Dimension and Fundamental Groups of Asymptotic Cones

José Burillo

November 5, 1998

Asymptotic cones were first used by Gromov in his paper [6], where he constructed *limit spaces* of nilpotent groups in order to prove that groups with polynomial growth are virtually nilpotent. Gromov does not use the term *asymptotic cone*, which was introduced later by Van den Dries and Wilkie in [12], when they gave a nonstandard interpretation of Gromov's results. Ultrafilters appear in [12] for the first time in this context. Later Gromov gave an extensive treatment of asymptotic cones in [7]. From this paper several authors have used asymptotic cones to obtain interesting results, for instance, in identifying quasi-isometry classes of 3-manifolds ([8]) or relating asymptotic cones with Dehn functions of finitely presented groups (see [2] and [11], whose results are stated in section 2).

The purpose of this paper is to develop some of the results stated in [7], in particular those describing the asymptotic cone of the Baumslag–Solitar groups and of *Sol*. According to [2], these spaces are not simply connected, since their Dehn functions are exponential, so our primary goal is to study their fundamental groups. It will be proved that these fundamental groups are uncountable and nonfree (section 9), by constructing subgroups isomorphic to the fundamental group of the Hawaiian earring. These subgroups are constructed by finding subspaces in the asymptotic cones which are homotopically equivalent to the Hawaiian earring, and which induce injections in the fundamental group level (section 8). Crucial to prove these facts is the computation of the covering dimension of these asymptotic cones (section 7), which is done using a more general theorem on dimensions of spaces which admit certain maps into well-known spaces (section 6).

The author would like to express his most sincere gratitude to Professor Stephen M. Gersten for his guidance and encouragement in the development of this project.

1 Asymptotic Cones

The definition of asymptotic cone relies greatly on the concept of ultrafilter. In [12], Van den Dries and Wilkie extended Gromov's definition to general finitely generated groups, and it is here where ultrafilters were introduced to insure the existence of the limits involved. The reader should regard the ultrafilter as a technical tool to construct the spaces, and let his own intuition prevail over the technicalities of the definition.

Definition 1 *An* ultrafilter *in the set of natural numbers* $\mathbf{N}$ *is a family* $\mathcal{F} \subset 2^{\mathbf{N}}$ *satisfying the following properties:*

(1) $\emptyset \notin \mathcal{F}$,

(2) if $A \in \mathcal{F}$ *and* $A \subset B$ *then* $B \in \mathcal{F}$,

(3) if $A, B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$, *and*

(4) if $A \cup B = \mathbf{N}$, *with* $A \cap B = \emptyset$, *then either* $A \in \mathcal{F}$ *or* $B \in \mathcal{F}$.

The only examples of ultrafilters that can be described are the *principal* ultrafilters, i.e., the set of subsets of $\mathbf{N}$ containing a fixed element n_0. To prove that nonprincipal ultrafilters exist one must use Zorn's Lemma (see [12]). Nonprincipal ultrafilters contain all cofinite subsets of $\mathbf{N}$. The definition of ultrafilter leads to the definition of ultralimit, which, for nonprincipal ultrafilters, generalizes the usual notion of limit of a sequence.

Definition 2 *We say that the limit of the sequence* (x_n) *with respect to the ultrafilter* $\mathcal{F}$ *is* $x \in \mathbf{R}$, *or that* x *is the* $\mathcal{F}$*-limit of* (x_n), *denoted*

$$\lim_{\mathcal{F}} x_n = x,$$

when, for all $\varepsilon > 0$, *the set*

$$\{n \in \mathbf{N} \,|\, x_n \in (x - \varepsilon, x + \varepsilon)\}$$

is an element of $\mathcal{F}$.

Ultralimits are the tool used in the definition of asymptotic cone to ensure the existence of limits which otherwise could be undefined. With a short compactness argument we can prove that the ultralimit of a bounded sequence always exists.

Let (X, d) be a metric space. We want to construct a space that can be thought of as the space that would be seen by an observer that is infinitely far. To obtain this effect we will take sequences of points in X and define the distance between two sequences (x_n) and (y_n) by the limit of $\frac{1}{n}d(x_n, y_n)$. To avoid the nonexistence problem for this limit we will use (nonprincipal) ultrafilters. Let $\mathcal{F}$ be a nonprincipal ultrafilter. Fix a basepoint $x_0 \in X$, and consider the space LBS (linearly bounded sequences) of sequences $\mathbf{x} = (x_n)$ of points in X, for which there exists a constant $K \in [0, \infty)$ such that

$$d(x_0, x_n) \leq Kn.$$

Define now a pseudo-distance $\mathbf{d}$ in LBS as follows: given two sequences $\mathbf{x} = (x_n)$ and $\mathbf{y} = (y_n)$, satisfying

$$d(x_0, x_n) \leq K_x n \qquad \text{and} \qquad d(x_0, y_n) \leq K_y n,$$

define

$$\mathbf{d}(\mathbf{x}, \mathbf{y}) = \lim_{\mathcal{F}} \frac{d(x_n, y_n)}{n}.$$

Note that the sequence of real numbers $\frac{1}{n} d(x_n, y_n)$ is bounded by $K_x + K_y$, so the ultralimit always exists. Now to make LBS a metric space, we identify two sequences at distance zero: define the equivalence relation

$$\mathbf{x} \equiv \mathbf{y} \Leftrightarrow \mathbf{d}(\mathbf{x}, \mathbf{y}) = 0.$$

The distance between equivalence classes will still be denoted by $\mathbf{d}$.

Definition 3 *The asymptotic cone* $\operatorname{Con}_{\mathcal{F}} X$ *of* X *with respect to the ultrafilter* $\mathcal{F}$ *is defined as the metric space* $(LBS/\equiv, \mathbf{d})$.

Examples:

1. If X is a bounded metric space, i.e., it has finite diameter —for instance when X is compact— then the asymptotic cone is a single point.

2. When $X = \mathbf{Z}^n$ with the word metric associated to the usual generators, then $\operatorname{Con}_{\mathcal{F}} \mathbf{Z}^n$ is $\mathbf{R}^n$ with the L^1 metric.

3. If X is the Cayley graph of a finitely presented nilpotent group, then $\operatorname{Con}_{\mathcal{F}} X$ is $\mathbf{R}^n$ but with a nonriemannian Carnot metric.

4. If X is a δ-hyperbolic geodesic metric space, then the asymptotic cone is an $\mathbf{R}$-tree; see [7] and [2].

For more details and properties of asymptotic cones see [12] and [7].

2 Fundamental Groups

It was first asked in [12] whether it is possible to find a metric space whose asymptotic cone is not simply connected. Several results have advanced in this direction, at least for the Cayley graphs of finitely presented groups.

