(3.1) Camps escalars i vectorials

1 Sigui c un escalar, f, g camps **escalars** i F, G camps vectorials suficientment diferenciables. Proveu que

(a) i) $\operatorname{grad}(cf+g) = c \operatorname{grad} f + \operatorname{grad} g$.

ii) $\operatorname{rot}(cF+G) = c \operatorname{rot} F + \operatorname{rot} G$

iii) $\operatorname{div}(cF+G) = c \operatorname{div} F + \operatorname{div} G$

(b) i) $\operatorname{grad}(f\cdot g) = f \operatorname{grad} g + g \operatorname{grad} f$

ii) $\operatorname{rot}(f\cdot F) = (\operatorname{grad} f)\times F + f\cdot \operatorname{rot} F$

iii) $\operatorname{div}(f\cdot F) = \langle \operatorname{grad} f, F\rangle + f\cdot \operatorname{div} F$

iv) $\operatorname{div}(F\times G) = \langle \operatorname{rot} F, G\rangle - \langle F, \operatorname{rot} G\rangle$

(c) i) $\operatorname{rot}(\operatorname{grad} f) = 0$

ii) $\operatorname{div}(\operatorname{rot} F) = 0$

(a) i) Usem la linealitat de les derivades parcials: $(cf+g)_x = c f_x + g_x$, etc.

ii) iii) $\}$ ídem.

(b) i) $\operatorname{grad}(f\cdot g) = ((f\cdot g)_x, (f\cdot g)_y, (f\cdot g)_z) = (f_x\cdot g + f\cdot g_x,\ f_y\cdot g + f\cdot g_y,\ f_z\cdot g + f\cdot g_z) =$
$= g\cdot(f_x, f_y, f_z) + f\cdot(g_x, g_y, g_z) = g\cdot \operatorname{grad} f + f\cdot \operatorname{grad} g$.

ii) $F = (P, Q, R)$

$\operatorname{rot}(f\cdot F) = ((f R)_y - (f\cdot Q)_z,\ (f\cdot P)_z - (f\cdot R)_x,\ (f\cdot Q)_x - (f\cdot P)_y) =$
$= (f_y\cdot R - f_z\cdot Q,\ f_z\cdot P - f_x\cdot R,\ f_x\cdot Q - f_y\cdot P) + f\cdot(R_y - Q_z, P_z - R_x, Q_x - P_y) =$
$= (\operatorname{grad} f)\times F + f\cdot \operatorname{rot} F$.

iii) $\operatorname{div}(f\cdot F) = (f\cdot P)_x + (f\cdot Q)_y + (f\cdot R)_z = f_x\cdot P + f_y\cdot Q + f_z\cdot R + f\cdot(P_x + Q_y + R_z) =$
$= \langle \operatorname{grad} f, F\rangle + f\cdot \operatorname{div} F$.

iv) $F = (P, Q, R)$, $G = (S, T, U)$

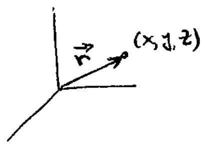
$\operatorname{div}(F\times G) = (Q\cdot U - R\cdot T)_x + (R\cdot S - P\cdot U)_y + (P\cdot T - Q\cdot S)_z =$
$= Q_x\cdot U - R_x\cdot T + R_y\cdot S - P_y\cdot U + P_z\cdot T - Q_z\cdot S + Q\cdot U_x - R\cdot T_x + R\cdot S_y - P\cdot U_y + P\cdot T_z - Q\cdot S_z =$
$= (R_y - Q_z)\cdot S + (P_z - R_x)\cdot T + (Q_x - P_y)\cdot U + P(T_z - U_y) + Q\cdot(U_x - S_z) + R\cdot(S_y - T_x) =$
$= \langle \operatorname{rot} F, G\rangle - \langle F, \operatorname{rot} G\rangle$.

(c) i) $\operatorname{rot}(\operatorname{grad} f) = (f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy}) = (0,0,0)$, si f és C^2.

ii) $\operatorname{div}(\operatorname{rot} F) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z = 0$, si F és C^2.

② Troben

(a) $\operatorname{grad}\left(3r^2-4\sqrt{r}+\frac{6}{\sqrt[3]{r}}\right)$.



Notació: $\vec r=(x,y,z)$, $\quad r=\|\vec r\|=\sqrt{x^2+y^2+z^2}$

Tenim $r^\alpha=(x^2+y^2+z^2)^{\alpha/2}$

$\longrightarrow$ derivem: $(r^\alpha)_x=\frac{\alpha}{2}(x^2+y^2+z^2)^{\frac{\alpha}{2}-1}\cdot 2x=\alpha r^{\alpha-2}x$,

i també $(r^\alpha)_y=\alpha r^{\alpha-2}y$, $\quad (r^\alpha)_z=\alpha r^{\alpha-2}z$.

$\Rightarrow \boxed{\operatorname{grad}(r^\alpha)=\alpha r^{\alpha-2}\vec r}$

Llavors,

$$\operatorname{grad}\left(3r^2-4\sqrt{r}+\frac{6}{\sqrt[3]{r}}\right)=\operatorname{grad}\left(3r^2-4r^{1/2}+6r^{-1/3}\right)=\left(6-2r^{-3/2}-2r^{-4/3}\right)\vec r.$$

(b) $$\operatorname{grad}(r^2e^{-r})=e^{-r}\operatorname{grad}(r^2)+r^2\operatorname{grad}(e^{-r})=e^{-r}2\vec r-r^2\frac{e^{-r}}{r}\vec r=(2-r)e^{-r}\vec r.$$

Calculem: $(e^{-r})_x=-e^{-r}r_x=-e^{-r}r^{-1}x$, $\quad (e^{-r})_y,\ (e^{-r})_z$ semblants $\Rightarrow \operatorname{grad}(e^{-r})=-\frac{e^{-r}}{r}\vec r$

③ Calculem

(a) $\operatorname{div} r^3(x,y,z)=\operatorname{div}(r^3\vec r)$

En general, $\boxed{\operatorname{div}(r^\alpha\vec r)=\langle\operatorname{grad}(r^\alpha),\vec r\rangle+r^\alpha\operatorname{div}\vec r\underset{\text{probl. 2}}{=}\langle\alpha r^{\alpha-2}\vec r,\vec r\rangle+r^\alpha\cdot 3=(\alpha+3)r^\alpha}$

