, and they have the structure of a cyclic group of order k. Hence, for the rotation index we also assume $\mathfrak{R}^i \equiv \mathfrak{R}^{i \bmod k}$.

Rotations can be defined backwards, as retrograde rotations,

$$\mathfrak{R}^{-1} a^i = a^{i-1} = \tau_{-A_{i-1}} a_i | \mathfrak{s}^{-1} \mu^i = \tau[\mathfrak{n}\mu^i]_{-1} a_i | \mathfrak{s}^{-1} \mu^i \tag{56}$$

Then, an arbitrary anticlockwise rotation is evaluated, by taking into account Eq. 41, as

$$\mathfrak{R}^{-j} a^i = a^{i-j} = a_{i-j} | \mathfrak{s}^{-j} \mu^i = \tau[\mathfrak{n}\mu^i]_{-1}^{-j} a_i | \mathfrak{s}^{-j} \mu^i \tag{57}$$

If we take into account equations 28 (remember that $\mathfrak{p}\mu = \overline{\mu}$) and 30, we may write $\tau[\mathfrak{n}\mu^i]_{-1}^{-j} = \tau[\mathfrak{p}\mathfrak{n}\mu^i]_0^j = \tau[\mathfrak{r}\mu^i]_0^j$, allowing us to write the translation in Eq. 57 in terms of the prograde intervals of the retrograde mode as

$$\mathfrak{R}^{-j} a^i = a^{i-j} = \tau[\mathfrak{r}\mu^i]_0^j a_i | \mathfrak{s}^{-j} \mu^i \tag{58}$$

which is a formula similar to Eq. 55, with the only difference that retrograde rotations take translations in the root by the retrograde mode.

By definition, the chords are invariant under rotations, i.e.,

$$\{a\} = \{\mathfrak{R}^i a^0\}, \forall i \in \mathbb{Z}_k \tag{59}$$

Therefore, rotations induce a constraint between translations and shifts. Furthermore, according to Eq. 11, which expresses an inverted chord in two equivalent ways, the relationship $\{\mathfrak{R}^i(a|\mathfrak{n}\mu)\} = \{\mathfrak{R}^j(a|\mathfrak{p}\mu)\}$ is satisfied for any couple of indices $i,j \in \mathbb{Z}_k$.

[21] In an n-TET system, the translations and inversions are made by intervals $x \in \mathbb{Z}_n$, being $\tau_x = \tau_1^x$. Since translations are generated by τ_1, they satisfy $\tau_1^n = \tau_0$ and are isomorphic to $\mathbb{Z}_n$. On the other hand, an inversion by x is obtained as $\mathfrak{I}_x = \tau_x \mathfrak{I}_0$. Since $\mathfrak{I}_0^2 = \epsilon$, then $\mathfrak{I}_0$ generates a group isomorphic to $\mathbb{Z}_2$. Therefore, such a group of operations on a chord is generated by τ_1 and $\mathfrak{I}_0$. It is is a semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ with $2n$ elements $\{\tau_0, \tau_1, \tau_1^2, \ldots, \tau_1^{n-1}, \mathfrak{I}_0, \tau_1 \mathfrak{I}_0, \tau_1^2 \mathfrak{I}_0, \ldots, \tau_1^{n-1} \mathfrak{I}_0\}$ satisfying $\mathfrak{I}_0 \tau_1 \mathfrak{I}_0 = \tau_1{}^{-1}$. Similarly to the operations on the root of a directed chord, the operations on a chord are isomorphic to the dihedral group D_n.

4.3.5 ***Drifts along edges.*** Theorem 4.2 and Corollary 4.3 can be expressed by using a more appropriate notation by defining the *drift* operator, transforming directed chords along single edges of the chord network,

$$\mathfrak{D}_i \equiv \mathfrak{R}^{-i}\mathfrak{t}\mathfrak{R}^i; \quad i \in \mathbb{Z}_k, k > 2 \tag{60}$$

When acting on the directed chord $a_0|\mu = a_0|[A_0,...,A_{k-1}]$, bearing in mind Eq. 36, we get

$$\mathfrak{D}_i(a_0|\mu) = \mathfrak{R}^{-i}(a_i|\mathfrak{t}\mathfrak{s}^i\mu) = a_0|\mathfrak{t}_i\mu; \quad i = 0,...,k-2 \tag{61}$$

$$\mathfrak{D}_{k-1}(a_0|\mu) = \mathfrak{R}^{-(k-1)}(a_{k-1}|\mathfrak{t}\mathfrak{s}^{k-1}\mu) = a_0 + A_0 - A_{k-1}|\mathfrak{t}_{k-1}\mu \tag{62}$$

THEOREM 4.6 *For $i = 0,...,k-2$ the operator $\mathfrak{D}_i$ transforms a directed chord $a_0|\mu$ sideways along one edge in the same cell a_0, while for $i = k-1$ the directed chord is sent forward along one edge towards the non-congruent cell $a_0 + A_0 - A_{k-1}$.*

It is a direct consequence of Theorem 4.2 and of the fact that the transpositions $\{\mathfrak{t}_0,...,\mathfrak{t}_{k-2}\}$ generate the symmetric group $\mathcal{S}_k$.

THEOREM 4.7 *For $j = 2,...,k$, given a subset of $j-1$ consecutive numbers $\{i, i+1,...,i+j-2\}$ in $\mathbb{Z}_k$ and a permutation $\sigma \in \mathcal{S}_{j-1}$, then $(\mathfrak{D}_{\sigma(i+j-2)} \cdots \mathfrak{D}_{\sigma(i+1)}\mathfrak{D}_{\sigma(i)})^j = \mathfrak{E}$.*

Proof. Let $\mathfrak{S} = \mathfrak{D}_{\sigma(i+j-2)} \cdots \mathfrak{D}_{\sigma(i+1)}\mathfrak{D}_{\sigma(i)}$. When $\mathfrak{S}^j$ is applied to a directed chord $a_0|[A_0,...,A_{k-1}]$, with regard to the mode part $\mathfrak{c} = \mathfrak{t}_{\sigma(i+j-2)} \cdots \mathfrak{t}_{\sigma(i+1)}\,\mathfrak{t}_{\sigma(i)}$, Proposition 4.1 guarantees the identity result. With regard to the root, if $\mathfrak{S}$ does not contain $\mathfrak{D}_{k-1}$, the root is also maintained. Otherwise, if $\mathfrak{S}$ contains $\mathfrak{D}_{k-1}$, since $\mathfrak{c}$ is a cycle of length j involving the mode intervals $A_i, A_{i+1},...,A_{i+j-1}$, the first and the last mode intervals are involved in each iteration and, in particular, from the first application of the operator $\mathfrak{D}_{k-1}$ onward, the root is modified according to Eq. 62. Each time $\mathfrak{D}_{k-1}$ is applied, the first mode interval runs a cycle $A_{\rho(0)}, A_{\rho(1)},...,A_{\rho(j-1)}$ of length j, with $\rho \in \mathcal{S}_j$. Similarly, the last mode interval runs a shifted cycle $A_{\rho(m)}, A_{\rho(m+1)},...,A_{\rho(m+j-1)}$ for certain $m \neq 0$ in $\mathbb{Z}_k$. Therefore, after j applications of $\mathfrak{D}_{k-1}$, the root a_0 is transformed according to a series of translations $\tau[A_{\rho(0)} - A_{\rho(m)}] \cdots \tau[A_{\rho(j-1)} - A_{\rho(m+j-1)}] = 0$, since all the mode intervals involved in one cycle appear once as positive and once as negative. ■

