

COMBINATORIAL AND METRIC PROPERTIES OF THOMPSON'S GROUP T

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ABSTRACT. We discuss metric and combinatorial properties of Thompson's group T , such as the normal forms for elements and uniqueness of tree pair diagrams. We relate these properties to those of Thompson's group F when possible, and highlight combinatorial differences between the two groups. We define a set of unique normal forms for elements of T arising from minimal factorizations of elements into convenient pieces. We show that the number of carets in a reduced representative of T estimates the word length, that F is undistorted in T , and that cyclic subgroups of T are undistorted. We show that every element of T has a power which is conjugate to an element of F and describe how to recognize torsion elements in T .

1. INTRODUCTION

Thompson's groups F , T and V are a remarkable family of infinite, finitely-presentable groups studied for their own interesting properties as well as for their connections with questions in logic, homotopy theory, geometric group theory and the amenability of discrete groups.

Cannon, Floyd and Parry give an excellent introduction to these groups in [5]. These three groups can be viewed either algebraically, combinatorially, or analytically. Algebraically, each has both finite and infinite presentations. Combinatorially, an element in each group can be viewed as a *tree pair diagram*; that is, as a pair of finite binary rooted trees with the same number of leaves, with a numbering system pairing the leaves in the two trees. Analytically, an element of each group can be viewed as a self map of the unit interval:

- in F as a piecewise linear homeomorphism,
- in T as a homeomorphism of the unit interval with the endpoints identified, and thus of S^1 ,
- in V as a right-continuous bijection which is locally orientation preserving.

Thompson's group F in particular has been studied extensively. The group F has a standard infinite presentation in which every element has a unique normal form, and a standard two-generator finite presentation. Fordham [8] presented a method of computing the word length of $w \in F$ with respect to the standard finite generating set directly from the tree pair diagram representing w . Regarding F as a diagram group, Guba [10] also obtained an effective geometric method for computing the word metric with respect to the standard finite generating set. Belk and Brown [1] have similar results which arise from

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viewing elements of F as forest diagrams. The amenability of F is one open question which has motivated continuing research into F for many years. Either answer to this question would be interesting, for if F is not amenable it would be a familiar finitely presented non-amenable group without free non-abelian subgroups, whereas if F is amenable it would be a familiar finitely presented, amenable, but not elementary-amenable group.

In this paper, we discuss analogues for T of some properties of F , such as normal forms for elements. We consider metrically how F is contained as a subgroup of T , and show that the number of carets in a reduced tree pair diagram representing $w \in T$ estimates the word length of w with respect to a particular generating set. We show that cyclic subgroups of T are undistorted and that every element of T has a power which is conjugate to an element of F . The groups T and V , unlike F , contain torsion elements, and we describe how to recognize these torsion elements from their tree pair diagrams.

2. BACKGROUND ON THOMPSON'S GROUPS F AND T

2.1. Presentations and tree pair diagrams. Thompson's groups F and T both have representations as groups of piecewise-linear homeomorphisms. The group F is the group of orientation-preserving homeomorphisms of the interval $[0, 1]$, where each homeomorphism is required to have only finitely many discontinuities of slope, called *breakpoints*, have slopes be powers of two and have the coordinates of the breakpoints all lie in the set of dyadic rationals. Similarly, the group T consists of orientation-preserving homeomorphisms of the circle S^1 satisfying the same conditions where we represent the circle S^1 as the unit interval $[0, 1]$ with the two endpoints identified.

Cannon, Floyd and Parry give an excellent introduction to Thompson's groups F , T and V in [5]. We refer the reader to this paper for full details on results mentioned in this section. Since more readers have some familiarity with F than with T , we first give a very brief review of the group F , and then a slightly more detailed review of T . Algebraically, F has well known infinite and finite presentations. With respect to the infinite presentation

$$\langle x_i, i \geq 0 \mid x_j x_i = x_i x_{j+1}, i < j \rangle$$

for F , group elements have simple normal forms which are unique. It is easy to see that F can be generated by x_0 and x_1 , which form the standard finite generating set for F , and yield the finite presentation

$$\langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

A convenient representation for an element w in F is a tree pair diagram, as discussed in [5]. A *tree pair diagram* is a pair of finite rooted binary trees with the same number of leaves. By convention, the leaves of each tree are thought of as being numbered from 0 to n reading from left to right. A node of the tree together with its two downward directed edges is called a *caret*. The *left side* of the tree consists of the root caret, and all carets connected to the root by a path of left edges; the *right side* of the tree is defined analogously. A caret is called a *left caret* if its left leaf lies on the left side of the tree. A caret is called a *right caret* if it is not the root caret and its right leaf lies on the right side of the tree. All other carets are called *interior*. A caret is called *exposed* if it

contains two leaves of the tree. For $w \in F$, we write $w = (T_-, T_+)$ to express w as a tree pair diagram, and refer to T_- as the *source* tree and T_+ as the *target* tree. These trees arise naturally from the interpretation of F as a group of homeomorphisms. Thinking of w as a homeomorphism of the unit interval, the source tree represents a subdivision of the domain into subintervals of width $1/2^n$, and the target tree represents another such a subdivision of the range. The homeomorphism then maps the i^{th} subinterval in the domain linearly to the i^{th} subinterval in the range.

Such a tree pair diagram is not unique. A new diagram can always be produced from a given tree pair diagram representing w simply by adding carets to the i^{th} leaf of both trees. We impose a natural reduction condition: if $w = (T_-, T_+)$ and both trees contain a caret with two exposed leaves numbered i and $i + 1$, then we remove these carets, thus forming a representative for w with fewer carets and leaves. A tree pair diagram which admits no such reductions is called a *reduced tree pair diagram*, and any element of F is represented by a unique reduced tree pair diagram. When we write $w = (T_-, T_+)$ below, we are assuming that the tree pair diagram is reduced unless otherwise specified.

The group T also has both a finite and an infinite presentation. The infinite presentation is given by two families of generators, $\{x_i, i \geq 0\}$, the same generators as in the infinite presentation of F , a family $\{c_i, i \geq 0\}$ of torsion elements, and the following relators:

- (1) $x_j x_i = x_i x_{j+1}$, if $i < j$
- (2) $x_k c_{n+1} = c_n x_{k+1}$, if $k < n$
- (3) $c_n x_0 = c_{n+1}^2$
- (4) $c_n = x_n c_{n+1}$
- (5) $c_n^{n+2} = 1$.

This new family of generators c_n (of order $n + 2$), is simple to describe. c_n corresponds to the homeomorphism of the circle obtained as follows. Both domain and range can be thought of as the unit interval with the endpoints identified. We subdivide the interval into $n + 1$ subintervals by successively halving the rightmost subinterval; or in other words inserting endpoints at $\frac{1}{2}, \frac{3}{4}, \dots, \frac{2^{n+1}-1}{2^{n+1}}$. Then the homeomorphism maps $[0, 1/2]$ linearly to $[\frac{2^n-1}{2^n}, \frac{2^{n+1}-1}{2^{n+1}}]$, and so on around each circle. For example, the element c corresponds to the homeomorphism of S^1 given by

$$c(t) = \begin{cases} \frac{1}{2}t + \frac{3}{4} & \text{if } 0 \leq t < \frac{1}{2} \\ 2t - 1 & \text{if } \frac{1}{2} \leq t < \frac{3}{4} \\ t - \frac{1}{4} & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$$

See Figure 1 for graphs of the homeomorphisms corresponding to c_1 and c_2 .