Per tant, $\operatorname{div}(r^3\vec r)=6r^3$

(Obs. $\operatorname{div}\vec r=3$ a $\mathbb{R}^3$, però $\operatorname{div}\vec r=2$ a $\mathbb{R}^2$)

(b) $$\operatorname{div}\Big(r\,\underbrace{\operatorname{grad}\left(\tfrac{1}{r^3}\right)}_{=\,-3r^{-5}\vec r}\Big)=-3\operatorname{div}(r^{-4}\vec r)=3r^{-4}$$

④

Proveu que si A és un camp vectorial constant, llavors $\operatorname{rot}(A\times X)=2A$, $\operatorname{div}(A\times X)=0$ on $X=(x,y,z)$. En el moviment de rotació d'un sòlid rígid, al voltant d'un eix paral·lel a A que passi per l'origen de coordenades, amb velocitat angular $\omega=|A|$ s'obté que $A\times X$ és el vector velocitat.

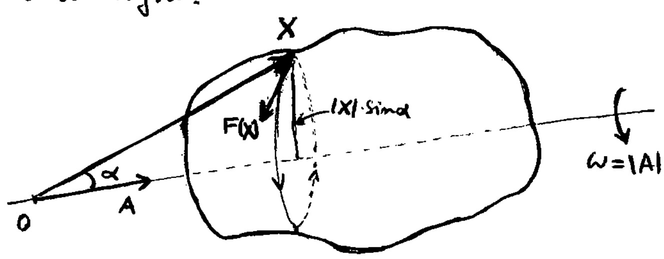
Escrivint $A=(a,b,c)$, $X=(x,y,z)$ $\rightsquigarrow$ $A\times X=\begin{vmatrix} i & j & k\\ a & b & c\\ x & y & z\end{vmatrix}=(bz-cy,\ cx-az,\ ay-bx)$

Calculem:

$$\operatorname{rot}(A\times X)=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ bz-cy & cx-az & ay-bx\end{vmatrix}=(2a,2b,2c)=2A$$

$$\operatorname{div}(A\times X)=0+0+0=0.$$

Sòlid rígid:



En un punt donat X del sòlid rígid, el vector velocitat $F(X)$ ha de complir:

- és ortogonal a A i a X
- té norma $\omega|X|\sin\alpha=|A\times X|$

$\Rightarrow F(X)=\pm A\times X$ (no es determina el sentit de gir)

⑤ Proven que els camps vectorials següents no deriven de potencial:

(a) $F = (\underbrace{y\cos x}_{P}, \underbrace{x \sin y}_{Q})$, $\underbrace{P_y = \cos x}$ $\neq$ $\underbrace{Q_x = \sin y}$ $\longrightarrow$ no deriva de potencial.

(b) $F = (x^2+y^2, -2xy, z)$

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & z \end{vmatrix} = (0,0,-4y) \neq (0,0,0) \longrightarrow \text{no deriva de potencial.}$$

⑥ Troben les constants a, b, c de forma que $F = (x+2y+az, bx-3y-z, 4x+cy+2z)$ sigui irrotacional, i troben en aquest cas una funció potencial de F.

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = (c+1, a-4, b-2) = (0,0,0)$$

$$\Rightarrow \underline{\underline{a=4,\ b=2,\ c=-1}}$$

Amb aquestes constants, $F = (\underbrace{x+2y+4z}_{P}, \underbrace{2x-3y-z}_{Q}, \underbrace{4x-y+2z}_{R})$.

Per trobar una funció potencial, fem:

$$f(x,y,z) = \int_0^x P(t,0,0)\,dt + \int_0^y Q(x,t,0)\,dt + \int_0^z R(x,y,t)\,dt = \int_0^x t\,dt + \int_0^y (2x-3t)\,dt + \int_0^z (4x-y+2t)\,dt =$$

$$= \left[\frac{t^2}{2}\right]_{t=0}^{t=x} + \left[2xt - \frac{3t^2}{2}\right]_{t=0}^{t=y} + \left[4xt - yt + t^2\right]_{t=0}^{t=z} = \underline{\underline{\frac{x^2}{2} + 2xy - \frac{3y^2}{2} + 4xz - yz + z^2}},$$

i es comprova que $\underline{\nabla f = F}$.

* Una altra possibilitat:

$$\frac{\partial f}{\partial x} = P \longrightarrow f(x,y,z) = \int P(x,y,z)\,dx = \frac{x^2}{2} + 2xy + 4xz + \varphi(y,z).$$

$$\frac{\partial f}{\partial y} = 2x + \frac{\partial \varphi}{\partial y} = Q \longrightarrow \varphi(y,z) = \int (-3y-z)\,dy = -\frac{3y^2}{2} - yz + \psi(z)$$

$$\frac{\partial f}{\partial z} = 4x + \frac{\partial \varphi}{\partial z} = 4x - y + \psi'(z) = R \longrightarrow \psi(z) = \int 2z\,dz = z^2 + C$$

$\Rightarrow f(x,y,z) = \cdots$ (com abans, + const.)

[__Obs__: $F = A\vec{r}$, amb A matriu simètrica, i hem obtingut $f = \frac{1}{2}\langle A\vec{r}, \vec{r}\rangle$

⑦ Proven que els camps vectorials

(a) $F = (2xyz^3, x^2z^3, 3x^2yz^2)$

(b) $F = (y^2 - 2xyz^3, 3 + 2xy - x^2z^3, 6z^3 - 3x^2yz^2)$

són irrotacionals i troben una funció potencial U de F tal que $U(1,-2,2) = 4$.

(a)

$$\text{rot } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} = (0,0,0).$$

un potencial: $f(x,y,z) = \int_0^x 0\,dt + \int_0^y 0\,dt + \int_0^z 3x^2yt^2\,dt = x^2yz^3$, $\nabla f = F$

Com que $f(1,-2,2) = -16$, prenem $U(x,y,z) = f(x,y,z) + 20 = \underline{\underline{x^2yz^3 + 20}}$

(b) $\text{rot } F = \cdots = (0,0,0)$.

$$f(x,y,z) = \int_0^x 0\,dt + \int_0^y (3 + 2xt)\,dt + \int_0^z (6t^3 - 3x^2yt^2)\,dt = 3y + xy^2 + \frac{3z^4}{2} - x^2yz^3, \quad \nabla f = F.$$

$$f(1,-2,2) = 38 \leadsto U(x,y,z) = f(x,y,z) - 34 = \cdots$$

(3.2) Teoremes integrals.