COROLLARY 4.8 *For all $i \in \mathbb{Z}_k$, the drift operators satisfy*

$$\text{(i)}\ \mathfrak{D}_i^2 = \mathfrak{E}, \quad \text{(ii)}\ (\mathfrak{D}_j\mathfrak{D}_i)^2 = \mathfrak{E} \Leftrightarrow |i-j| \neq 1, \quad \text{(iii)}\ (\mathfrak{D}_{i+1}\mathfrak{D}_i)^3 = \mathfrak{E}$$

Condition (i) is obvious; (ii) is a consequence of (i), since the transpositions of the mode intervals are disjoint; and (iii) is a consequence of the above theorem.

Therefore, the elemental polygons composing one cell of k-chords are squares (only if $k > 3$) and hexagons, associated with the simple circuits (closed paths where no vertex is repeated) of the constraints (ii) and (iii) of Corollary 4.8, for indices in $\{0,...,k-2\}$. Then, one node of a chord cell is connected to the neighbor $k-1$ nodes shared with the neighbor cells with the above drifts and the drift $\mathfrak{D}_{k-1}$ connects the node with a non-congruent cell. In the chord cell there are other simple circuits, such as allowed concatenations of the above ones or paths corresponding to k consecutive shifts of the mode $(\mathfrak{D}_{k-2} \cdots \mathfrak{D}_0)^k$ involving $k(k-1)$ edges. In general, throughout the chord network, we will find the simple circuits described in Theorem 4.7.

Since the Cayley graph is Hamiltonian, in the permutahedron *we will always find Hamiltonian circuits*, i.e., closed paths that go through each vertex of the chord cell exactly once.

In addition, it is desirable to find the shortest circuits that run along several chord cells allowing changes of tonality, likewise the maximally smooth cycles alternating major and minor triads of the 12-TET scale. Each simple circuit begins and ends in the same chord, so that the total length of the path is equivalent to a number of translations along the chord network satisfying $\sum_i c_i A_i = 0 \bmod n$ for certain values $c_i \in \mathbb{N}$, although the path runs step-by-step along several directed chords of different modes and mode classes (i.e., by directed chords that cannot be achieved only by translations and mode shifts) until becomes closed. It is relevant the case where translations are done always along the same vector interval direction, i.e., when there is a minimum positive integer j, $2 \leq j < n$, satisfying $jA_i = 0 \bmod n$. Since, for $j = n$ there is always a simple circuit, we may speak of a *shortcut circuit*. We study the case $jA_0 = 0 \bmod n$.

PROPOSITION 4.9 *The drifts operators satisfy* $\mathfrak{D}_{i+1}=\mathfrak{R}^{-1}\mathfrak{D}_i\mathfrak{R}$, *hence* $\mathfrak{D}_{i+j}=\mathfrak{R}^{-j}\mathfrak{D}_i\mathfrak{R}^j$ *for* $i,j\in\mathbb{Z}_k$.

Proof. For $i=0,\ldots,k-3$ it is an immediate consequence of Eq. 36, since these drifts do not change the root. For $i=k-2$ and $k-1$, the relationship is also satisfied, since

$$\mathfrak{R}^{-1}\mathfrak{D}_{k-2}\mathfrak{R}(a_0|[A_0,\ldots,A_{k-1}])=\mathfrak{R}^{-1}\mathfrak{D}_{k-2}(a_0+A_0|[A_1,\ldots,A_0])=\mathfrak{R}^{-1}(a+A_0|[A_1,\ldots,A_0,A_{k-1}])$$
$$=(a+A_0-A_{k-1}|[A_{k-1},A_1,\ldots,A_0])=\mathfrak{D}_{k-1}(a_0|[A_0,\ldots,A_{k-1}])$$
$$\mathfrak{R}^{-1}\mathfrak{D}_{k-1}\mathfrak{R}(a_0|[A_0,\ldots,A_{k-1}])=\mathfrak{R}^{-1}\mathfrak{D}_{k-1}(a_0+A_0|[A_1,A_2,\ldots,A_{k-1},A_0])=\mathfrak{R}^{-1}(a+A_0+A_1-A_0|[A_0,A_2,\ldots,A_{k-1},A_1])$$
$$=(a+A_0+A_1-A_0-A_1|[A_1,A_0,A_2,\ldots,A_{k-1}])=\mathfrak{D}_0(a_0|[A_0,\ldots,A_{k-1}])$$

■

The graph in Eq. 63 describes the first translation $\tau[A_0](a|\mu)$ depending on whether we refer this path to the edges of the adjacent chord cell a_0+A_0 (upper row) or whether it starts running along the edge connecting the cells a_0 and a_0+A_0 (lower row). The translation from the first chord (left column) to the last chord (right column) involves $k-1$ steps.

$$\begin{array}{ccccccc} a+A_0|\mathfrak{s}\mu & \xrightarrow{\mathfrak{D}_{k-2}} & a+A_0|\mathfrak{t}_{k-2}\mathfrak{s}\mu & \xrightarrow{\mathfrak{D}_{k-3}} & \cdots & \xrightarrow{\mathfrak{D}_0} & a+A_0|\mu \\ \mathfrak{R}\uparrow & & & \tau[A_0] & & & \mathfrak{R}\uparrow \\ a|\mu & \xrightarrow[\mathfrak{D}_{k-1}]{} & a+A_0-A_{k-1}|\mathfrak{t}_{k-1}\mu & \xrightarrow[\mathfrak{D}_{k-2}]{} & \cdots & \xrightarrow[\mathfrak{D}_1]{} & a+A_0-A_{k-1}|\mathfrak{s}^{-1}\mu \end{array} \quad (63)$$

In the first case, since $\mathfrak{D}_0\cdots\mathfrak{D}_{k-2}=\mathfrak{s}^{-1}$, $\mathfrak{D}_0\cdots\mathfrak{D}_{k-2}\mathfrak{R}(a_0\mid\mu)=\mathfrak{s}^{-1}\tau[A_0](a_0\mid\mathfrak{s}\mu)=\tau[A_0](a_0\mid\mu)$, so that $(\mathfrak{D}_0\cdots\mathfrak{D}_{k-2}\mathfrak{R})^j(a_0\mid\mu)=\tau[jA_0](a_0\mid\mu)=a_0\mid\mu$. In the second case, by Proposition 4.9, $\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1}(a_0|\mu)=\tau[A_0](a_0|\mu)$, hence $(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})^j(a_0|\mu)=\tau[jA_0](a_0|\mu)=a_0|\mu$. Since chords are invariant under rotations, the path generates a simple circuit along $j(k-1)$ chords.