Using the first three relators, we see that only the generators x_0, x_1 and c_1 are required to generate the group, since the other generators can be obtained from these three. In the following, we will use c to denote the generator c_1 . The group T is finitely presented using the following relators, both with respect to the infinite generating set and the finite generating set $\{x_0, x_1, c\}$:

- (1) $[x_0 x_1^{-1}, x_0^{-1} x_1 x_0] = 1$
- (2) $[x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] = 1$
- (3) $x_1 c_3 = c_2 x_2$, (that is $x_1 (x_0^{-2} c x_1^{-2}) = (x_0^{-1} c x_1) (x_0^{-1} x_1 x_0)$)

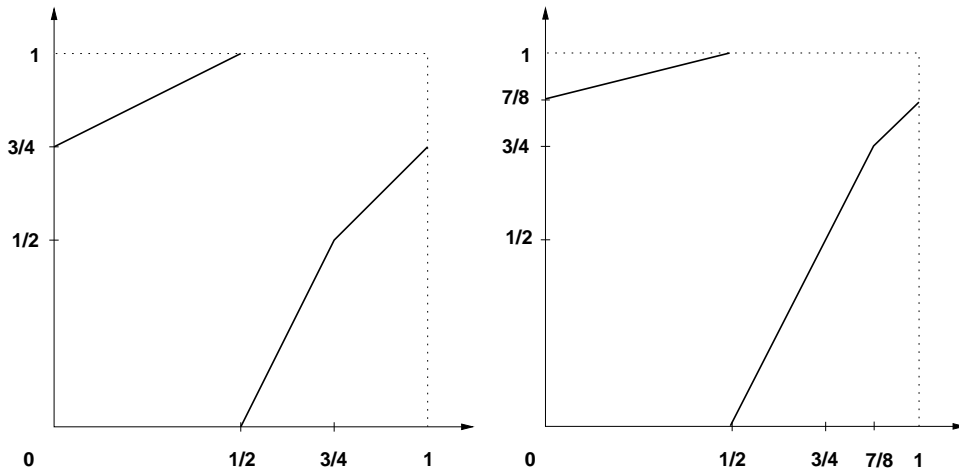


FIGURE 1. The graphs of the homeomorphisms corresponding to the elements c_1 and c_2 .

- (4) $c_1 x_0 = c_2^2$, (that is, $c x_0 = (x_1^{-1} c x_0)^2$)
- (5) $x_1 c_2 = c$, (that is, $x_1 (x_0^{-1} c x_1) = c$)
- (6) $c^3 = 1$.

As with Thompson's group F , we will frequently work with the more convenient infinite set of generators when constructing normal forms for elements and performing computations in the group. We will need to express elements with respect to the finite generating set when discussing word length.

Just as for F , tree pair diagrams serve as convenient representations for elements of T . However, since elements of T represent homeomorphisms of the circle rather than the interval, the tree pair diagram must also include a bijection between the leaves of the source tree and the leaves of the target tree to fully encode the homeomorphism. Since this bijection can at most cyclically shift the leaves, it is determined by the image of the leftmost leaf in the source tree. Since by convention this leaf in the source tree is already thought of as leaf 0, this information is recorded by writing a 0 under the image leaf in the target tree. Hence, for $w \in T$, a *marked tree pair diagram* representing w is a pair of finite rooted binary trees with the same number of leaves, together with a mark (the numeral 0) on one leaf of the second tree. As usual, we write $w = (T_-, T_+)$ to express w as a tree pair diagram, and refer to T_- as the *source* tree and T_+ (the one with the mark) as the *target* tree. We remark that to extend this to V , since now the bijection of the subintervals may permute the order in any way, the marking required on the target tree to record the bijection consists of a number on every leaf of the target tree. Just as for F , one can always obtain new diagrams for elements of T by adding carets to old diagrams. However, when adding the carets, placement is guided by the marking. The leaves of the source tree are thought of as numbered from 0 to n reading from left to right, whereas the marking of the target tree specifies where leaf number 0 of that tree is, and other leaves

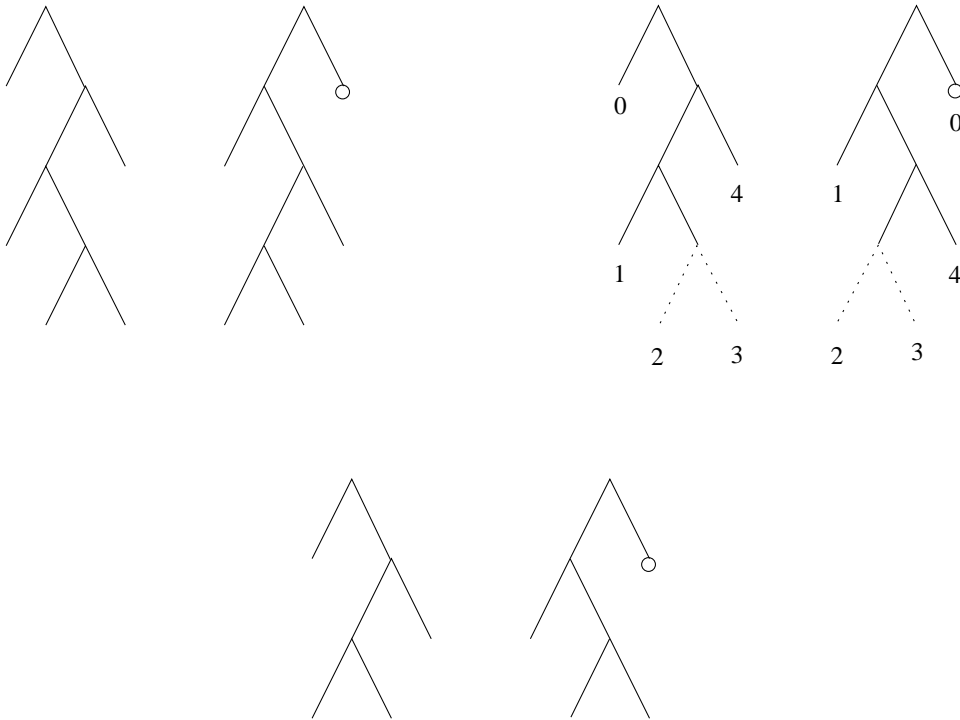


FIGURE 2. An example of a caret reduction in T . The diagram on the top left is reducible, observe that the two dotted carets on the top right are paired to each other, since the numbering matches. On the bottom, the reduced diagram.

are numbered from 1 to n reading from left to right cyclically wrapping back to the left once you reach the rightmost leaf. With this numbering in mind, carets can be added as before to leaf i of both trees. If $i \neq 0$, the mark stays where it is. Otherwise, if $i = 0$, the mark on the new target tree is placed on the left leaf of the added caret. So for T , we have a similar reduction condition: if $w = (T_-, T_+)$ and both trees contain a caret with two exposed leaves numbered i and $i + 1$, then we remove these carets and renumber the leaves, thus forming a representative for w with fewer carets and leaves.

A tree pair diagram which admits no such reductions is again called a *reduced tree pair diagram*, and any element of T is represented by a unique reduced tree pair diagram. In T as well as in F , when we write $w = (T_-, T_+)$ below, we are assuming that the tree pair diagram is reduced unless otherwise specified. Checking whether or not a tree pair diagram is reduced is slightly more difficult in T than in F . The process of checking for possible reductions is illustrated in Figure 2. A marked tree pair diagram for an element of T is shown on the top left. In the top right tree pair diagram, the underlying numbering of the leaves of both trees determined by the marking is written explicitly, revealing the reducible carets. The bottom tree pair diagram shows the resulting reduced diagram.

Note that the torsion generators c_i have particularly simple tree pair diagrams. In the diagram for c_i , both source and target trees consist of the root caret plus i right carets. The mark 0 is placed on the rightmost leaf of the target tree.