Theorem 4 (Gromov [7], Drutu [2]) *If the asymptotic cone of a finitely presented group is simply connected for every nonprincipal ultrafilter, then the group has a polynomial Dehn function.* □

A partial converse of this result has been given by Papasoglu:

Theorem 5 (Papasoglu [11]) *If a finitely presented group has a quadratic Dehn function, then the asymptotic cone is simply connected for every nonprincipal ultrafilter.* □

In view of the result by Gromov and Drutu, any finitely presented group whose Dehn function is higher than polynomial will have a nonsimply connected fundamental group. The purpose of this paper is to study some of these asymptotic cones, in particular, those of the Baumslag–Solitar groups and of the 3-manifold *Sol*. We will prove that these asymptotic cones have uncountable, nonfree fundamental groups. In order to prove these results, we will first prove that these asymptotic cones have covering dimension 1 (section 7), and use this fact to prove that they contain as a subgroup an uncountable nonfree group, namely, the fundamental group of the Hawaiian earring (section 8).

3 Maps with Metrically Parallel Fibers

Not every continuous map between metric spaces induces a map in their asymptotic cones. The fact that in the definition of asymptotic cone only linearly bounded sequences are used, induces the following restriction: only maps for which there exist constants A and B such that

$$d(f(x), f(x')) \leq Ad(x, x') + B$$

are guaranteed to induce well-defined maps in the asymptotic cones. In particular, Lipschitz maps and distance-decreasing maps (i.e: Lipschitz with constant 1) induce maps in the corresponding asymptotic cones. Quasi-isometries induce bi-Lipschitz homeomorphisms between asymptotic cones.

Even when the maps between metric spaces induce maps in the asymptotic cones, there is no relation between the fibers of the map and the fibers of the induced map in the asymptotic cones. An example can be seen with the map log between the metric spaces $[1, \infty)$ and $[0, \infty)$, which is bijective and distance decreasing, but since

$$\lim_{n\to\infty} \frac{\log n}{n} = 0,$$

the induced map in the asymptotic cones collapses the whole space in one point. To avoid this situations we introduce the concept of metrically parallel fibers, which will make sure that for the induced map in the asymptotic cones, its fibers are the asymptotic cones of the fibers of the original map.

Definition 6 *Let*

$$f : X \longrightarrow Y$$

be a map between metric spaces. We say that f has metrically parallel *fibers when, given $x \in X$ and $y \in f(X) \subset Y$, there exists $x' \in f^{-1}(y)$ satisfying*

$$d(x, x') = d(f(x), f(x')) = d(f(x), y).$$

Examples:

1. Isometries are maps with metrically parallel fibers.
2. The map

   $$\mathbf{H}^2 \longrightarrow \mathbf{R}$$

   that sends a point $z \in \mathbf{H}^2$ (with the upper half plane model) to $\operatorname{Im} z \in \mathbf{R}$ has metrically parallel fibers, since two horospheres are at constant distance.

Proposition 7 *Let*

$$\mathbf{f} : \operatorname{Con}_{\mathcal{F}} X \longrightarrow \operatorname{Con}_{\mathcal{F}} Y$$

be a map induced by a map f with metrically parallel fibers. Let $\mathbf{y} = (y_n) \in \operatorname{Con}_{\mathcal{F}} Y$ and let $\mathbf{x} = (x_n) \in \operatorname{Con}_{\mathcal{F}} X$ with $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Then there exists a sequence (x'_n), also representing $\mathbf{x}$, with $f(x'_n) = y_n$.

Proof We just need to choose $x'_n \in f^{-1}(y_n)$ according to the definition of metrically parallel fibers, that is, satisfying

$$d(x_n, x'_n) = d(f(x_n), y_n).$$

Since $\mathbf{f}(\mathbf{x}) = \mathbf{y}$, we have

$$\lim_{\mathcal{F}} \frac{d(f(x_n), y_n)}{n} = \lim_{\mathcal{F}} \frac{d(x_n, x'_n)}{n} = 0,$$

so (x_n) represents $\mathbf{x}$. We need to check that (x_n) is linearly bounded:

$$\begin{aligned} d(x_0, x'_n) &\leq d(x_0, x_n) + d(x_n, x'_n) \\ &= d(x_0, x_n) + d(f(x_n), y_n) \\ &\leq d(x_0, x_n) + d(f(x_n), f(x_0)) + d(f(x_0), y_n), \end{aligned}$$

so the constant for (x'_n) can be taken as the sum of the constants for (x_n), $(f(x_n))$ and (y_n). □

Corollary 8 *If f has metrically parallel fibers and induces a map $\mathbf{f}$ in the asymptotic cones, then:*

(1) If $y_0 \in Y$ is the basepoint, and $\mathbf{y_0} = (y_0) \in \mathrm{Con}_{\mathcal{F}} Y$, then

$$\mathbf{f}^{-1}(\mathbf{y_0}) = \mathrm{Con}_{\mathcal{F}} f^{-1}(y_0).$$

(2) The fibers of $\mathbf{f}$ are also metrically parallel.

Proof Assertion (1) is clear since every point $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) = \mathbf{y_0}$ admits a representative (x_n) with $f(x_n) = y_0$. For (2), take $\mathbf{x} = (x_n) \in \mathrm{Con}_{\mathcal{F}} X$ and $\mathbf{y} = (y_n) \in \mathrm{Con}_{\mathcal{F}} Y$. Then there exist x'_n, for all n, with

$$f(x'_n) = y_n \qquad \text{and} \qquad d(x_n, x'_n) = d(f(x_n), y_n).$$

It is easy to see that (x'_n) is linearly bounded, and then it defines a point $\mathbf{x}'$. Finally,

$$\mathbf{d}(\mathbf{x}, \mathbf{x}') = \lim_{\mathcal{F}} \frac{d(x_n, x'_n)}{n} = \lim_{\mathcal{F}} \frac{d(f(x_n), y_n)}{n} = \mathbf{d}(\mathbf{f}(\mathbf{x}), \mathbf{y}).$$

□

4 Ultrametric Spaces

We will use ultrametric spaces as the fibers for our maps in section 6.