The generalization of the above result to any interval satisfying $jA_i=0 \bmod n$ is straightforward, since it suffices to apply the ith rotation to the directed chord $(a|\mu)$ at the beginning of the path, and the ith inverse rotation at the end, in order to recover the original mode. Therefore,

THEOREM 4.10 *If* $jA_i=0 \bmod n$, $2\le j<n$, *there exist shortcut circuits in the chord network running* $j(k-1)$ *chords and edges, so that* $(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})^j(a_i|\mu^i)=\tau[jA_i](a_i|\mu^i)=a_i|\mu^i$.

Furthermore, by taking into account Proposition 4.9, we may write
$\ldots(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})=$
$\ldots(\mathfrak{R}\mathfrak{R}^2\mathfrak{R}^{-2}\mathfrak{D}_1\mathfrak{R}^2\mathfrak{R}^{-2}\ldots\mathfrak{R}^2\mathfrak{R}^{-2}\mathfrak{D}_{k-1})(\mathfrak{R}\mathfrak{R}\mathfrak{R}^{-1}\mathfrak{D}_1\mathfrak{R}\mathfrak{R}^{-1}\ldots\mathfrak{R}\mathfrak{R}^{-1}\mathfrak{D}_{k-1})(\mathfrak{R}\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})=$
$\ldots(\mathfrak{D}_3\cdots\mathfrak{D}_0\mathfrak{D}_1)(\mathfrak{D}_2\cdots\mathfrak{D}_{k-1}\mathfrak{D}_0)(\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})$. Therefore,

COROLLARY 4.11 *The shortcut circuit of the above theorem can be run without rotations, by consecutive application of drifts with decreasing indices in* $\mathbb{Z}_k$ *(this is also valid for any simple circuit),*

$$\underbrace{\ldots(\mathfrak{D}_3\cdots\mathfrak{D}_0\mathfrak{D}_1)(\mathfrak{D}_2\cdots\mathfrak{D}_{k-1}\mathfrak{D}_0)(\mathfrak{D}_1\cdots\mathfrak{D}_{k-1})}_{j(k-1)\text{ steps}}(a_i|\mu^i)=a_i|\mu^i$$

5. Chord network

5.1 *Referring a chord to different cells*

For a directed chord $x_0|\mu$, the condition of invariance of the chord, $\{x_0|\mu\}=\{\mathfrak{R}(x_0|\mu)\}$, according to Eq. 54 implies $\{x_0|\mu\}=\{\tau[\mu]_0\,x_0|\mathfrak{s}\mu\}$. By applying the opposite translation $\tau^{-1}[\mu]_0=\tau[-\mu]_0$ in both sides of the foregoing equation, we get

$$\{\tau^{-1}[\mu]_0\,x_0|\mu\}=\{x_0|\mathfrak{s}\mu\} \quad (64)$$

Thus, starting from the chord $\{x_0\,|\,\mu\}$ we may reach the chord $\{x_0\,|\,\mathfrak{s}\mu\}$ either from a shift or from a translation. In a similar way as in the geometric representation of Fig. 3, the translated chord in the new cell is placed at the same relative position than the chord $\{x_0|\mu\}$, although the shifted chord remains in the original chord cell x_0 in a different relative position. Then, the same chord can be expressed from two roots, which are the center of two contiguous cells sharing the chord of Eq. 64.

According to equations 53 and 59, a chord $\{x_0|\nu\}$ is invariant by rotations, hence

$$\{x_0|\nu\} = \{\mathfrak{R}^j(x_0|\nu)\} = \{\tau[\nu]_0^j\, x_0|\mathfrak{s}^j\nu\}, j \in \mathbb{Z}_k \tag{65}$$

For each chord, the value $j=0$ provides the chord written as a directed chord with root x_0, while values $j=1,\dots,k-1$ provide the same chord from the other roots. Then, each chord $\{x_0\,|\,\nu\}$ can be expressed according to a directed chord referred to one of the $k-1$ surrounding cells of x_0 centered in $\tau[\nu]_0^j\, x_0$, although from a different shift of the mode. Thus, by generalizing Eq. 64, we get the following result,

THEOREM 5.1 *The chords $\{x_0|\mathfrak{s}^j\nu\}$, $j\in\mathbb{Z}_k$, are equivalent to the following translations,*

$$\{x_0|\mathfrak{s}^j\nu\} = \{\tau^{-1}[\nu]_0^j\, x_0|\nu\} \tag{66}$$

The translated chords become referred to directed chords in neighbor chord cells according to the successive notes of the relative mirror chord $\{x_0|\mathfrak{n}\nu\}$, read anticlockwise.

Similarly, by taking into account Eq. 58, the retrograde rotations of a chord satisfy

$$\{x_0|\nu\} = \{\mathfrak{R}^{-j}(x_0|\nu)\} = \{\tau^{-1}[\mathfrak{p}\nu]_0^j\, x_0|\mathfrak{s}^{-j}\nu\} \tag{67}$$

By applying $\tau[\mathfrak{p}\nu]_0^j$ in both sides, we get an equivalent result to the previous one,

THEOREM 5.2 *The chord $\{x_0|\mathfrak{s}^{-j}\nu\}$, $j\in\mathbb{Z}_k$, is equivalent to the following translation*[22],

$$\{x_0|\mathfrak{s}^{-j}\nu\} = \{\tau[\mathfrak{p}\nu]_0^j\, x_0|\nu\} = \{\tau[\nu]_{-1}^{-j}\, x_0|\nu\} \tag{68}$$

Now, the translated chords are referred to directed chords in neighbor chord cells according the successive notes of the relative inverted chord $\{x_0|\mathfrak{p}\mu\}$.

