COUNTING PRIMITIVE ELEMENTS IN FREE GROUPS

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Abstract

In this paper it is proved that the set of primitive elements of a nonabelian free group has density zero, i.e. the ratio of primitive elements in increasingly large balls is arbitrarily small. Two notions of density (natural and exponential density) are defined and some of their properties are studied. A class of subsets of the free group (graphical sets) is defined restricting the occurrence of adjacent letters in the reduced word for an element, and the relation between graphical sets and the set of primitive elements is studied and used to prove the above result.

Primitive elements in free groups are those that can be part of a free basis of the group. There has been a large body of research concerning primitive elements, dating back to the original paper by Whitehead [W]. It is well known, for instance, what a primitive element looks like in the free group of rank 2 (Cohen et. al., [CMZ] and Hoare, [H]). In this paper we will be interested in their distribution inside the free group with relation to the concentric balls of elements. In particular, we will prove that the set of primitive elements is increasingly sparse in subsequent balls, or, using our terminology, that its *natural density* is zero, and we will give more precise bounds of their number in each ball, using the *exponential density.*

To achieve these goals, the two notions of density are defined. Natural density, as its name indicates, seems to be the logical definition of density, but it lacks precision when related to groups with exponential growth. In these groups, subsets with natural density zero include among them those whose exponential rate of growth is smaller than the one of the total. These subsets will be compared to each other using this exponential growth rate, whose ratio with the total one we have called exponential density. Thus, for groups with exponential growth, interesting subsets will have natural density zero but relevant values for its exponential density, while for groups with subexponential growth, subsets will have exponential density one but smaller (and interesting) values for their natural densities.

The study of densities of subsets has a long tradition in the ambit of number theory (see, for instance, [NZ], Chapter 11). There are well-known results concerning the (natural) densities of some subsets of $\mathbb{Z}$ and of $\mathbb{Z}^2$. Here, density is computed by taking a suitable ball (the intervals $[0, n]$ or $[-n, n]$ in $\mathbb{Z}$ or the square of vertices $(\pm n, \pm n)$ in $\mathbb{Z}^2$, for instance), finding the density of the subset inside this ball, and taking the limit (if it exists) when n goes to infinity. The fascinating Prime Number Theorem (see [HW], for instance) states that the number of prime numbers in the interval $[0, n]$ is asymptotically equal to $n/\log n$. This implies that the density of the set of prime numbers in $\mathbb{Z}$ is zero, so prime numbers are increasingly sparse. Another example is the density of the set of pairs of relatively prime integers in the integer lattice of the plane. It is a well-known result that this density is $6/\pi^2$ (for a proof, see [A] or [HW]). This result is also usually stated (loosely) the following way: taking two integers at random, the probability of getting a pair of relatively prime numbers is $6/\pi^2$. This surprising value is also the density of the set in $\mathbb{Z}$ of square-free integers, see [NZ].

It is interesting to remark that a consequence of all these results is the heuristic fact that it is increasingly more difficult to find bases as the algebraic structure of the set is relaxed, as it could be expected. From the elementary linear algebra fact that any nonzero element can be completed to a basis of a vector space (probability one), to the intermediate values of the probability of finding a primitive element in a free abelian group ($6/\pi^2$ for rank 2) to probability zero in nonabelian free groups, as will be proved in this paper.

To study the set of primitive elements of the free group of rank p, F_p, and inspired by the definition of the Whitehead graph of an element of the free group, we define *graphical sets*. Graphical sets are sets of elements of F_p for which there is a graph which specifies which letters can be adjacent to each other. The vertices of such a graph are the generators of F_p and their inverses, and an edge goes from x to y if the subword xy is allowed. As expected, the graphical set corresponding to the complete graph (of $2p$ vertices) is the whole group, having density one. It will be proved that the graphical set corresponding to any noncomplete graph has density zero. From Whitehead's cut vertex lemma, which forces Whitehead graphs of primitive elements to miss some edges, we will deduce our main result.

The paper is organized as follows: in Section 1 we give the two notions of density, their first properties, and examples of subsets with strange behavior. In section 2, we give some general results about densities, sometimes restricting our attention to a restricted class of groups for which some further properties of the density can be proved. Sections 3 and 4 are dedicated to subsets of the free group. In section 3 we compute the exponential density of the collection of graphical subsets of the free group. And in section 4, we obtain as a corollary the fact that the set of primitive elements in the free group has natural density zero (and we give some upper and lower bounds for its exponential density).

1 Introduction and notation

Let G be a finitely generated group and X a finite set of generators for G.

The *ball* of radius n (centered at the identity) with respect to X, denoted $B_X(n)$, is the set of elements in G that can be written as a word on X having length at most n. In particular, $B_X(0) = \{1\}$. The *sphere* of radius n with respect to X, denoted $S_X(n)$, is precisely $\{1\}$ for $n = 0$ and $B_X(n) \backslash B_X(n-1)$ otherwise.

It is known that the limit $\lim_{n\to\infty} |B_X(n)|^{1/n}$ always exists and it is called the *exponential growth rate* of G with respect to X (see [GH]). In general, this limit depends on the set of generators X. But the fact of being 1 or larger than 1 does not depend on X (see [GH]) and it is an invariant of the group G. A group is said to have *exponential* growth when its exponential growth rate is larger than one, and *subexponential* growth when the exponential growth rate is one. Furthermore, the growth of G is called *polynomial* when $|B_X(n)|$ is bounded above by a polynomial on n.

1.1 Definition Let G be a finitely generated group with finite generating set X. For any subset $S \subseteq G$ we define the *natural density* of S with respect to X, denoted $\delta_X(S)$, as

$$\delta_X(S) = \limsup_{n\to\infty} \frac{|S \cap B_X(n)|}{|B_X(n)|}.$$

And we define the *exponential density* of S with respect to X, denoted $d_X(S)$, as

$$d_X(S) = \limsup_{n\to\infty} \left(\frac{|S \cap B_X(n)|}{|B_X(n)|} \right)^{1/n}.$$

Observe that both natural and exponential densities of any subset S in any finitely generated group G are real numbers between 0 and 1, measuring the relative size of S as a subset of G. When $|G| = \infty$, finite sets have natural density 0 and the whole G has natural density 1. Observe also that if $a = \lim_{n\to\infty} |B_X(n)|^{\frac{1}{n}}$ is the exponential growth rate of G with respect to X (and because of the existence of a as a limit), we have

$$d_X(S) = \frac{1}{a} \limsup_{n\to\infty} |S \cap B_X(n)|^{1/n}.$$

So, the empty set is the unique one with exponential density 0, finite subsets have exponential density equal to $\frac{1}{a}$, and those subsets with positive natural density have exponential density 1.

