COMMENSURATIONS AND SUBGROUPS OF FINITE INDEX OF THOMPSON'S GROUP F

JOSÉ BURILLO, SEAN CLEARY AND CLAAS E. RÖVER

Abstract. We determine the abstract commensurator $\mathrm{Com}(F)$ of Thompson's group F and describe it in terms of piecewise linear homeomorphisms of the real line and in terms of tree pair diagrams. We show $\mathrm{Com}(F)$ is not finitely generated and determine which subgroups of finite index in F are isomorphic to F. We show that the natural map from the commensurator group to the quasi-isometry group of F is injective.

Introduction

Thompson's groups have been extensively studied since their introduction by Thompson in the 1960s, despite the fact that Thompson's account [10] appeared only in 1980. They have provided examples of infinite finitely presented simple groups, as well as some other interesting counterexamples in group theory (see for example, Brown and Geoghegan [3]). Cannon, Floyd and Parry [5] give an excellent introduction to Thompson's groups where many of the basic results used below are proven carefully.

Automorphisms for Thompson's group F were studied by Brin [2], where a key theorem by McCleary and Rubin [8] is used to realize each automorphisms as conjugation by a piecewise linear map. Here, we generalize from automorphisms to commensurations, which are isomorphisms between two subgroups of finite index. These form a group (under a natural equivalence relation involving passing to smaller yet still finite-index subgroups), called the commensurator group.

We classify finite-index subgroups of F, and then we extend Brin's results from automorphisms to commensurations, again realizing every commensuration as conjugation by a piecewise linear homeomorphism of the real line. These maps exhibit a particular structure, satisfying an affinity condition in the neighborhood of ∞ which we use to find the algebraic structure of the commensurator of F.

Commensurators have proven to be an effective tool for investigating quasi-isometries of a group to itself, and for effectively analyzing rigidity, particularly of lattices. In the case of F, the only quasi-isometries of F known previously were automorphisms. This paper provides a wide array of examples of quasi-isometries, since all commensurations are quasi-isometries, and we prove in Section 5 that the commensurator group embeds into the quasi-isometry group in the case of F.

Our approach is algebraic, but we note that elements of the commensurator of F can be represented by marked, infinite, eventually periodic, binary tree pair diagrams. We also

The authors are grateful for the hospitality of the Centre de Recerca Matemàtica. The first author acknowledges support from MEC grant #MTM2006-13544-C02. The second author acknowledges support from the National Science Foundation and from PSC-CUNY Research Award #69034.

note that recently Bleak and Wassink [1] have independently described the finite-index subgroups of F, using different methods.

The paper is organized as follows. In Section 1 we give the necessary definitions, and in Section 2 the first basic results for the finite-index subgroups of F. In Section 3 the main result about the commensurator is stated and proved, and in Section 4 its algebraic structure is given. The proof of the embedding of the commensurator group into the quasi-isometry group is given in Section 5.

Acknowledgements. The authors thank Matt Brin and Dan Margalit for useful conversations and are grateful for the hospitality of the Centre de Recerca Matemàtica. The first author acknowledges support from MEC grant #MTM2006-13544-C02. The second author acknowledges support from the National Science Foundation and from PSC-CUNY Research Award #69034.

1. Definitions

Let P denote the group of all homeomorphisms f from $\mathbb{R}$ to itself that

(1) are piecewise linear with a discrete (but possibly infinite) set of breakpoints (discontinuities of the derivative of f),
(2) use only slopes that are integral powers of 2,
(3) have their breakpoints in the set $\mathbb{Z}[\frac{1}{2}]$and
(4) satisfy $f(\mathbb{Z}[\frac{1}{2}]) \subset \mathbb{Z}[\frac{1}{2}]$.

It is easy to check that each element f of P actually satisfies $f(\mathbb{Z}[\frac{1}{2}]) = \mathbb{Z}[\frac{1}{2}]$ and that P has a subgroup of index two which contains only the order preserving elements. We denote this subgroup by P_+. The quotient P/P_+ is generated by the image of the homeomorphism $\tau : t \mapsto -t$.