For the chords generated by an inverted mode, we get some similar results,

COROLLARY 5.3 *The chords $\{x_0|\mathfrak{s}^j\mathfrak{p}\nu\}$, $j\in\mathbb{Z}_k$, satisfy*

$$\{x_0|\mathfrak{s}^j\mathfrak{p}\nu\} = \{\tau^{-1}[\mathfrak{p}\nu]_0^j\, x_0|\mathfrak{p}\nu\} \tag{69}$$

The translated chords are referred to directed chords in neighbor chord cells according the successive notes, anticlockwise direction, of the chord $\{x_0|-\mathfrak{p}\nu\} = \{x_0|\mathfrak{r}\nu\}$, obtained from the retrograde mode.

COROLLARY 5.4 *The chords $\{x_0|\mathfrak{s}^{-j}\mathfrak{p}\nu\}$, $j\in\mathbb{Z}_k$, satisfy*[23]

$$\{x_0|\mathfrak{s}^{-j}\mathfrak{p}\nu\} = \{\tau[\nu]_0^j\, x_0|\mathfrak{p}\nu\} = \{\tau[\mathfrak{p}\nu]_{-1}^{-j}\, x_0|\mathfrak{p}\nu\} \tag{70}$$

The translated chords become referred to directed chords in neighbor chord cells according the successive notes of the chord $\{x_0|\nu\}$.

5.2 *Co-cycles, co-cells, and congruent cells*

For $\mu\in M(k,n)$, we write Eq. 66 as $\{\tau^{-1}[\mu]_0^j\, x_0\,|\,\mu\} = \{x_0\,|\,\mu^j\}, j\in\mathbb{Z}_k$. In the chord cell x_0 the family of chords described by the right-hand side member of this equation is composed of k different chords, which form a *co-cycle* (Cohn 1996). The co-cycles of the cell x_0 are then generated by the modes of the mode class μ^S applied to the same root.

Alternatively, on the left-hand side of this equation, each chord of the co-cycle in the cell x_0 is referred to one of the $k-1$ surrounding cell of x_0 with centers in $\tau^{-1}[\mu]_0^j\, x_0$, respectively, in addition to the chord $\{x_0|\mu\}$. We call them *co-cells* of the chord $\{x_0|\mu\}$, where the chords of the co-cycle are placed *in the same relative position* as $\{x_0|\mu\}$ with regard to the cell x_0.

Two chords $\{a\}$ and $\{b\}$ in the cell x_0 are in the same co-cycle if, and only if, for any interval u the translated chords $\{\tau_u a\}$ and $\{\tau_u b\}$ are in the same co-cycle in the cell x_0+u. On the other hand, in the cell x_0 a directed chord $x_0|\mu$ is placed in a vertex shared by k co-cells. In each one of these cells this chord is generated by other shifts μ^m of the mode[24]. Since the cells sharing the chord are those corresponding to

[22] The translation may be rewritten according to Eq. 39, so that it is carried out by the same mode ν.

[23] Here, the translation may be also rewritten according to Eq. 39, so that it is carried out by the same mode $\mathfrak{p}\nu$.

[24] In a non-degenerate Tonnetz, these shifts are always different, otherwise, for $p\neq q$, the shifts μ^p and μ^q may coincide.

the k rotations of the directed chord, according to Eq. 53, the chord $\{x_0|\mu\}$ in the cell $x_j = \tau[\mu]_0^j\, x_0$ (which is the root of the jth prograde rotation of the directed chord) is generated by the shift μ^j, for $j = 1,...,k-1$. Then, the shift of the mode is $m = j$ for the cell x_j.

For any mode class $\nu^S \in \mathcal{C}(\mu)$, the co-cycles $\{x_0|\nu^i\}$, $i = 0,...,k-1$, belong to the cell x_0. By collecting all the modes of the mode classes in $\mathcal{C}(\mu)$ we get the symmetric modes $\mathcal{S}(\mu)$. Then, the *congruent* cells of the cell x_0 are those of the notes composing the chords $\{x_0|\nu\}$ with $\nu \in \mathcal{S}(\mu)$. According to §3.8, this makes a number of $2^k - 2$ cells, that are adjacent to the facets of the cell x_0.

5.3 *Dependent operations on chords*

Equations 66 and 68 show that a mode shift can be computed from a dependent translation by mode intervals of a directed chord, which is an operation that, in this form, is well defined for chords, on condition of referring the directed chord to another cell. For example, in the following case, the mode may be shifted according to Eq. 66 as follows, $\{a_0|\mu^0\} = \{a_1|\mu^1\} \Rightarrow \{\tau^{-1}[\mu^0]_0\,(a_0|\mu^0)\} = \{\tau^{-1}[\mu^0]_0\,(a_1|\mu^1)\}$. In the second member we get the shift of $\{a_0|\mu^0\}$ in the cell a_0, $\{\tau^{-1}[\mu^0]_0\,(a_1|\mu^1)\} = \{a_1 - A_0|\mu^1\} = \{a_0|\mu^1\}$, while, in the first member, the shift is expressed as a translation to the cell $a_0 - A_0$, such as $\{\tau^{-1}[\mu^0]_0\,(a_0|\mu^0)\} = \{a_0 - A_0|\mu^0\}$. It corresponds to a chord placed in the same relative cell position than $\{a_0|\mu^0\}$.

Therefore, we may use the alternative operations described in equations 66 and 68 to compute the chords obtained as shifts of a directed chord rotation.

THEOREM 5.5 *For any rotation of the directed chord $a_i\,|\,\mu^i$, $i \in \mathbb{Z}_k$, the chords corresponding to the shifted directed chords are obtained from the following translations on the chord $\{a\}$,*

$$\{\mathfrak{s}^j\,(a_i|\mu^i)\} = \tau^{-1}[\mu^i]_0^j\,\{a\}, \quad \{\mathfrak{s}^{-j}\,(a_i|\mu^i)\} = \tau[\mu^i]_{-1}^{-j}\,\{a\} \tag{71}$$

We have a similar situation for inversions on the root of a directed chord. The inversion of the root can be obtained as a translation also depending on the root, that can be expressed as $\{\iota x|\mu\} = \tau_x^{-2}\{x|\mu\}$.