Definition 9 *A metric space X is said to be* ultrametric *if it satisfies the* ultrametric inequality: *for every three points $x, y, z \in X$, we have*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

The ultrametric inequality is a stronger version of the triangle inequality, which has enormous consequences on the topology of the space. The proof of the following proposition is completely elementary:

Proposition 10 *Let X be an ultrametric space. Then:*

(1) *Every triangle is isosceles, i.e., for every three points $x, y, z \in X$, two of the three distances $d(x, y)$, $d(y, z)$, and $d(x, z)$ are equal.*

(2) *If $y \in B(x, r)$, then $B(y, r') \subset B(x, r)$ for all $r' \leq r$.*

(3) *If $y \notin B(x, r)$, then $B(y, r') \cap B(x, r) = \emptyset$ for all $r' \leq r$.*

(4) *Every open ball is closed, and every closed ball is open.*

(5) *Every point has a neighborhood basis that consists of sets which are open and closed.* □

The next result will provide us with many examples of ultrametric spaces, in particular the ones we will be interested in:

Proposition 11 *Let (X, d) be a metric space. Construct the new metric $d' = \log(1+d)$. Then, the asymptotic cone of (X, d') is an ultrametric space, for any nonprincipal ultrafilter.*

Proof Let $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$ and $\mathbf{z} = (z_n)$ be three points in $\mathrm{Con}_{\mathcal{F}}(X, d')$. We need to prove that

$$\mathbf{d}'(\mathbf{x}, \mathbf{z}) \leq \max\{\mathbf{d}'(\mathbf{x}, \mathbf{y}), \mathbf{d}'(\mathbf{y}, \mathbf{z})\}.$$

Since there is nothing to prove if all three distances are equal, assume that $\mathbf{d}'(\mathbf{x}, \mathbf{y}) < \mathbf{d}'(\mathbf{y}, \mathbf{z})$. We want to see then that $\mathbf{d}'(\mathbf{x}, \mathbf{z}) = \mathbf{d}'(\mathbf{y}, \mathbf{z})$.

We can also assume that $\mathbf{d}'(\mathbf{y}, \mathbf{z})$ is not zero, because if it is, then the inequality is trivial. So for all n in some set in $\mathcal{F}$, we have that $d(y_n, z_n)$ is not zero. It is enough to prove now that

$$\lim_{\mathcal{F}} \frac{d(x_n, z_n)}{d(y_n, z_n)} = 1.$$

From the triangle inequality we have

$$|d(y_n, z_n) - d(x_n, y_n)| \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n),$$

or

$$1 - \frac{d(x_n, y_n)}{d(y_n, z_n)} \leq \frac{d(x_n, z_n)}{d(y_n, z_n)} \leq 1 + \frac{d(x_n, y_n)}{d(y_n, z_n)}.$$

So we only need to prove that

$$\lim_{\mathcal{F}} \frac{d(x_n, y_n)}{d(y_n, z_n)} = 0.$$

Call $a = \mathbf{d}'(\mathbf{x}, \mathbf{y})$ and $b = \mathbf{d}'(\mathbf{y}, \mathbf{z})$. We assumed that $a < b$, so

$$\lim_{\mathcal{F}} \frac{\log(1 + d(x_n, y_n))}{n} < \lim_{\mathcal{F}} \frac{\log(1 + d(y_n, z_n))}{n}.$$

Fix some $\delta > 0$ such that $\delta < a$, $\delta < b$ and $\delta < \frac{b-a}{2}$. We have that there exists a set $U \in \mathcal{F}$ such that for all $n \in U$, we have

$$a - \delta < \frac{\log(1 + d(x_n, y_n))}{n} < a + \delta,$$

and

$$b - \delta < \frac{\log(1 + d(y_n, z_n))}{n} < b + \delta,$$

and rewriting the inequalities,

$$e^{(a-\delta)n} - 1 < d(x_n, y_n) < e^{(a+\delta)n} - 1,$$

and

$$e^{(b-\delta)n} - 1 < d(y_n, z_n) < e^{(b+\delta)n} - 1.$$

So, now, we have

$$\lim_{\mathcal{F}} \frac{d(x_n, y_n)}{d(y_n, z_n)} \leq \lim_{\mathcal{F}} \frac{e^{(a+\delta)n} - 1}{e^{(b-\delta)n} - 1} = 0,$$

because since $\delta < \frac{b-a}{2}$, we have $a + \delta < b - \delta$. □

Metrics of the form $\log(1 + d)$ are called *log-metrics*.

The main example to which this result applies is the horosphere of a hyperbolic space: the metric induced on a horosphere by the hyperbolic metric is a log-metric. Given two points in $\mathbf{H}^n$, we always have a totally geodesic hyperbolic plane that contains them, so we only need to study $\mathbf{H}^2$. And two horospheres are always isometric, so we can assume that the two points are in the horosphere $\{z \in \mathbf{C} \mid \operatorname{Im} z = 1\}$ in the upper-half plane model for $\mathbf{H}^2$. Let $x + i$ and $x' + i$ be these points. Then the distance in $\mathbf{H}^2$ between them is

$$\log\left(1 + \frac{|x - x'|}{2}\left(|x - x'| + \sqrt{|x - x'|^2 + 4}\right)\right),$$

and, since

$$\lim_{x\to\infty}\frac{\log\left(1+\frac{x}{2}\left(x+\sqrt{x^2+4}\right)\right)}{2\log(1+x)}=1,$$

we see that the asymptotic cone of the horosphere with the metric induced by the hyperbolic metric is isometric to the asymptotic cone of the same horosphere but with the metric

$$d'(x+i,x'+i)=2\log(1+|x-x'|)$$

which is a log-metric.

So inside $\mathrm{Con}\,\mathbf{H}^2$ we have ultrametric subspaces. We will see that this subspaces are the fibers of a map into $\mathbf{R}$, and these fibers will be metrically parallel, so we will be able to apply the main result in section 6.

5 Dimension

The dimension theory that will be used in this paper will be the covering dimension, which, due to a Theorem of Morita, for normal topological spaces (hence for metric spaces) coincides with the large inductive dimension. Recall the definition of covering dimension:

Definition 12 *Let X be a set and let $\mathcal{A}$ be a family of subsets of X. We say that $\mathcal{A}$ has* order n *if $n+1$ is the largest number of sets in $\mathcal{A}$ that have nonempty intersection. If no such number exists, we say that the order of $\mathcal{A}$ is ∞. The order of $\mathcal{A}$ is denoted by $\mathrm{ord}\,\mathcal{A}$.*

Definition 13 *To every normal topological space X one assigns its* covering dimension, *denoted $\dim X$, subject to the following conditions:*

(1) $\dim X \le n$ *if every open cover $\mathcal{U}$ of X has a refinement $\mathcal{V}$ with $\mathrm{ord}\,\mathcal{V}\le n$,*

(2) $\dim X = n$ *if $\dim X\le n$ and $\dim X > n-1$, and*

(3) $\dim X=\infty$ *if $\dim X>n$ for all $n=-1,0,1,2,\ldots$.*

For more details in the properties of covering dimension, see [3] and [10]. In particular, we will use the following characterization of covering dimension based on sequences of open coverings. Recall that if $\mathcal{U}$ is a covering of a metric space X, its mesh is defined as

$$\mathrm{mesh}\,\mathcal{U}=\sup\{\mathrm{diam}\,U\,|\,U\in\mathcal{U}\}.$$

Theorem 14 *A metric space X has covering dimension at most n if and only if there exists a sequence of open coverings $\mathcal{U}_k$, $k=1,2,\ldots$, such that:*

(1) $\mathcal{U}_{k+1}$ *is a refinement of* $\mathcal{U}_k$*, for all* k*,*

(2) $\mathrm{ord}\,\mathcal{U}_k \leq n$ *for all* k*, and*

(3) $\lim_{k\to\infty} \mathrm{mesh}\,\mathcal{U}_k = 0$*.* □

A proof of this result can be found in [10].