Clearly, if $S \subseteq S'$ then $d_X(S) \leq d_X(S')$ and $\delta_X(S) \leq \delta_X(S')$.

Suppose that G is a group with subexponential growth. Then, its exponential growth rate is 1 and so, $d_X(S) = \limsup_{n\to\infty} |S \cap B_X(n)|^{1/n} = 1$ for every non-empty subset $S \subseteq G$. Thus, in these groups the exponential density measures nothing and the only relevant definition is the natural density.

Suppose now that G is a group with exponential growth rate $a > 1$. And suppose that S is a subset with exponential density strictly less than 1. Then, there exist $\epsilon > 0$ such that $|S \cap B_X(n)| < (a-\epsilon)^n$ for $n \gg 0$. And this implies that the natural density of S is zero, $\delta_X(S) = 0$. So, natural density is only sensible among those subsets that have exponential density one. Contrarily, exponential density is sensible among sets with natural density 0. In this sense, natural density begins to mesure sets non-trivially when exponential density stops doing it (and this happens in groups with exponential growth when the subset rises asymptotically to the same size as the whole group).

In general, both natural and exponential densities of a subset $S \subseteq G$ depend on the set of generators X. Additionally, the limits in the definitions may not exist forcing us to take $\limsup$ instead of $\lim$ (essentially the same theory can be developed taking $\liminf$ in the definitions of densities). The following examples illustrate the strange consequences of this situation. However, the interesting subsets will be those for which the values of their densities do not depend on the set of generators used to compute them, and for which the limits in the definitions do exist.

1.2 Example Consider the group $G = \mathbb{Z}^2$, seen as the lattice of integer points in the real plane, and the two sets of generators $X = \{(1,0),(0,1)\}$ and $Y = \{(1,0),(1,1)\}$. Let S be the set of points with positive coordinates,

$$S = \{(x,y) \in \mathbb{Z}^2 \mid x, y > 0\}.$$

The balls of radius n with respect to any basis of G all contain $2n^2 + 2n + 1$ elements (so, G has polynomial growth). Easy computations for the cardinals of $S \cap B_X(n)$ and $S \cap B_Y(n)$ show that the natural density of S with respect to X is $\frac{1}{4}$, while with respect to Y is $\frac{3}{8}$. So, in general, natural density does depend on the set of generators.

1.3 Example Let's consider the free group of rank 2, $F_2 =< a, b >$, and $X = \{a, b\}$. In F_2 we have the set of positive words

$$S = \{x_1 \cdots x_r \mid r \geq 0, x_i = a, b\}.$$

It is easy to see that the spheres of radius n with respect to any free basis of F_2 all contain $4 \cdot 3^{n-1}$ words, so the corresponding balls contain $2 \cdot 3^n - 1$ words (so, F_2 has exponential growth). Also, the number of positive words of length n is 2^n, that is, $|S \cap S_X(n)| = 2^n$. And $|S \cap B_X(n)| = 2^{n+1} - 1$. Hence, the exponential density of S with respect to X is $\frac{2}{3}$ (it is easy to extend this computation to an arbitrary finite rank). In Example 3.5 we will see that there exists another basis with respect to which S has different density, showing that, in general, exponential density also depends on the set of generators.

1.4 Example For any infinite, finitely generated group G and any finite system of generators X, there is always a subset $S \subset G$ such that $\delta_X(S) = 1$ and also $\delta_X(G \setminus S) = 1$. To construct such a set, we observe that we can find two sequences of positive integers (n_i) and (m_i), with $i \in \mathbb{N}$, satisfying $n_i < m_i < n_{i+1}$, with $n_1 = 1$, and in such a way that

$$S = \bigcup_{i \geq 1} \bigcup_{n_i < j \leq m_i} S_X(j)$$

has the desired property. Choosing the sequences (n_i) and (m_i) appropriately we can have

$$\frac{|S \cap B_X(n_i)|}{|B_X(n_i)|} < \frac{1}{i} \qquad \text{and} \qquad \frac{|S \cap B_X(m_i)|}{|B_X(m_i)|} > 1 - \frac{1}{i},$$

and then also

$$\frac{|(G \setminus S) \cap B_X(n_i)|}{|B_X(n_i)|} \geq 1 - \frac{1}{i} \qquad \text{and} \qquad \frac{|(G \setminus S) \cap B_X(m_i)|}{|B_X(m_i)|} \leq \frac{1}{i}.$$

The idea in this example is that the set S is the union of some spheres, $S_X(n)$, and the accumulation or the absence of spheres at given intervals makes the sequence of ratios $|S \cap B_X(n)|/|B_X(n)|$ oscillate wildly between 0 and 1.

Modifying the sequences (n_i) and (m_i) in the above example, and assuming that G has exponential growth, it is easy to find a set S which satisfies the same property for the exponential density, i.e. satisfying $d_X(S) = 1$ and $d_X(G \setminus S) = 1$.

The above example shows that, in general, natural density is not additive, even on pairwise disjoint sets, as one could conceivably expect. Another example of this fact, which will be used later in the paper, is the following:

1.5 Example Consider the free group F_p of rank $p \geq 2$, a free basis X, and take the two disjoint sets

$$S_1 = \bigcup_{n \text{ odd}} S_X(n), \qquad S_2 = \bigcup_{n \text{ even}} S_X(n).$$

Straightforward computations show that

$$\delta_X(S_1) = \delta_X(S_2) = \frac{(2p-2)(2p-1)}{(2p-1)^2 - 1},$$

while $S_1 \cap S_2 = \varnothing$, $S_1 \cup S_2 = F_p$ and $\delta_X(S_1) + \delta_X(S_2) \neq 1 = \delta_X(F_p)$.

2 Properties of densities

Let G be a finitely generated group and X a finite set of generators. The purpose now is to show that the pathological behaviors seen in the previous section do not happen if we look at reasonably evenly distributed sets inside G. Our first example is given by nets.

Recall that a subset $S \subseteq G$ is called a *net* if there exists a constant $C > 0$ such that every element in the group is at X-distance at most C from an element of S (i.e. for every $g \in G$ there exist $s \in S$ with $|g^{-1}s|_X \leq C$, where $|\ \ |_X$ means X-length). Note that this notion does not depend on the set of generators because for every other finite generating system Y,

$|g^{-1}s|_Y \leq M|g^{-1}s|_X$ where M is the maximum of the Y-lengths of the elements in X (and so, a net with constant C with respect to X is also a net, now with constant CM, with respect to Y).