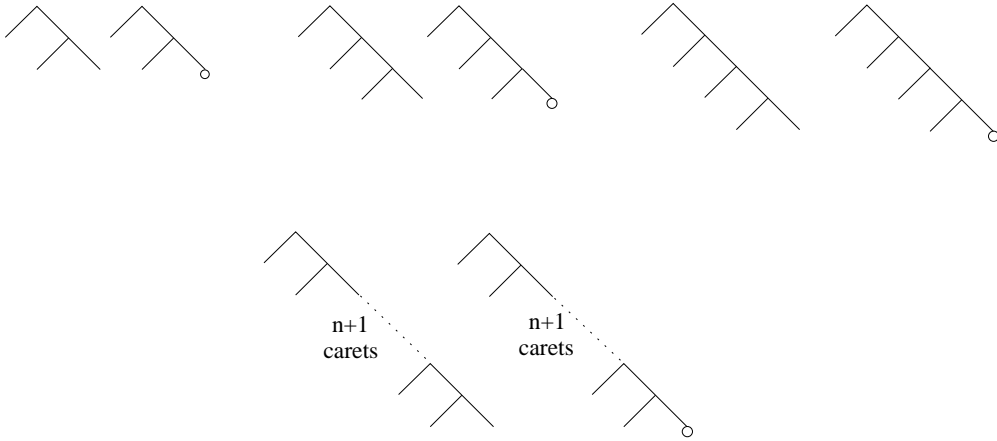


FIGURE 3. Tree pair diagrams representing the elements c_1 , c_2 and c_3 on top, plus c_n on the bottom.

In Figure 3 we can see the tree pair diagrams of the first generators c_n , together with a general one.

When $w \in F$ or $w \in T$, we denote the number of carets in either tree of a tree pair diagram representing w by $N(w)$. When p is a word in the generators of F or T , then p represents an element w in either F or T , and we write $N(p)$ interchangeably with $N(w)$.

2.2. Group Multiplication in F and T . Group multiplication in F and T corresponds to composition of homeomorphisms, which we can interpret on the level of tree pair diagrams as well. First, we consider $u, v \in F$, where $u = (T_-, T_+)$ and $v = (S_-, S_+)$. To compute the tree pair diagram corresponding to the product vu , we create unreduced representatives (T'_-, T'_+) and (S'_-, S'_+) of the two elements in which $T'_+ = S'_-$. Then the product is represented by the possibly unreduced tree pair diagram (T'_-, S'_+) . Observe that the multiplication is written following the conventions on composition of homeomorphisms, so the product vu has as a source diagram that of u , and as a target diagram that of v , i.e., the diagram on the left is the source of u and the diagram on the right is the target of v .

To multiply tree pair diagrams representing elements of T we follow a similar procedure. We let $u, v \in T$, where $u = (T_-, T_+)$ and $v = (S_-, S_+)$. To compute the tree pair diagram corresponding to the product vu , we create unreduced representatives (T'_-, T'_+) and (S'_-, S'_+) of the two elements in which $T'_+ = S'_-$ as trees. The product vu will be represented by the pair (T'_-, S'_+) of trees. To decide which leaf in S'_+ to mark with the zero, we just note that it should be the leaf which is mapped onto by the zero leaf in T'_- . To identify this leaf, we find the zero leaf in T'_+ . Since $T'_+ = S'_-$ as trees, this leaf viewed as a leaf in S'_- will be labelled m . Then the leaf labelled m in S'_+ will be the new zero leaf in the tree pair diagram (T'_-, S'_+) for vu . Alternately, we can follow the composition in both pairs of trees to see how the leaves map to each other. This constructed tree pair diagram will represent vu and is not necessarily reduced. For an example of this multiplication, see Figures 4, 5 and 6.

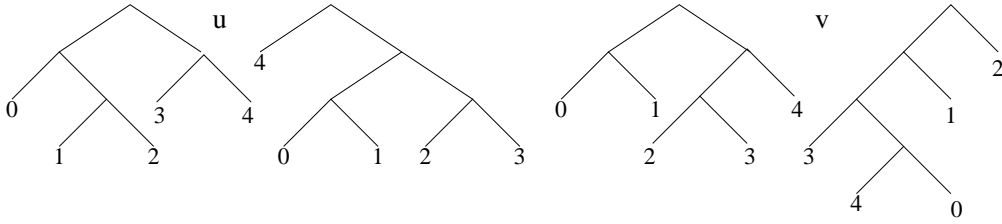


FIGURE 4. The tree pair diagram for sample elements u and v in T .

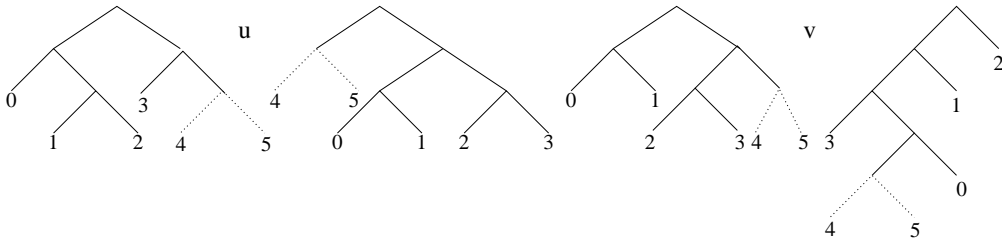


FIGURE 5. Unreduced versions of u and v for the multiplication vu in T , with carets added to perform the multiplication indicated with dashes. Now the target tree of u has the same shape as the source tree of v , allowing the composition.

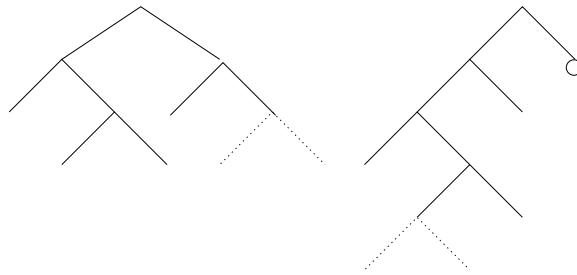


FIGURE 6. The tree pair diagram representing the product vu obtained from Figure 5. The dotted carets must be erased to find the reduced diagram.

3. WORDS AND DIAGRAMS

3.1. **Normal forms and tree pair diagrams in F .** With respect to the infinite presentation for F given above, every element of F has a unique normal form. To describe these, we first observe that any w can be written in the form

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k} x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

where $r_i, s_i > 0$, $0 \leq i_1 < i_2 \dots < i_k$ and $0 \leq j_1 < j_2 \dots < j_l$. However, this expression is not unique. Uniqueness is guaranteed by the addition of the following condition: when both x_i and x_i^{-1} occur in the expression, so does x_{i+1} or x_{i+1}^{-1} , as discussed by Brown and Geoghegan [2]. When we refer to elements of F in normal form, we mean this unique normal form.

If the normal form for $w \in F$ contains no generators with negative exponents, we refer to w as a *positive word* and similarly, we say a normal form is a *negative word* if there are no generators with positive exponents.

We call any word which has the form

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k} x_{j_1}^{-s_1} \dots x_{j_2}^{-s_2} x_{j_l}^{-s_l}$$

where $r_i, s_i > 0$, $0 \leq i_1 < i_2 \dots < i_k$ and $0 \leq j_1 < j_2 \dots < j_l$, a word in *pq form*, where p is the positive part of the normal form and q the negative part. The normal form for an element of F is the shortest word among all words in *pq form* representing the given element.

To any (not necessarily reduced) tree pair diagram (T_-, T_+) for an element of F we may associate a word in *pq form* representing the element, using the *leaf exponents* in the target and source trees. When the leaves of a finite rooted binary tree are numbered from left to right, beginning with zero, the leaf exponent of leaf k is the integer length of the longest string of left edges of carets which originates at leaf k and does not reach the right side of the tree. A tree pair diagram then gives the word

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

precisely when leaf i_k in T_+ has exponent r_k , leaf j_k in T_- has leaf exponent s_k , and generators which do not appear in the word correspond to leaves with exponent zero. We think of this word as the *pq factorization* of the element given by the particular tree pair diagram. We call a tree an *all-right tree* if it consists of a root caret together with only right carets. Note that if we let R be the all-right tree with the same number of carets as T_- or T_+ , then (T_-, R) is a diagram for q and (R, T_+) is a diagram for p . On the other hand, any word in *pq form* can be translated into a tree pair diagram. It can be obtained by taking diagrams for p (respectively q), which will have all right source (respectively target) trees. Then, if one diagram has fewer carets, one adds right carets to its all right tree, and of course a corresponding string to its other tree, to make both diagrams have exactly the same all right tree. Furthermore, under this correspondence for F , reduced tree pair diagrams correspond exactly to normal forms. Figure 7 is an example of this correspondence, and more details can be found in [5, 6, 7].