(8) Troben la circulació de $(2xy+z^3, x^2, 3xz^2)$ a través de la corba $\alpha(t) = (\cos t^2, \sin t^2, t^2)$, $0 \le t \le \sqrt{\pi}$.

Comprovem si $F = (2xy+z^3,\ x^2,\ 3xz^2)$ és conservatiu (és a dir, deriva de potencial):

$$\operatorname{rot} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = (0,0,0),$$

trobem un potencial, $f(x,y,z) = \int_0^x 0\,dt + \int_0^y x^2\,dt + \int_0^z 3xt^2\,dt = x^2y + xz^3$,

que compleix $\nabla f = F$.

Com que F deriva del potencial f,

$$\int_\alpha \langle F, dl\rangle = f(\alpha(\sqrt{\pi})) - f(\alpha(0)) = f(-1,0,\pi) - f(1,0,0) = -\pi^3.$$

(9) Troben $\int_C (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$, essent C la corba de $(0,0)$ a $(2,1)$ satisfent l'equació $x^4 - 6xy^3 = 4y^2$.

$F = (P,Q) = (10x^4 - 2xy^3, -3x^2y^2)$, compleix $P_y = Q_x = -6xy^2$.

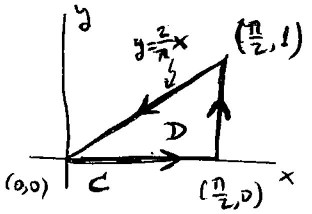
Trobem un potencial, $f(x,y) = \int_0^x P(t,0)\,dt + \int_0^y Q(x,t)\,dt = \int_0^x 10t^4\,dt + \int_0^y (-3x^2t^2)\,dt =$

$= 2x^5 - x^2y^3$, $\nabla f = F$.

Llavors,

$$\int_C P\,dx + Q\,dy = f(2,1) - f(0,0) = 60.$$

(10) Troben $\int_C (y - \sin x)\,dx + \cos x\,dy$, essent C el triangle de vèrtexs $(0,0)$, $(\frac{\pi}{2},0)$, $(\frac{\pi}{2},1)$ recorregut en el sentit positiu.



Podem aplicar el teo. de Green:

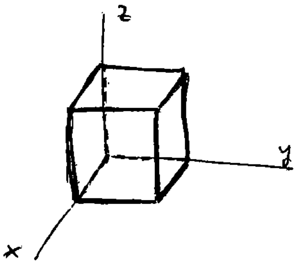
$$\int_{C^+} \underbrace{(y-\sin x)}_{P}\,dx + \underbrace{\cos x}_{Q}\,dy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = -\int_D (\sin x + 1)\,dx\,dy =$$

$$= -\int_0^{\pi/2} (\sin x + 1)\,dx \int_0^{\frac{2}{\pi}x} dy = -\frac{2}{\pi}\int_0^{\pi/2} (x\sin x + x)\,dx =$$

$$= -\frac{2}{\pi}\left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\pi/2} = -\frac{2}{\pi}\left(1 + \frac{\pi^2}{8}\right) = -\frac{2}{\pi} - \frac{\pi}{4}.$$

[Nota: sense aplicar el teo. de Green, caldria calcular la integral de línia al llarg dels 3 segments.]

(11) Troben el flux de $(4xz, -y^2, yz)$ a través del cub unitat.



$W = [0,1]\times[0,1]\times[0,1]$

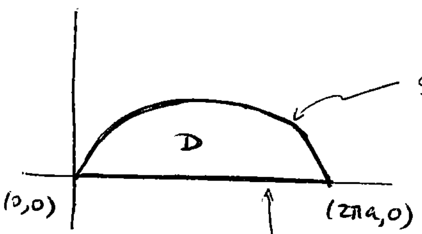
$S = \partial W$ superfície regular a trossos (amb 6 trossos).

Apliquem el teo. de la divergència. Per al camp $F = (4xz, -y^2, yz)$, el flux sortint del cub és:

$$\int_{S^+} \langle F, dS\rangle = \int_W \operatorname{div} F \, dx\,dy\,dz = \int_W (4z - y)\, dx\,dy\,dz = \int_0^1 dx \int_0^1 dy \int_0^1 (4z-y)\,dz =$$

$$= \int_0^1 4z\,dz - \int_0^1 y\,dy = \underline{3/2}.$$

(14) Troben l'àrea limitada per un arc de la cicloide i l'eix x, fent servir la fórmula de Green-Riemann.



cicloide: $\alpha(t) = (a(t - \sin t), a(1-\cos t))$, $0 \le t \le 2\pi$.

segment: $\beta(t) = (t, 0)$, $0 \le t \le 2\pi a$.

La frontera ∂D, recorreguda en sentit positiu (antihorari), està formada per la cicloide $\alpha(t)$ recorreguda en sentit invers, i pel segment $\beta(t)$.

Llavors,

$$A(D) = \frac{1}{2}\int_{\partial D^+} (-y)\,dx + x\,dy = -\frac{1}{2}\int_\alpha + \frac{1}{2}\int_\beta .$$

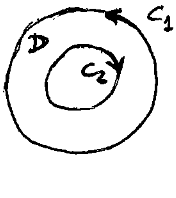
$$\int_\alpha (-y)dx + x\,dy = \int_0^{2\pi} (-y(t)\,x'(t) + x(t)\,y'(t))\,dt = \int_0^{2\pi} \left(-a^2(1-\cos t)^2 + a^2(t - \sin t)\sin t\right) dt =$$

$$= a^2 \int_0^{2\pi} (t \sin t + 2\cos t - 2)\,dt = -6\pi a^2$$

$\int_\beta (-y)dx + x\,dy = 0$, ja que $(-y, x) \perp (1,0)$ per a $y = 0$.

Per tant, $A(D) = -\frac{1}{2}(-6\pi a^2) + 0 = \underline{\underline{3\pi a^2}}.$

(15) Verifiquen el teorema de Green-Riemann integrant $(2x^3-y^3)dx+(x^3+y^3)dy$, al llarg de la vora de la regió del pla determinada per $a^2\leq x^2+y^2\leq b^2$.