2.1 Proposition *Let G be a finitely generated group, and let $S \subseteq G$. If S is a net then $\delta_X(S) > 0$ independently of the finite system of generators X. Furthermore, we also have that $\liminf_{n\to\infty} \frac{|S\cap B_X(n)|}{|B_X(n)|} > 0$.*

Proof. Let X be a finite system of generators for G and suppose that S is a net with constant C with respect to X. If $g \in G$, denote by $B_{X,g}(n)$ the ball of radius n centered at g. Using the triangle inequality we have the inclusion

$$B_X(n-C) \subset \bigcup_{a\in S\cap B_X(n)} B_{X,a}(C),$$

so taking cardinals, and noting that $|B_{X,a}(C)| = |B_X(C)|$, we have

$$|B_X(n-C)| \leq |S \cap B_X(n)||B_X(C)|.$$

Dividing then by $|B_X(n)|$ and taking $\liminf$ we obtain

$$\delta_X(S) \geq \liminf_{n\to\infty} \frac{|S\cap B_X(n)|}{|B_X(n)|} \geq \frac{1}{|B_X(C)|} \liminf_{n\to\infty} \frac{|B_X(n-C)|}{|B_X(n)|}.$$

Now, using the inequality $|B_X(n)| \leq |B_X(n-C)||B_X(C)|$, we obtain

$$\delta_X(S) \geq \frac{1}{|B_X(C)|} \liminf_{n\to\infty} \frac{|B_X(n-C)|}{|B_X(n)|} \geq \frac{1}{|B_X(C)|^2} > 0.$$

Then, both $\delta_X(S)$ and its inferior bound depend on the generators, but the fact that they are strictly positive is indeed independent of X. □

The converse of this last result is obviously not true. One can easily find sets with strictly positive density but which are not nets. Example 1.2 above is one of them. Another interesting example is the following.

2.2 Example The set of pairs (m,n) of relatively prime integers, as stated in the introduction, has positive density as a subset of $\mathbb{Z}^2$ and seems to be evenly distributed, but a closer look reveals that it has arbitrarily large holes and hence it is not a net. The proof of this fact is an easy exercise on the Chinese Remainder Theorem: choosing n^2 different prime numbers p_{ij} with $i,j = 0,\ldots,n-1$, and solving the two systems of congruences

$$x + i \equiv 0 \mod p_{ij}, \text{ for } i,j = 0,\ldots,n-1,$$

and

$$y + j \equiv 0 \mod p_{ij}, \text{ for } i, j = 0, \dots, n-1,$$

we obtain a point (x, y), which is the lower left corner of an $n \times n$ square of points in the integer lattice of the plane, none of which has relatively prime coordinates. The first example of a 2×2 square satisfying this condition is the one formed by the points (20,14), (20,15), (21,14) and (21,15).

Natural density becomes more reasonably behaved when we require an extra condition. Assume that for the finite generating system X we have that

$$\lim_{n\to\infty} \frac{|B_X(n+1)|}{|B_X(n)|} = 1.$$

It is easy to see that if a group satisfies this condition then its exponential growth rate is one, i.e., it has subexponential growth. It is not known whether the converse of this result is true. Partially, this condition is known to be true for groups of polynomial growth, i.e. virtually nilpotent groups. This fact is consequence of results of Pansu [P].

2.3 Proposition *Let G be a finitely generated group, and let X be a finite system of generators for G, satisfying*

$$\lim_{n\to\infty} \frac{|B_X(n+1)|}{|B_X(n)|} = 1.$$

Then the natural density with respect to X is both left- and right-invariant.

Proof. Clearly, to see the left-invariance for δ_X, it is enough to prove that

$$\delta_X(xS) = \delta_X(S)$$

for every $x \in X^{\pm 1}$. The inclusion

$$x(S \cap B_X(n)) \subset xS \cap B_X(n+1)$$

gives

$$|S \cap B_X(n)| = |x(S \cap B_X(n))| \leq |xS \cap B_X(n+1)|.$$

Dividing by $|B_X(n)|$ and taking $\limsup$ we obtain

$$\delta_X(S) \leq \delta_X(xS) \lim_{n\to\infty} \frac{|B_X(n+1)|}{|B_X(n)|} = \delta_X(xS).$$

This shows that δ_X is left invariant. Right-invariance can be proved in a similar way. □

Note that when a group satisfies this extra condition, then the sequence of balls is a sequence of Følner sets for the group, and hence the group is amenable, i.e., it admits a left-invariant, finitely additive probabilistic measure. So this last result could be somewhat expected, but it is interesting to remark that natural density is not additive, so it does not in general coincide with the left-invariant measure provided by the definition of amenability.

2.4 Proposition *Let G be a finitely generated group and let $H \leq G$ be a subgroup of finite index. For every finite system X of generators for G, satisfying*

$$\lim_{n\to\infty} \frac{|B_X(n+1)|}{|B_X(n)|} = 1,$$

the natural density of H with respect to X is

$$\delta_X(H) = \frac{1}{[G:H]},$$

independently of X. Furthermore, the limit

$$\lim_{n\to\infty} \frac{|H \cap B_X(n)|}{|B_X(n)|}$$

does exist.

Proof. Let $X = \{g_1, g_2, \ldots, g_p\}$ be any finite set of generators for G satisfying the condition above, and let Γ be the Cayley graph for G with respect to X. Let Γ' be the Schreier graph of H in G, also with respect to X. Recall that Γ' has as vertices the right cosets for H, and two cosets aH and bH are joined by a directed edge from aH to bH (labelled by g_i) when $g_i aH = bH$. Clearly, there is a canonical map from Γ to Γ' where every element of the group is sent to its coset. It is easy to see that using this map, Γ is a covering space of Γ'.

The graph Γ' is connected and has $[G : H]$ vertices. Choose T to be a maximal subtree in Γ', and let C be the diameter of T (i.e. the maximal distance between any two of its vertices). For an element $h \in H$, due to the lifting properties of covering spaces, we can find a unique subtree of Γ which maps isomorphically to T and which contains h as a vertex. Denote such subtree by T_h. Obviously two of these subtrees are disjoint, and their union covers the whole group, meaning that every vertex of Γ belongs to exactly one T_h.