THEOREM 5.6 *The chord obtained as a root inversion of the ith rotation $a_i\,|\,\mu^i$ is the result of the following translation on the chord $\{a\}$, $\{\iota(a_i|\mu^i)\} = \tau^{-2}[a_i]\,\{a\}$.*

Similarly, by taking into account Eq. 43 and the above equation, we have,

THEOREM 5.7 *The chord obtained as the relative mirror chord of $a_i|\mu^i$ is computed from the following composition on the chord $\{a\}$, $\{\mathfrak{n}(a_i\,|\,\mu^i)\} = \mathfrak{M}\,\tau^{-2}[a_i]\;\{a\}$, which is equivalent to the relative inverted chord $\{\mathfrak{p}(a_i|\mu^i)\} = \mathfrak{I}\,\tau^{-2}[a_i]\,\{a\}$.*

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5.4 *Translations by mode intervals*

5.4.1 Single translations. We study in detail the transformations on the Tonnetz between chords of the same mode class. As we have seen, these transformations can be interpreted as consecutive single translations of chords. Let us remember that the Tonnetz is non-degenerate, i.e., the interval vectors have different length. According to the preceding section, the chord $\{x_0|\mu^0\}$ in the cell x_0 belongs to the co-cycle formed by k chords ($k \geq 3$) in the form $\{x_0|\mu^i\}$, $i \in \mathbb{Z}_k$. In the cell x_0, these chords can be expressed from directed chords with root x_0 and shifts of the mode class μ^S and they may also be expressed as translations to the co-cells by consecutive intervals of a mode of the class. Among these translations, there are only two translations depending on a *single* mode interval, the first and the last, yielding a chord of the co-cycle in the same cell. According to Eq. 71, these translations are

$$\tau[-\mu^i]_0\,\{x_0|\mu^i\} = \{x_0|\mu^{i+1}\}, \quad \tau[\mu^i]_{-1}\,\{x_0|\mu^i\} = \{x_0|\mu^{i-1}\} \tag{72}$$

Other translations by a single interval lead to a chord in another cell.

For the first relationship of Eq. 72, if the translation is carried out by an arbitrary mode interval $[-\mu^i]_j = [-\mu^{i+j}]_0 = -A_{i+j}$, for $0 < j \leq k-1$, by expressing the chord from the jth prograde rotation of $x_0|\mu^i$, i.e., $\{\mathfrak{R}^j\,(x_0|\mu^i)\} = \{x_j|\mu^{i+j}\}$, the first expression of Eq. 71 allows us to use a translation as a shift, so that

$$\tau[-\mu^{i+j}]_0\{x_0|\mu^i\} = \tau[-\mu^{i+j}]_0\{\mathfrak{R}^j\,(x_0|\mu^i)\} = \tau[-\mu^{i+j}]_0\{x_j|\mu^{i+j}\} = \{x_j|\mu^{i+j+1}\} \tag{73}$$

Therefore, the chord $\{x_j\,|\,\mu^{i+j+1}\}$ in the cell x_j is not a rotation of any chord in the cell x_0. Hence, the following $k-1$ translations lead to a chord which is not in the co-cycle of $\{x_0\,|\,\mu^i\}$ and is placed in the neighbor cell of x_0, $x_j = x_0 + A_0 + \ldots + A_{j-1}$,

$$\tau[-\mu^{i+j}]_0\,\{x_0|\mu^i\} = \{x_j|\mu^{i+j+1}\}, \quad 0 < j \leq k-1 \tag{74}$$

Similarly, for the second relationship of Eq. 72, if the translation is carried out by the mode interval $[\mu^i]_{-j} = [\mu^{i-j+1}]_{-1} = A_{i-j}$, for $1 < j \leq k$, by expressing the chord from the $(j-1)$th retrograde rotation of $x_0|\mu^i$, i.e., $\{\mathfrak{R}^{-(j-1)}(x_0|\mu^i)\} = \{x_{k-j+1}|\mu^{i-j+1}\}$, and taking into account Eq. 71, we get

$$\tau[\mu^{i-j+1}]_{-1}\{\mathfrak{R}^{-(j-1)}(x_0|\mu^i)\} = \tau[\mu^{i-j+1}]_{-1}\{x_{k-j+1}|\mu^{i-j+1}\} = \{x_{k-j+1}|\mu^{i-j}\} \tag{75}$$

Therefore, the chord $\{x_{k-j+1}|\mu^{i-j}\}$ in the cell x_{k-j+1} cannot be expressed as a rotation of a directed chord in the cell x_0. Then, the following $k-1$ translations lead to a chord that is not in the co-cycle of $\{x_0\,|\,\mu^i\}$ and is placed in the neighbor cell of x_0, $x_{k-j+1} = x_0 - A_{k-1} - \ldots - A_{k-j+1}$,

$$\tau[\mu^{i-j+1}]_{-1}\,\{x_0|\mu^i\} = \{x_{k-j+1}|\mu^{i-j}\}, \quad 1 < j \leq k \tag{76}$$

Equations 72, 74 and 76 can be summarized as follows,

Theorem 5.8 *If $\mu = [A_0, \ldots, A_{k-1}]$, the chords resulting from applying a single translation by a mode interval to the chord $\{x_0|\mu\}$ are,*

$$\tau[-A_0]\,\{x_0|\mu\} = \{x_0|\mu^1\}; \qquad \tau[A_{k-1}]\,\{x_0|\mu\} = \{x_0|\mu^{-1}\} \tag{77}$$

$$\tau[-A_l]\,\{x_0|\mu\} = \{x_l|\mu^{l+1}\}, \quad 1 \leq l \leq k-1; \quad \tau[A_l]\,\{x_0|\mu\} = \{x_{l+1}|\mu^l\}, \quad 0 \leq l \leq k-2 \tag{78}$$

The translations of Eq. 78 cannot be rooted in the cell x_0 and describe the chords from congruent cells. On the left-hand side, the chords are rooted in a note not belonging to the original chord, while on the right-hand side they are rooted in a note of the original chord.

Corollary 5.9 *Single translations by a mode interval of the chord $\{x_0|\mu\}$ are composed of the chords $\{x_l|\mu^{l+1}\}$ with $0 \leq l \leq k-1$, and the chords $\{x_{l+1}|\mu^l\}$ with $0 \leq l \leq k-1$.*

To complete the study of single translations of a chord $\{x_0\,|\,\mu\}$, we will investigate when two single translations lead to chords sharing a common chord cell, either the same cell x_0 or a different one, and when they drive to different cells, that is, that they cannot share a common root. This will be done by comparing the pairs of possible positive and negative translations, i.e., by $\tau[-A_p]$ and $\tau[A_q]$, by $\tau[A_p]$ and $\tau[A_q]$, and by $\tau[-A_p]$ and $\tau[-A_q]$.