$D=\{a^2\leq x^2+y^2\leq b^2\}$

$\partial D = C_1 \cup C_2$

Per tenir ∂D orientada positivament (∂D^+), considerem les parametritzacions

$$\begin{cases} C_1: & \alpha(t)=(b\cos t,\ b\sin t),\quad 0\leq t\leq 2\pi. \\ C_2: & \beta(t)=(a\cos t,\ -a\sin t),\quad 0\leq t\leq 2\pi. \end{cases}$$

$F=(P,Q)=(2x^3-y^3,\ x^3+y^3)$, hem de comprovar: $\underbrace{\int_{\partial D^+} P\,dx+Q\,dy}_{\int_{C_1}+\int_{C_2}} = \int_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy.$

Calculem:

$$\int_{C_1} P\,dx+Q\,dy = \int_0^{2\pi}\left[\left(2(b\cos t)^3-(b\sin t)^3\right)\cdot(-b\sin t)+\left((b\cos t)^3+(b\sin t)^3\right)b\cos t\right]dt =$$

$$= b^4\int_0^{2\pi}\left(-2\cos^3 t\sin t+\sin^4 t+\cos^4 t+\sin^3 t\cdot\cos t\right)dt = b^4\cdot\left(\frac{3\pi}{4}+\frac{3\pi}{4}\right)=\boxed{\frac{3\pi b^4}{2}}$$

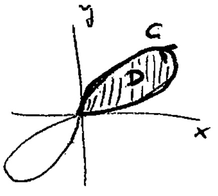
$$\boxed{\begin{aligned} &\int_0^{2\pi}\cos^3 t\sin t = -\frac{\cos^4 t}{4}\Big]_0^{2\pi}=0,\quad \int_0^{2\pi}\sin^3 t\cdot\cos t=\frac{\sin^4 t}{4}\Big]_0^{2\pi}=0.\\ &\int_0^{2\pi}\sin^4 t\,dt = 4\int_0^{\pi/2}\sin^4 t\,dt = 2\,B\left(\tfrac{5}{2},\tfrac{1}{2}\right)=2\,\frac{\Gamma(5/2)\cdot\Gamma(1/2)}{\Gamma(3)}=\\ &\qquad = \frac{2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2!}=\frac{3\pi}{4}.\\ &\int_0^{2\pi}\cos^4 t\,dt = 4\int_0^{\pi/2}\cos^4 t\,dt = 4\int_0^{\pi/2}\sin^4 dt \end{aligned}}$$

$$\int_{C_2} P\,dx+Q\,dy = \int_0^{2\pi}\left[\left(2(a\cos t)^3-(-a\sin t)^3\right)(-a\sin t)+\left((a\cos t)^3+(-a\sin t)^3\right)\cdot(-a\cos t)\right]dt =$$

$$= a^4\int_0^{2\pi}\left(-2\cos^3 t\sin t-\sin^4 t-\cos^4 t+\sin^3 t\cos t\right)dt = -\frac{3\pi a^4}{2}$$

$$\int_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int_D(3x^2+3y^2)\,dx\,dy \underset{\text{(polars)}}{=} 2\pi\int_a^b 3r^2\cdot r\,dr = 6\pi\,\frac{r^4}{4}\Big]_a^b=\frac{3\pi}{2}(b^4-a^4).$$

(17) Calculen l'àrea d'un pètal de rosa $r=3\sin 2\theta$, usant el teorema de Green-Riemann.



$C:\quad r=3\sin 2\theta,\quad 0\leq\theta\leq\pi/2.$

$\rightarrow$ ve parametritzada per $\alpha(\theta)=(r(\theta)\cdot\cos\theta,\ r(\theta)\cdot\sin\theta)=(\overbrace{3\sin 2\theta\cdot\cos\theta}^{x(\theta)},\ \overbrace{3\sin 2\theta\cdot\sin\theta}^{y(\theta)})$, $0\leq\theta\leq\pi/2$.

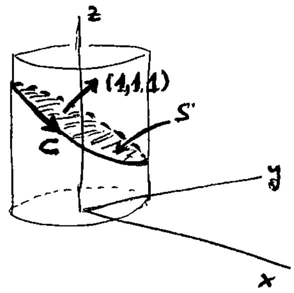
(orientació positiva).

$$A(D)=\frac{1}{2}\int_C(-y)\,dx+x\,dy=\frac{1}{2}\int_0^{\pi/2}\Big[-3\sin 2\theta\cdot\sin\theta\cdot(6\,\cancel{\cos 2\theta\cdot\cos\theta}-3\sin 2\theta\cdot\sin\theta)+$$

$$+3\sin 2\theta\cdot\cos\theta\,(6\,\cancel{\cos 2\theta\cdot\sin\theta}+3\sin 2\theta\cdot\cos\theta)\Big]\,d\theta=$$

$$=\frac{9}{2}\int_0^{\pi/2}\sin^2 2\theta\cdot(\sin^2\theta+\cos^2\theta)\,d\theta=\frac{9}{2}\int_0^{\pi/2}\frac{1-\cos 4\theta}{2}\,d\theta=\frac{9}{2}\cdot\frac{\pi}{4}=\frac{9\pi}{8}$$

(19) Fen servir el teorema de Stokes per calcular la integral de $-y^3dx + x^3dy - z^3dz$ a través de la intersecció del cilindre $x^2+y^2=1$ i el pla $x+y+z=1$, orientada amb el vector normal $(1,1,1)$.



$C = \{x^2+y^2=1,\ x+y+z=1\}$

$S = \{x^2+y^2 \leq 1,\ x+y+z=1\}$, amb l'orientació donada per $N = \frac{1}{\sqrt{3}}(1,1,1)$; la vora de S és C.

L'orientació de C és la compatible amb N.