Using the trees T_h, their diameter C, and the triangle inequality we now have the following inclusions:

$$B_X(n-C) \subseteq \bigcup_{h \in H \cap B_X(n)} V(T_h) \subseteq B_X(n+C),$$

where $V(T_h)$ is the set of vertices of T_h. Taking cardinals,

$$|B_X(n-C)| \leq |H \cap B_X(n)|[G:H] \leq |B_X(n+C)|.$$

Now, dividing by $|B_X(n)|$, we obtain

$$\frac{|B_X(n-C)|}{|B_X(n)|} \leq \frac{|H \cup B_X(n)|}{|B_X(n)|}[G:H] \leq \frac{|B_X(n+C)|}{|B_X(n)|},$$

and the condition we are requiring on X implies that the limits in the first and third terms of the above inequality do exist are equal to 1, since C is fixed. Hence,

$$\lim_{n\to\infty} \frac{|H \cup B_X(n)|}{|B_X(n)|} = \frac{1}{[G:H]}.$$

In particular,

$$\delta_X(H) = \frac{1}{[G:H]}. \quad \square$$

Example 1.5 provides a counterexample for this result if the hypothesis is dropped. The set of words of even X-length in the free group F_p is a subgroup of index 2. But, as proven before, if $p \geq 2$, it has natural density $\frac{(2p-2)(2p-1)}{(2p-1)^2-1}$, which is never equal to $\frac{1}{2}$.

The following results about exponential density will be useful later. This lemma concerns finite unions of subsets.

2.5 Lemma *Let G be a group with a finite set of generators X, and let S_i, $i = 1, \ldots, k$, be a finite collection of subsets of G. Then,*

$$d_X(\bigcup_{i=1}^{k} S_i) = \max_i \{d_X(S_i)\}.$$

Proof. Write $d_X(S_i) = \alpha_i$, $i = 1, \ldots, k$, and we may assume that $\alpha_1 = \max_i \ \alpha_i$. Fix $\epsilon > 0$. We know that, for big enough n,

$$|S_i \cap B_X(n)| \leq (\alpha_i + \epsilon)^n \leq (\alpha_1 + \epsilon)^n.$$

So,

$$\left|\left(\bigcup_{i=1}^{k} S_i\right) \cap B_X(n)\right| \leq \sum_{i=1}^{k} |S_i \cap B_X(n)| \leq k(\alpha_1 + \epsilon)^n.$$

Thus, $d_X(\cup_{i=1}^{k} S_i) \leq \alpha_1 + \epsilon$, and this is valid for every $\epsilon > 0$. Hence, $d_X(\cup_{i=1}^{k} S_i) \leq \alpha_1$. The other inequality is clear. □

The following proposition is already stated and proved in [GH] but we state it here again for its relevance to the subject and its usefulness later in the paper.

2.6 Proposition *Let G be a group with a finite set of generators X, and let $S \subseteq G$ be a subset. Then, if $|S| = \infty$, we have*

$$\limsup_{n\to\infty} |S \cap B_X(n)|^{\frac{1}{n}} = \limsup_{n\to\infty} |S \cap S_X(n)|^{\frac{1}{n}}.$$

Furthermore, if S is the empty set both limits are 0, and if S is finite and non-empty the limits are 1 and 0, respectively.

3 The density of graphical sets in the free group

From now on, we will compute the exponential density of several subsets in the free group. First, we need to fix some notation and terminology.

Throughout the rest of the paper let p be a positive integer and F_p be a free group of rank p. Recall that F_p has exponential growth rate $2p-1$. Let X be a basis for F_p that will be fixed for all the paper. The $2p$ elements in $X^{\pm 1}$ will be denoted by $a_1, \ldots, a_{2p}$ and when we take an element $a_i \in X^{\pm 1}$ we will not specify if it is an element of the basis or one of their inverses. We fix also an (arbitrary) total order in $X^{\pm 1}$, $a_1 < \cdots < a_{2p}$, to use from now on. Most of the concepts defined below depend on the basis chosen for F_p. However, we will not need to reflect this dependence on their names because we will always refer to X, the basis of F_p fixed for the rest of the paper.

Let $w \in F_p$.

Concepts as the *reduced* word for w, the *length* of w, denoted $|w|$, and *cancellation* in the product of two words are the usual ones. By a *cyclic* word we mean a conjugacy class of words in F_p. We say that a word $w \in F_p$ is *cyclically reduced* if it has the minimum length among all the elements in its conjugacy class. Clearly, every $w \in F_p$ is of the form $w = \alpha^{-1}\tilde{w}\alpha$ for some $\alpha \in F_p$ and some cyclically reduced word $\tilde{w} \in F_p$. Here, $\tilde{w}$ is called a *cyclic reduction* of w and it is unique up to cyclic permutation of its letters.

Let $a, b \in X^{\pm 1}$, $a \neq b^{-1}$, and let $w = x_1 \cdots x_n$ with $x_i \in X^{\pm 1}$ be the reduced expression of w. We say that w *contains* $a \cdot b$ if $ab = x_i x_{i+1}$ for some $i = 1, \dots, n-1$. Clearly, the trivial word and length 1 words contain nothing. And no word contains $a \cdot a^{-1}$, $a \in X^{\pm 1}$.

The usual definition of a graph and the basic concepts of graph theory will be used. We will deal only with directed graphs, possibly having loops (i.e. edges with same initial and terminal vertices), having no multiple edges (i.e. at most one edge is allowed from a given vertex to another) and having $X^{\pm 1}$ as the set of vertices. From now on, we will refer to all these properties with the single word "graph".

Let Z and Z' be two graphs.

We will use the notation (a, b) to refer to the edge beginning at the vertex a and ending at the vertex b, for $a, b \in X^{\pm 1}$. We use EZ (a subset of $(X^{\pm 1})^2$) to denote the set of edges of Z. An edge (a, b) is said to be *redundant* when $ab = 1$; otherwise, it is called *irredundant*. We say that Z is *irredundant* when EZ does not contain redundant edges. We say that Z is *simpler* than Z', denoted $Z \leq Z'$, when $EZ \subseteq EZ'$.

The *inverse* graph of Z is the new graph $\overline{Z}$ with the following set of edges:

$$E\overline{Z} = \{(a, b) \in (X^{\pm 1})^2 \mid (a, b^{-1}) \in EZ\}.$$

Observe that, when inverting, redundant edges become loops, and loops become redundant edges. So, Z is irredundant if and only if $\overline{Z}$ has no loops. Furthermore, note that $\overline{\overline{Z}} = Z$.

The *adjacency* matrix of Z, denoted $\mathrm{ad}(Z)$, is a $2p$-sized square matrix whose (i, j) entry is 1 if Z contains the edge (a_i, a_j) and 0 otherwise, $i, j = 1, \dots, 2p$. Note that $Z \leq Z'$ if and only if $\mathrm{ad}(Z) \leq \mathrm{ad}(Z')$, where the inequality between matrices has entry-by-entry meaning.

