METRIC PROPERTIES OF HIGHER-DIMENSIONAL THOMPSON'S GROUPS

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Abstract. Higher-dimensional Thompson's groups nV are finitely presented groups described by Brin which generalize dyadic self-maps of the unit interval to dyadic self-maps of n-dimensional unit cubes. We describe some of the metric properties of higher-dimensional Thompson's groups. We give descriptions of elements based upon tree-pair diagrams and give upper and lower bounds for word length in terms of the size of the diagrams. Though these upper and lower bounds are somewhat separated, we show that there are elements realizing the lower bounds and that the fraction of elements which are close to the upper bound converges to 1, showing that the bounds are optimal and that the upper bound is generically achieved.

Introduction

Thompson's groups serve as a wide range of interesting examples of unusual group-theoretic behavior. The family of Thompson's groups includes the original groups described by Thompson (commonly denoted F, T and V) as well as generalizations in many different directions. The bulk of these generalizations includes groups which can be regarded as self-maps of the unit interval. Brin [2, 3] describes higher-dimensional groups nV, generalizations of Thompson's group V, which are described naturally in terms of dyadic self-maps of n-dimensional cubes. Little is known about these groups aside from their simplicity. Brin describes these elements geometrically in terms of dyadic interpolations of collections of dyadic blocks, and gives presentations for the group $2V$.

We describe elements in higher-dimensional Thompson's groups as being given by tree-pair diagrams. Though tree-pair diagrams usually take advantage of the natural left-to-right ordering of subintervals of the unit interval, by using several types of carets, Brin gives a a description of elements of nV via tree-pair diagrams. A natural question is how the size of these tree-pair diagrams corresponds to the word length of elements with respect to finite generating sets.

We give upper and lower bounds for the word lengths of elements of higher-dimensional Thompson's groups nV with respect to finite generating sets, in terms of the size of tree-pair descriptions of elements. An element with a minimal tree-pair description of size N has word length between $\log N$ and $N \log N$ (up to the standard affine equivalences) with respect to the standard finite generating set. This of course also thus holds for any finite generating set.

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1. Background on higher-dimensional Thompson's groups

Brin [2] describes higher-dimensional Thompson's groups nV, giving presentations and showing that the group $2V$ is simple. Brin [2, 3] defines nV as a subgroup of the homeomorphism group of the n-fold product of the Cantor set with itself, and then shows there are a number of equivalent characterizations. The one we use here is Brin's characterization [3] in terms of equivalence classes of labelled tree pair diagrams. Here we summarize the relevant properties of the the groups nV for use in this article.

We denote the unit interval $[0, 1]$ as I, and correspondingly I^n denotes the n-dimensional unit cube.

An element of V is given by two finite rooted binary trees with the same number of leaves and a permutation, which represents a bijection between the leaves of the two trees. Hence an element of V can be seen as a triple (T_+, π, T_-), where T_+ and T_- are trees with k leaves, and $\pi \in \mathcal{S}_k$.

Each binary tree can be seen as a way of subdividing the interval I into dyadic subintervals of the type $[\frac{i}{2^r}, \frac{i+1}{2^r}]$, where $r > 0$ and $0 \leq i < 2^r$. A rooted binary tree gives instructions for successive halvings of subintervals to obtain a particular dyadic subdivision. Given two binary trees with n leaves, and a permutation in $\mathcal{S}_n$, an element in V is represented as a left-continuous map of the interval into itself, sending each interval in the first subdivision to a corresponding interval in the second subdivision, as specified by the given permutation. We note that though not every dyadic subdivision of I can be obtained by a successive halving process described by trees, every dyadic subdivision has a refinement which can be obtained by a successive halving process.

For more details on the group V, including presentations and a proof of its simplicity, see Cannon, Floyd and Parry [6] as well as Brin [2].

To obtain an element in $2V$ we will define a partition of I^2 into dyadic rectangles of the type

$$\left[\frac{i}{2^r}, \frac{i+1}{2^r}\right] \times \left[\frac{j}{2^s}, \frac{j+1}{2^s}\right].$$

Though again, not every such partition can be realized by successive halving processes, every such partition has a refinement which can be realized in that way. For the two-dimensional successive halving process, there are two possible halvings that can occur at each stage to a specified dyadic rectangle— a horizontal or a vertical subdivision. We can obtain a refinement of any dyadic partition of I^2 via iterated horizontal and vertical subdivisions. This gives a natural means of describing elements of $2V$ as pairs of dyadic subdivisions, each with m rectangles, and a permutation in S_m giving the bijection between the rectangles.