$\mathbf{F} = (-y^3, x^3, -z^3)$

$$\int_C \langle F, dl\rangle = \int_S \langle \text{rot } F, dS\rangle = \int_S \langle \text{rot } F, N\rangle\, dS = \sqrt{3}\int_S (x^2+y^2)\, dS = \sqrt{3}\int_D (x^2+y^2)\sqrt{3}\, dx\, dy =$$

rot $\mathbf{F} = (0,0,3x^2+3y^2)$
$N = \frac{1}{\sqrt{3}}(1,1,1)$

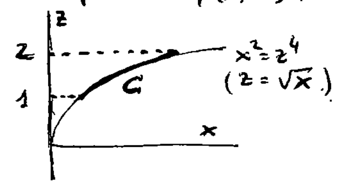
S ve parametritzada per $\Phi(x,y) = (x,y,1-x-y)$, $(x,y)\in D = \{x^2+y^2\leq 1\}$
$\|\Phi_x \wedge \Phi_y\| = \sqrt{3}$

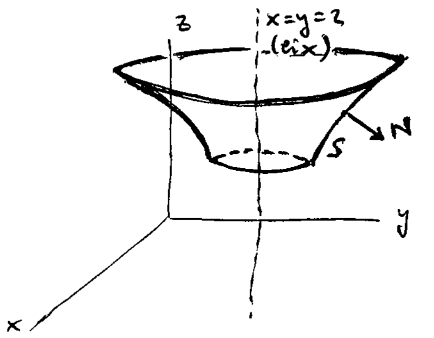
$$= 3\cdot 2\pi\int_0^1 r^2\cdot r\, dr = \frac{3\pi}{2}$$

polars

(20) Calculeu el flux de $\mathbf{F} = (ze^{-x}y,\ y^2e^{-x},\ 1)$ a través de $S = \{(x-2)^2+(y-2)^2 = z^4,\ 1\leq z\leq 2\}$, orientada pel camp de vectors normals $N = (x-2, y-2, -2z^3)$

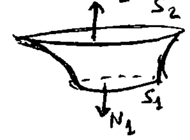
Obs. La superfície S s'obté com a translació de $\tilde{S} = \{x^2+y^2=z^4,\ 1\leq z\leq 2\}$, la qual és la superfície de revolució generada per $C = \{(x,z):\ x^2=z^4,\ 1\leq z\leq 2,\ x\geq 0\}$ quan gira al voltant de l'eix z.





$N = (x-2, y-2, -2z^3)$ és un vector normal a S (no unitari), orientat cap a l'exterior.

Com veiem, la superfície S no és sup. tancada, però podem afegir-li unes "tapes" S_1 i S_2 per a completar una sup. tancada:



$S_1 = \{(x-2)^2+(y-2)^2\leq 1,\ z=1\}$, amb vector normal exterior $N_1 = (0,0,-1)$

$S_2 = \{(x-2)^2+(y-2)^2\leq 16,\ z=2\}$, " " " $N_2 = (0,0,1)$.

Llavors, $S\cup S_1\cup S_2 = \partial W$, amb $W = \{(x-2)^2+(y-2)^2\leq z^4,\ 1\leq z\leq 2\}$.

Pel teo. de la divergència, $\int_W \text{div } F = \int_{\partial W^+}\langle F, dS\rangle = \int_S\langle F, dS\rangle + \int_{S_1}\langle F, dS\rangle + \int_{S_2}\langle F, dS\rangle$.

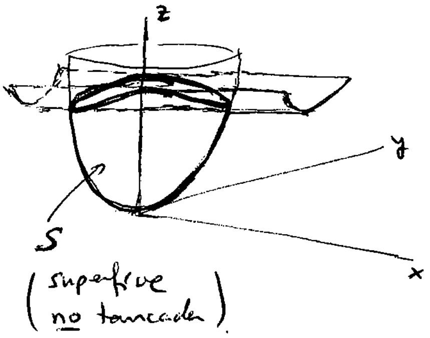
Tenim: $\text{div } F = -ze^{-x}y + 2ye^{-x} + 0 = 0 \Rightarrow \int_W \text{div } F = 0$.

$\int_{S_1}\langle F, dS\rangle = \int_{S_1}\langle F, N_1\rangle dS = \int_{S_1}(-1)\, dS = -A(S_1) = -\pi$

$\int_{S_2}\langle F, dS\rangle = \int_{S_2}\langle F, N_2\rangle dS = \int_{S_2} 1\cdot dS = A(S_2) = 16\pi$

$\Rightarrow \int_S\langle F, dS\rangle = 0 - (-\pi) - 16\pi = -15\pi$

(21) Es considera la superfície S definida per $z = x^2+4y^2$, $z \le 3y^2+1$, orientada pel camp normal $N=(-2x,-8y,1)$. Calculeu el flux del camp $F=(1,0,2)$ a través de S.



$z = x^2+4y^2$ paraboloide el·líptic (no de revolució)

$z = 3y^2+1$ cilindre parabòlic.

$S = \{ z = x^2+4y^2,\ z \le 3y^2+1 \}$, parametritzada per

$\Phi(x,y) = (x, y, \underbrace{x^2+4y^2}_{f(x,y)})$, $(x,y) \in D = \{(x,y) : x^2+y^2 \le 1\}$ $\quad [z = x^2+4y^2 \le 3y^2+1]$

$f(x,y)$, gràfica.

$\Phi_x \wedge \Phi_y = (-f_x, -f_y, 1) = (-2x, -8y, 1) = N$.

Calculem el flux directament:

$$\int_S \langle F, dS\rangle = \int_D \langle F\circ\Phi, \Phi_x\wedge\Phi_y\rangle\, dx\,dy = \int_D (-2x+2)\,dx\,dy \underset{(\text{polars})}{=} \int_0^1 dr \int_0^{2\pi} (-2r\cos\theta+2)\, r\, d\theta = \underline{2\pi}$$

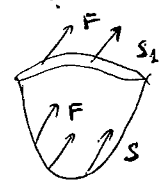
Obs. També es pot aplicar el teo. de la divergència, considerant

$W = \{ x^2+4y^2 \le z \le 3y^2+1 \}$, $\partial W = S \cup S_1$, essent $S_1 = \{ z = 3y^2+1,\ z \ge x^2+4y^2 \}$.

("tapa superior")

$$\int_W \underbrace{\operatorname{div} F}_{\overset{\shortparallel}{0}} = \int_{\partial W^+} \langle F, dS\rangle = \int_{S^+} \langle F, dS\rangle + \int_{S_1^+} \langle F, dS\rangle$$

normals exteriors a W.