We observe that if an exposed caret has leaves numbered i and $i + 1$, then leaf $i + 1$ must have leaf exponent zero, since it is a right leaf. If both trees in a tree pair diagram have exposed carets with leaves numbered i and $i + 1$, then the corresponding normal form, computed via leaf exponents, contains the generators x_i to both positive and negative indices, but no instances of the generator x_{i+1} . This is precisely the situation when the normal form can be reduced by a relator of F . Thus the condition that the normal form is unique is exactly the condition that the tree pair diagram is reduced. This correspondence will be extended to elements of T in the next section.

3.2. Tree pair diagrams for elements of T . We now discuss the relationship between words in T and tree pair diagrams. The relationship is more complicated in T than it is in F . The representation of elements of T by marked tree pair diagrams suggests a way to decompose an element of T into a product of three elements: the positive and negative parts together with a torsion part in the middle, as described in [5].

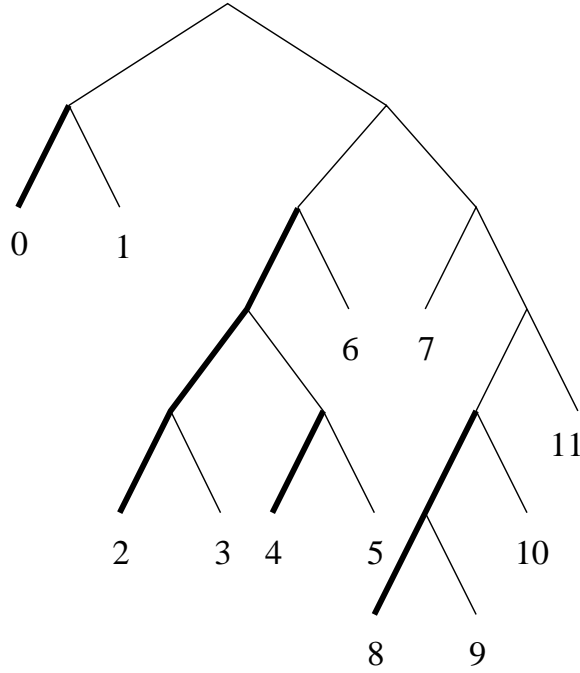


FIGURE 7. The correspondence between normal forms and reduced diagrams via leaf exponents. The thick edges are used to count the leaf exponent in the diagram. Hence, if a leaf labelled i has r_i thick edges, that is, if it has a string of r_i left edges going up without reaching the right side, then the i -th leaf exponent is r_i and the generator appearing in the normal form is $x_i^{r_i}$. The element depicted here is $x_0x_2^3x_4x_8^2$.

Definition 3.1. Let the marked tree pair diagram (T_-, T_+) represent $g \in T$. If T_- and T_+ each have $i + 1$ carets, then we let R be the all-right tree which has $i + 1$ carets. We can write g as a product pc_i^jq , where:

- (1) p , a positive word in the generators of F , is the normal form for the element of F with tree pair diagram (R, T_+) , ignoring the marking on T_+ .
- (2) c_i^j is a cyclic permutation of the leaves of R , with $1 \leq j \leq i + 2$, and
- (3) q , a negative word, is the normal form for the element of F represented by (T_-, R) .

Then the word $w = vc_i^ju$ is called the pcq factorization of g associated to the marked tree pair diagram (T_-, T_+) . In the special case where $g \in F \subset T$, the pcq factorization will just be the usual pq factorization, as we consider the c part of the word to be empty.

Figure 8 illustrates an example of an element of T decomposed in this way.

The following theorem follows from the existence of these decompositions, and an algebraic proof of this result is found in [5].

Theorem 3.2 ([5], Theorem 5.7). Any element $x \in T$ admits an expression of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} c_i^j x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

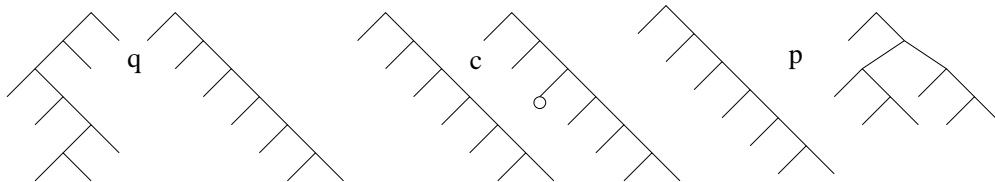


FIGURE 8. Three tree pair diagrams representing the word $x_1x_2c_3^5x_2^{-2}x_1^{-1}x_0^{-2}$ factorized as pcq .

where $0 \leq i_1 < i_2 < \dots < i_n$ and $0 \leq j_1 < j_2 < \dots < j_m$ and either $1 \leq j < i + 2$ or c_i^j is not present.

We refer to any word satisfying the hypotheses of Theorem 3.2 as a word in pcq form for an element of T (just as words of this form with no c_i^j term are called words in pq form in the group F). Neither proof of the existence of pcq forms gives an easy explicit method for transforming a general word in the generators $x_i^{\pm 1}, c_i$ into pcq form without resorting to drawing tree pair diagrams, so we will outline such a method below. We recall that the four types of relators we are using in T are:

- (1) $x_jx_i = x_ix_{j+1}$, if $i < j$
- (2) $x_kc_{n+1} = c_nx_{k+1}$, if $k < n$
- (3) $c_nx_0 = c_{n+1}^2$
- (4) $c_n = x_nc_{n+1}$

Lemma 3.3 (Pumping Lemma). *The generators x_i and c_j of T satisfy the following identities*

$$c_n^m = x_{n-m+1}c_{n+1}^m \quad c_n^m = c_{n+1}^{m+1}x_{m-1}^{-1}$$

if $1 \leq m < n + 2$.

Proof. This follows immediately from the relators of the types (2) and (3). For instance, for the first identity, we have that

$$c_n^m = c_n^{m-1}c_n = c_n^{m-1}x_nc_{n+1}$$

by an application of relator (4). Now, several repeated applications of relator (2) allow the x_n to switch with the c_n^{m-1} to obtain the desired result. The second identity is the first one taking inverses, and noticing that c_n has order $n + 2$ to avoid negative exponents for the c . \square

We consider a word $w \in T$ written in the generators $\{x_i, c_j\}$, and we describe explicitly an algebraic method of rewriting it in pcq form. The idea is to first combine occurrences of multiple c generators into a power of a single one, and then to move the x generators to the appropriate side of it. Consider first a subword of the type

$$c_n^m w(x_i) c_k^l,$$

where $w(x_i)$ is a word on the generators x_i only, and which may very well be empty. We will apply relators to reduce this subword to a word of the form $w_1(x_i)c_j^h w_2(x_i)$, where w_1 has only positive x generators and w_2 consists of only negative ones. By the relators (1) we can assume that w is of the form pq , that is, with all positive generators on the

left and in increasing index, and all negative ones on the right and in decreasing index. The goal is to move all the positive x generators to the left of c_n^m and all negative ones to the right of c_k^l . Although these moves may change the indices and powers of the c generators, they merely change a power of a single c generator to another power of a different single c generator. To move all the positive generators to the left of c_n^m , we only need to use relators of the type (2), assuming the index of c is high enough. If it is not, by repeated applications of the first identity of the pumping lemma, the index can be increased arbitrarily, adding only positive generators to the *left* of c_n^m . When the index is high enough, use relators (2) to move all positive generators of w past c_n^m . If it happens that in the process of moving a positive generator to the left of c_n^m it gets reduced to x_0 , then use a relator (3) to eliminate the x_0 . It may be necessary to combine the two c generators obtained into one after this swallowing of x_0 , as we see in an example:

$$c_4^3 x_1 = c_4^2 x_0 c_5$$

At this point x_0 cannot be moved farther, but use relator (3) to obtain

$$c_4 c_5^3 = x_5 c_5^4$$

the last equality being an application of the pumping lemma to c_4 . Observe that the goal of moving a positive generator to the left of c_n^m has been achieved.