The dimension of an ultrametric space (see section 4) is well known, and not too difficult to deduce from the previous characterization:

Proposition 15 *If* X *is an ultrametric space, then* $\dim X = 0$*.* □

Our goal is to find the dimension of some asymptotic cones by finding suitable maps into known spaces with particular fibers. It is not true that for a general map the dimension of the source space is equal to the dimension of the target space plus the dimension of the fibers. A classical result states that this is true if the map is closed, but this is not the situation on our case. We will strengthen the hypotheses to accomodate it to our purposes: we will require that the map has metrically parallel (see section 3), ultrametric fibers. The theorem is stated and proved in the next section.

6 The Dimension Theorem

The purpose of this section is to prove the main dimension theorem of this paper. This theorem will allow us to compute the dimension of some asymptotic cones.

Theorem 16 *Let*

$$f : X \longrightarrow Y$$

be a continuous map between metric spaces whose fibers are ultrametric and metrically parallel. Then $\dim X \leq \dim Y$*.*

Proof If $\dim Y = \infty$ there is nothing to prove, so assume that $\dim Y = n$. By the characterization of covering dimension by sequences of coverings (see section 5) we have a sequence $\mathcal{U}_k$, for $k \in \mathbf{N}$ of coverings of Y satisfying:

1. $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$,
2. $\mathrm{ord}\,\mathcal{U}_k \leq n$, and
3. $\lim_{k\to\infty} \mathrm{mesh}\,\mathcal{U}_k = 0$.

We want to see that X has a sequence of coverings satisfying the same properties (see Figure). Call $r_k = \mathrm{mesh}\,\mathcal{U}_k$ and assume (taking a subsequence of the sequence of coverings if necessary) that $r_{k+1} \leq \frac{1}{9} r_k$. Let U_k^α be the open sets in $\mathcal{U}_k$, with α in some index set I_k. Choose $y_k^\alpha \in U_k^\alpha$, and let $F_k^\alpha = f^{-1}(y_k^\alpha)$.

We know F_k^α is an ultrametric space, so choose a covering of F_k^α whose elements are open balls of radius $5r_k$, pairwise disjoint. Let $B_k^{\alpha\beta}$ be these balls, with β a new index in a set J_k^α. We have that $B_k^{\alpha\beta}$ is an open ball of radius $5r_k$ in an ultrametric space, so every two points in $B_k^{\alpha\beta}$ are at distance less than $5r_k$, and every point in $B_k^{\alpha\beta}$ can be thought of as the center of the ball. Observe that $B_k^{\alpha\beta}$ is an open set in F_k^α.

Let $V_k^{\alpha\beta}$ be the $2r_k$-open neighborhood of $B_k^{\alpha\beta}$ in X, i.e., the set of points of X for which there exists a point in $B_k^{\alpha\beta}$ at distance less than $2r_k$. Clearly $V_k^{\alpha\beta}$ is an open set of X. Take now $W_k^{\alpha\beta} = V_k^{\alpha\beta} \cap f^{-1}(U_k^\alpha)$. Define

$$\mathcal{W}_k = \{W_k^{\alpha\beta}, \alpha \in I_k, \beta \in \bigcup_{\alpha \in I_k} J_k^\alpha\}.$$

We claim that the $\mathcal{W}_k$ are the coverings of X satisfying all the properties.

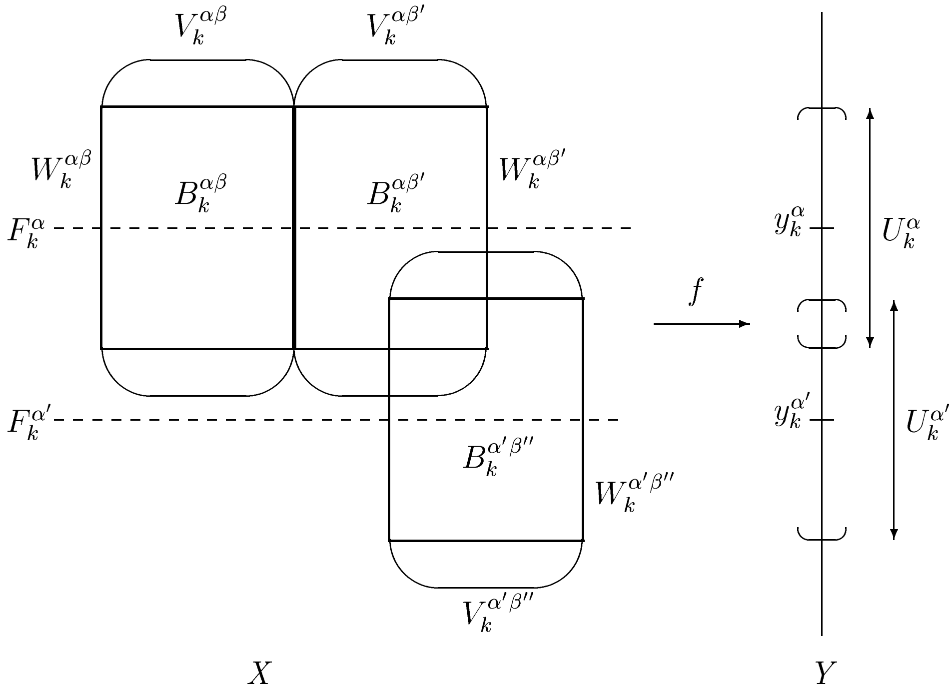


Figure: The cover $\mathcal{W}_k$.

