GROWTH OF POSITIVE WORDS IN THOMPSON'S GROUP F

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Abstract. Although it is well known that the growth of Thompson's group F is exponential, the exact growth function is still unknown. Elements of its submonoid of positive words can be described using a binary rooted tree, whose norm can be computed assigning weights to each caret. Combining this fact with a combinatorial argument, the growth function of the submonoid is computed and thus providing a first step in the computation of the growth function of the group, as well as a lower bound for the growth rate for the group.

Introduction

The fascinating properties of Thompson's group F have made it a preferred object of study in many branches of mathematics. Even though it is finitely presented, it also admits an infinite presentation which is much more useful for practical purposes due to simplicity and symmetry of its relators. This duality causes F to combine finiteness properties with other ones which would seem more apropriate for groups of infinite type, where the term "infinite type" is meant here to be used naïvely. For instance, F has an infinitely dimensional $K(G,1)$, described in [1], but which has only two cells in every dimension. Another example of this tendency is the fact that F admits free abelian subgroups of infinite rank, and subgroups isomorphic to $F\times\mathbb{Z}$ and to $F\times F$, which are all quasi-isometrically embedded, i.e. embedded in a non-distorted way ([2]). For a detailed description of F and many of its properties see [1] and also [3], a wonderful survey with proofs of many of the properties of F.

A long-standing open problem concerns the amenability of F. It was for a long time a finitely presented candidate for a counterexample for the Von Neumann conjecture (a group is either amenable or has a free non-abelian subgroup), and still is, but with less interest now that Sapir and Olshanskii have such a counterexample ([6]). In any case, the amenability of F is still an outstanding problem which has been open for at least thirty years. Somewhat related to amenability is the concept of growth of a group. It is well known that Thompson's group has exponential growth (see [2] and [3] for proofs), which leaves the amenability question undecided, but the exact growth function is not known. Several authors have shown interest in the particular growth function for F, see [4] for instance. In this paper we will give a partial answer to that question, constructing a growth function for the submonoid of positive words of F.

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Thompson's group and the submonoid of positive words

Thompson's group F is defined by the infinite presentation

$$\mathcal{P} = \langle x_i,\, i \geq 0 \,|\, x_j x_i = x_i x_{j+1},\ \text{if } i < j \rangle$$

although as it can be easily seen from the relators above, only the two generators x_0 and x_1 are relevant, and they can be used to construct the following finite presentation for F:

$$\mathcal{F} = \left\langle x_0, x_1 \,|\, [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \right\rangle$$

We will work mostly with the infinite presentation, but it is well understood that notions that require the group to be finitely presented, like the growth function for instance, will always make reference to the finite presentation $\mathcal{F}$. When we talk about the length of a word, or the word metric, or the distance between two words, we are always referring to the finite presentation $\mathcal{F}$. Especially, we will consider the *norm* of an element $x \in F$, written $|x|$, as the length of the shortest possible word in the generators x_0 and x_1 which represents x.

Every element of Thompson's group admits a normal form of the type

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}, \qquad \text{with } i_1 < i_2 < \ldots < i_n \neq j_m > \ldots > j_2 > j_1,$$

unique under certain conditions (see [1]) which will not be needed in this paper. The subset P of *positive* words, i.e. words of the type

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n}, \qquad \text{with } i_1 < i_2 < \ldots < i_n,$$

form a submonoid of F, for which $\mathcal{P}$ is a monoid presentation. This submonoid of positive words is the object of study of this paper, where we will compute the exact number of positive words which have a given norm.

The group F admits a geometric interpretation as the group of piecewise-linear, orientation-preserving homeomorphisms of the interval $[0,1]$ which have slopes (in the linear parts) which are always powers of 2, and whose breaking points (points where they are not differentiable) are always dyadic integers (of the form $a/2^m$, with $a, m \in Z$). For a detailed exposition of this interpretation, see [3].

The most celebrated and useful interpretation of F uses rooted binary trees. An element of F, as a piecewise-linear homeomorphism of $[0,1]$, can be determined by two subdivisions of $[0,1]$ into the same number of intervals, by just agreeing that intervals are mapped linearly into each other preserving the order. Slopes are determined by the length of the two intervals. As an example, an element of F can be represented by:

$$\{[0,1/4],[1/4,1/2],[1/2,1]\} \longrightarrow \{[0,1/2],[1/2,3/4],[3/4,1]\}$$

which has three linear pieces, one of slope 2, one of slope 1 and one of slope $1/2$. This homeomorphism corresponds to the element x_0 of F.

