

Lower Bounds of Isoperimetric Functions for Nilpotent Groups

José Burillo

ABSTRACT. In this paper we prove that Heisenberg groups and the groups of unipotent upper triangular matrices are not combable, by giving lower bounds for the isoperimetric inequalities in higher dimensions, and provide a counterexample to a conjecture by Gromov.

Introduction

Isoperimetric inequalities have recently been a useful tool in the study of the complexity of the word problem for finitely presented groups, since Gromov's breakthrough paper [5] in hyperbolic groups and the extensive treatment of automatic groups in [2]. The two important results in this direction state that a finitely presented group is hyperbolic if and only if it has linear Dehn function, and that an automatic group has quadratic Dehn function. The term *Dehn function* was coined by Gersten [3] to denote the best possible isoperimetric function. The second of these results, due to Thurston, has been crucial in deciding which groups are automatic, due to the difficulty in verifying the conditions stated on the definition of an automatic group. It is worth noticing, however, that this result depends only on the geometric properties of automatic groups, and not on the logic ones, and the notion which causes it has been distilled in the definition of combing and combable groups. Let G be a group, and let \mathcal{A} be a set of semigroup generators for G . Then a *combing* for G is a section σ of the canonical evaluation map from the free semigroup generated by \mathcal{A} in G , which satisfies a synchronous k -fellow traveller property for some $k > 0$. A group is combable if it admits a combing. A combable group is finitely presented, and an automatic group is combable. In the definition given in [2, Ch.3] there is an implicit bound on the length of the word $\sigma(g)$, linear on $|g|$, while in the definition given by Gersten [4] this bound is not required. This difference makes that combable groups, in the sense of [2], satisfy quadratic isoperimetric inequalities, while Gersten combable groups are only known to satisfy an exponential isoperimetric inequality. The definition we are going to use in this paper is the one in [2].

1991 *Mathematics Subject Classification*. Primary 20F32; Secondary 20F18, 57M07, 53C15.
This paper is in final form and no version of it will be submitted for publication elsewhere.

It was proved by Thurston [2, Ch.8], using geometric methods, and by Gersten [3], using combinatorial methods, that the integral 3-dimensional Heisenberg group presented by

$$\langle a, b, c \mid [a, c] = [b, c] = 1, [a, b] = c \rangle$$

has cubic Dehn function, and hence it cannot be automatic, nor combable. The question of whether a nilpotent group can be automatic was answered by Holt, proving in [2, Ch.8] that a nilpotent automatic group is virtually abelian, but since Holt's proof makes strong use of the logic properties of automatic groups, it leaves open the question of when a nilpotent group can be combable. In this paper we give negative answer to that question for two particular families of groups, namely, the integral Heisenberg groups and the groups of unipotent upper triangular matrices with integer entries. The latter class of groups will provide a counterexample to a conjecture stated by Gromov about the smallest dimension where the isoperimetric inequality fails to be euclidean.

The techniques used in the proofs are higher dimensional generalizations of Thurston's proof for the 3-dimensional Heisenberg group, and in the same line of his proof of the fact that $SL(n, \mathbb{Z})$ is not a combable group (and hence not automatic). In the latter case, the proof goes by constructing a higher dimensional Lipschitz cycle with known volume and proving that every Lipschitz chain bounding this cycle must have very large volume. The key theorem used in these arguments is the Riemannian isoperimetric inequality in higher dimensions [2, Th.10.3.5]. This result led to the definition by Gromov [6, 5.D] of the *filling volume functions* FV_k . Let M be an n -connected Riemannian manifold, and let z be a Lipschitz k -cycle in M . The filling volume for z is the minimal volume of a Lipschitz $(k+1)$ -chain c with $\partial c = z$. Then the filling volume function $FV_{k+1}(l)$ is defined as the maximum of the filling volumes of all k -cycles z with $\text{vol}_k z \leq l$. Then the Riemannian isoperimetric inequality in higher dimensions can be reformulated in terms of the filling volume functions as follows: If a combable group acts properly discontinuously, cocompactly and by isometries in an n -connected Riemannian manifold M , then the filling volume functions for M satisfy

$$FV_{k+1}(l) \prec l^{\frac{k+1}{k}}$$

for $k \leq n$. In [6, 5.D] Gromov conjectures the values of the filling volume functions for the Heisenberg groups, of which we are going to prove the lower bounds, and outlines a proof of the fact that the Heisenberg groups are not combable, which is somewhat different from ours.