The *complete* graph, denoted by K_{2p}, is the graph having precisely all possible non-loop edges (the subscript here is just to specify the number of vertices). Observe that $\overline{K}_{2p}$ is the graph with precisely all irredundant edges, now including loops. Note also that $\mathrm{ad}(K_{2p})$ is the $2p$-sized square matrix with ones everywhere except in the diagonal, where there are zeroes. And that $\mathrm{ad}(\overline{K}_{2p})$ is the matrix with ones everywhere, including the diagonal, except in p pairs of symmetric entries, where there are zeros. In particular, $\mathrm{ad}(K_{2p})$ and $\mathrm{ad}(\overline{K}_{2p})$ are symmetric matrices.

3.1 Definition A *graphical* set is a subset of F_p of the form

$$S(Z) = \{w \in F_p \mid \text{ if } w \text{ contains } a \cdot b \text{ then } (a,b) \in EZ\} \subseteq F_p,$$

where Z is a graph. In the opposite direction, let $w \in F_p$ be a word. The *local* graph of w, denoted Z_w, is the simplest graph Z such that $w \in S(Z)$.

The local graph of a word $w \in F_p$ locally describes w according to the fact that, for every $a, b \in X^{\pm 1}$, we have that $(a,b) \in EZ_w$ if and only if w contains $a \cdot b$. For example, Z_{a_i} is the graph with no edges, and $Z_{a_i^n}$ is the graph with a single loop at a_i, for $n \geq 2$, $i = 1, 2, \ldots, 2p$. Clearly, Z_w is irredundant for every $w \in F_p$.

The following are elementary properties of graphical sets. Let Z be a graph. Then we have:

(i) $\{1\} \cup X^{\pm 1} \subseteq S(Z)$,

(ii) $S(Z)$ is closed under taking subwords (in general, it is not a subgroup),

(iii) if $Z = \cup_i Z_i$ is the decomposition of Z in connected components then $S(Z) = \cup_i S(Z_i)$,

(iv) $S(Z)$ does not change if we delete the redundant edges of Z (this is why we called such edges redundant),

(v) for an irredundant graph Z, we have that $S(Z)$ is finite if and only if Z has no nontrivial (oriented) closed paths,

(vi) if Z and Z' are irredundant then, $S(Z) \subseteq S(Z')$ if and only if $Z \leq Z'$,

(vii) if Z and Z' are irredundant then, $S(Z) = S(Z')$ if and only if $Z = Z'$,

(viii) $S(Z) = F_p$ if and only if Z contains every irredundant edge.

A graph Z is said to be *symmetric* when $(a,b) \in EZ$ implies $(b,a) \in EZ$. A graphical set is called *symmetric* when the corresponding graph is symmetric, i.e. whenever the subword ab is allowed, ba is allowed too.

Let Z be an irredundant graph. Reading the sequence of vertices gives a natural bijection between the set of oriented paths in Z and the set $S(Z) \setminus \{1\}$. Words in $S(Z)$ of length $n \geq 1$ beginning with a_i and ending with a_j correspond to length $n-1$ paths in Z starting at the vertex a_i and ending at the vertex a_j. And the number of such paths (or words) coincide exactly with the (i,j) entry of the $(n-1)$-th power of the adjacency matrix $\mathrm{ad}(Z)$ (see [B]). This is the key point in the proof of the next theorem, where we compute the exponential density of a graphical set in terms of the corresponding irredundant graph Z.

3.2 Theorem *Let Z be an irredundant graph. If Z has no nontrivial closed paths, then the exponential density of $S(Z)$ is $1/(2p-1)$. Otherwise,*

$$d_X(S(Z)) = \frac{\rho(\mathrm{ad}(Z))}{2p-1}$$

where $\rho(\mathrm{ad}(Z))$ is the spectral radius of the adjacency matrix of Z.

Proof. If Z has no nontrivial closed paths then $S(Z)$ is finite and $d_X(S(Z)) = \frac{1}{2p-1}$.

Suppose now that $S(Z)$ is infinite. Let M be the adjacency matrix of Z and write $M^n = (m_{n;i,j})$, $n \geq 0$. As we observed, for $n \geq 1$ there is a bijection between $S(Z) \cap S_X(n)$ and the set of oriented paths in Z having length $n-1$. And the cardinal of these sets is

$$|S(Z) \cap S_X(n)| = \sum_{i,j=1,\dots,2p} m_{n-1;i,j}.$$

Now, applying corollary 5.6.14 in [HJ] to the matrix l_1-norm, we have that

$$\lim_{n\to\infty} \left(\sum_{i,j=1,\dots,2p} m_{n-1;i,j} \right)^{\frac{1}{n-1}} = \rho(M).$$

Thus, by Proposition 2.6,

$$\begin{aligned}
d_X(S(Z)) &= \frac{1}{2p-1} \limsup_{n\to\infty} |S(Z) \cap S_X(n)|^{\frac{1}{n}} \\
&= \frac{1}{2p-1} \limsup_{n\to\infty} \left(\sum_{i,j=1,\dots,2p} m_{n-1;i,j} \right)^{\frac{1}{n}} \\
&= \frac{1}{2p-1} \limsup_{n\to\infty} \left(\left(\sum_{i,j=1,\dots,2p} m_{n-1;i,j} \right)^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}} \\
&= \frac{\rho(M)}{2p-1}. \quad \square
\end{aligned}$$

3.3 Remark Note that the previous argument in fact shows that, for every irredundant graph Z with nontrivial closed paths, the limit $\lim_{n\to\infty} |S(Z) \cap B_X(n)|^{\frac{1}{n}}$ does exist and it is equal to $\rho(\mathrm{ad}(Z))$. So, for graphical sets, the limit in the definition of exponential density is an honest limit and not a lim sup.

By looking at the corresponding graph and computing its spectral radius, it is easy to extend Example 1.3 to an arbitrary rank, and give another basis with respect to which the same set has different exponential density.

3.4 Corollary *With respect to a given free basis X, the set of X-positive words in F_p has exponential density $\frac{p}{2p-1}$.*

3.5 Example In Example 1.3, we considered the free group on $X = \{a, b\}$ and the set S of positive words with respect to X. We proved that $d_X(S) = \frac{2}{3}$, and announced that $d_Y(S) \neq \frac{2}{3}$ for some other basis Y.