To see that $\mathcal{W}_k$ is an open covering of X, choose $x \in X$. Then $f(x) \in U_k^\alpha$ for some $\alpha \in I_k$, and, since mesh$\,\mathcal{U}_k = r_k$, then $d(f(x), y_k^\alpha) \le r_k$. The fibers of

f are metrically parallel, so there exists $x' \in F_k^\alpha$ with $d(x,x') = d(f(x), y_k^\alpha) \leq r_k < 2r_k$, and there exists $\beta \in J_k^\alpha$ with $x' \in B_k^{\alpha\beta}$. It is clear now that $x \in W_k^{\alpha\beta}$, for these choices of α and β.

We will see now that $\operatorname{mesh} \mathcal{W}_k \leq 9r_k$. Take a $W_k^{\alpha\beta} \in \mathcal{W}_k$, and take $x, y \in W_k^{\alpha\beta}$. By the construction, there exist two points $x', y' \in B_k^{\alpha\beta}$ such that

$$d(x,x') \leq 2r_k \qquad \text{and} \qquad d(y,y') \leq 2r_k.$$

Since $d(x',y') \leq 5r_k$, we obtain $d(x,y) \leq 9r_k$, as desired.

To see that $\operatorname{ord} \mathcal{W}_k \leq \operatorname{ord} \mathcal{U}_k$, we need the following fact: if we have two different sets $W_k^{\alpha\beta}$ and $W_k^{\alpha\beta'}$ based on the same U_k^α, then $W_k^{\alpha\beta} \cap W_k^{\alpha\beta'} = \emptyset$. If $x \in W_k^{\alpha\beta} \cap W_k^{\alpha\beta'}$, then there exist two points $y \in B_k^{\alpha\beta}$ and $y' \in B_k^{\alpha\beta'}$ with

$$d(x,y) \leq 2r_k \qquad \text{and} \qquad d(x,y') \leq 2r_k,$$

so $d(y,y') \leq 4r_k < 5r_k$, and then $B_k^{\alpha\beta} = B_k^{\alpha\beta'}$ and $W_k^{\alpha\beta} = W_k^{\alpha\beta'}$. Now, if we have

$$W_k^{\alpha_1\beta_1} \cap \ldots \cap W_k^{\alpha_m\beta_m} \neq \emptyset,$$

then

$$U_k^{\alpha_1\beta_1} \cap \ldots \cap U_k^{\alpha_m\beta_m} \neq \emptyset,$$

with $\alpha_i \neq \alpha_j$ if $i \neq j$. Then, since $\operatorname{ord} \mathcal{U}_k \leq n$, we have $m \leq n+1$ and $\operatorname{ord} \mathcal{W}_k \leq n$.

It only remains to be seen that $\mathcal{W}_{k+1}$ is a refinement of $\mathcal{W}_k$. So choose $W_{k+1}^{\alpha\beta} \in \mathcal{W}_{k+1}$. Then $W_{k+1}^{\alpha\beta} \subset f^{-1}(U_{k+1}^\alpha)$. Since $\mathcal{U}_{k+1}$ is a refinement of $\mathcal{U}_k$, there exists $U_k^{\alpha'}$ with $U_{k+1}^\alpha \subset U_k^{\alpha'}$, and then $W_{k+1}^{\alpha\beta} \subset f^{-1}(U_k^{\alpha'})$. Imagine that there exist $\beta', \beta'' \in J_k^\alpha$ with

$$W_{k+1}^{\alpha\beta} \cap W_k^{\alpha'\beta'} \neq \emptyset \qquad \text{and} \qquad W_{k+1}^{\alpha\beta} \cap W_k^{\alpha'\beta''} \neq \emptyset.$$

Let

$$x' \in W_{k+1}^{\alpha\beta} \cap W_k^{\alpha'\beta'} \qquad \text{and} \qquad x'' \in W_{k+1}^{\alpha\beta} \cap W_k^{\alpha'\beta''}.$$

We have then

$$d(x',x'') \leq \operatorname{mesh} \mathcal{W}_{k+1} \leq 9r_{k+1} \leq r_k,$$

and there exist $y' \in B_k^{\alpha'\beta'}$ and $y'' \in B_k^{\alpha'\beta''}$ such that

$$d(x',y') < 2r_k \qquad \text{and} \qquad d(x'',y'') < 2r_k.$$

Then,

$$d(y', y'') \leq d(y', x') + d(x', x'') + d(x'', y'') < 2r_k + r_k + 2r_k = 5r_k,$$

and $B_k^{\alpha'\beta'} = B_k^{\alpha'\beta''}$. So $W_{k+1}^{\alpha\beta}$ can only be included in one of the sets of $\mathcal{W}_k$, and $\mathcal{W}_{k+1}$ is then a refinement of $\mathcal{W}_k$. This concludes the proof of $\dim X \leq \dim Y$. □

7 Applications

7.1 Hyperbolic spaces

The map that collapses $\mathbf{H}^n$ into a geodesic along its horospheres has metrically parallel fibers. Since all the fibers in the asymptotic cone are ultrametric, this map satisfies all the conditions of the theorem, and the dimension of the asymptotic cone of $\mathbf{H}^n$ is 1.

7.2 Baumslag–Solitar Groups

More interesting is the case of the asymptotic cone of the Cayley graph of a Baumslag–Solitar group $BS_{p,q}$ given by the presentation

$$< a, b \mid ab^p a^{-1} = b^q >,$$

with $|p| \neq |q|$. For a description of the Cayley graph of the Baumslag–Solitar groups, see [4]. This Cayley graph admits a map into a (directed) tree T of constant valence $p + q$, and the map has metrically parallel fibers. Also, these fibers are horospheres in a hyperbolic plane, because the preimage of a geodesic in the tree is a hyperbolic plane. So again this map induces a map in the asymptotic cones which satisfies the conditions of the theorem, so $\dim \operatorname{Con} BS_{p,q} \leq \dim \operatorname{Con} T$. To see that $\operatorname{Con} T$ has dimension 1, we only need to see that T embeds isometrically into $\mathbf{H}^2$, so its asymptotic cone embeds into $\operatorname{Con} \mathbf{H}^2$, which has dimension 1.

7.3 *Sol*

The 3-manifold *Sol* can be thought of as the manifold obtained when we equip $\mathbf{R}^3$ with the Riemannian metric

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$

It is well-known that *Sol* admits two perpendicular foliations by hyperbolic planes. In the coordinates above, one foliation is given by the xz-planes and the other one by the yz-planes. If we map *Sol* into a hyperbolic plane by projecting all the planes of the xz foliation into one of them along the y direction, then the fibers are the horospheres of the hyperbolic planes of the other foliation, and the map obtained induces a map in the asymptotic cones that satisfies all the conditions of the theorem in the previous section. Thus, $\dim \operatorname{Con} Sol \leq \dim \operatorname{Con} \mathbf{H}^2 = 1$.