Let $f \in P$. We call f *integrally affine* if $f(t) = \varepsilon t + p$ for some integer p and $\varepsilon \in \{\pm 1\}$. We say f is *periodically affine* if $f(t + p) = f(t) + q$ for some non-zero $p, q \in \mathbb{R}$ and *integrally periodically affine* if p and q are integers. Note that all integrally affine maps are integrally periodically affine with $q = \pm p$ depending on whether f is in P_+ or not.

When $\mathcal{P}$ is any of the above properties, then we call f *eventually* $\mathcal{P}$ if f satisfies $\mathcal{P}$ for all $t \in \mathbb{R}$ with $|t| > M$ for some $M > 0$; here $|t|$ denotes the absolute value of t. For example, $f \in P_+$ is eventually integrally affine if there exist $l, r \in \mathbb{Z}$, $M \in \mathbb{R}$, $M > 0$, so that $f(t) = t + r$ for all $t > M$ and $f(t) = t + l$ for all $t < -M$. Notice that l and r may well be different.

It is well-known that Thompson's group F is isomorphic to the subgroup of P_+ consisting of all eventually integrally affine elements (see [5]). It is easy to see that the commutator subgroup F' of F consists of all eventually trivial elements of P_+(those where eventually $f(t) = t$). This group is denoted by $BPL_2(\mathbb{R})$ by Brin [2], where B stands for bounded support.

2. Finite-index Subgroups of F

Let f be an element of F. Since f is eventually integrally affine, there are two integers l, r and a real number $M > 0$ such that $f(t) = t + r$ for $t > M$ and $f(t) = t + l$ for $t < -M$. The two numbers l and r are precisely the two components of the image of f in $\mathbb{Z} \times \mathbb{Z}$ under the abelianization map. The subgroups of finite index of F are in one-to-one correspondence with those of its abelianization $\mathbb{Z} \times \mathbb{Z}$ by the following result.

***Proposition* 2.1.** *Let H be a subgroup of F of finite index. Then H contains F', the commutator subgroup of F, and hence H is normal in F. Moreover, $H' = F'$.*

Proof. Since F is finitely generated, H has only finitely many conjugates in F and the intersection of all of them, K say, is normal and of finite index in F. We consider $K \cap F'$, which is thus normal and of finite index in F'. Hence, since F' is simple and infinite, we conclude that $K \cap F' = F'$ and $F' \subset K \subset H$.

Hence H is normal in F. The final claim follows from the fact that H' is contained in F' but also characteristic in H and hence normal in F, whence $F' \subset H'$. □

From this fact we deduce that the finite-index subgroups of F are in bijection with those of $\mathbb{Z} \times \mathbb{Z}$. There is a distinguished family among these—the subgroups $p\mathbb{Z} \times q\mathbb{Z}$. We denote by $[p, q]$, $p, q \in \mathbb{Z}$, the preimage in F under the abelianization homomorphism of the subgroup $p\mathbb{Z} \times q\mathbb{Z}$ of $\mathbb{Z} \times \mathbb{Z}$. Thus $F = [1, 1]$ and $F' = [0, 0]$.

3. The Commensurator Group

As mentioned before, a *commensuration* of a group G is an isomorphism $\alpha : \mathrm{Com}(A) \to B$, where A and B are subgroups of G of finite index. Two commensurations α and β are equivalent if they agree on some subgroup of finite index in G. In view of this, the product $\beta \circ \alpha$ of two commensurations

$$\alpha : \mathrm{Com}(A) \to B \quad \text{and} \quad \beta : \mathrm{Com}(C) \to D$$

is defined on $\alpha^{-1}(B \cap C)$. The set of all commensurations of G modulo the above equivalence relation, together with this composition, forms a group called the *commensurator of* G which we denote by $\mathrm{Com}(G)$. If G is a subgroup of the group H, then the (relative) commensurator of G in H, $\mathrm{Com}_H(G)$, consists of all elements h of H for which $G \cap G^h$ has finite index in both G and G^h; here $G^h = h^{-1}Gh$.

The main result of this paper is the following.

***Theorem* 3.1.** *The commensurator of F is isomorphic to $\mathrm{Com}_P(F)$, which consists of all eventually integrally periodically affine elements (of P).*

The strategy of the proof is to find a large group where F is a subgroup, and in such a way that every commensuration can be seen as a conjugation by an element of the large group. The group P plays this role in the case of F.