To prove this, take $Y = \{a, ab\}$. Note that (with respect to Y) S is a subset of the graphical set $S(Z)$, where

$$EZ = \{(a, a), (a, ab), (ab, a), (ab, ab), (a^{-1}, ab), (ab, a^{-1})\}$$

(observe that $a^{-1}(ab)$ and $(ab)a^{-1}$ are the unique possible length two non positive Y-subwords of positive X-words). Precisely,

$$S = \{w \in S(Z) \,|\, w \text{ does not terminate with } a^{-1}\}.$$

Let A be the minor of $\mathrm{ad}(Z)$ corresponding to the vertices a, a^{-1} and ab. By looking at the expression $A^{n-1}A = A^n$, we see that

$$|S(Z) \cap S_Y(n-1)| \leq |S \cap S_Y(n)| \leq |S(Z) \cap S_Y(n)|.$$

Now, taking the n-th root and the limit when n tends to infinity, we obtain that $\lim_{n\to\infty} |S \cap S_Y(n)|^{1/n} = \rho(A) = 2.2469796\ldots$. Thus,

$$d_Y(S) = \frac{\rho(A)}{3} = 0.7489932 \neq \frac{2}{3}.$$

The following proposition emphasizes the fact that strict inclusion between graphical sets imply strict inequality between their exponential densities. This will be crucial in the next section.

3.6 Proposition *Let Z and Z' be two irredundant graphs with nontrivial closed paths. If $S(Z) \subseteq S(Z')$ then $d(S(Z)) \leq d(S(Z'))$, and the inequality is strict whenever the inclusion is strict.*

Proof. We already observed that for irredundant graphs, $S(Z) \subseteq S(Z')$ is equivalent to $Z \leq Z'$ and to $\mathrm{ad}(Z) \leq \mathrm{ad}(Z')$, and the same is true for strict inclusion and inequalities. Now, Theorem 3.2, and Corollary 8.1.19 in [HJ] together with the subsequent exercise are enough to end the proof. □

Suppose now that Z is a symmetric graph with $2p$ vertices (and so, $S(Z) \subseteq F_p$ a symmetric graphical set). The *degree* of a vertex $v \in VZ$ is the number of vertices adjacent to it, that is, the number of $u \in VZ$ such that (u, v) (and so (v, u)) belong to EZ. We denote by ∂_{max} (resp. $\overline{\partial}$) the maximum (resp. average) of the degrees of vertices in Z. Using several well-known results in spectral graph theory, one can obtain the following corollary of Theorem 3.2. It provides bounds for the exponential density of a symmetric graphical set in terms of the maximum and average number of neighbours allowed for a given $a \in X^{\pm 1}$, and also in terms of the total number of length two subwords allowed.

3.7 Proposition *Let Z be a symmetric graph with $2p$ vertices, $VZ = X^{\pm 1}$, and $S(Z) \subseteq F_p$ the corresponding symmetric graphical set. The exponential density of $S(Z)$ with respect to X satisfies the following upper and lower bounds:*

(i) $\dfrac{\overline{\partial}}{2p-1} \leq d_X(S(Z))$,

(ii) $\dfrac{\sqrt{\partial_{max}}}{2p-1} \leq d_X(S(Z)) \leq \dfrac{\partial_{max}}{2p-1}$,

(iii) $\dfrac{|EZ|}{2p(2p-1)} \leq d_X(S(Z)) \leq \sqrt{\dfrac{|EZ|}{2p(2p-1)}}$.

Proof. (i) Let $A = \mathrm{ad}(Z)$, a $2p$-sized matrix over the real numbers. Because of its symmetry, A diagonalizes orthogonally, and easy computations using an orthogonal basis give the Rayleigh inequality:

$$\frac{uAu^T}{uu^T} \leq \rho(A),$$

for every vector $u \in \mathbb{R}^{2p}$, $u \neq 0$. Taking $u = (1, \dots, 1) \in \mathbb{R}^{2p}$, we obtain $\overline{\partial} \leq \rho(A)$ and so, $\overline{\partial}/(2p-1) \leq d_X(S(Z))$ (note that uA is the vector of degrees of vertices in Z).

(ii) The first inequality follows immediately from the fact $\sqrt{\partial_{max}} \leq \rho(A)$, proven in [N] and [LP]. By Theorem 8.3.1 in [HJ], there exist $v \neq 0$ with real nonnegative entries such that $Av^T = \rho(A)v^T$. So,

$$\rho(A)uv^T = uAv^T \leq \partial_{max}uv^T,$$

where $u = (1, \dots, 1)$. Hence, $\rho(A) \leq \partial_{max}$ and we have the second inequality.

(iii) Observe that $|EZ|$ coincides with the sum of degrees of vertices in Z. So, the first inequality is obvious from (i). On the other hand, in [Wi] it is proven that

$$\rho(A) \leq \sqrt{|EZ|(1-\frac{1}{|VZ|})},$$

from which the second inequality follows. □

Before developing Theorem 3.2 about the exponential density of graphical sets, Richard Z. Goldstein suggested to us that the natural density of the set of words $w \in F_p$ with $Z_w = \overline{K}_{2p}$ should be 1, by natural probabilistic feeling. It seemed reasonable to think that the probability for a word taken at random of containing all possible $a \cdot b$ with $a, b \in X^{\pm 1}$, $a \neq b^{-1}$, is 1. We prove this now by computing the exact exponential density of the complement of such set of words.

3.8 Theorem *Let X be a free basis of the free group, F_p, of rank $p \geq 2$, and let S be the following set*

$$S = \{w \in F_p \mid Z_w < \overline{K}_{2p}\} \subseteq F_p.$$

Then,

$$d_X(S) = \frac{\gamma_p}{2p-1} < 1,$$

where $\gamma_p \in (2p-2, 2p-1)$ is the largest root of the polynomial

$$x^3 - (2p-2)x^2 - (2p-1)x + (2p-2).$$

In particular, $\delta_X(S) = 0$.

Proof. Clearly, S is the union of the graphical sets $S(\overline{K}_{2p} \setminus \{e\})$ where e ranges over the set of edges of $\overline{K}_{2p}$. So, by Theorem 3.2 and Proposition 2.5,

$$d_X(S) = \frac{1}{2p-1} \max_{e \in E\overline{K}_{2p}} \{\rho(\mathrm{ad}(\overline{K}_{2p} \setminus \{e\}))\}.$$

Relabeling the vertices (i.e. conjugating the adjacency matrix by a suitable permutation matrix) if necessary, we can assume that the maximum in the expression above is one of the two spectral radii of the following two $2p$-sized matrices:

$$A_p = \begin{pmatrix} 1 & 0 & \boxed{0} & 1 & & 1 & 1 \\ 0 & 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & 0 & & 1 & 1 \\ 1 & 1 & 0 & 1 & & 1 & 1 \\ & & & & \ddots & & \\ 1 & 1 & 1 & 1 & & 1 & 0 \\ 1 & 1 & 1 & 1 & & 0 & 1 \end{pmatrix}, \quad B_p = \begin{pmatrix} \boxed{0} & 0 & 1 & 1 & & 1 & 1 \\ 0 & 1 & 1 & 1 & & 1 & 1 \\ 1 & 1 & 1 & 0 & & 1 & 1 \\ 1 & 1 & 0 & 1 & & 1 & 1 \\ & & & & \ddots & & \\ 1 & 1 & 1 & 1 & & 1 & 0 \\ 1 & 1 & 1 & 1 & & 0 & 1 \end{pmatrix},$$

where the boxed entry corresponds to the deleted edge. Let us now compute these two spectral radii.