Subdividing $[0, 1]$ into intervals of length $1/2^m$ can be achieved by successive subdivisions of the corresponding intervals into two equal halves. For example, both subdivisions in the element above can be obtained subdividing $[0, 1]$ into two pieces and then one of these pieces into two more, the piece $[0, 1/2]$ for the source and the piece $[1/2, 1]$ for the target. Successive subdivisions of the interval into two equal pieces can be identified as rooted binary trees. A rooted binary tree is a tree which emanates from a root with two branches (the *left* and *right* branches), and each vertex either has two more branches (going "down") or none at all. A *caret* is formed by a vertex together with the two edges going down from it. The *children* carets from a given caret are the two carets immediately below, and they will be referred to as its left and right children. A caret is called a *right* caret if it is on the "rightmost side" of the tree, i.e., if it is obtained from the root through a sequence of right children only. Left carets are defined the same way. A caret is called *interior* if it is not a left or a right caret. For a complete description of this interpretation see [3] and [5].

So an element of the group can be represented by two rooted binary trees with the same number of carets, one for the source and one for the target. The rooted tree representation for an element and its normal form are totally equivalent, since one can be easily determined from the other, see [3] and [5]. In particular, for positive elements of F, the target tree is redundant, because it has only right carets, which corresponds to the fact that the normal form only has the positive part. So positive elements can be represented with only one rooted tree, and this representation will be used extensively in the rest of the paper.

Length of a word

In his unfortunately unpublished thesis [5], S. Blake Fordham constructed an algorithm to compute the exact norm of an element of F from the rooted tree diagram. The method consists in describing different types of pairs of carets (one caret from the source and one from the target), and assigning a weight to each pair of carets. The weight of the element is the sum of the weights of all the pairs of carets, and he proves that this weight satisfies all the properties of a distance and hence must coincide with it. We will not describe this algorithm here, but only make reference to the particular case of a positive word, which is the only one which will be used in this paper.

As said before, a positive word can be understood as one rooted binary tree. So we will assign a weight to each caret of the tree in such a way that the total weight of the tree will correspond to the norm of the element. The weight of a caret is defined the following way:

(1) The root caret has weight zero, and for the purposes of this computation will not be considered as either a left or a right caret.
(2) Right carets which have a right child have weight two. These are all right carets except the last one, and in Fordham's notation correspond to the types $\mathcal{R}_i$ and $\mathcal{R}_{ni}$.
(3) The right caret with no right children (i.e. the last right caret, type $\mathcal{R}_\varnothing$ in [5]) has weight zero.
(4) Interior carets which have a right child (type $\mathcal{I}_R$ in [5]) have weight three.
(5) All other carets (all left carets and all interior carets with no right children) have weight one.

We have then the following result:

Theorem. *(S. B. Fordham, [5]) For a positive word $x \in P$, represented by a rooted binary tree as defined above, the total weight of the tree coincides with its norm $|x|$ with respect to the generators x_0 and x_1.*

The growth function of the monoid P

This last result is the key piece which allows us to reduce the problem of computing the growth function for the monoid P to a combinatorial calculation for the number of trees which have a given weight. This combinatorial argument is reminiscent of the computation of the classical Catalan numbers, see [7]. Catalan numbers compute, in our terminology, how many rooted binary trees have exactly n carets, i.e. as if all carets had weight one. If a_n is the n-th Catalan number, i.e. the number of rooted binary trees with n carets, then one considers that a tree with n carets has, as descendants of the root caret, a left tree with i carets and a right tree with $n-1-i$ carets, for some number i. One obtains then that the following recurrence is satisfied:

$$a_n = \sum_{i=0}^{n-1} a_i a_{n-1-i} \qquad a_0 = 1$$

from which the well-known values of the Catalan numbers and their generating function

$$a_n = \frac{1}{n+1}\binom{2n}{n} \qquad\qquad \sum_{n=0}^{\infty} a_n x^n = \frac{1-\sqrt{1-4x}}{2x}$$

can be deduced. For details in this calculations and for generalities about sequences, recurrences and their generating functions see [7].

We will use the same method to compute the number of rooted trees with given weight, with some variations due to the different weights that correspond to different types of carets. Let p_n be the number of rooted binary trees which have weight n according to the above definition of weight. It is clear that p_n is also the number of positive words of the monoid P with norm n. The result is as follows:

Theorem. *The sequence $(p_n)_n$ satisfies the recurrence:*

$$p_n = 2p_{n-1} + p_{n-2} - p_{n-3}$$

with $p_0 = 1$, $p_1 = 2$, and $p_2 = 4$. Its generating function is

$$P(x) = \sum_{i=0}^{\infty} p_n x^n = \frac{1-x^2}{1-2x-x^2+x^3},$$

and its growth rate

$$\lim_{n\to\infty} {p_n}^{1/n} = 2.2469796\ldots$$

is the largest root of the polynomial

$$x^3 - 2x^2 - x + 1.$$

Proof. The idea is to follow the method used to compute Catalan numbers adapting it to trees with weights. The recurrence will be more complicated because of the different weights that the trees have, but it will not be difficult to compute its generating function. The other parts of the statement (the recurrence for the p_n and the growth rate) can be easily deduced from the generating function, in particular, following standard methods (see [7]) it is easy to see that the sequence $(p_n)_n$ satisfying this recurrence has $P(x)$ as a generating function and viceversa. The growth rate is also easily deduced. So we will only need to find the generating function.