8 The Hawaiian Earring as a Subspace

The Hawaiian earring has been an interesting object of study due to its surprising properties. The Hawaiian earring is a union of countably many circles whose radii tend to zero, and which are all tangent at a common point. The fact that the radii tend to zero makes the topology of the Hawaiian earring different from the one in an infinite wedge of spheres, since an open set containing the common point must contain all but finitely many of the circles. This fact also makes the fundamental group of the Hawaiian earring much more complicated than just the free product of infinitely many copies of $\mathbf{Z}$, since one can construct many more loops which are continuous in the Hawaiian earring, in particular, loops that induce nontrivial loops in infinitely many circles. For details in the fundamental group of the Hawaiian earring, see [9] and [1]. The only result we will use is the following:

Proposition 17 *The fundamental group of the Hawaiian earring is uncountable and nonfree.* □

The proof of this result can be found in [1].

Our goal is to prove that the fundamental groups of the asymptotic cones of the Baumslag–Solitar groups and of *Sol* are also uncountable and nonfree, by finding suitable subspaces with the same topological properties that those of the Hawaiian earring, and proving that these subspaces induce injections on the fundamental group level. In order to prove these results, we will first prove that the Hawaiian earring is a classifying space for spaces with covering dimension one, i.e., that if the Hawaiian earring is found as a subspace of a larger space, then the fundamental group of this space will contain the fundamental group of the Hawaiian earring as a subgroup.

The motivation for this theorem is the theorems that characterize dimension by extensions of maps to spheres. An example is the following theorem, which is Theorem 3.2.10 in [3]:

Theorem 18 *A normal topological space satisfies the inequality* $\dim X \leq n \geq 0$ *if and only if for every closed subspace* A *of* X *and every continuous map*

$$f : A \longrightarrow S^n$$

there exists a continuous extension

$$F : X \longrightarrow S^n$$

to the whole space X. □

Our goal is to prove a similar theorem extending a map from a closed subspace into the Hawaiian earring, to the space X, when X is one-dimensional. Let E be the Hawaiian earring, with basepoint O, and let C_n be the circles in the Hawaiian earring, all tangent to each other at O, with C_n having radius $1/n$.

Theorem 19 *Let X be a normal topological space with $\dim X \leq 1$. Then for every closed subspace A and every continuous map*

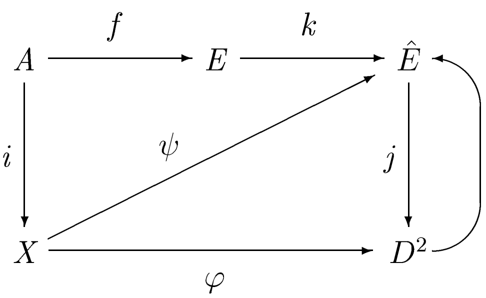
$$f : A \longrightarrow E$$

there exists a continuous extension

$$F : X \longrightarrow E.$$

Proof Let $\hat{E}$ be the *filled Hawaiian earring*, that is, a one-point union of infinitely many disks whose radii tend to zero. The topology of $\hat{E}$ is the topology induced on $\hat{E}$ as a subspace of D^2 (see Figure 1).

Let $\hat{C}_n$ be the n-th disk in $\hat{E}$, whose boundary is C_n, the n-th circle of the Hawaiian earring. We have then that $\hat{E}$ is a deformation retract of the disk D^2 (see also Figure 1) Let j be the inclusion of $\hat{E}$ in D^2, and let r be the retraction. Let also i be the inclusion from A into X and k the inclusion of the Hawaiian earring E into $\hat{E}$:



The map $j \circ k \circ f$ is a map from A to D^2, which by Tietze's extension theorem can be extended to a map φ from X to D^2. Compose φ with r to obtain a map ψ from X to $\hat{E}$. Since $\varphi \circ i = j \circ k \circ f$, we have that $\psi \circ i = r \circ \varphi \circ i = r \circ j \circ k \circ f = k \circ f$. So we have a map ψ from X to $\hat{E}$ which extends the composition of f with k to X. We will modify the map ψ to a map from X to E which will satisfy our requirements.

Let $Y_n = \psi^{-1}(\hat{C}_n)$, and let $B_n = \psi^{-1}(C_n)$. Then Y_n is a metric space of dimension at most one, being a subspace of X, and B_n is a closed set in Y_n. The map ψ sends B_n to C_n. Then we can use the theorem on characterizations of dimension by maps to spheres to find a map F_n from Y_n to C_n, such that restricted to B_n gives ψ. Let F be the map defined as the union of all the F_n, i.e., if $x \in X$ is in Y_n, then $F(x) = F_n(x)$. The map F goes from X to the Hawaiian earring E, and it is well defined because $Y_n \cap Y_m = \psi^{-1}(O)$. If we restrict F to the union of all the B_n, we obtain ψ, and clearly A is included in the union of all the B_n. So the map F is an extension of our original map f. The proof that F is continuous is straightforward and it is left to the reader.□

Corollary 20 *Let*

$$i : A \longrightarrow X$$

be the inclusion of a subspace A in a metric space X of covering dimension 1. If A is homotopically equivalent to the Hawaiian earring E, then the fundamental group of X admits $\pi_1(E)$ as a subgroup. In particular, $\pi_1(X)$ is uncountable and nonfree.

Proof Let

$$f : A \longrightarrow E$$

be the homotopy equivalence. By the previous theorem, the map f can be extended to

$$F : X \longrightarrow E.$$

Then, since $F \circ i = f$, we have that the induced maps in the fundamental groups also satisfy $F_* \circ i_* = f_*$. Since f is a homotopy equivalence, then f_* is an isomorphism, and then the map i_* is injective. □

9 Construction of Subspaces

We will construct subspaces of certain asymptotic cones which are homtopically equivalent to the Hawaiian earring. Since the asymptotic cones will be 1-dimensional, this will imply that the fundamental group of these asymptotic cones will contain the fundamental group of the Hawaiian earring as a subgroup, and hence they will be uncountable and nonfree.

9.1 Baumslag–Solitar Groups

For simplicity we will construct the subspace in the asymptotic cone of $BS_{1,2}$. The cases for the other Baumslag–Solitar groups are analogous.