It is easy to check that the $p-1$ vectors $(0, 0, \dots, 0, 0, -1, 1, 0, 0, \dots, 0, 0)$ with the nonzero coordenates in positions $2i-1$ and $2i$, $i = 1, 3, \dots, p$, are linearly independent and are right eigenvectors of A_p of eigenvalue 1. Also, the $p-2$ vectors $(-1, -1, 0, 0, \dots, 0, 0, 1, 1, 0, 0, \dots, 0, 0)$ with the two positive coordenates in positions $2i-1$ and $2i$, $i = 3, \dots, p$, are linearly independent and are right eigenvectors of A_p of eigenvalue -1. It is also easy to check that if α_1 and α_2 are the two (real) roots of the polynomial $x^2 - (2p-2)x - (2p-2)$, then the vector

$$\left(-\frac{2p-3}{4p-5}(\alpha_1 - 1), -\frac{2p-2}{4p-5}(\alpha_1 - 1), 1, \dots, 1\right)$$

is a right eigenvector of A_p of eigenvalue α_2 (and similarly with α_1 and α_2 interchanged). So, the characteristic polynomial of A_p is

$$\chi_{A_p}(x) = (x-1)^{p-1}(x+1)^{p-2}(x^2 - (2p-2)x - (2p-2))(x - \beta),$$

for some real number β. By looking at the trace of A_p we see that $\beta = 1$. Thus, the spectral radius of A_p is $\alpha_1 = p - 1 + \sqrt{p^2 - 1}$.

Similarly, the $p-1$ vectors $(0, 0, \dots, 0, 0, -1, 1, 0, 0, \dots, 0, 0)$ with the nonzero coordenates in positions $2i-1$ and $2i$, $i = 2, 3, \dots, p$, are linearly independent and are right eigenvectors of B_p of eigenvalue 1. And the $p-2$ vectors $(0, 0, -1, -1, 0, 0, \dots, 0, 0, 1, 1, 0, 0, \dots, 0, 0)$ with the two positive coordenates in positions $2i-1$ and $2i$, $i = 3, \dots, p$, are linearly independent and are right eigenvectors of B_p of eigenvalue -1. So, the characteristic polynomial of B_p is of the form

$$\chi_{B_p}(x) = (x-1)^{p-1}(x+1)^{p-2}(x^3 + ax^2 + bx + c)$$

for some real numbers a, b and c.

To determine a, b and c we will look at the trace, the sum of the principal $(2p-1)$-minors, and the determinant of B_p, three matrix invariants that are well known to be respectively equal to $-a_{2p-1}$, $-a_1$, and a_0, where a_i is the coefficient of degree i in $\chi(x)$. Standard computations show that, for all $p \geq 2$, the minor of B_p obtained by deleting the first row and column is $(-1)^{p-1}$, while the one obtained by deleting the second row and column is $2(-1)^{p-1}(p-1)$. With this information, it is straightforward to verify that the trace of B_p is $2p-1$, the sum of the $2p$ principal $(2p-1)$-minors of B_p is $(-1)^{p-1}(4p-3)$, and the determinant of B_p is $2(-1)^{p-1}(p-1)$. On the other hand, $-a_{2p-1} = 1-a$, $-a_1 = (-1)^p(b-c)$ and $a_0 = (-1)^{p-1}c$. Now, solving the corresponding three equations we obtain $a = -(2p-2)$, $b = -(2p-1)$, and $c = 2p-2$. Thus,

$$\chi_{B_p}(x) = (x-1)^{p-1}(x+1)^{p-2}(x^3 - (2p-2)x^2 - (2p-1)x + (2p-2)).$$

Some more computations show now that $\chi_{B_p}(-2) < 0$, $\chi_{B_p}(0) > 0$, $\chi_{B_p}(2p-2) < 0$ and $\chi_{B_p}(2p-1) > 0$. So, the spectral radius of B_p is the largest root of the degree 3 part, say $\gamma_p \in (2p-2, 2p-1)$.

Since $\chi_{B_p}(p-1+\sqrt{p^2-1}) < 0$, we have $p-1+\sqrt{p^2-1} < \gamma_p$ and the maximum above equals γ_p. Hence, $d_X(S) = \frac{\gamma_p}{2p-1}$ as wanted. □

3.9 Corollary *Let S be the set of words $w \in F_p$ such that $Z_w = \overline{K}_{2p}$. Then, S has natural density 1.*

4 The density of primitive words in F_p

¿From Theorems 3.2 and 3.8, we will deduce that primitive elements in F_p has natural density zero. To do this, we will make use of the classical Whitehead cut vertex lemma, which gives a strong necessary condition on a word w to be primitive. Also, we need the next result.

4.1 Proposition *Let F_p be the free group of rank p, freely generated by X. Let $S' \subseteq F_p$ be a non-empty subset of F_p and let*

$$S = \{x^{-1}wx \,|\, w \in S', x \in F_p\}.$$

If S' contains only cyclically reduced words, then

$$d_X(S) = \max\{d_X(S'),\ (2p-1)^{-\frac{1}{2}}\}.$$

Proof. Because of S' being non-empty, $|S| = \infty$ and we can use Proposition 2.6 and compute the density of S with the limit using spheres.

Consider first the case where $|S'| = 1$, say $S' = \{w\}$ where w is a cyclically reduced word. If the X-length of w is k, then w has exactly

$$(2p-2)(2p-1)^{\frac{n-k}{2}-1} = \frac{2p-2}{2p-1}\sqrt{2p-1}^{\,n-k}$$

conjugates of length n, if $n > k$ and $n \equiv k \mod 2$, and 0 otherwise. So,

$$\begin{aligned} d_X(S) &= \frac{1}{2p-1}\limsup_{n\to\infty} |S \cap S_X(n)|^{\frac{1}{n}} \\ &= \frac{1}{2p-1}\lim_{n\to\infty} \sqrt{2p-1}^{\frac{n-k}{n}} \\ &= (2p-1)^{-\frac{1}{2}} \\ &= \max\{d_X(S'),\ (2p-1)^{-\frac{1}{2}}\}. \end{aligned}$$

The case where $|S'| < \infty$ can be easily checked using the previous case and Lemma 2.5.