5.4.2 Translations towards one cell. According to Eq. 78, a translation by a single mode interval of a chord $\{x_0\,|\,\mu\}$ in the cell x_0 leads to a cell different from x_0 in $2k-2$ out of $2k$ cases. Nevertheless, different single translations by a mode interval may lead to a common neighbor cell. Let us see in what cases the families of translated chords

$$\tau[-A_p]\,\{x_0|\mu\} = \{x_p|\mu^{p+1}\}, \quad p \in \mathbb{Z}_k; \qquad \tau[A_q]\,\{x_0|\mu\} = \{x_{q+1}|\mu^q\}, \quad q \in \mathbb{Z}_k \tag{79}$$

produce two chords that can be referred to the same root.

Proposition 5.10 *The following chord translations lead to a common neighbor cell,*

$$\tau[-A_p]\,\{x_0|\mu\} = \{x_p|\mu^{p+1}\}, \quad \tau[A_{p-1}]\,\{x_0|\mu\} = \{x_p|\mu^{p-1}\}, \quad p \in \mathbb{Z}_k \tag{80}$$

Proof. We already know that for $p = 0$ and $q = k-1$ the translations leave the chords in the cell x_0 according to Eq. 77. It is also obvious that for $q = p-1$ mod k the translations by $\tau[-A_p]$ and $\tau[A_{p-1}]$ lead to the cell x_p. This includes the case $p = 0$. ∎

In addition to the particular cases of Eq. 80, we study the other possible cases.

Theorem 5.11 *The following chord translations lead to the same co-cycle into a non-congruent cell,*

$$\tau[-A_p]\,\{x_0|\mu\} = \{x_p + A_{p+1}|\mu^{p+2}\}, \quad \tau[A_{p+1}]\,\{x_0|\mu\} = \{x_p + A_{p+1}|\mu^p\}, \quad p \in \mathbb{Z}_k \tag{81}$$

Proof. In the general case general, we ask for values p,q so that both chords of Eq. 79 will have rotations

$$\{\mathfrak{R}^{i}\,\tau[-A_p]\,x_0|\,\mu\},\quad \{\mathfrak{R}^{-j}\,\tau[A_q]\,x_0|\,\mu\} \tag{82}$$

with a common root for some values i,j, such that $0\leq i<k$ and $0\leq j<k$. Then, by equations 55 and 57, $\{\mathfrak{R}^{i}\,\tau[-A_p]\,x_0|\,\mu\}=\{\tau[\mu]_0^{i}\,\tau[-A_p]\,x_0|\mathfrak{s}^{i}\,\mu\}$, $\{\mathfrak{R}^{-j}\,\tau[A_q]\,x_0|\,\mu\}=\{\tau[-\mu]_{-1}^{-j}\,\tau[A_q]\,x_0|\mathfrak{s}^{-j}\,\mu\}$. Thus, by relating the part corresponding to the root of both equations, after reordering terms[25],

$$A_{k-j}+A_{k-j+1}+\cdots+A_{k-1}+A_0+\cdots+A_{i-2}+A_{i-1}=A_p+A_q \bmod n \tag{83}$$

The intervals on both sides are positive and on the left there are at most $2k-2$ terms[26]. Then, the values i,j allowing the above relationship are as follows.

The simplest case corresponds to $i=1$ and $j=1$, so that either $p=0, q=k-1$ or $p=k-1, q=0$. Other cases are those involving all the intervals of one full k-mode, i.e., any shift of μ, since $A_0+\cdots+A_{k-1}=0 \bmod n$. Then, Eq. 83 is satisfied if, and only if,

$$A_{i-2}+\underbrace{A_{i-1}+\cdots+A_{k-1}+A_0+\cdots+A_{i-2}}_{0 \bmod n}+A_{i-1}=A_p+A_q \bmod n \tag{84}$$

so that the underbraced terms do not contribute to the total sum. Hence, comparing to Eq. 83, we get $k-j=i-2 \bmod k$. Therefore, since the indices of the mode intervals are different, one of the following cases is hold (the equalities are $\bmod k$):

(a) $p=i-1$, $q=k-j$, that is, $q=p-1, i=p+1, -j=p-1$;

(b) $p=k-j$, $q=i-1$, that is, $q=p+1, i=p+2, -j=p$.

Case (a) is similar to Eq. 80. Case (b) provides the family of couples of chords

$$\{\tau[-A_p]\,x_0|\mu\}=\{x_p|\mu^{p+1}\},\quad \{\tau[A_{p+1}]\,x_0|\mu\}=\{x_{p+2}|\mu^{p+1}\},\quad p\in\mathbb{Z}_k \tag{85}$$

According to Eq. 82 and to the corresponding values i, j, they can be written respectively from a common root as $\{\tau[-A_p]\ x_0\mid\mu\}=\{\mathfrak{R}^{p+2}\ \tau[-A_p]\ x_0\mid\mu\}=\{\tau[\mu]_0^{p+2}\ x_0-A_p\mid\mu^{p+2}\}=\{x_0+A_0+\cdots+A_{p-1}+\cancel{A_p}+A_{p+1}|\mu^{p+2}\}=\{x_p+A_{p+1}|\mu^{p+2}\}$, and $\{\tau[A_{p+1}]\,x_0|\mu\}=\{\mathfrak{R}^{p}\,\tau[A_{p+1}]\,x_0|\mu\}=\{\tau[\mu]_0^{p}\,x_0+A_{p+1}|\mu^{p}\}=\{x_0+A_0+\cdots+A_{p-1}+\cancel{A_p}+A_{p+1}|\mu^{p}\}=\{x_p+A_{p+1}|\mu^{p}\}$. ■

5.4.3 Translations towards different cells. Chords belong to different cells if they cannot be referred by rotations to the same cell. We study the other two possibilities of single translations. First, let us see whether the families of translated chords,

$$\tau[A_p]\,\{x_0|\mu\}=\{x_{p+1}|\mu^{p}\},\quad p\in\mathbb{Z}_k;\qquad \tau[A_q]\,\{x_0|\mu\}=\{x_{q+1}|\mu^{q}\},\quad q\in\mathbb{Z}_k \tag{86}$$

assuming $p\neq q$, provide two chords that can be referred to the same root. By following the same procedure as the previous section, after rotating, the roots must match. Then, $A_{k-j}+A_{k-j+1}+\cdots+A_{k-1}+A_0+\cdots+A_{i-2}+A_{i-1}+A_p=A_q \bmod n$.