1. Homogeneous nilpotent Lie groups

Let G be a simply connected nilpotent Lie group.

DEFINITION 1.1. The group G is said to be *homogeneous* [6, 5.A'2] if its Lie algebra L admits a grading:

$$L = L^1 \oplus L^2 \oplus \dots \oplus L^d$$

with $[L^i, L^j] \subset L^{i+j}$.

For every $t \in (0, \infty)$ consider the linear map in L defined by

$$\begin{aligned} a_t : L &\longrightarrow L \\ a_t|_{L^i} &= t^i \text{Id}. \end{aligned}$$

Since G is simply connected, there exists a 1-parameter group of *scaling automorphisms* A_t with $dA_t(1) = a_t$. These scaling automorphisms can be used to produce lower bounds on the filling volume functions for G , constructing a cycle which requires large filling and scaling it to produce a family of cycles with this property:

PROPOSITION 1.2. *Let G be a homogeneous simply connected nilpotent Lie group, and let A_t be the scaling automorphisms. If there exist constants $M, r, s > 0$, a Lipschitz k -chain c and a closed G -invariant k -form η in G such that:*

1. $\text{vol}_{k-1} A_t(\partial c) \leq M t^r$,

2. $\int_c \eta > 0$, and

3. $A_t^* \eta = t^s \eta$,

then $FV_k(l) \succ l^{s/r}$.

PROOF. Let d_t be any k -chain with $\partial d_t = A_t(\partial c) = \partial A_t(c)$. Since η is G -invariant, its norm is constant at every point, and

$$\int_{d_t} \eta \leq \|\eta\| \text{vol}_k d_t,$$

and, since η is closed, by Stokes' theorem

$$\int_{d_t} \eta = \int_{A_t(c)} \eta.$$

Then, we find the lower bound

$$\text{vol}_k d_t \geq \frac{1}{\|\eta\|} \int_{A_t(c)} \eta = \frac{1}{\|\eta\|} \int_c A_t^* \eta = \frac{t^s}{\|\eta\|} \int_c \eta.$$

which gives the required bound for the filling volume function.

2. Heisenberg groups

Let H_{2n+1} be the real $(2n+1)$ -dimensional Heisenberg group. This group can be seen as \mathbb{R}^{2n+1} with coordinates (x, y, z) , where

$$x = (x_1, x_2, \dots, x_n)$$

$$y = (y_1, y_2, \dots, y_n),$$

with the left-invariant metric obtained when H_{2n+1} is regarded as the group of upper triangular $(n+2) \times (n+2)$ matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & I & y^T \\ 0 & 0 & 1 \end{pmatrix}.$$

If

$$\omega = dz - \sum_{i=1}^n x_i dy_i$$

is the non-closed invariant 1-form, then the metric can be expressed as:

$$ds^2 = \sum_{i=1}^n dx_i^2 + \sum_{i=1}^n dy_i^2 + \omega^2.$$

The form ω defines a contact distribution [1] in H_{2n+1} , since $\omega \wedge (d\omega)^n$ is nowhere vanishing. The maximal dimension of an integral submanifold for this distribution is n .

It is clear that the Heisenberg groups are nilpotent and homogeneous, and the scaling automorphisms are given by

$$A_t(x, y, z) = (tx, ty, t^2z).$$

Note that $A_t^*\omega = t^2\omega$.

THEOREM 2.1. *The filling volume functions for H_{2n+1} have the following lower bounds:*

1. $FV_{k+1}(l) \succ l^{\frac{k+1}{k}}$, for $k < n$.
2. $FV_{n+1}(l) \succ l^{\frac{n+2}{n}}$.
3. $FV_{k+1}(l) \succ l^{\frac{k+2}{k+1}}$, for $n < k \leq 2n$.

PROOF. We want to construct a Lipschitz chain c and a form η to use Proposition 1.2.

For the cases (1) and (3) we take for the chain c a cube $[0, 1]^{k+1}$, on the variables:

1. x_1, \dots, x_{k+1} , and
3. $x_1, \dots, x_{k-n}, y_1, \dots, y_n, z$,

and for η their respective volume forms.