Moving the negative x generators to the left is even easier. Using the second identity in the pumping lemma, we can increase the index in c_k^l as much as necessary to be able to move all negative x generators in w to the right of c_k^l using the relators (2) rewritten as $c_{n+1} x_{k+1}^{-1} = x_k^{-1} c_n$. After this process, we will have a word consisting of positive x generators, two powers of c generators, and negative x generators. One only needs to combine the powers of the two c generators into a power of a single generator, by increasing the smallest index to reach the largest. To do this, if the smallest is on the left, use the first identity in the pumping lemma, and if it is on the right, use the second one. This way no x generator will be added in between the two c generators and when they have the same index they can be combined into one. Notice that positive x generators appear only to the left of the c generator, and negative generators only to the right.

After repeated applications of this process to subwords of the type cwc , we will have all occurrences of the c generators combined into a power of a single one. Our original word is now of the type

$$w_1(x_i) c_n^m w_2(x_i),$$

and w_1 and w_2 may again be assumed, after using relators (1), to be in pq form. We only need to move the positive generators in w_2 to the left of c_n^m and the negative x generators of w_1 to the right of c_n^m , still maintaining a power of a single c generator in the middle. But notice that the very first step in our algorithm described how to do precisely this. Furthermore, if the pumping lemma is needed to move a positive generator to the left, recall that new positive generators may appear in the word, but only to the left of the power of the c generator. Hence, after moving each positive generator, all positive generators in the word are to the left of c_n^m .

Now move each negative generator to the right, and notice that the only cost of this is to add more negative x generators to the right of the power of c . When this is finished, the

word has only positive generators to the left of a power of a single c and negative ones to the right. Reorder them if necessary using relators (1).

We will work an example as an illustration. Consider the word

$$x_0^{-1}c_1x_3c_3^2x_1^{-1}$$

The process starts by trying to switch the x_3 to the left of c_1 . Since 3 is too large for c_1 according to the relator (2), we must increase c_1 . Using the pumping lemma, we have $c_1 = x_1c_2 = x_1x_2c_3$. Hence our word is now the following, as we can apply the switch (2) to c_3x_3 .

$$x_0^{-1}x_1x_2c_3x_3c_3^2x_1^{-1} = x_0^{-1}x_1x_2^2c_4c_3^2x_1^{-1}$$

We need to merge $c_4c_3^2$. Increase the index by doing $c_3^2 = c_4^3x_1^{-1}$ to obtain

$$x_0^{-1}x_1x_2^2c_4c_4^3x_1^{-2} = x_0^{-1}x_1x_2^2c_4^4x_1^{-2}.$$

The only movement left is to move the initial x_0^{-1} to the right side, using (2) several times to obtain $x_0^{-1}c_4^4 = c_5^4x_4^{-1}$. There has been no need this time to increase the index of c_4^4 but if it had been needed, we could have done it by using the pumping lemma (2nd identity) again. The final result is

$$x_2x_3^2c_5^4x_4^{-1}x_1^{-2}$$

which is in pcq form.

The relationship between words in pq form and tree pair diagrams in F is different than the relationship between pcq forms and tree pair diagrams in T . In F , every tree pair diagram has a pq factorization associated to it, and any word in pq form is in fact the pq factorization associated to a (not necessarily unique) tree pair diagram. Given any word in pq form, then we can form a tree pair diagram for this element as follows. We consider reduced tree pair diagrams for p and q , and construct a tree pair diagram for the product pq as described in Section 2.2. The middle trees of the four trees involved in the product are all-right trees. The all-right trees in this decomposition may not have the same number of carets, so in forming the diagram for pq we simply enlarge the smaller of the two of these all-right trees (as well as the other tree in that diagram). Since only right carets are ever added during this process, all of whose leaves have leaf exponent zero, this results in a tree pair diagram whose pq factorization is precisely the word pq we began with.

In T , the correspondence between pcq factorizations and general pcq words is not so straightforward. Although each tree pair diagram has a pcq factorization associated to it, general words in algebraic pcq form are not always the pcq factorizations associated to a tree pair diagram. The difficulty arises when the tree pair diagram for c does not have as many carets as those for p or q , as adding right carets to enlarge c appropriately necessitates adding generators to the normal forms for p and q , so the tree pair diagram one obtains by multiplying as in F will not necessarily have the original word as its factorization. For example, the word x_1c_1 is in algebraic pcq form, yet it is not the pcq factorization associated to some tree pair diagram. There is a different representative for this element of T which is the pcq factorization associated to the reduced tree pair diagram for this group element: $x_1c_2x_1^{-1}$. We prefer to work with words which are pcq factorizations associated to tree pair diagrams, which will lead us to unique normal forms.

We can algebraically characterize the words of type pcq which are pcq factorizations associated to tree pair diagrams. The important condition is that the reduced tree pair diagram for c should have at least as many carets as those for p and q . We say that words in T with this property satisfy the *factorization condition*.

Theorem 3.4. *For elements in $T \setminus F$, the word*

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} c_i^j x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

where $i_1 < i_2 < \dots < i_n$, $j_1 < j_2 < \dots < j_m$, and $1 \leq j < i + 2$, is the pcq factorization associated to a tree pair diagram if and only if the number of carets in the reduced tree pair diagram for c_i^j is greater than or equal to the number of carets in the reduced tree pair diagram for either of the words $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n}$ or $x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$ in F .

Proof. Given a tree pair diagram, by construction, the pcq factorization associated to it satisfies the factorization condition. Given a word that satisfies the factorization condition, we can easily construct the corresponding tree pair diagram as described above. We see that the tree pair diagram for the c part has enough carets so that only right carets need to be added to the trees in the diagrams for p and q , so that the diagram constructed will indeed have the original word as its pcq factorization. \square

We can compute the number of carets of a word $w \in F$ algebraically from the normal form of w [4].

Proposition 3.5 ([4]). *Given a positive word in $w \in F$ in the form*

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n},$$

then the number of carets $N(w)$ in either tree of a reduced tree diagram representing w is

$$N(w) = \max\{i_k + r_k + \dots + r_n + 1, \text{ for } k = 1, 2, \dots, n\}.$$

When $w \in F$ is not a positive word, $N(w)$ is the maximum of the two numbers obtained by applying Proposition 3.5 to the positive and negative parts of the normal form for w . When considering elements of T , we recall that the number of carets in a tree pair diagram for c_i^j is equal to $i + 1$. Thus it is always possible to decide algebraically when $w \in T$ written in pcq form corresponds to a tree pair diagram, using Proposition 3.5 to count the carets for the positive and negative parts of the word.

4. NORMAL FORMS IN T

In T , we will declare the words in pcq form which are pcq factorizations associated to reduced diagrams to be the normal forms for elements of T , similar to the approach used in F . However, it is no longer true that these words cannot be shortened by applying a relation. As we saw with the normal form $x_1 c_2 x_1^{-1}$ in T , a word may be the shortest word representing an element which satisfies the factorization condition, yet there may be shorter words we can obtain by applying a relator which do not satisfy the factorization condition.

Thus, when algebraically characterizing the normal form for elements of T , we restrict ourselves to words of pcq form which satisfy the factorization condition, regardless of

whether or not a relator may reduce the length of the word. We next need to specify algebraic conditions which characterize the pcq forms that correspond to normal forms.

Theorem 4.1. *Let w be a pcq factorization for an element $g \in T$ associated to a marked tree pair diagram in which each tree has $i + 1$ carets, where the c part of the word is c_i^j with $1 \leq j < i + 2$. A reduction of a pair of carets from the tree pair diagram occurs only if the word w satisfies one of the following conditions:*

- (1) *There exists a pair of generators x_{k-j} and x_k^{-1} , with $j \leq k < i$, and neither of the two generators x_{k-j+1} and x_{k+1}^{-1} appear. The reduction corresponds to applying the relation*

$$x_{k-j}c_i^jx_k^{-1} = c_{i-1}^j$$

after applying relations from F in the p and q parts of the word, if necessary, to make x_{k+j} and x_k^{-1} adjacent to c_i^j .