We can represent elements of $2V$ also with pairs of binary trees, representing the successive halving processes to represent the refined subdivisions. Since there are two types of subdivisions, vertical and horizontal, we will consider binary trees which contain two types

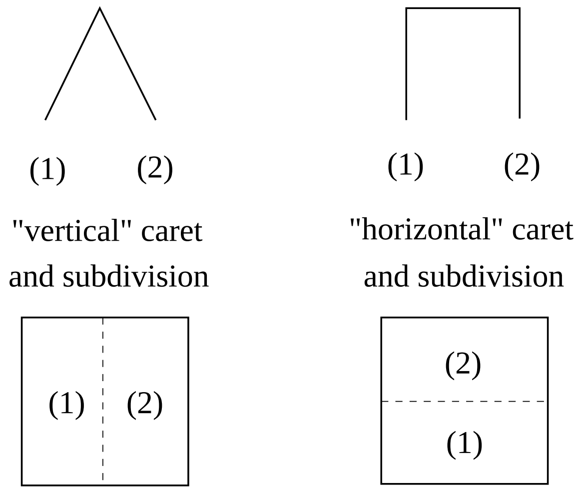

FIGURE 1. The two types of carets and their corresponding subdivisions

of carets, "vertical" and "horizontal" carets, represented by "triangular" and "square" carets, respectively.

As with other tree-based definitions of Thompson's groups, there are numerous representatives of a given group element, with natural equivalence relations coming from expansions and contractions of leaves of trees. To multiply two elements, we typically find representatives for each of them where the trees are compatible for a natural multiplication via composition.

Here, we define the group $2V$ as the set of equivalence classes of triples, (T_+, π, T_-) where the trees are labelled to indicate the successive halving process and π is a permutation between the leaves of the trees (where each leaf represents a dyadic rectangle.)

To fix a labeling convention, we will represent a vertical subdivision with a traditional triangular caret, where the left and right leaves naturally represent the left and right rectangles. A horizontal subdivision will be represented by a square caret, and in it, the left leaf represents the bottom rectangle, and the right leaf represents the top rectangle, as shown in Figure 1.

We say a representative of an element is *reduced* if for each caret c in T_- of with two leaves, the permutation π does not map both leaves of c to leaves of the same caret of the same type in T_+ in the same order. We say a representative of an element is of *minimal size* if it has the minimal number of carets among all representatives in its equivalence class, and we denote by $N(x)$ the number of carets in a minimal size representative of an element x. Unlike V, but very similarly to $F(n, m)$ [7], we note that there may be multiple distinct representatives of an element of minimal size.

See Figure 2 for an example of a partition of I^2 and its corresponding binary tree. Every dyadic partition of the unit square has a refinement which can be obtained with a binary tree with these two types of carets.

For the general groups nV, the partitions are divisions of the unit n-cube in n-rectangles of dyadic lengths, and the corresponding binary trees have n types of carets. For simplicity,