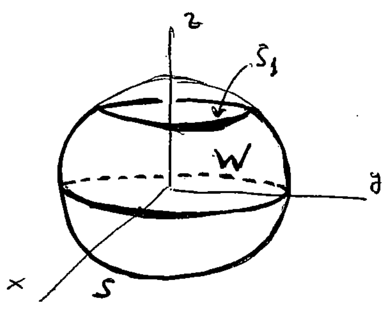


Com que en el nostre cas S ve orientada per N, normal interior, resulta:

$$\int_S \langle F, dS\rangle = -\int_{S^+} \langle F, dS\rangle = \int_{S_1^+} \langle F, dS\rangle = \int_D \langle F\circ\Psi, \Psi_x\wedge\Psi_y\rangle\, dx\,dy = \int_D 2\,dx\,dy = \underline{2\pi}$$

$\Psi(x,y) = (x, y, 3y^2+1)$,
domini: $3y^2+1 \ge x^2+4y^2 \to D$.
$\Psi_x \wedge \Psi_y = (0, -6y, 1)$, normal exterior.

(24) Sigui S la superfície definida per $x^2+y^2+z^2=1$, $z \le 1/2$, orientada pel camp de vectors normals $N=(x,y,z)$. Sigui F el camp vectorial definit per $F(x,y,z) = (x,y,-2z)$. Trobeu el flux de F a través de S.



$S = \{ x^2+y^2+z^2 = 1,\ z \le 1/2 \}$ orientada per $N = (x,y,z)$ (normal exterior).

Considerem $W = \{ x^2+y^2+z^2 \le 1,\ z \le 1/2 \}$,

tenim $\partial W = S \cup S_1$, essent $S_1 = \{ x^2+y^2+z^2 \le 1,\ z = 1/2 \} =$
$= \{ x^2+y^2 = \frac{3}{4},\ z = \frac{1}{2} \}$
(cercle de radi $\sqrt{3}/2$)

Tenim $\operatorname{div} F = 1+1-2 \equiv 0$. Aplicant el teo. de la divergència,

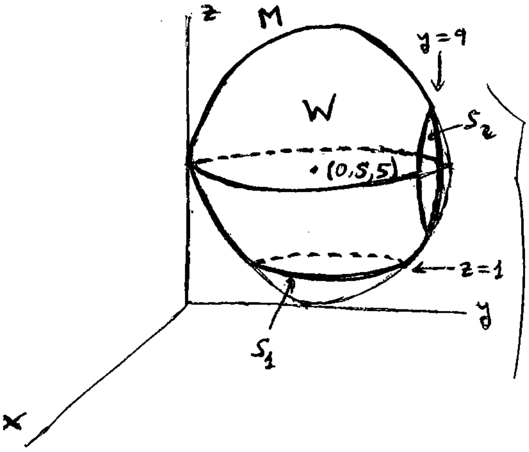
$$0 = \int_W \operatorname{div} F = \int_{\partial W^+} \langle F, dS\rangle = \int_S \langle F, dS\rangle + \int_{S_1} \langle F, dS\rangle$$

$N_1 = (0,0,1)$ normal ext. a S_1

$z \equiv 1/2$ sobre S_1

$$\Rightarrow \int_S \langle F, dS\rangle = -\int_{S_1} \langle F, dS\rangle = -\int_{S_1} \langle F, N_1\rangle\, dS = \int_{S_1} 2z\, dS = A(S_1) = \pi\left(\frac{\sqrt{3}}{2}\right)^2 = \underline{\frac{3\pi}{4}}$$

(22) Sigui M el subconjunt de $\mathbb{R}^3$ definit per $M=\{x^2+(y-5)^2+(z-5)^2=25,\ z\geq 1,\ y\leq 9\}$.
Sigui $F(x,y,z)=(-x,0,x+z)$, determineu el flux de F a través de M, orientat pel camp de vectors normals $N=(2x,2(y-5),2(z-5))$.



Considerem $W=\{x^2+(y-5)^2+(z-5)^2\leq 25,\ z\geq 1,\ y\leq 9\}$.

Tenim $\partial W = M \cup S_1 \cup S_2$,

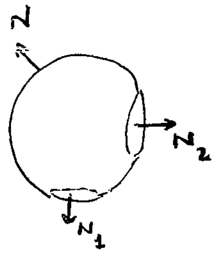
$S_1=\{x^2+(y-5)^2+(z-5)^2\leq 25,\ z=1\}=\{x^2+(y-5)^2\leq 9,\ z=1\}$.

$S_2=\{x^2+(y-5)^2+(z-5)^2\leq 25,\ y=9\}=\{x^2+(z-5)^2\leq 9,\ y=9\}$.

(S_1 i S_2 són cercles de radi 3).

$\operatorname{div} F\equiv 0 \longrightarrow$ pel teo. de la divergència,

$$0=\int_W \operatorname{div} F=\int_{\partial W^+}\langle F,dS\rangle=\int_M\langle F,dS\rangle+\int_{S_1}\langle F,dS\rangle+\int_{S_2}\langle F,dS\rangle,$$



amb M, S_1 i S_2 orientades per la normal exterior.

En el cas de M, ens venia orientada ja per la normal exterior $N=(2x,2(y-5),2(z-5))$.

• Calculem:
$$\int_{S_1}\langle F,dS\rangle=\int_{S_1}\langle F,N_1\rangle\,dS=\int_{S_1}(-x-z)\,dS=\int_{D_1}(-x-1)\,dx\,dy=$$

$N_1=(0,0,-1)$

parametrització:
$\Phi(x,y)=(x,y,1)$,
$D_1=\{x^2+(y-5)^2\leq 9\}$
$\Phi_x\wedge\Phi_y=(0,0,1)$, $\|\Phi_x\wedge\Phi_y\|=1$

$$=\int_0^3 dr\int_0^{2\pi}(-r\cos\theta-1)\,dr\,d\theta=-9\pi$$

polars centrades al $(0,5)$:
$\begin{cases}x=r\cos\theta\\ y=5+r\sin\theta.\end{cases}$

• D'altra banda, $\displaystyle\int_{S_2}\langle F,dS\rangle=\int_{S_2}\langle F,N_2\rangle=0$

$N_2=(0,1,0)$
$\langle F,N_2\rangle\equiv 0$.