Suppose now that $|S'| = \infty$. Write $l = \limsup_{n\to\infty} |S' \cap S_X(n)|^{\frac{1}{n}}$ and write $a = \max\{l,\ \sqrt{2p-1}\}$. By Proposition 2.6, it remains to prove that $d_X(S) = \frac{a}{2p-1}$. Clearly, S contains S' and the set of conjugates of any given $w \in S'$. So, by the above argument, $(2p-1)d_X(S) \geq a$. To see the other inequality, let us argue in a similar way. Given a (cyclically reduced) word $w \in S'$ of length k, and given a positive integer $n > k$, the number of words in $S \cap S_X(n)$ cyclically reducing to w is precisely $(2p-2)(2p-1)^{\frac{n-k}{2}-1}$ if $n \equiv k \mod 2$, and 0 otherwise. And there are at most k conjugates of w having length k (that may lie outside S'). So,

$$\begin{aligned} |S \cap S_X(n)| &\leq \sum_{k=1}^{n} k|S' \cap S_X(k)|(2p-2)(2p-1)^{\frac{n-k}{2}-1} \\ &\leq \sum_{k=1}^{n} k|S' \cap S_X(k)|\sqrt{2p-1}^{\,n-k} \end{aligned}$$

Now, fix $\epsilon > 0$. We know that there exists k_0 such that $|S' \cap S_X(k)| \leq (l+\epsilon)^k \leq (a+\epsilon)^k$ for $k \geq k_0$. So, there exist a constant M such that, for every $n \geq k_0$, we have

$$\begin{aligned} |S \cap S_X(n)| &\leq \sum_{k=1}^{n} k|S' \cap S_X(k)|\sqrt{2p-1}^{\,n-k} \\ &\leq M + n\sum_{k=k_0}^{n} (a+\epsilon)^k\sqrt{2p-1}^{\,n-k} \\ &\leq M + n\sum_{k=k_0}^{n} (a+\epsilon)^n \\ &\leq M + n^2(a+\epsilon)^n. \end{aligned}$$

Hence,

$$\begin{aligned}(2p-1)d_X(S) &= \limsup_{n\to\infty} |S \cap S_X(n)|^{\frac{1}{n}} \\ &\leq \limsup_{n\to\infty} (M + n^2(a+\epsilon)^n)^{\frac{1}{n}} \\ &= a + \epsilon.\end{aligned}$$

And this is valid for every $\epsilon > 0$. Thus, $(2p-1)d_X(S) \leq a$, as wanted. □

Let Z be a graph and $v \in VX$ a vertex. We say that v is a *cut vertex* of Z if the graph obtained by deleting v together with all its adjacent edges is disconnected. In particular, every non-isolated vertex of a non-connected graph is a cut vertex. Observe that every non-connected graph contains a cut vertex (with the only exception of the one with two vertices and no edges). Clearly, the connectedness or the existence of a cut vertex are not preserved in general by inverting.

Let w be a word of F_p. The *Whitehead graph* of w, denoted W_w, is the graph with the following set of edges:

$$EZ_w = \{(a, b^{-1}) \in (X^{\pm 1})^2 \mid w \text{ contains } a \cdot b\}.$$

This definition does not entirely agree with the original one given by Whitehead in [W]. Let $w = x_1 \cdots x_n$ be the reduced expression of w. Whitehead's original graph of w contains as many edges from a_i to a_j as the number of times that $a_i a_j^{-1}$ appears as a subword of w. Furthermore, it considers w cyclically that is, it contains also an edge from x_n to x_1^{-1}. Our simplified version of Whitehead graph will be enough for our purposes.

Note that, as defined, $W_w = \overline{Z}_w$ for every $w \in F_p$. So, W_w may contain redundant edges but it does not contain loops.

The most important result concerning Whitehead graphs is the following, known as the Whitehead cut vertex lemma.

4.2 Theorem *[Whitehead] Let w be a cyclically reduced word of the free group F_p. If w is primitive then W_w has a cut vertex.*

For a proof, see the original article by Whitehead [W], or the modern versions [GT] and [St]. Theorem 4.2, with our particular definition of Whitehead graph, is an immediate corollary of Theorem 2.4 in [St]. Clearly, the converse to the cut vertex lemma is not true as can be tested, for example, with the words a^2 and $(aba)^2$ in $F_2 =< a, b >$.

We have now all the necessary ingredients to prove our theorem about the densities of the set of primitive words.

4.3 Theorem *Let F_p be the free group of rank $p \geq 2$, and let $S \subseteq F_p$ be the set of primitive words. For every free basis X of F_p, we have*

$$\frac{2p-3}{2p-1} \leq d_X(S) \leq \frac{\gamma_p}{2p-1} < 1.$$

Consequently, $\delta_X(S) = 0$.

Proof. Let $X = \{a_1, \ldots, a_p\}$ be a free basis for F_p.

The first inequality is clear because every word w beginning with a_1 and having no other occurence of $a_1^{\pm 1}$ is clearly primitive. And there are precisely $(2p-2)(2p-3)^{n-2}$ such words in $S_X(n)$. So,

$$d_X(S) \geq \frac{2p-3}{2p-1}.$$

For the other inequality, consider S' the set of cyclically reduced primitive elements. By Proposition 4.1, we know that

$$d_X(S) = \max\{d_X(S'), (2p-1)^{-\frac{1}{2}}\}.$$

So, it only remains to prove that $d_X(S') \leq \frac{\gamma_p}{2p-1}$. Take a word $w \in S'$. By Theorem 4.2, W_w has a cut vertex. In particular, $W_w < K_{2p}$, and so $Z_w = \overline{W}_w < \overline{K}_{2p}$. Thus, S' is contained in the set defined in Theorem 3.8. Consequently, $d_X(S') \leq \frac{\gamma_p}{2p-1} < 1$, ending the proof. □

4.4 Remark During the process of writing this manuscript, Vladimir Shpilrain and Alexei Myasnikov announced us that they have a better upper bound for the exponential density of the set of primitive elements. Concretely, in [MS] they prove that the exponential density of the primitive elements in the free group F_p is bounded above by $\frac{2p-2}{2p-1}$.

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