The case (2) requires a more complicated chain. This chain is the higher dimensional analog of the chain constructed in [2, 8.1] to prove the cubic lower bound for the Dehn function of H_3 .

Let $K = \{1, \dots, n\}$. Inside the affine subspace of dimension n defined by the equations

$$\begin{aligned} x_l &= 0, & l \in K, \\ z &= 1, \end{aligned}$$

construct the n -cube Q defined by $-1 \leq y_l \leq 1$ for $l \in K$. Let $I, J \subset K$, with $I \cap J = \emptyset$ and $I \cup J = K$, and consider a map $\epsilon : I \rightarrow \{-1, 1\}$. The data I, J, ϵ determine a cell $c_{I, \epsilon}$ of dimension $m = \text{card } J$ in the boundary of Q , defined by

$$\begin{aligned} y_i &= \epsilon(i) & \text{if } i \in I, \\ -1 \leq y_j &\leq 1 & \text{if } j \in J, \\ x_l &= 0 & \text{for } l \in K, \\ z &= 1. \end{aligned}$$

Since we want to consider only cells in the boundary of Q , we must impose that $I \neq \emptyset$, since $I = \emptyset$ would give us the whole cube Q . However, J can be empty, in which case $I = K$ and the 2^n possible maps $\epsilon : K \rightarrow \{-1, 1\}$ describe the 2^n vertices of Q .

FIGURE 1. The chain d for $n = 2$.

For each cell $c_{I,\epsilon}$ of dimension m , we are going to construct a simplex $\sigma_{I,\epsilon}$, of dimension $n - m$, inside the coordinate n -affine subspace $y_1 = \dots = y_n = z = 0$. The simplex $\sigma_{I,\epsilon}$ is defined by the equations

$$\begin{aligned} \sum_{i \in I} \epsilon(i)x_i &\leq 1, \\ \epsilon(i)x_i &\geq 0 \quad \text{for } i \in I, \\ x_j &= 0 \quad \text{for } j \in J. \end{aligned}$$

Let $\tau_{I,\epsilon}$ be the face of the simplex which satisfies

$$\sum_{i \in I} \epsilon(i)x_i = 1.$$

Construct then $d_{I,\epsilon} = c_{I,\epsilon} \times \tau_{I,\epsilon}$. We have that $d_{I,\epsilon}$ is an n -cell in \mathbb{R}^{2n+1} , and $d_{I,\epsilon} \cap Q = c_{I,\epsilon}$. The union of Q with all the n -cells $d_{I,\epsilon}$ form an n -chain d in \mathbb{R}^{2n+1} , which is contained in the hyperplane $z = 1$, and whose boundary consists of the

FIGURE 2. The chain c for $n = 2$, with the chain d on top

$(n - 1)$ -cells $c_{I,\epsilon} \times \pi_{I,\epsilon}$. Notice that, unlike it is shown in Figure 2, the cells $d_{I,\epsilon}$ are orthogonal to the cube Q .

Project the chain d in the coordinate x -plane, i.e. making $y_l = 0$, for $l \in K$, and $z = 0$. The projection of Q is just the origin, while the projection of $d_{I,\epsilon}$ is the simplex $\sigma_{I,\epsilon}$. Joining each point of d with its projection by a segment we obtain the chain c . The chain c is then the union of the cone from the origin to Q with the $(n + 1)$ -cells obtained from the projection of each $d_{I,\epsilon}$.

Let

$$\eta = dy_1 \wedge \dots \wedge dy_n \wedge \omega = dy_1 \wedge \dots \wedge dy_n \wedge dz.$$

It is clear then that $A_t^* \eta = t^{n+2} \eta$, since z scales by a factor of t^2 . And the value of

$$\int_c \eta$$

is equal to the volume of the cone of Q .

Note finally that each cell in the boundary of c is contained in an affine subspace of dimension n , and each of these subspaces are integral submanifolds of the contact distribution defined by the 1-form ω . Then, in each of these subspaces, the metric scales by t under the action of A_t , and the volume by t^n . It is clear now that the n -volume of the boundary of c is of the order of t^n .