Like in the previous case, this relationship has a geometric meaning. On the right-hand side there is one single positive interval. On the left-hand side, in addition to A_p, there are $i+j$ consecutive intervals, with $i+j\leq 2k-2$. Hence, to fulfill that relationship there must be one cycle plus one interval. If there is no cycle, then $p=q$, which is not the case. Thus, the above relationship should be either $\underbrace{A_p+\cdots+A_{k-1}+A_0+\cdots+A_{p-1}}_{0 \bmod n}+A_p=A_q \bmod n$, or $A_p+\underbrace{A_{p+1}+\cdots+A_{k-1}+A_0+\cdots+A_p}_{0 \bmod n}=A_q \bmod n$, by yielding, in both cases, to $p=q$, against the prior assumption. Therefore, we conclude:

THEOREM 5.12 *Translations of one chord by different single positive intervals lead to different cells.*

[25]This is a way to test whether two chords $\{x|\mu\}$ and $\{x'|\mu'\}$ belong to the same co-cycle. Since it is not sure that both chords may be expressed as directed chords with the same root (in which case we should only to compare whether both modes belong to the same class), we write both chords as generated by the same mode $\mu=[A_0,\ldots,A_{k-1}]$, e.g., $\{x'|\mu'\}=\{y|\mu\}$. Then, both chords are in the same co-cycle if, and only if, there exist two indices i,j so that the interval difference among the roots x and y can be expressed as a series $A_{k-j}+A_{k-j+1}+\cdots+A_{k-1}+A_0+\cdots+A_{i-2}+A_{i-1}$, i.e., as a fraction of any shift of the mode μ.

[26]Let us remember that this equality has a geometric meaning. According to Eq. 16, the series of intervals $A_0,\ldots,A_{i-1}$ ($i<k$) are associated with independent directions. Similarly, the series $A_{k-j},\ldots,A_{k-1}$ ($j<k$) are independent intervals. A number of k consecutive terms are dependent and form a cycle, hence they can be suppressed. In that case, the remaining terms on the left-hand side are at most $k-2$ in number and are independent. In addition, the intervals A_p and A_q must be included among those terms.

Second, let us see whether the families of translated chords

$$\tau[-A_p]\{x_0|\mu\}=\{x_p|\mu^{p+1}\},\quad p\in\mathbb{Z}_k;\qquad \tau[-A_q]\{x_0|\mu\}=\{x_q|\mu^{q+1}\},\quad q\in\mathbb{Z}_k \tag{87}$$

provide two chords that can be referred to the same root, also by assuming $p\neq q$. By following a similar procedure, after rotating, the roots must match. Then, $A_{k-j}+A_{k-j+1}+\cdots+A_{k-1}+A_0+\cdots+A_{i-2}+A_{i-1}+A_q=A_p \bmod n$. Thus, we reach a similar result as the first case, leading to $p=q$, against the former assumption[27].

THEOREM 5.13 *Translations of one chord by different single negative intervals lead to different cells.*

5.5 *Honeycomb of trichords*

The relationships obtained in the above sections are used to interpret the honeycomb of trichords depicted in Fig. 3 for the general case associated with a mode $\mu=[A,B,C]$, as well as those of Fig. 4 for the particular case $\mu=[4,3,5]$. This is an example of how to proceed in a more general case.

Each chord is shared by three cells, and can be rooted in three different notes labeling the center of the congruent cells, i.e., the roots of the three rotations of the directed chord. Each cell has 6 principal facets: 3 front facets that are orthogonal to the vector intervals $+A$,$+B$,$+C$, and 3 back facets that are orthogonal to the vector intervals $-A,-B,-C$. Each cell is surrounded by $2^3-2=6$ cells, as many as facets, all of them, principal facets.

As explained in §3.9, voice leading along facets in the tonal network is associated with paths that maintain the notes on either side of the interval to which the facet is orthogonal. This corresponds to paths along edges of the chord network. According to §4.2.3, in such a path it is possible to remain in the same cell or to divert to another cell through the edge connecting to a non-congruent cell. If we apply the transformations of Theorems 4.2 and 4.7 to the actual chord network, between the cells $a+A$ and $a+B$, the connected directed chords are $a\,|\,\mu^0$ and[28] $\mathfrak{D}_0(a\,|\,\mu^0)=a|-\mu^2$, which share the root a and the third note $a+A+B=a-C$ (the fifth in the case of $\mu=[4,3,5]$). Between the cells $a+A+B$ and $a+A+C$, the connected directed chords are $a\,|\,\mu^0$ and $\mathfrak{D}_1(a\,|\,\mu^0)=a|-\mu^1$, sharing the root a and the second note $a+A$ (the major third). Between the cell $a=a+A+B+C$ and the non-congruent cell $a+A+B+A=a+A-C$, the connected chords are $a|\mu^0$ and $\mathfrak{D}_2(a|\mu^0)=a+A-C|-\mu^0$. They share the second note $a+A$ and the third note $a+A+B$ (the major third and the fifth). Therefore, the movements along the edges of the chord network can be interpreted as being: $\mathfrak{D}_0$ a displacement towards the neighbor chord on the edge in the direction orthogonal to the interval C (which preserves the fifth interval), $\mathfrak{D}_1$ a displacement in the direction orthogonal to the interval A (which preserves the major third interval), and $\mathfrak{D}_2$ a displacement in the direction orthogonal to the interval B (which preserves the minor third interval) towards a non-congruent cell. These operators match and generalize for an arbitrary mode Lewin's (1987) basic operators P, R, and the leading tone exchange L of the Neo-Riemmanian triadic progressions.

The chords in one cell may be translated by mode intervals to neighbor cells and to the same cell. In the latter case, a translation is equivalent to a shift of the mode by maintaining the root. There are two mode classes, i.e., major and minor chords, and, therefore, two co-cycles that translations cannot mix, except in the case of a degenerate Tonnetz with a single mode class. Thus, for the single translations of the chord $\{a|\mu^0\}$ towards the same co-cycle in the same or another cell, from Eq. 80 we get[29] the following six chords, which are referred to the same cell a and to the neighbor cells $a+A$ and $a-C$ (the symbol † indicates translations that do not change the shift of the mode, therefore they are chords referred to co-cells of $\{a|\mu^0\}$),

$$\begin{array}{lll} p=0; & \tau_{-A}\{a|\mu^0\}=\{a|\mu^1\}; & \tau_C\{a|\mu^0\}=\{a|\mu^2\} \\ p=1; & \tau_{-B}\{a|\mu^0\}=\{a+A|\mu^2\}; & \tau_A\{a|\mu^0\}=\{a+A|\mu^0\}^\dagger \\ p=2; & \tau_{-C}\{a|\mu^0\}=\{a-C|\mu^0\}^\dagger; & \tau_B\{a|\mu^0\}=\{a-C|\mu^1\} \end{array} \tag{88}$$

and from Eq. 81 we get the following translations, which are the same chords as in Eq. 88, but now referred

[27] However, if the Tonnetz is degenerate and the interval vectors A_p and A_q have the same length, the cells x_0+A_p and x_0+A_q, and the cells x_0-A_p and x_0-A_q have respective roots with a similar value.