- (2) *The generator x_{i-j} appears, and x_{i-j+1} does not. The reduction corresponds to applying*

$$x_{i-j}c_i^j = c_{i-1}^j$$

after possibly using relations from F as in (1).

- (3) *There exists a pair of generators $x_{k+i-j+2}$ and x_k^{-1} for $0 \leq k < j-2$ and neither one of the generators $x_{k+i-j+1}$ or x_{k+1}^{-1} appear. The reduction corresponds to applying*

$$x_{k+i-j+2}c_i^jx_k^{-1} = c_{i-1}^{j-1}$$

after possibly applying relations from F .

- (4) *The generator x_{j-2}^{-1} appears, and the generator x_{j-1}^{-1} does not appear. The reduction corresponds to*

$$c_i^jx_{j-2}^{-1} = c_{i-1}^{j-1}$$

after possibly applying relations from F .

Proof. Let $g \in T$ be represented by a tree pair diagram (T_-, T_+) . If both trees have an exposed caret whose leaves are identically numbered, then we call that a *reducible caret*, as it must be removed in order to obtain the reduced tree pair diagram representing g . We now consider algebraic conditions corresponding to a reducible caret in a tree pair diagram.

In the tree pair diagram (T_-, T_+) for $g \in T$, there are two ways of labelling the leaves in the target tree T_+ . The first labelling corresponds to the order in which the intervals in the subdivisions determined by these trees are paired in the homeomorphism, and is called the cyclic labelling. The cyclic labelling gives the marked leaf in the target tree the number zero, and the other leaves are given increasing labels from left to right around the leaves of the tree. The second labelling ignores the marking and puts the leaves in increasing order from left to right, beginning with zero. The first labelling is used to determine which leaves in T_- are paired with which leaves in T_+ , and the second labelling is used in the computation of leaf exponents to determine the powers of the generators that appear in the word. Observe Figure 9 where the labelling is specified for the four cases.

Suppose that the tree pair diagram for $g \in T$ is not reduced. The four cases above correspond to the following four possible locations of a reducible caret relative to the marked leaf in the target tree.

- (1) Case (1) corresponds to the case when the left leaf of the reducible caret is to the left of the marked leaf in T_+ , but the reducible caret is not the rightmost caret in T_- .
- (2) Case (2) corresponds to the special case when the reducible caret is a right caret in T_- (in which case necessarily the left leaf is to the left of the marked leaf in T_+). Leaf exponents from right carets will always be zero and thus right carets cannot contribute generators to the normal form. They may still result in an exposed reducible caret, which occurs exactly in this case, and the reduction will only affect the q part of the normal form.
- (3) Case (3) corresponds to the case when the left leaf of the reducible caret is either to the right of or coincides with the marked leaf in T_+ , but the reducible caret is not the rightmost caret in T_+ .
- (4) Case (4) corresponds to the special case when the reducible caret is a right caret in T_+ (in which case it cannot be to the left of the marked caret in T_+). As in Case (2), the exposed caret in this case is a right caret and does not contribute a generator to the normal form, but may still be reduced. This cancellation affects only the p part of the normal form.

To see that these are all the possibilities, observe that k , the number of the left leaf of the reducible caret in T_- , achieves all possible values in the cases above:

- If $0 \leq k < j - 2$ we are in case (3).
- If $k = j - 2$ we are in case (4).
- Observe that $k = j - 1$ is impossible because the leaves $j - 1$ and j are at the two ends of the tree. With a cyclic ordering the last and first leaves do not form a caret.
- If $j \leq k < i$ we are in case (1).
- If $k = i$ we are in case (2).

Check these cases with Figure 9 to see that these are all the possibilities. □

The conditions in Theorem 4.1 together with the factorization condition algebraically characterize our normal forms. The normal forms for elements in F have already been characterized, so we restrict to elements not in F in our description.

Theorem 4.2. *Any element $g \in T$ which is not an element of F admits an expression of the form pcq where*

$$p = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} \quad c = c_i^j \quad q = x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

$0 \leq i_1 < i_2 < \dots < i_n$, $0 \leq j_1 < j_2 < \dots < j_m$, and $1 \leq j < i + 2$. Among all the words in this form representing an element, there is a unique one satisfying the following conditions, and it is the normal form.

- The factorization condition, which we now state as $i + 1 \geq \max\{N(p), N(q)\}$.

- *The word does not admit any reductions. Namely, this word satisfies the following conditions:*
 - *If there exists a pair of generators x_{k-j} and x_k^{-1} simultaneously, for $j \leq k < i$, then one of the generators x_{k-j+1} or x_{k+1}^{-1} must appear as well.*
 - *If there is a generator x_{i-j} , then x_{i-j+1} must exist too.*
 - *If there exists a pair of generators $x_{k+i-j+2}$ and x_k^{-1} for $0 \leq k < j - 2$, then one of the generators $x_{k+i-j+1}$ or x_{k+1}^{-1} must appear as well.*
 - *If there exists a generator x_{j-2}^{-1} , then a generator x_{j-1}^{-1} must also appear.*

Proof. We claim that the conditions above precisely describe a set of unique normal forms for T . A pcq word satisfying the factorization condition is the pcq factorization associated to a marked tree pair diagram. However, if the pcq word satisfies all four reduction conditions, we have just shown in the previous theorem that this diagram is in fact the unique reduced diagram, and hence the word is in fact a normal form. \square

We remark that the Pumping Lemma together with the reductions in Theorem 4.1 give an explicit way of algebraically transforming any word in the generators of T into a normal form. Namely, given any word, we rewrite it in pcq form using the process described following the Pumping Lemma. If the resulting word does not satisfy the factorization condition, then we iterate the Pumping Lemma until we obtain a word for which the factorization condition is satisfied. The Pumping Lemma increases the number of carets for c and the number of carets for one of the words p and q . Once a word is obtained which satisfies the factorization condition, there must be a corresponding tree pair diagram for the element. Now, if the word satisfies any of the reduction conditions in Theorem 4.1, we apply them successively using the relations described there. This method thus produces the unique normal form.

5. THE WORD METRIC IN T

5.1. Estimating the word metric. For metric questions concerning T , we must consider a finite generating set instead of the one used to obtain the normal form for elements. We now approximate the word length of an element of T with respect to the generating set $\{x_0, x_1, c_1\}$, using information contained in the normal form and the tree pair diagram. These estimates are similar to those for the word metric in F with respect to the generating set $\{x_0, x_1\}$ found in [3] and [4].

Theorem 5.1. *Let $w \in T$ have normal form*

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} c_i^j x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}.$$

We define

$$D(w) = \sum_{k=1}^n r_k + \sum_{l=1}^m s_l + i_n + j_m + i.$$

Let $|w|$ denote the word metric in T with respect to the generating set $\{x_0, x_1, c_1\}$. There exists a constant $C > 0$ so that for every $w \in T$,

$$\frac{D(w)}{C} \leq |w| \leq C D(w)$$

and similarly, for $N(w)$ the number of carets in a reduced tree pair diagram representing w ,

$$\frac{N(w)}{C} \leq |w| \leq C N(w).$$

Proof. These inequalities follow from the correspondence between the normal form and the tree pair diagram for an element $w \in T$. It is clear, from Proposition 3.5, that $N(w) \geq \sum_{k=1}^n r_k$, $N(w) \geq \sum_{l=1}^m s_l$, $N(x) \geq i_n$, and $N(w) \geq j_m$. The inequality $N(w) \geq i$ is clear from the fact that c_i has $i + 1$ carets. These inequalities prove that

$$D(w) \leq 5 N(w).$$

We rewrite the generators x_i and c_j in terms of x_0 , x_1 and c and look at the lengths of these words to obtain the inequality

$$|w| \leq C D(w)$$

for some constant $C > 0$. Combining the two inequalities above we have

$$|w| \leq C' N(w).$$

To obtain lower bound on the word length, one only needs to observe that the tree pair diagram for each generator has either two or three carets. If u is a word in x_0 , x_1 and c with length n , then as these generators are multiplied together, each product may add at most 3 carets to the tree pair diagram. Thus the diagram for u will have at most $3n$ carets. It then follows that

$$N(w) \leq 3|w|.$$

Combining this with the above inequality, we obtain the desired bounds. \square

We use Theorem 5.1 to show that the inclusion of F in T is a quasi-isometric embedding. This means that there are constants $K > 0$ and C so that for any $w, z \in F$ we have

$$\frac{1}{K} d_F(w, z) - C \leq d_T(w, z) \leq K d_F(w, z) + C$$

where d_F and d_T represent the word metric in F and T respectively, with regard to the generating set $\{x_0, x_1\}$ of F and $\{x_0, x_1, c\}$ of T .