COROLLARY 2.2. *The integral $(2n + 1)$ -dimensional Heisenberg group is not combable.*

PROOF. The integral $(2n + 1)$ -Heisenberg group acts properly discontinuously, cocompactly and by isometries in H_{2n+1} . The $(n + 1)$ -chain $A_t(c)$ has $(n + 1)$ -volume of the order of t^{n+2} , while its boundary has n -volume of the order of t^n and diameter of the order of t , contradicting the higher-dimensional isoperimetric inequality that a combable group satisfies [2, Th.10.3.5].

3. Upper triangular groups and Gromov's conjecture

Let $T(n, K)$ be the group of unipotent upper-triangular matrices with coefficients in K , where $K = \mathbb{Z}$ or \mathbb{R} . Since $T(3, \mathbb{Z})$ is equal to the integral 3-dimensional Heisenberg group, and $T(m, \mathbb{Z})$ is a retract of $T(n, \mathbb{Z})$ when $m \leq n$, it follows that the Dehn function of $T(n, \mathbb{Z})$ is at least cubic, if $n \geq 3$. Then:

PROPOSITION 3.1. *For $n \geq 3$, the groups $T(n, \mathbb{Z})$ are not combable.*

Let G be a simply connected nilpotent Lie group, and let L be its Lie algebra. In [6, 5.D], Gromov proposes the following conjecture: If $L = [L, L] \oplus T$, and k is the maximal dimension of an integral submanifold of the distribution obtained by left translating T , then the first dimension where the filling volume function fails to be euclidean is $k + 1$, i.e:

$$FV_{i+1}(l) \sim l^{\frac{i+1}{i}} \quad \text{if } i < k,$$

and

$$FV_{k+1}(l) \sim l^{\frac{k+2}{k}}.$$

In the case of the $(2n+1)$ -dimensional Heisenberg groups, the distribution obtained left translating T is precisely the contact distribution defined by the form ω , whose integral submanifolds have dimension at most n , and the lower bounds agree with Gromov's conjecture. But the case of the groups $T(n, \mathbb{R})$ provides a counterexample.

Let E_{ij} be the matrix with all entries equal to zero except for the (i, j) entry, which is equal to one. Then the matrices E_{ij} , for $1 \leq i < j \leq n$, form a basis for the Lie algebra $t(n, \mathbb{R})$ of $T(n, \mathbb{R})$, and a basis for T can be taken as $\{E_{i, i+1}, 1 \leq i \leq n-1\}$. Observe that the matrices $E_{i, i+1}$ and $E_{j, j+1}$ commute if $|i - j| > 1$. Then, if $n \geq 4$, the subspace T contains an abelian subalgebra of $t(n, \mathbb{R})$, generated by the matrices $E_{2i-1, 2i}$, for $1 \leq i \leq \lfloor n/2 \rfloor$. Since an abelian subalgebra always leads to an integrable distribution by left translation, by Frobenius theorem $T(n, \mathbb{R})$ admits a submanifold of dimension $\lfloor n/2 \rfloor$ which is integral to the distribution defined by T . However, FV_2 is already cubic, hence non-euclidean, contradicting Gromov's conjecture.

References

1. Arnol'd, V.I., *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York-Berlin-Heidelberg, 1978.
2. Epstein, D. B. A., Cannon, J. W., Holt, D. F., Levy, S. V. F., Paterson, M. S., Thurston, W. P., *Word Processing in Groups*, Jones and Bartlett, Boston-London, 1992.
3. Gersten, S.M., *Dehn functions and l_1 -norms of finite presentations*, Algorithms and Classification in Combinatorial Group Theory (G. Baumslag, C.F. Miller III, eds.), Springer-Verlag, New York-Berlin-Heidelberg, 1992.
4. Gersten, S. M., *Bounded cocycles and combings of groups*, Internat. J. Algebra Comput. **2** (1992), no. 3, 307-326.
5. Gromov, M., *Hyperbolic groups*, Essays in Group Theory (S.M. Gersten, ed.), MSRI series vol. 8, Springer-Verlag, New York-Berlin-Heidelberg, 1987.
6. Gromov, M., *Asymptotic invariants of infinite groups*, Geometric Group Theory II (G. Niblo, M. Roller, eds.), London Mathematical Society Lecture Note Series, Cambridge University Press, 1993.

DEPT. OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112.
E-mail address: burillo@math.utah.edu