Consider a rooted binary tree T with weight n. Starting at any given vertex, we can consider the subtree that hangs from it. We will say that a subtree of T is a *left*, *right* or *interior* subtree in the obvious situations: interior if all its carets are interior (and obviously when it hangs from an interior vertex), and right or left if it contains right or left carets. The whole T is the only subtree which is right and left at the same time. Note that to compute the weight of an interior subtree all carets are counted as interior, i.e., all have weights 1 or 3 depending on their right children. If the tree is a left subtree, then left and interior carets are counted as usual, but its right carets have to be counted as interior since they will ultimately end up being interior in the complete tree T.

So consider T, of weight n, and consider the two subtrees that hang from the root caret. These two subtrees are obviosuly a left subtree carrying some weight i (for some i, $0 \leq i \leq n$) and a right subtree carrying weight $n-i$. So to compute p_n we only need to compute how many left subtrees we can have that carry weight i and how many right subtrees carry weight $n-i$, for all possible i. If we define two new sequences $(l_i)_i$ and $(r_j)_j$, where l_i is the number of rooted binary trees such that when acting as left subtrees have weight i, and r_j the corresponding definition for right subtrees, then we have that

$$p_n = \sum_{i=0}^{n} l_i r_{n-i}.$$

Note that the extreme cases $i=0$ and $i=n$ account for the possibility of either the left and the right subtree being empty, but they should be counted as $l_0 = 1$ and $r_0 = 1$. We will develop now recurrences for the l_i and the r_j.

Consider now a left subtree with weight i. From its root caret (which now carries weight 1 since it will ultimately be a left caret) hang two subtrees, a left subtree carrying some weight and, hanging from the right, an interior subtree, all whose carets are treated as interior carets. Defining a new sequence $(c_k)_k$ where c_k is the number of rooted binary trees that carry a weight k when considered as interior trees, the same argument as before produces the recurrence

$$l_i = \sum_{j=0}^{i-1} l_j c_{i-1-j}, \qquad c_0 = 1$$

A recurrence for the r_j can be constructed the same way, with one small difference, caused by the fact that the last right caret counts zero while the previous right carets count 2. So given a right subtree with weight j, we must separate the case that its root caret has no right children (in which case it has weight zero) from the case that it has right children (when it has weight 2). If the root caret has no right children, all the weight of this right subtree is concentrated on the interior subtree that hangs from the left side of the root. And if the root caret has right children, then there are two subtrees, one interior subtree (possibly empty) and one right subtree (obviously nonempty now), whose combined weight is $j-2$. So the recurrence becomes

$$r_j = c_j + \sum_{k=1}^{j-2} r_k c_{j-2-k}.$$

The term c_j corresponds to the case where all the tree hangs from the left side, and the rest is the general case. Observe that the sum starts now at $k=1$ because, in the general case, the right subtree is nonempty and must carry at least a weight 1.

Finally, interior trees are the ones that allow us to compute the generating functions, because in an interior tree, both subtrees hanging from the root are also interior trees, so the recurrence only involves the sequence $(c_k)_k$. The computation is analogous to the one for right trees, distinguishing the two cases motivated by the different weights of the root caret depending on its right children. The recurrence is

$$c_k = c_{k-1} + \sum_{m=1}^{k-3} c_m c_{k-3-m}, \qquad c_0 = 1.$$

Observe that it also has two parts, depending on the existence of right children in the root caret.

Armed with all these recurrences we can now compute their generating functions by the standard method (see [7]). Let $P(x)$, $L(x)$, $R(x)$ and $C(x)$ be the four generating functions. From all the recurrences we obtain the relations

$$x^3C(x)^2 - (x^3 - x + 1)C(x) + 1 = 0$$
$$L(x) = \frac{1}{1 - xC(x)}$$
$$R(x) = \frac{(1-x^2)C(x)}{1 - x^2C(x)}$$
$$P(x) = L(x)R(x)$$

which give

$$P(x) = \frac{(1-x^2)C(x)}{(1-xC(x))(1-x^2C(x))} = \frac{1-x^2}{1-2x-x^2+x^3},$$

which concludes the proof.

As an example, the first few terms of the sequence are

$$\begin{array}{llll} p_0 = 1 & p_3 = 9 & p_6 = 101 & p_9 = 1146 \\ p_1 = 2 & p_4 = 20 & p_7 = 227 & p_{10} = 2575 \\ p_2 = 4 & p_5 = 45 & p_8 = 510 & p_{11} = 5786, \end{array}$$

and, for instance, the nine positive words of norm three are, expressed in their normal forms as well as in the shortest possible form in x_0 and x_1:

$$x_0^3 \qquad x_0^2x_1 \qquad x_0x_1^2 \qquad x_1^3 \qquad x_0^2x_2 = x_0x_1x_0$$
$$x_0^2x_3 = x_1x_0^2 \qquad x_0x_1x_3 = x_1x_0x_1 \qquad x_0x_2^2 = x_1^2x_0 \qquad x_2 = x_0^{-1}x_1x_0$$

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