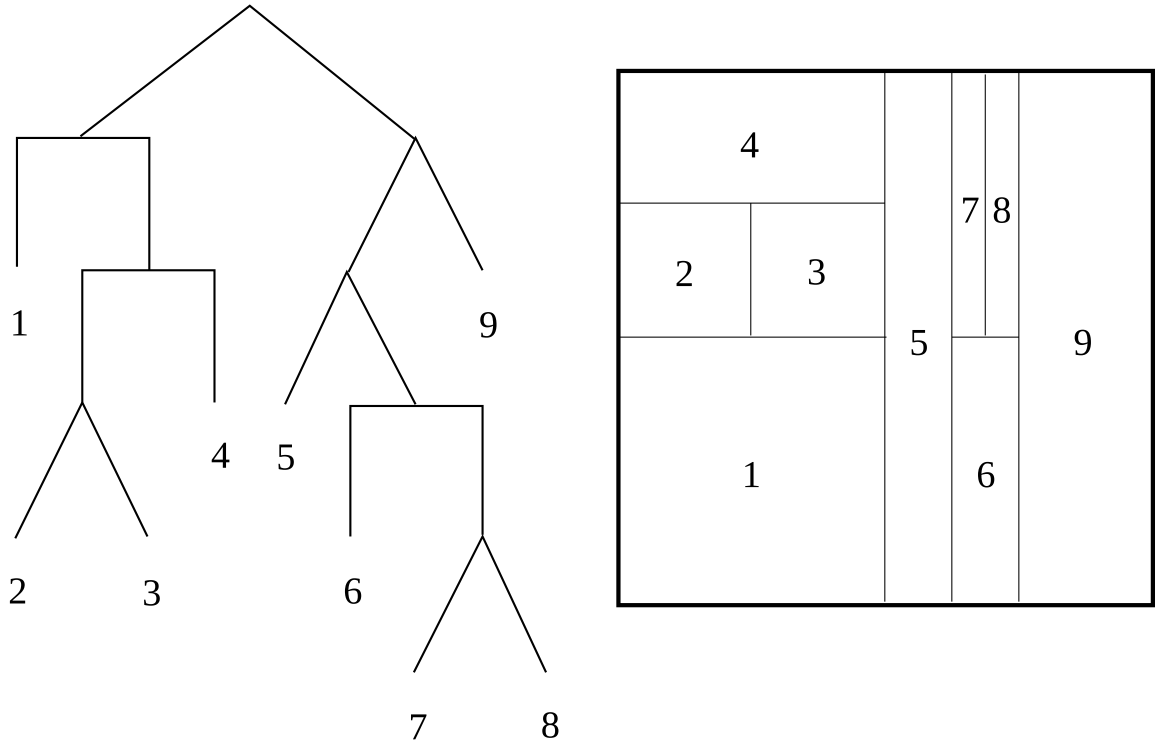

Figure 2. An example of a binary tree with the two types of carets, and its corresponding partition of I^2.

we state our results for $2V$, but all of the results below extend naturally to the cases for nV, with $n > 1$.

The fact that we have vertical and horizontal subdivisions brings new relations to the group. These relations arise when both types of subdivisions are combined in different orders to obtain different descriptions of the same dyadic partition of I^2. The obvious relation (and the one from which all other relations are deducible) is the combination of one subdivision of each type in the two possible orders, as illustrated in Figure 3.

Subdividing in both the vertical and horizontal directions once, but in the two possible orders, gives the same dyadic partitions, but according to our convention the resulting rectangles are numbered in different way. This fact makes using tree diagrams to express the multiplication of elements tricky and quite unwieldy for large elements. Even though the leaves of the two-caret-type tree diagrams are ordered in a natural way, this order is not apparent in the square, and it is not preserved when different diagrams represent the same partition. Thus, there are (non-minimal) diagrams representing the identity whose leaves are ordered in different ways— an example is illustrated in Figure 4.

2. Families of generators

Brin [2, 3] showed that these groups nV are finitely generated and gave several generating sets for $2V$. As it is common with groups of the Thompson family, there is an infinite

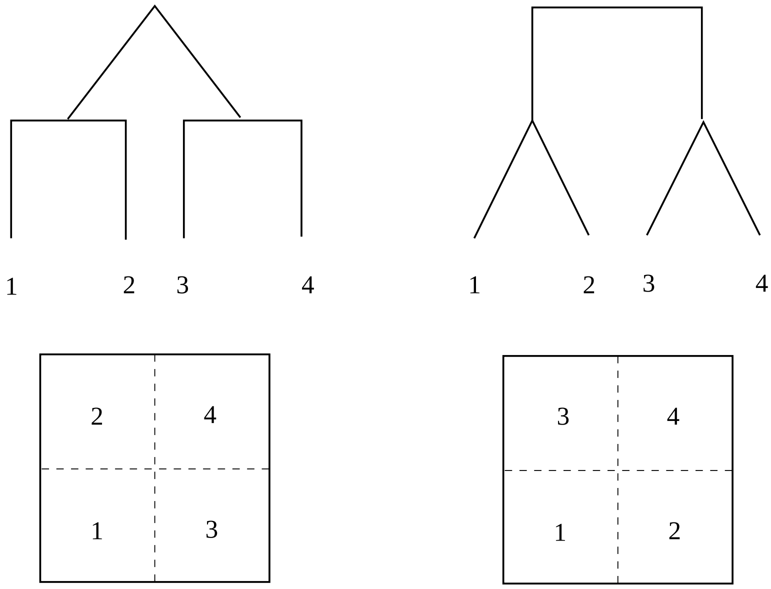

FIGURE 3. The relation obtained when performing vertical and horizontal subdivisions