In order to explain this strategy, we need some definitions and one of the main results of McCleary and Rubin [8]. Let $(L, <)$ be a dense linear order. By *interval* we mean a nonempty open interval. A subgroup G of $\mathrm{Aut}(L)$ is *locally moving* if for every interval I there exists a nontrivial element $g \in G$ which acts as the identity on $L \setminus I$. Finally,

G is *n-interval-transitive* if for every pair of sequences of intervals $I_1 < \cdots < I_n$ and $J_1 < \cdots < J_n$ there exists $g \in G$ such that $I_k^g \cap J_k \neq \varnothing$ for $1 \leq k \leq n$. Below, $\overline{L}$ denotes the Dedekind completion of L which is assumed to have no endpoints.

***Theorem* 3.2.** (McCleary–Rubin [8]) *Assume $(L_i, <)$ is a dense linear order without endpoints and let $G_i \subset \mathrm{Aut}(L_i)$ be locally moving and 2-interval transitive, $i = 1, 2$. Suppose that $\alpha\mathrm{Com}(G)_1 \to G_2$ is an isomorphism. Then there is a monotonic bijection $\tau\mathrm{Com}(\overline{)}L_1 \to \overline{L}_2$ which induces α, that is, $g^\alpha = \tau^{-1}g\tau$ for every $g \in G_1$; and τ is unique.*

Being locally moving and having 2-interval transitivity are local properties in the sense that a group inherits these from any of its subgroups.

Proof of Theorem 3.1. View $\mathbb{Z}[\frac{1}{2}]$ as a dense linear order and F as the eventually integrally affine elements of P_+. Let $\alpha : \mathrm{Com}(A) \to B$ be a commensuration of F. By Proposition 2.1, both A and B contain F' which is (obviously) locally moving and 2-interval transitive (see [2, Lemma 2.1]). So Theorem 3.2 tells us that α is induced by conjugation with a unique element of $\mathrm{Homeo}(\mathbb{R})$. This yields an injective homomorphism $\Psi\mathrm{Com}(\mathrm{Com}())F \to \mathrm{Homeo}(\mathbb{R})$.

Next, we show that the image of Ψ is in fact contained in P. By Proposition 2.1, each commensuration of F induces an automorphism of F'. In other words, the image of Ψ is contained in $N_{\mathrm{Homeo}(\mathbb{R})}(F')$, the normalizer of F' in $\mathrm{Homeo}(\mathbb{R})$. But this normalizer is equal to P by Theorem 1 of Brin [2]. The existence and uniqueness statements in Theorem 3.2 now imply that Ψ is an isomorphism between $\mathrm{Com}(F)$ and $\mathrm{Com}_P(F)$, which proves the first part of Theorem 3.1.

Let $\alpha \in \mathrm{Com}(F)$ and choose positive integers p and q so large that α is defined on the subgroup $[p, q]$, that is $[p, q]^\alpha$, the image of $[p, q]$ under α, is contained in F. By what was said above, we can view α as conjugation by an element of P. So for $f \in [p, q]$ we find $f^\alpha = \alpha^{-1}f\alpha$ to be eventually integrally affine. Suppose for a moment that α is order preserving and that $f(t) = t + kq$ for $t \gg 0$, where $k \in \mathbb{Z}$. Then

$$f^\alpha(t) = (\alpha \circ f \circ \alpha^{-1})(t) = \alpha(f(\alpha^{-1}(t))) = \alpha(\alpha^{-1}(t) + kq) = t + r$$

must hold for some $r \in \mathbb{Z}$. In other words, $\alpha^{-1}(t + r) = \alpha^{-1}(t) + s$ for some integers r and s and all $t \gg 0$. Since f was arbitrary, we may assume that $k \neq 0$, which implies that $s \neq 0$, and hence also $r \neq 0$. Therefore α^{-1}, and hence α, must be integrally periodically affine near infinity. A similar calculation holds for $t \ll 0$ and also when α is order reversing. Consequently, each commensuration of F must be eventually integrally periodically affine.