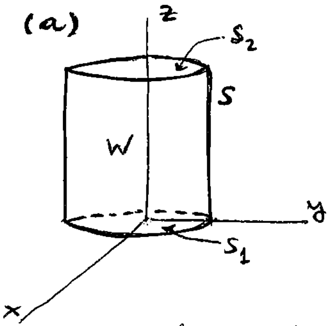
Per tant,
$$\int_M\langle F,dS\rangle=-\int_{S_1}\langle F,dS\rangle-\int_{S_2}\langle F,dS\rangle=-(-9\pi)-0=\underline{\underline{9\pi}}$$

25 Donat el camp $F=(2x,y,3z)$ i la superfície $S=\{(x,y,z): x^2+y^2=4,\ 0\le z\le 5\}$ orientada segons el vector normal $N=(x,y,0)$,

(a) Trobeu el flux del camp F a través de S.

(b) Trobeu la circulació del camp F a través de ∂S.

(a)



$$W=\{x^2+y^2\le 4,\ 0\le z\le 5\}$$

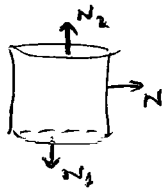
$$\partial W = S\cup S_1\cup S_2, \text{ amb } S_1=\{z=0,\ x^2+y^2\le 4\},$$
$$S_2=\{z=5,\ x^2+y^2\le 4\}.$$

Aplicant el teo. de la divergència,

$$\int_W \operatorname{div}F\,dx\,dy\,dz = \int_{\partial W^+}\langle F,dS\rangle = \int_S\langle F,dS\rangle+\int_{S_1}\langle F,dS\rangle+\int_{S_2}\langle F,dS\rangle,$$

on cal orientar les superfícies per la normal exterior a W:

$N=(x,y,0)$ per a S; $N_1=(0,0,-1)$ per a S_1, $N_2=(0,0,1)$ per a S_2.



Calculem:

$$\int_W \operatorname{div}F\,dx\,dy\,dz = 6\operatorname{vol}(W) = 6\cdot\pi\cdot 2^2\cdot 5 = 120\pi$$

$\operatorname{div}F = 2+1+3=6$

$$\int_{S_1}\langle F,dS\rangle = \int_{S_1}\langle F,N_1\rangle\,dS = 0$$

$\langle F,N_1\rangle = 0$ sobre S_1

$$\int_{S_2}\langle F,dS\rangle = \int_{S_2}\langle F,N_2\rangle\,dS = 15\,A(S_2) = 15\cdot\pi\cdot 2^2 = 60\pi.$$

$\langle F,N_2\rangle = 3z = 15$

Per tant, $\int_S\langle F,dS\rangle = 120\pi - 0 - 60\pi = \underline{60\pi}$.

(b) $\partial S = C_1\cup C_2$

Aplicant el teo. de Stokes, $\int_{\partial S}\langle F,dl\rangle = \int_S\langle \operatorname{rot}F,dS\rangle = \underline{0}$

$$\operatorname{rot}F = \begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x & y & 3z\end{vmatrix} \equiv 0$$

(28)

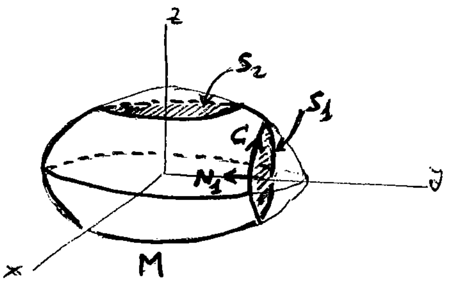
Sigui M el subconjunt de $\mathbb{R}^3$ definit pel sistema $9x^2+6y^2+9z^2=9$, $y\le 1$, $z\le 2/3$.

(a) Sigui C la corba d'intersecció de M amb el pla $y=1$, i $F=(z,y,-x)$. Considerant en C l'orientació pel camp de vectors tangents $(-z,0,x)$, calculeu la circulació de F a través de C.

(b) Calculeu el flux del camp $G=(0,1,0)$, orientant M de manera compatible amb l'orientació de C en l'apartat anterior.

$9x^2+6y^2+9z^2=9$

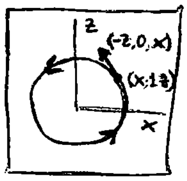
el·lipsoide de semieixos $1, \sqrt{3/2}, 1$.
$\quad (x) \quad (y) \quad (z)$



(a) $C = M \cap \{y=1\} = \{x^2+z^2=\frac{1}{3},\ y=1\}$,
circumferència de radi $1/\sqrt{3}$

(obs. $\frac{1}{\sqrt{3}} < \frac{2}{3} \Rightarrow C$ no talla la intersecció de M amb $z=2/3$)

Orientació de C:



pla $y=1$
(vist des de l'esquema)

Parametritzem C: $\alpha(t)=\left(\frac{1}{\sqrt{3}}\cos t,\ 1,\ \frac{1}{\sqrt{3}}\sin t\right)$, $0\le t\le 2\pi$.

Llavors, $\displaystyle\int_C \langle F, dl\rangle = \int_0^{2\pi}\langle F(\alpha(t)), \alpha'(t)\rangle\, dt = \int_0^{2\pi}\left(-\frac{1}{3}\right)dt = \boxed{-\frac{2\pi}{3}}$

$F(\alpha(t)) = \left(\frac{1}{\sqrt{3}}\sin t,\ 1,\ -\frac{1}{\sqrt{3}}\cos t\right)$
$\alpha'(t) = \left(-\frac{1}{\sqrt{3}}\sin t,\ 0,\ \frac{1}{\sqrt{3}}\cos t\right)$

• També es pot aplicar el teorema de Stokes, considerant que la corba C és la vora de $S_1=\{y=1,\ x^2+z^2\le\frac{1}{3}\}$, tros de pla orientat per $N_1=(0,0,-1)$.