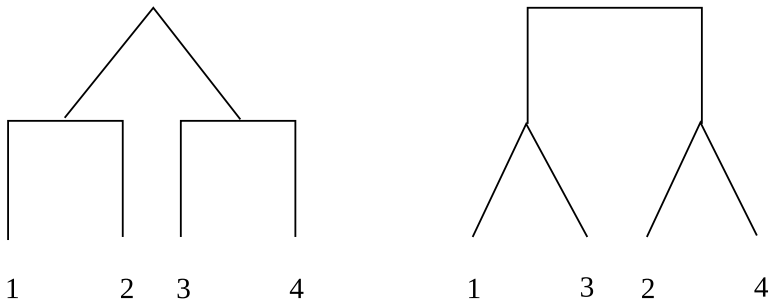

FIGURE 4. A tree-pair diagram for the identity element which has a non-identity permutation

presentation, which is useful for its symmetry and regularity, which has a finite subpresentation. For the purposes of word length, we work with a standard finite generating set described by Brin. We will next define the families of generators for $2V$. As it is customary in Thompson's groups, see [6], the generators are built on a backbone of an all-right tree with only vertical carets.

(1) The generators A_n involve only vertical subdivisions, and they are the traditional generators of Thompson's group F.
(2) The generators B_n have the one non-right caret replaced by a horizontal one.
(3) The generators C_n have the last caret of the all-right tree of horizontal type, and they are used to build horizontal carets on the right-hand-side of the tree, as will be seen later.
(4) The generators π_n and $\overline{\pi}_n$ are permutations built on an all-right tree with only vertical carets. These generators are exactly equal to those for the subgroup V appearing as the purely vertical elements.

Theorem 2.1 (Brin [2, 3])**.** *The families A_n, B_n, C_n, π_n and $\overline{\pi}_n$ generate $2V$.*