It remains to show that each eventually integrally periodically affine $\beta \in P$ induces a commensuration of F by conjugation. Suppose $\beta(t + p) = \beta(t) + q$ for $t \gg 0$ and $\beta(t + p') = \beta(t) + q'$ for $t \ll 0$, with $p, q, p', q' \in \mathbb{Z} \setminus \{0\}$. Let $U = [p', p]$ if β is order preserving and set $U = [p, p']$ otherwise. Then for $f \in U$, we have

$$f^\beta(t) = \begin{cases} \beta(\beta^{-1}(t) + kp) = t + kq, & t \gg 0 \\ \beta(\beta^{-1}(t) + k'p') = t + k'q', & t \ll 0 \end{cases}$$

where $k, k' \in \mathbb{Z}$ depend on f. Together with a similar argument for β^{-1} one easily sees that $U^\beta = [q', q]$ or $[q, q']$, depending on whether β is order preserving or not. Theorem 3.1 is thus established. □

We immediately obtain the following corollaries from this result.

***Corollary* 3.3.** *A subgroup U of F of finite index is isomorphic to F if and only if $U = [p, q]$ for some positive integers p and q.*

Proof. Suppose U is a subgroup of finite index in F. If U is isomorphic to F, then there exists an eventually integrally periodically affine $\alpha \in P$ with $F^\alpha = U$ and calculations as above show that U must be of the form $[p, q]$. On the other hand, the final paragraph of the proof of the theorem read with $p = p' = 1$ shows that $[q', q]$ is isomorphic to F for every choice of positive integers q and q'. This completes the proof. □

Finally, since each subgroup of finite index in F contains $[p, q]$ for some positive integers p and q by Proposition 2.1, we have the following results.

***Corollary* 3.4.** *Every finite-index subgroup of F is virtually F.*

***Corollary* 3.5.** *A group is commensurable with F if and only if it is a finite extension of F.*

4. The Structure of Com(F)

Descriptions of elements of Com(F) as conjugations in P allow us to study its structure as a group. An element α of Com(F) is eventually integrally periodically affine, so there exist positive integers p, p', q, q' and a real number M such that

$$\alpha(t + p) = \alpha(t) + q, \text{ for } t > M$$
$$\alpha(t + p') = \alpha(t) + q', \text{ for } t < -M.$$

We need a lemma about affine functions, whose proof is elementary and left to the reader.

***Lemma* 4.1.** *Let $f\mathrm{Com}(\mathbb{R}) \to \mathbb{R}$ be an integrally periodically affine map, and assume that there are integers i, i', j, j' such that for all $t \in \mathbb{R}$ we have*

$$f(t + i) = f(t) + j \quad \text{and} \quad f(t + i') = f(t) + j'.$$

Then we have

$$f(t + r) = f(t) + s,$$

where
$$r = \gcd(i, i') \quad \text{and} \quad s = \gcd(j, j').$$

Furthermore, we have

$$\frac{i}{j} = \frac{i'}{j'}.$$

From this lemma, we see that the integers p, p', q, q' for element of Com(F) depend only on the element.

We recall that Com(F) has a subgroup of index 2, denoted Com$^+$(F), formed by the commensurations induced by conjugations by piecewise-linear maps which preserve the orientation of $\mathbb{R}$.

***Proposition* 4.2.** *There exists a surjective homomorphism* $\Phi \mathrm{Com}(\mathrm{Com}^+())F \to \mathbb{Q}^* \times \mathbb{Q}^*$ *defined by*

$$\Phi(f) = \left(\frac{p}{q}, \frac{p'}{q'}\right).$$

Here $\mathbb{Q}^*$ denotes the multiplicative group of the positive rational numbers.

The map is obviously well-defined due to the lemma above, and it is very easy to see that it is a homomorphism of groups. The two components of the map capture the behavior at both ends, eventually near $-\infty$ and eventually near $+\infty$. The two numbers p/q and p'/q' measure the "rate of growth" of the map at both ends.

A corollary of this result is that, as expected, $\mathrm{Com}(F)$ is infinitely generated.

5. Commensurations as Quasi-isometries

Let G be a finitely generated group. Quasi-isometries of G can be naturally composed, and there is a natural notion of equivalence class of quasi-isometries. Two quasi-isometries are considered equivalent if they are a bounded distance apart in the sense that f and g are considered equivalent if there exists a number $M > 0$ such that $d(f(t), g(t)) \leq M$ for all t in G.

Equivalence classes of quasi-isometries form elements of the group of quasi-isometries $QI(G)$ of G. It is well known that the commensurator group admits a map to the quasi-isometry group, since all commensurations give maps between finite index subgroups which are canonically quasi-isometric to the ambient group. The result we want to prove in this section is that for Thompson's group F, this map is one-to-one.