When considering whether the inclusion of a finitely generated subgroup H into a finitely generated group G is a quasi-isometric embedding, we can instead equivalently show that the distortion function is bounded. The distortion function is defined by

$$h(r) = \frac{1}{r} \max\{|x|_H : x \in H \text{ and } |x|_G \leq r\}.$$

Word length in F is comparable to the number of carets in the reduced tree pair diagram representing the word, as seen in [4, 8]. This, combined with Theorem 5.1 easily shows that the distortion function is bounded, and thus proves the following corollary.

Corollary 5.2. *The inclusion of F in T is a quasi-isometric embedding.*

5.2. Comparing word length in F and T . Although Corollary 5.2 shows that F is quasi-isometrically embedded in T , in fact the word length of many elements of F does not change when these elements are considered as elements of T , with respect to natural finite generating sets. As an example of this phenomenon, we characterize one type of element of F whose word length is unchanged when viewed as an element of T , using the generating set $\{x_0, x_1\}$ for F and $\{x_0, x_1, c_0\}$ for T . These are elements $w \in F$ for which $N(w)$ exceeds the word length $|w|_F$. Fordham [8] computes $|w|_F$ by assigning an integer weight between zero and four to each pair of carets in the tree pair diagram representing w . In a given word there are at most two weights of zero. Here we investigate words in which most weights are one. Such words, for example, are represented by tree pair diagrams with no interior carets having right children.

Theorem 5.3. *If $w \in F$ with $N(w) \geq |w|_F + 1$ then $|w|_T = |w|_F$, where word length is computed with respect to the generating set $\{x_0, x_1\}$ for F and $\{x_0, x_1, c_0\}$ for T .*

This theorem is proved by taking a word in the generators of T , and analyzing how each generator changes the intermediate tree pair diagram as one builds up the final tree pair diagram for w . Carefully controlling the process allows one to obtain an upper bound on the $N(w)$ in terms of the length of the word. If the word is actually shorter than $|w|_F$, then this bound, considered together with the lower bound given by the hypothesis, yields a contradiction. We immediately obtain the following corollary, since $|x_0^n|_F = |x_1^n|_F = n$, while $N(x_0^n) = n + 1$ and $N(x_1^n) = n + 3$.

Corollary 5.4. *The elements x_0^n and x_1^n have word length n in both F and T with respect to the finite generating sets $\{x_0, x_1\}$ and $\{x_0, x_1, c_0\}$ respectively.*

5.3. Cyclic subgroups of T . We can use the interpretation of elements of T as homeomorphisms of the circle to understand how F is contained in T and that the cyclic subgroups of T are quasi-isometrically embedded.

Theorem 5.5. *Every element in T has a power which is conjugate to an element in F .*

Proof. Ghys and Sergiescu [9] and Lioussé [12] show that every element in T has rational rotation number as a homeomorphism of S^1 . Recall that given a homeomorphism f of S^1 , its rotation number is defined as the limit

$$\lim_{n \rightarrow \infty} \frac{f^n(x) - x}{n},$$

for some point $x \in S^1$, and it is independent of x . A standard result in dynamics is that a homeomorphism of the circle has rational rotation number if and only if it has a periodic orbit (see for example Katok and Hasselblatt [11]). Thus, every element in T has a periodic orbit. That is, given $f \in T$, there exists a dyadic rational $x \in [0, 1]$ such that $f^n(x) = x$ for some n .

Given a dyadic rational x , we consider the rotation of the circle $t_x(p) = p + x$, where we view S^1 as the unit interval with the endpoints identified. Then t_x is an element of T , and conjugation of $w \in T$ by t_x , for the appropriate choice of x yields an element with zero as a fixed point. Thus the conjugated element must lie in F . \square

We use Theorem 5.5 to show that cyclic subgroups of T are quasi-isometrically embedded. We again bound the distortion function, as in the proof of Corollary 5.2.

Theorem 5.6. *Let $x \in T$ be a non-torsion element. Then the cyclic subgroup $\langle x \rangle$ generated by x is quasi-isometrically embedded in T .*

Proof. If the element x lies in F , the result can be deduced from Corollary 5.2, since Burillo [3] shows that all cyclic subgroups in F are quasi-isometrically embedded, and F is quasi-isometrically embedded in T . If $x \notin F$, it has a power w^n which is conjugate to an element of F by some element $z \in T$. Since x is not a torsion element, this power is not conjugate to the identity element of F .

The cyclic subgroup generated by $zw^n z^{-1}$ is quasi-isometrically embedded in T . Conjugation by z changes the number of carets by at most a factor of $2N(z)$, and thus we see that $\langle x \rangle$ must also be quasi-isometrically embedded in T . \square

6. TORSION ELEMENTS

Although the group F is torsion free, both T and V contain torsion elements. It is easy to construct torsion elements in T or V by choosing any binary tree S and making any marked tree pair diagram with S as both source and target tree. If the labelling of the target tree is the same as the labelling of the source tree, we get an unreduced representative of the identity; otherwise, we get a non-trivial torsion element. If this is an element of T , the tree pair diagram has pcq factorization in which $q = p^{-1}$. In fact, any torsion element can be represented by such a tree pair diagram, though its reduced marked tree pair diagram may well not have the same source and target trees, corresponding to the fact that although it has a pcq word where $q = p^{-1}$, the normal form may well not have this special balanced appearance.

Proposition 6.1. *If $f \in F, T$ or V is a torsion element, then it can be represented by a (marked) tree pair diagram with the same source and target trees.*

Before proving Proposition 6.1, we establish some notation which links the analytic and combinatorial interpretations of these groups. For $f \in F, T$, or V , if (T_-, T_+) is a marked tree pair diagram representing f , then it is sometimes convenient to denote the tree T_+ by $f(T_-)$. The tree T_- corresponds to a certain subdivision of the circle, which maps under f linearly to another subdivision of the circle. This subdivision is represented by the tree T_+ , and the marking describes where each subinterval of the circle is mapped. The element f can be thought of as mapping the leaves of T_- to the leaves of $f(T_-) = T_+$, where the marking defines this mapping of the leaves. If f does not have a tree pair diagram in which the tree T appears as the source tree, then the symbol $f(T)$ has no meaning.

Given two rooted binary trees T and T' , we say that T' is an *expansion* of T if T' can be obtained from T by attaching the roots of additional trees to some subset of the leaves of T . We observe that if $(T, f(T))$ is a marked tree pair diagram for f , and T' is an expansion of T , then there is always a tree pair diagram $(T', f(T'))$ for f , and $f(T')$ is an expansion of $f(T)$. Given two rooted binary trees S and T , by the *minimal common expansion* of

S and T we mean the smallest rooted binary tree which is an expansion of both S and T . Using this language, if $(T, f(T))$ and $(S, g(S))$ are marked tree pair diagrams for f and g respectively, the process described in Section 2.3 for creating a tree pair diagram for the product gf could be summarized as follows. If E is the minimal common expansion of $f(T)$ and S , then there are tree pair diagrams $(f^{-1}(E), E)$ for f , $(E, g(E))$ for g , and $(f^{-1}(E), g(E))$ for gf (with appropriate markings).