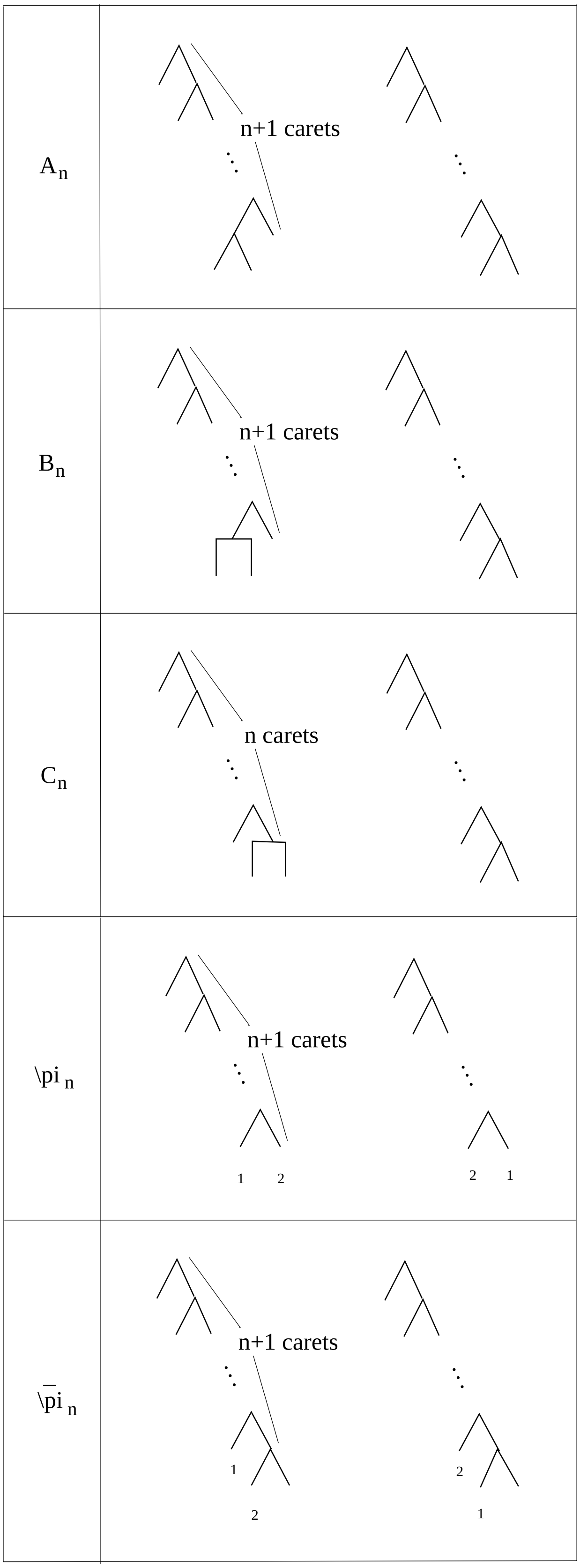

FIGURE 5. Generators of $2V$

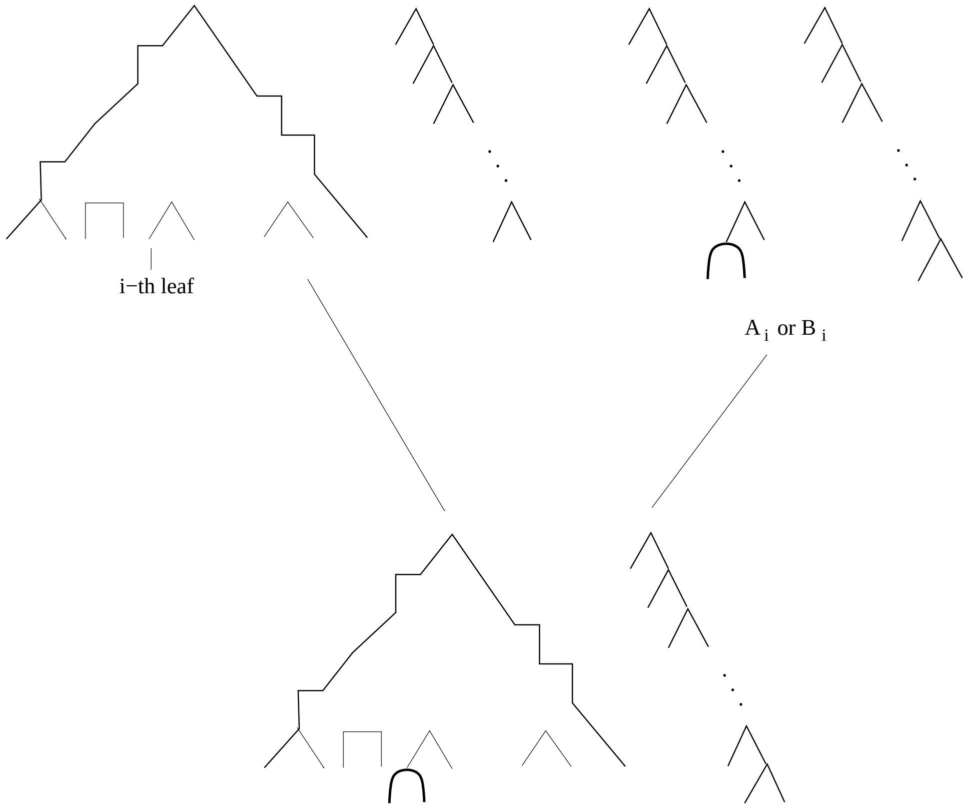

FIGURE 6. The building process: attaching a caret to the i-th leaf by multiplying by A_i or B_i.

Proof: Since our proof is actually quite different from that of Brin's, we will include it here, and use aspects of it later for metric considerations.

An element of $2V$ is given by a triple (T_+, π, T_-), where the two trees are composed of the two types of carets. First, we subdivide a given element into three elements: (T_+, id, R_k), (R_k, π, R_k), and (R_k, id, T_-), where R_k is the all-right tree with k leaves and only vertical carets. Clearly (R_k, π, R_k) is product of the permutation generators, as it is the case in V already.

To obtain the element (T_+, id, R_k), we will concentrate first on the backbone of the tree T_+; that is, the sequence of carets in the right-hand-side. If this sequence of carets has horizontal carets in the positions $m_1, m_2, \dots, m_p$, then the product of the generators $C_{m_1} C_{m_2} \dots C_{m_p}$ produces exactly a backbone with horizontal carets in the desired positions. We denote this backbone by K- that is, K is the subtree of T_+ which consists only of right carets. Thus, at this stage, we have constructed the element (K, id, R_{m_p}).

Once the backbone is constructed, then each new caret is obtained by a generator of the type A_i or B_i. To attach a vertical caret from the leaf labelled i, we only need to multiply by A_i on the right. Similarly, to attach a horizontal caret to leaf i, we multiply by B_i on the right. This multiplication process is shown in Figure 6, with an example illustrated in Figure 7.

This proves that the element (T_+, id, R_k) is product of the generators A_i, B_i and C_i, and using inverses, we have that the entire group is generated by the full family of generators. $\square$

An element of the type (T_+, id, R_k) is called a *positive* element of $2V$. Positive elements can always be written as products of the generators A_i, B_i and C_i without using their inverses.