To construct this subspace consider two hyperbolic planes included in the 2-complex associated to the presentation of $BS_{1,2}$. Let Y be the union of these two hyperbolic planes. The intersection of these hyperbolic planes is the complement of a horoball. Let H be the common horosphere which bounds the two horoballs in both hyperbolic planes (see figure 3). Take the basepoint x_0 in H. Take pairs of points $x_n, y_n \in H$ such that $d(x_0, x_n) = d(x_0, y_n) = n$. Join x_n and y_n by geodesics γ_n and γ'_n, each one in a different sheet. The sequences of points (x_n) and (y_n) define points $\mathbf{x}_1$ and $\mathbf{y}_1$ in $\mathrm{Con}_{\mathcal{F}}\, Y$ which are in $\mathrm{Con}_{\mathcal{F}}\, H$ and such that $\mathbf{d}(\mathbf{x}_0, \mathbf{x}_1) = \mathbf{d}(\mathbf{x}_0, \mathbf{y}_1) = 1$. The geodesics γ_n and γ'_n define two geodesics joining $\mathbf{x}$ and $\mathbf{y}$, each one each one contained in the asymptotic cone of a different sheet. We obtain in this way a loop A_1 in $\mathrm{Con}_{\mathcal{F}}\, Y$.

Construct a similar loop in $\mathrm{Con}_{\mathcal{F}}\, Y$ but now choose the points x_n and y_n with $d(x_0, x_n) = d(x_0, x_n) = n/2$, to obtain a loop A_2 passing through two points $\mathbf{x}_2$, $\mathbf{y}_2$ that satisfy $\mathbf{d}(\mathbf{x}_0, \mathbf{x}_2) = \mathbf{d}(\mathbf{x}_0, \mathbf{y}_2) = 1/2$. Repeating this process we can obtain loops A_k, for all k, such that A_k passes through two points $\mathbf{x_k}$ and $\mathbf{y_k}$ with $\mathbf{d}(\mathbf{x}_0, \mathbf{x}_k) = \mathbf{d}(\mathbf{x}_0, \mathbf{y}_k) = 1/k$. These loops accumulate to the basepoint $\mathbf{x_0}$; thus their union is close to the Hawaiian earring. To make it homotopically equivalent to the Hawaiian earring, we only have to take a segment joining all the loops in one of the sheets, which is induced by segments in the vertical geodesic in $\mathbf{H}^2$ starting at the basepoint. Let A be the space consisting of the union of all the loops A_k and of this segment (see Figures 2 and 3). It is clear that the space A is homotopically equivalent to the Hawaiian earring. Thus, by the last corollary of the previous section, we can conclude the following result:

Corollary 21 *The fundamental group of the asymptotic cone of $BS_{p,q}$ is uncountable and nonfree.* □

It is important to remark that the geodesics γ_n and γ'_n in Y form a loop enclosing an area that grows exponentially in n. These loops are the geometric version of the loops constructed by Gersten in [5] to prove that the Baumslag–Solitar groups have an exponential Dehn function. This fact confirms the idea already stated in Drutu's result (see section 2) that loops that are homotopically trivial in the asymptotic cone correspond to loops that can be filled polynomially in the original space, since here loops that cannot be filled polinomially give loops in the asymptotic cone that are not trivial in the fundamental group. It is somewhat expected that loops with length linear in n that require fillings which are exponential in n will not be able to be filled in the asymptotic cone, since the restriction to linearly bounded sequences makes the exponential disks nonconvergent in the asymptotic cone.

9.2 *Sol*

We want to construct a subspace of $\mathrm{Con}_{\mathcal{F}}\, Sol$ which is homeomorphic to A. So we will need to construct the circles A_k in $\mathrm{Con}_{\mathcal{F}}\, Sol$, and each one will be a limit of circles in *Sol*.

Let $a \in \mathbf{R}$. Consider the four planes $x = a$, $x = -a$, $y = a$ and $y = -a$ in *Sol*. We want to join the four points $(a, a, 0)$, $(a, -a, 0)$, $(-a, a, 0)$ and $(-a, -a, 0)$ with geodesics contained in the four planes above (see Figure 4).

So for every $a \in \mathbf{R}$ we have a loop α_a that is the concatenation of these four geodesics. To obtain the loop A_k in $\mathrm{Con}_{\mathcal{F}}\, Sol$, we take the sequence of loops

$(\alpha_{e^{n/k}})_{n\in\mathbf{N}}$. This sequence of loops defines A_k in the asymptotic cone of *Sol*. We only need to join again all A_k with a segment to obtain a subspace which is homeomorphic to A. We obtain again the desired result:

Corollary 22 *The fundamental group of the asymptotic cone of Sol is uncountable and not free.* □

Note that the loops α_{e^n} have length linear in n because the horospheres are exponentially distorted, and the area required to fill them is exponential. So we find another example of a space with exponential Dehn function and nonsimply connected asymptotic cone, and the loops that require exponential filling induce nontrivial loops in the asymptotic cone.

References

[1] B. De Smit, The fundamental group of the Hawaiian earring is not free, Internat. J. Algebra Comput. 2 (1992) p. 33–37.

[2] C. Drutu, Cône asymptotique d'après Gromov. Relations avec des invariants de quasi-isométrie, thesis, Université de Paris-Sud, 1993.

[3] R. Engelking, *Dimension Theory*, North–Holland, Amsterdam, 1978.

[4] D.B.A. Epstein, J.W. Cannon, D.F. Holt, S.V.F. Levy, M.S. Paterson, W.P. Thurston, *Word Processing in Groups*, Jones and Bartlett, Boston, 1992.

[5] S.M. Gersten, Dehn functions and l^1-norms of finite presentations, in Algorithms and Classification in Combinatorial Group Theory (G. Baumslag, C.F. Miller III, eds.), Springer–Verlag, New York, 1992.

[6] M. Gromov, Groups of polynomial growth and expanding maps, Publ. Math. IHES 53 (1981) pp. 53–78.

[7] M. Gromov, Asymptotic invariants of infinite groups, in Geometric Group Theory II (G. Niblo, M. Roller, eds.), London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1993.

[8] M. Kapovich, B. Leeb, On asymptotic cones and quasiisometry classes of fundamental groups of 3-manifolds, Geom. Funct. Anal. 5(1995) pp. 582–603.

[9] J.W. Morgan, I. Morrison, A van Kampen theorem for weak joins, Proc. London Math. Soc (3), 53 (1986) pp. 562–576.

[10] J.I. Nagata, *Modern Dimension Theory*, Sigma Series in Pure Mathematics, vol. 2, Heldemann Verlag, Berlin, 1993.

[11] P. Papasoglu, On the asymptotic cone of groups satisfying a quadratic isoperimetric inequality, preprint, Paris, 1994.

[12] L. Van den Dries, A.J. Wilkie, On Gromov's theorem concerning groups of polynomial growth and elementary logic, J. Algebra 89 (1984) pp. 349–374.