[28] In this case we obtain the mode $-\mu^2$, which belongs to the co cycle of the mode $-\mu$, but in a general case with $k>3$, we may reach other co-cycles which are not in such a relation with the mode μ, since there will be more than two mode classes.

[29] In this case, $A_0=A$, $A_1=B$, $A_2=C$, $x_0=a$, $x_1=a+A$, and $x_2=a+A+B=a-C$.

to non-neighbor cells of $\{a|\mu^0\}$, i.e., $a{+}B$, $a{-}B$ and $a{-}C{+}A$,

$$\begin{array}{lll} p=0; & \tau_{-A}\{a|\mu^0\}=\{a+B|\mu^2\}; & \tau_B\{a|\mu^0\}=\{a+B|\mu^{0\dagger}\} \\ p=1; & \tau_{-B}\{a|\mu^0\}=\{a-B|\mu^0\}^\dagger; & \tau_C\{a|\mu^0\}=\{a-B|\mu^1\} \\ p=2; & \tau_{-C}\{a|\mu^0\}=\{a-C+A|\mu^1\}; & \tau_A\{a|\mu^0\}=\{a-C+A|\mu^2\} \end{array} \tag{89}$$

The relationships in equations 88 and 89 may be summarized and completed with the remaining rotations of each directed chord according to the following scheme,

$$\begin{array}{c|ccccc} & \{a|\mu^0\} & = & \{a+A|\mu^1\} & = & \{a-C|\mu^2\} \\ \hline \tau_A & \{a+A|\mu^0\} & = & \{a+2A|\mu^1\}^\ddagger & = & \{a-C+A|\mu^2\} \\ \tau_B & \{a+B|\mu^0\} & = & \{a-C|\mu^1\} & = & \{a-C+B|\mu^2\}^\ddagger \\ \tau_C & \{a+C|\mu^0\}^\ddagger & = & \{a-B|\mu^1\} & = & \{a|\mu^2\} \\ \tau_{-A} & \{a-A|\mu^0\}^\ddagger & = & \{a|\mu^1\} & = & \{a+B|\mu^2\} \\ \tau_{-B} & \{a-B|\mu^0\} & = & \{a+A-B|\mu^1\}^\ddagger & = & \{a+A|\mu^2\} \\ \tau_{-C} & \{a-C|\mu^0\} & = & \{a+A-C|\mu^1\} & = & \{a-2C|\mu^2\}^\ddagger \end{array} \tag{90}$$

The chords in black are referred to the cell a and its neighbor cells, where they form the three respective co-cycles. The initial chord and both translations share one note. These cells are 0 steps away from the initial chord. The chords in gray are referred to the cells in the second ring of cells from a, containing only two chords of a co-cycle. The two translations share one note that does not belong to the initial chord, although each translation has a common note with the initial chord. These cells are 1 step away from the initial chord. The chords marked with the symbol ‡, are the only single translations of the initial chord in their respective cells. However, these cells, also in the second ring, are 2 steps away from the initial chord. Similar relationships can be obtained with the inverse mode.

6. Conclusions

We provide a new formalism to study chord transformations over a non-degenerate Tonnetz, i.e., with a generating mode that has non-two equal intervals. The first part of the paper reviews the main concepts and definitions. Basic transformations and their geometric aspects are revisited, so that they can later be related to our algebra of k-chords. Geometry properties are described for trichords, which are easy to visualize, and afterwards they are extended to higher dimensions. In a second part, chord transformations are studied in operator form, either acting on the root, the mode, simultaneously on root and mode, or on the chord. The operators allow us to understand the structure of the chord cell, which, in general, is a permutahedron of order k, i.e., a $(k{-}1)$-dimensional polytope embedded in a k-dimensional space, whose vertices and edges are isomorphic to the Cayley graph of the symmetric group $\mathcal{S}_k$. Among the transformations along the chord network some of them are relevant, since they are frequently used in music, such as those between chords of the same co-cycle in one cell, or of a similar co-cycle in a co-cell, that can always be reduced to dependent translations by mode intervals.

For trichords, to change between co-cycles an inversion suffices, but for higher dimensional chords a novel operator is introduced, which, from a geometric point of view, is the only operator that can logically be defined: a step-by-step displacement along the chord network. Thus, a drift along one of the concurrent edges of the chord changes the co-cycle by combining mode transpositions and directed chord rotations. The drift operators $\mathfrak{D}_i$, for $i=0,\ldots,k-2$ transform a directed chord sideways along the $k{-}1$ edges in the same cell, while for $i=k{-}1$ the directed chord is sent forward along the edge connecting to a non-congruent cell. These operators generalize the basic operators P, R and L of the Neo-Riemmanian triadic progressions. The elemental polygons composing the chord cell and the chord network are squares (for $k>3$) and hexagons, which are associated with the simple circuits described in Theorem 4.7. Other simple circuits exist if, for a certain mode interval A_i, it is satisfied $j\,A_i=0 \bmod n$ with $j<n$. Then, there are shortcut circuits of $j(k{-}1)$ steps (for $j=n$ there is always a simple circuit) that can be expressed by consecutive application of drifts with decreasing indices in $\mathbb{Z}_k$. They generalize the maximally smooth cycles composing the four hexatonic systems for the trichords of the 12-TET scale. In the chord cell, as well as in the chord

network, there are always Hamiltonian circuits, i.e., closed paths that go through each vertex exactly once.

As an example, the model is explicitly applied to describe the chord network associated with the trichords of a non-degenerate Tonnetz and, in particular, with the mode that generates the major and minor triads. For k-chords in the 12-TET system, if $k=3$ there are 14 non-degenerate Tonnetze, but for $k>3$ there are only two, generated by the modes [1,2,3,6] and [1,2,4,5]. Nevertheless, the current model can easily be extended to degenerate Tonnetze. In higher dimensions, degenerate Tonnetze of particular interest are those generated by a mode such as $[A,A,B,B,C,C]$, in order to study the hexachords in the 12-tone scale with the mode [3,3,2,2,1,1], whose submodes, according to Lemma 2.2, generate, in addition to the major and minor triads, the major and minor seventh chords [4,3,4,1],[3,4,3,2], the augmented and diminished chords [4,4,4],[3,3,6], their seventh chords [4,4,3,1],[4,4,2,2],[3,3,3,3], the dominant seventh chord [4,3,3,2], etc. Therefore, a natural extension of the present work would be to study how the algebra of chords becomes restricted to less fine partitions of the octave, i.e., how it behaves with regard to submodes.

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