In the process of proving Brin's theorem, we have obtained the following result:

Theorem 2.2. *For an element of* $2V$*, with respect to the standard infinite generating set* $\{A_i, B_i, C_i, \pi_i, \overline{\pi}_n\}$*, we have:*

- *A positive element of* $2V$ *always admits an expression of the type*
$$C_{m_1} \dots C_{m_p} W_1(A_{i_1}, B_{i_1}) \dots W_r(A_{i_r}, B_{i_r}).$$
where the W_i *are words on the positive generators* A_i *and* B_i *and never their inverses, and also where*
$$m_1 < m_2 < \dots m_p \qquad i_1 < i_2 < \dots i_r$$
- *Any element of* $2V$ *always admits an expression* $P\Pi Q^{-1}$*, where* P *and* Q *are positive elements, and* Π *is a permutation on an all-right vertical tree, and thus a word in the* π_i *and* $\overline{\pi}_i$*.*

This expression will be used as a semi-normal form for elements of $2V$.

Brin also shows that several finite generating sets suffice to generate each nV. For our purposes, we consider the finite generating set $\{A_0, A_1, B_0, B_1, C_0, C_1, \pi_0, \pi_1, \overline{\pi}_0, \overline{\pi}_1\}$. This generating set is larger than the smallest ones used by Brin, but is more convenient for our methods below.

3. Metric properties

We will be interested of metric properties up to the standard affine equivalence, defined here.

Definition 3.1. *Given two functions*
$$f, g : G \longrightarrow \mathbb{R},$$
we say that $f \prec g$ *if there exists a constant* $C > 0$ *such that for all* $x \in G$*, we have that* $f(x) \leq Cg(x)$*. Also, we say that* f *and* g *are equivalent, written* $f \sim g$ *if* $f \prec g$ *and* $g \prec f$*.*

Elements of Thompson's groups admit representations as diagrams with binary trees in one form or another, perhaps with associated permutations or braids between the leaves. For several of these groups, the distance $|x|$ of an element x to the identity is closely related with the number of carets $N(x)$ of the minimal diagram— indicating that the complexity of the minimal (reduced) diagram is a good indication of the size of the element.