(la "tapa dreta")

$$\int_C\langle F,dl\rangle = \int_{S_1}\langle \mathrm{rot}\,F, dS\rangle = \int_{S_1}\langle \mathrm{rot}\,F, N_1\rangle\, dS = -2A(S_1) = -2\pi\left(\frac{1}{\sqrt{3}}\right)^2 = -\frac{2\pi}{3}$$

$\mathrm{rot}\,F=(0,2,0)$

(b) Hem d'orientar M per la normal exterior.
Considerem també $S_2=\{z=2/3,\ 9x^2+6y^2+9z^2\le 9\}$ (la "tapa superior"),
i llavors $M\cup S_1\cup S_2=\partial W$, amb el sòlid $W=\{9x^2+6y^2+9z^2\le 9,\ y\le 1,\ z\le\frac{2}{3}\}$

Pel teo. de la divergència,
$$\int_W \underset{\substack{\parallel\\0}}{\mathrm{div}\,G} = \int_{\partial W^+}\langle G, dS\rangle = \int_M + \int_{S_1} + \int_{S_2}$$

orientades per la normal exterior a W

$\displaystyle\int_{S_1}\langle G, dS\rangle = \frac{1}{2}\int_{S_1}\langle \mathrm{rot}\,F, dS\rangle = \frac{1}{2}\cdot\left(+\frac{2\pi}{3}\right) = \frac{\pi}{3}$

apartat (a), però cal invertir l'orientació

$\displaystyle\int_{S_2}\langle G, dS\rangle = \int_{S_2}\langle G, N\rangle\, dS = 0$

$N=(0,1,0)$

$\Rightarrow \displaystyle\int_M\langle G, dS\rangle = -\frac{\pi}{3}$

(*) [ex. final juny /09]

(a) Donada una superfície S amb vora C, proveu que si $F(x,y,z)=\vec{v}$ és un camp vectorial constant, es compleix la igualtat
$$2\int_S \langle \vec{v}, dS\rangle = \int_C \langle \vec{v}\wedge\vec{r}, dl\rangle,$$
essent $\vec{r}=(x,y,z)$, i suposant que S i C tenen orientacions compatibles.

(b) Per a la superfície $S=\{(x,y,z):\ z=x^2+y^2,\ z\le 1,\ x\ge 0,\ y\ge 0\}$, orientada en el punt $(1/2,1/2,1/2)$ pel vector $N=\frac{1}{\sqrt{3}}(1,1,-1)$, calculeu el flux del camp vectorial $F=(1,0,0)$ a través de S, aplicant el teorema de Stokes.

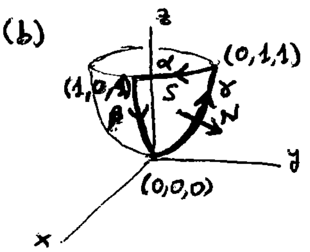
(c) El mateix flux de l'apartat (b), ara calculat directament (és a dir, usant la definició de flux).

(a) Escrivint $G(x,y,z)=\vec{v}\wedge\vec{r}=\begin{vmatrix} i & j & k\\ v_1 & v_2 & v_3\\ x & y & z\end{vmatrix}=(v_2z-v_3y,\ v_3x-v_1z,\ v_1y-v_2x)$,

tenim $\operatorname{rot} G=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ v_2z-v_3y & v_3x-v_1z & v_1y-v_2x\end{vmatrix}=(2v_1,2v_2,2v_3)=2\vec{v}.$

Pel teo. de Stokes, $\int_C \langle \underbrace{G}_{\vec{v}\wedge\vec{r}}, dl\rangle=\int_S\langle \underbrace{\operatorname{rot} G}_{2\vec{v}}, dS\rangle$

(b)



La vora de S és una corba regular a trossos C, amb 3 trossos que parametritzem d'acord amb l'orientació de S:

$$\begin{cases}\alpha(t)=(\sin t,\cos t,1), & 0\le t\le \pi/2,\\ \beta(t)=(1-t,0,(1-t)^2), & 0\le t\le 1,\\ \gamma(t)=(0,t,t^2), & 0\le t\le 1.\end{cases}$$

$F=(1,0,0)=\vec{v}\rightsquigarrow$ prenem $G=\vec{v}\wedge\vec{r}=(0,-z,y)$

Pel teo. de Stokes o l'apartat (a),

$$\int_S\langle F,dS\rangle=\frac12\int_C\langle G,dl\rangle=\frac12\left[\int_\alpha\langle G,dl\rangle+\int_\beta\langle G,dl\rangle+\int_\gamma\langle G,dl\rangle\right]=\frac12\left(1+0+\frac13\right)=\frac23$$

Calculem: $\int_\alpha\langle G,dl\rangle=\int_0^{\pi/2}\langle G(\alpha(t)),\alpha'(t)\rangle dt=\int_0^{\pi/2}\langle(0,-1,\cos t),(\cos t,-\sin t,0)\rangle dt=\int_0^{\pi/2}\sin t\,dt=1$

$\int_\beta\langle G,dl\rangle=\int_0^1\langle G(\beta(t)),\beta'(t)\rangle dt=\int_0^1\langle(0,-(1-t)^2,0),(-1,0,-2(1-t))\rangle dt=0$

$\int_\gamma\langle G,dl\rangle=\int_0^1\langle G(\gamma(t)),\gamma'(t)\rangle dt=\int_0^1\langle(0,-t^2,t),(0,1,2t)\rangle dt=\int_0^1 t^2dt=\frac13$

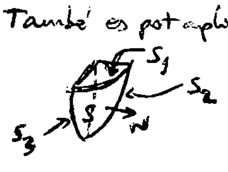
(c) Parametritzem S com una gràfica: $\Phi(x,y)=(x,y,x^2+y^2)$, $D=\{(x,y):x^2+y^2\le1,\ x,y\ge0\}$.

Tenim: $\Phi_x\wedge\Phi_y=(-2x,-2y,1)$; avaluant en el punt $(\frac12,\frac12,\frac12)$ obtenim $(-1,-1,1)$, orientació inversa a N.

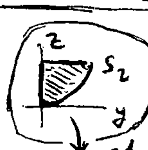
Per la definició de flux,

$$\int_S\langle F,dS\rangle=-\int_D\langle F(\Phi(x,y)),\Phi_x\wedge\Phi_y\rangle dx\,dy=-\int_D\langle(1,0,0),(-2x,-2y,1)\rangle dx\,dy=2\int_D x\,dx\,dy=$$
canvi a polars
$$=2\int_0^1 dr\int_0^{\pi/2} r\cos\theta\cdot r\,d\theta=2\int_0^1 r^2dr\cdot\int_0^{\pi/2}\cos\theta\,d\theta=2\cdot\frac13\cdot1=\frac23$$

(*) També es pot aplicar el teo. divergència, usant que $\operatorname{div} F\equiv 0$.



Com que F és tangent a S_1 i S_3, resulta:



$$\int_S\langle F,dS\rangle=-\int_{S_2}\langle F,dS\rangle=-\int_{S_2}\langle F,(-1,0,0)\rangle dS=A(S_2)=\int_0^1(1-y^2)dy=\frac23$$