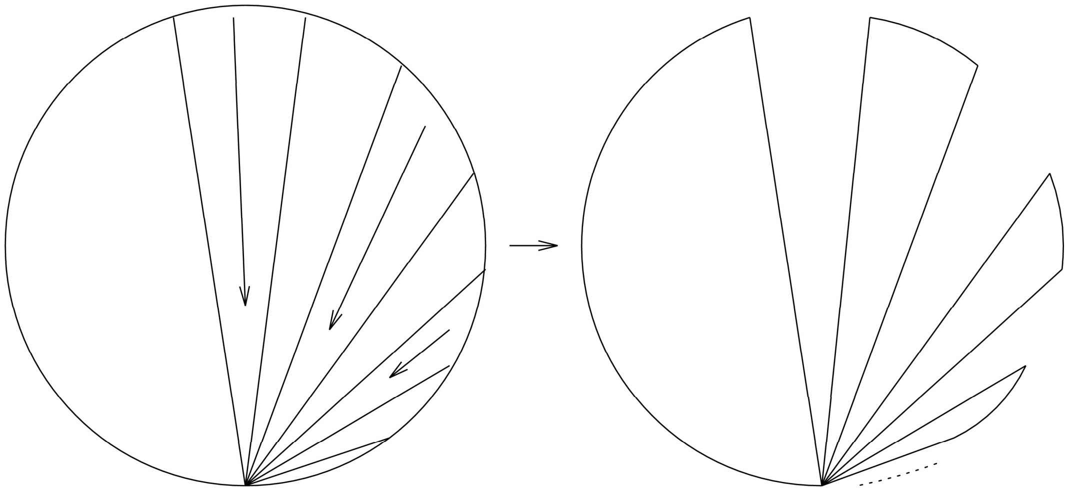


Figure 1: The retraction of D^2 into $\hat{E}$.

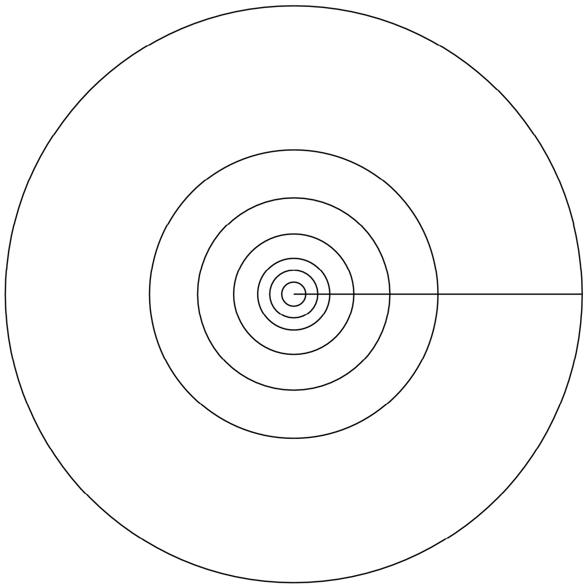


Figure 2: The subspace A.

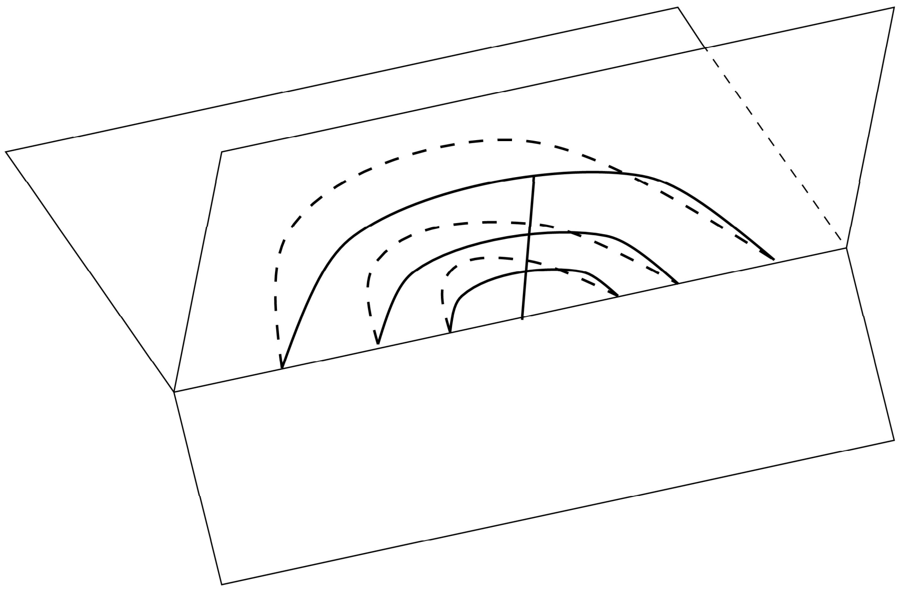


Figure 3: The subspace A inside $\mathrm{Con}_{\mathcal{F}}\, Y$.

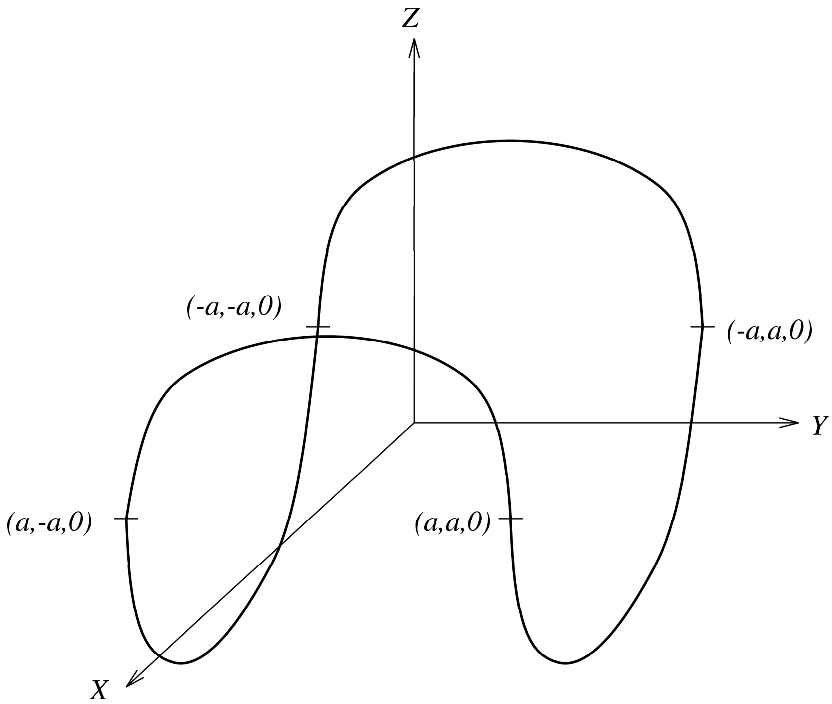


Figure 4: The loop α_a.