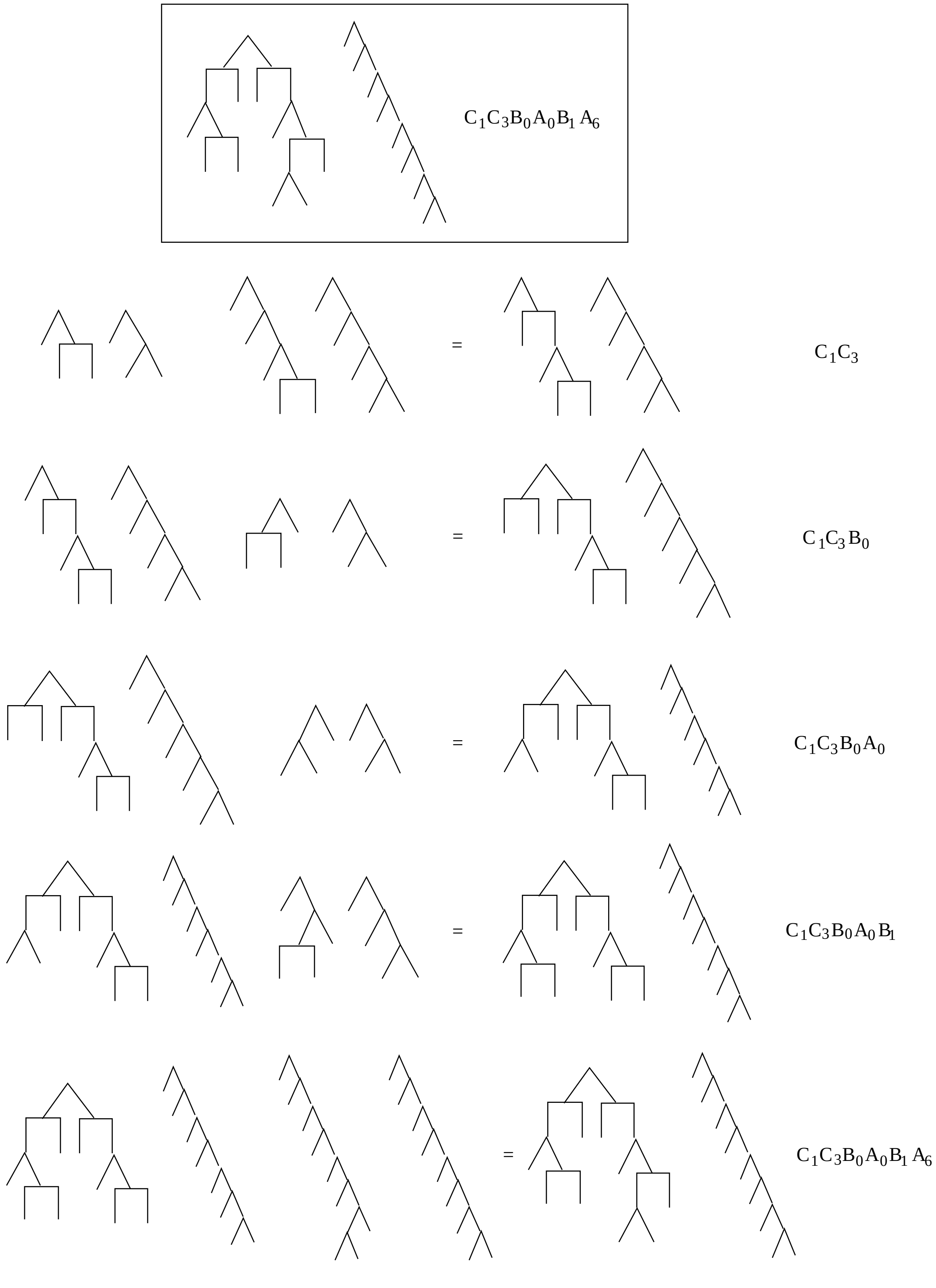

Figure 7. An example of construction of an element generator by generator.

In Thompson's groups F and T, the two functions are equivalent, for both groups we have that $|x| \sim N(x)$, see [4] and [5]. This result cannot hold in V, merely for counting reasons. The number of elements of V whose trees have N carets is at least the order of $N!$ (due to the permutations), while the number of distinct elements of length N can be at most exponential in N in any group.

The best possible bounding for the number of carets in terms of word length in the group V is the following inequality proved by Birget as Theorem 3.8 [1]. We have

$$N(x) \prec |x| \prec N(x) \log N(x),$$

and these bounds are optimal, in a sense similar to that which will be made precise in Section 4.

Since the lower bounds on word length are linear in the number of carets in the case of V we see that the standard inclusions $F \subset T \subset V$ are all quasi-isometric embeddings; that is, F is undistorted in T, and T is undistorted in V. This property will be no longer true for inclusions in $2V$, as we will show in Section 5.

In this section we will prove the analog for $2V$ to the inequality above. In the next section, we address optimality of the bounds.

Theorem 3.2. *In the group $2V$, the word length of an element $|x|$ and number of carets $N(x)$ in a minimal size tree-pair representative satisfy*

$$\log N(x) \prec |x| \prec N(x) \log N(x),$$

and the bounds cannot be improved.

Proof: We defer the proof of its optimality to the following section.

The upper bound is proven the standard way. We take a positive word P, and according to Proposition 2.2, write it as

$$C_{m_1} \dots C_{m_p} W_1(A_{i_1}, B_{i_1}) \dots W_r(A_{i_r}, B_{i_r}).$$

Then, we rewrite each generator using the following relations:

$$A_{i+1} = A_0^{-i+1} A_1 A_0^{i-1} \qquad B_{i+1} = A_0^{-i+1} B_1 A_0^{i-1} \qquad C_{i+1} = A_0^{-i+1} C_1 A_0^{i-1}$$

for all $i > 1$. Since the conjugating element is always A_0, cancellations ensure that the length stays approximately the same. The bound $N \log N$ appears because of the permutations, since $N \log N$ is the diameter of $\mathcal{S}_N$ with respect to the relevant transpositions.

For the lower bound, we note that if an element has k carets, when it is multiplied by a generator, it is possible that the multiplication could have up to $4k$ carets. If an element has, for instance, only horizontal subdivisions, when multiplied by A_1, the number of subdivisions (and hence the number of carets) becomes $4k$, since each one of the previous subdivision is divided in four, as illustrated in Figure 8

If each multiplication by a generator can multiply the number of carets by 4, an element of length ℓ could have up to 4^ℓ carets. Other generators may increase word length by additive factors, but the worst case is exactly the increase by a factor of 4. Hence $N(x) \leq 4^\ell$, so $\log N(x) \leq |x|$. □