Proof. Suppose that f is a torsion element. We begin by describing the construction of (marked) tree pair diagrams (A_n, B_n) for f^n for every $n \geq 1$. These tree pair diagrams are constructed inductively, viewing f^n as a product $(f^{n-1})(f)$. For $n = 1$, let (A_1, B_1) be the reduced marked tree pair diagram for f . Throughout this procedure, although markings are carefully carried through in either T or V , since our goal is merely to produce a tree pair diagram for f with the same source and target trees (regardless of marking), only the trees themselves are relevant for this argument. Hence we suppress mention of any markings throughout the construction. If $k \geq 2$, suppose the marked tree pair diagram (A_{k-1}, B_{k-1}) for f^{k-1} has been constructed. Let E_{k-1} be the minimal common expansion of the trees A_1 and B_{k-1} . Then f^k has tree pair diagram $(f^{-(k-1)}(E_{k-1}), f(E_{k-1}))$, and we let $B_k = f(E_{k-1})$ and $A_k = f^{-(k-1)}(E_{k-1})$.

By construction, A_{k+1} is an expansion of A_k for all $k \geq 1$. We claim also that B_{k+1} is an expansion of B_k for all $k \geq 1$. For $k = 1$, E_1 is by definition an expansion of A_1 , which implies that $B_2 = f(E_1)$ is an expansion of $B_1 = f(A_1)$. Suppose inductively that B_k is an expansion of B_{k-1} . Now E_k is an expansion of B_k and A_1 , so E_k is an expansion of B_{k-1} and A_1 . But E_{k-1} is the minimal common expansion of B_{k-1} and A_1 , so E_k is an expansion of E_{k-1} , which implies that $B_{k+1} = f(E_k)$ is an expansion of $B_k = f(E_{k-1})$.

Since there exists a positive integer m such that f^m is the identity, it follows that all tree pair diagrams for f^m must have the same source and target trees. Hence $A_m = B_m$, and then since A_m is an expansion of A_1 , B_m is an expansion of A_1 . But since E_{m-1} is the minimal common expansion of B_{m-1} and A_1 , the fact that B_m is an expansion of both B_{m-1} and A_1 implies that $B_m = f(E_{m-1})$ is an expansion of E_{m-1} . But they have the same number of carets, so in fact $f(E_{m-1}) = E_{m-1}$. In other words, the tree pair diagram $(E_{m-1}, B_m = f(E_{m-1}))$ is the desired tree pair diagram for f . \square

Corollary 6.2. *An element of T is torsion if and only if it is a conjugate of some c_i^j .*

Proof. If an element is torsion, then it admits a diagram with two equal trees. The pcq factorization associated with this diagram has the form $pc_i^j p^{-1}$, where p is a positive element of F . \square

A particularly natural torsion subgroup is the subgroup R of pure rotations, where by a pure rotation we mean a rotation by $d = \frac{a}{2^n}$ (where a is not divisible by 2). Such pure rotations were used in Section 5.3 to conjugate the fixed point of a homeomorphism to 0.

This subgroup is isomorphic to the group of dyadic rational numbers modulo 1, which has a 2-adic metric as follows: if $x = \frac{p}{2^l}$, $y = \frac{q}{2^m}$, and $z = |x - y| = \frac{r}{2^k}$, where p, q and r are odd, then $d(x, y) = 2^k$. With respect to this metric, the subgroup of rotations is quasi-isometrically embedded in T .

Proposition 6.3. *The subgroup R of the pure rotations, with the 2-adic metric, is quasi-isometrically embedded in T .*

Proof. We note that if $g \in T$ is the rotation by $\frac{a}{2^n}$ where a is not divisible by 2, then there are $2^n - 1$ carets in the reduced tree pair diagram representing g , so $N(g) = 2^n - 1$. Since we have shown that the word length of g in T is bi-Lipschitz equivalent to $N(g)$, the proposition follows. \square

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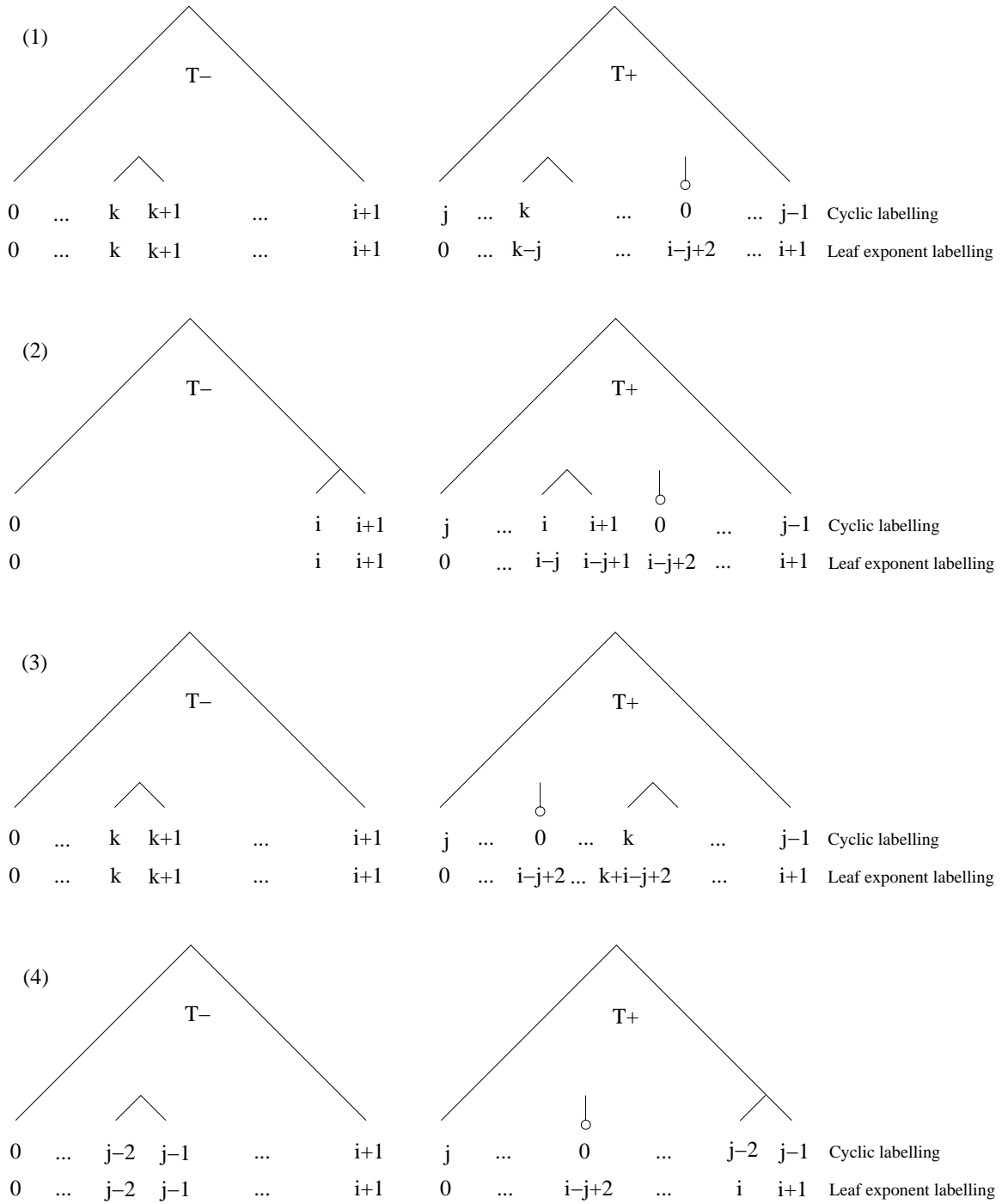


FIGURE 9. The four cases in Theorem 4.1. Observe the two labellings on the leaves of the trees, the cyclic labelling which indicates the correspondence of the leaves, and the leaf exponent labelling which indicates the corresponding generators in the normal form.