***Theorem* 5.1.** *The natural homomorphism* $\mathrm{Com}(F) \to QI(F)$ *is injective.*

We begin with an elementary lemma.

***Lemma* 5.2.** *Given an element* $\tau \in P$ *which is different from the identity, there exist two intervals* I *and* J *of the real line, whose endpoints are dyadic integers, with* $\tau(I) = J$*, and such that* $I \cap J = \varnothing$*.*

Proof. The case when the slope of τ is always 1 or -1 is trivial. For a map $t \mapsto t + k$ has a small interval (of length less than k) whose image is disjoint from it. If $\tau = -Id$ the result is trivial.

If the slope is not constantly equal to 1, it has a piece with slope $\pm 2^i$ with $i \neq 0$. Assume without loss of generality (by possibly taking τ^{-1} instead of τ) that $i > 0$. Hence there are two intervals $[a, b]$ and $[c, d]$ such that $\tau(a) = c$ and $\tau(b) = d$ and also $d - c = 2^i(b - a)$. It is possible that $[a, b]$ and $[c, d]$ overlap, but since $[c, d]$ is much larger than $[a, b]$ (at least twice the size), we can choose as J a small interval inside $[c, d]$ which is disjoint from $[a, b]$. By construction, the preimage I of J is in $[a, b]$, and hence I and J are disjoint. ☐

Proof of Theorem 5.1. We now take a nontrivial $\tau \in \mathrm{Com}(F)$. By the previous lemma, there exist intervals I and J satisfying the conditions stated above and, in addition, that I, and hence J, have endpoints of the form $k/2^j$ and $(k+1)/2^j$. We consider all elements of F whose support (that is, the part where they are not the identity) is contained in

I. Those elements form a subgroup which is isomorphic to F itself. Let f be one such element. Since its support is inside I, its image under the commensuration τ, that is, $f^\tau = \tau \circ f \circ \tau^{-1}$, has support inside J.

Hence, the distance (inside F) from f to f^τ is given by the distance from the identity to the element $f^\tau f^{-1}$. But this element has its support inside the disjoint union $I \cup J$, and the two parts are independent from each other (one given by f and the other one by f^τ). By work of Cleary and Taback [6], this subgroup—elements with support in $I \cup J$ which is a direct product of two clone subgroups in their terminology—is quasi-isometrically embedded in F. Hence, we can take elements f_n with support inside I with arbitrarily large norm, and hence $f_n^\tau f_n^{-1}$ has also arbitrarily large norm. This proves that the image of τ, a quasi-isometry, is not at bounded distance from the identity and the proof is complete. □

References

[1] Collin Bleak and Bronlyn Wassink. Finite index subgroups of R. Thompson's group F. Preprint,arXiv:0711.1014.

[2] Matthew G. Brin. The chameleon groups of Richard J. Thompson: automorphisms and dynamics. *Inst. Hautes Études Sci. Publ. Math.*, 84:5–33 (1997), 1996.

[3] Kenneth S. Brown and Ross Geoghegan. An infinite-dimensional torsion-free FP_∞ group. *Invent. Math.*, 77(2):367–381, 1984.

[4] José Burillo. Quasi-isometrically embedded subgroups of Thompson's group F. *J. Algebra*, 212(1):65–78, 1999.

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[6] Sean Cleary and Jennifer Taback. Tree automorphisms and quasi-isometries of Thompson's group F. preprint.

[7] Graham Higman. *Finitely presented infinite simple groups.* Department of Pure Mathematics, Department of Mathematics, I.A.S. Australian National University, Canberra, 1974. Notes on Pure Mathematics, No. 8 (1974).

[8] Stephen McCleary and Matatyahu Rubin. Locally moving groups and the reconstruction problem for chains and circles. Preprint, arXiv:math.LO/0510122.

[9] Melanie Stein. Groups of piecewise linear homeomorphisms. *Trans. Amer. Math. Soc.*, 332(2):477–514, 1992.

[10] Richard J. Thompson. Embeddings into finitely generated simple groups which preserve the word problem. In *Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976)*, volume 95 of *Stud. Logic Foundations Math.*, pages 401–441. North-Holland, Amsterdam, 1980.