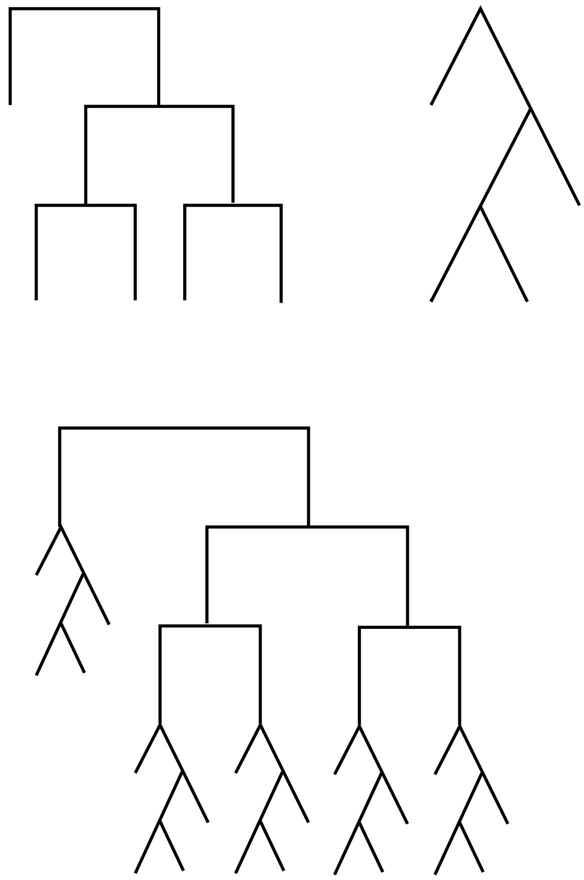

FIGURE 8. An illustration of why when two types of carets are involved, the number of carets gets multiplied. The tree on the top left has only horizontal subdivisions, so when joined with a tree with only vertical ones, these have to be put at every leaf.

4. Optimality of bounds

To show the optimality of the bounds in Theorem 3.2, we first describe some properties of the lower bound for word length in terms of the number of carets.

Lemma 4.1. *The word length of an element of* $2V$ *which is represented by a tree-pair diagram of depth* D *is at least* $D/3$

Proof: Right multiplying by a generator in the finite generating set can add levels to the tree-pair diagram. When we right multiply by a generator, we may need to add carets to T_- (and thus to T_+) if the carets in the generator are not already present in T_-. This can add add most 3 levels, which happens only in the case that we right-multiply an element (T_+, π, T_-) with T_- having no right subtree by any of A_1, B_1 or C_1 from the finite generating set. In that case, the right child of the root of T_- is a leaf lablelled n and we will need to add carets to it and the corresponding leaf from T_+ to obtain a representative compatible with multiplication by the generator. If leaf $\pi^{-1}(n)$ in T_+ is at maximal level, then the level could increase but only by the number of carets added to leaf $\pi^{-1}(n)$ in T_-. Thus, any product of less than $D/3$ generators cannot have depth D. $\square$

From this it follows that the cyclic subgroups generated by any single generator from the A_n or B_n families are undistorted in $2V$, for example.

As seen in the proof of Theorem 3.2, multiplying by a generator can increase the number of carets by a multiplicative factor. The powers of the element C_0 have exponentially many carets. In fact, the minimal number of carets to represent C_0^n is 2^n. We give further related specific examples later in Section 5. Any of these examples show that the lower bound of Theorem 3.2 is optimal.

To analyze the genericity of the upper bound in Theorem 3.2, we use arguments analogous to those for V. Birget [1] showed not only the $n \log n$ upper bound on the growth word length in V with respect to the number of carets n in minimal length representatives, but also that the fraction of elements close to this bound converges exponentially fast to 1. In 2V, we note analogous behavior:

Theorem 4.2. *Let H_n be the set of elements of $2V$ representable with n carets and having no representatives with fewer than n carets. The fraction of elements of H_n which have word length greater than $n \log n$ converges exponentially fast to 1.*

Proof: Here we use a counting argument, analogous to that used by Birget [1] to count elements in V. In $2V$, we consider the subset of representatives of elements which have only horizontal subdivisions in the domain and only vertical subdivisions in the range. Their tree-pair diagrams are guaranteed to be reduced and of minimal size by a straightforward argument. So to count the set of elements of this type of diagram size n, we have C_n choices (where C_n is the n-th Catalan number) for the first all-square tree of size n and C_n choices for the all-triangular tree of size n, and $n!$ choices for the permutation. This set of elements has size $C_n^2 n!$ which is, by the Stirling formula:

$$(C_n)^2 n! = \left(\frac{(2n)!}{(n+1)!n!} \right)^2 n! = \sqrt{\frac{2}{\pi}} \frac{16^n}{e^n} n^{(n-2)} (1 + o(1))$$

where the $o(1)$ term goes to zero as n increases.

The number of elements of word length n in any finitely generated group with respect to d generators is no more than $(2d)^n$. Thus we see that the ratio of elements that have word length less than $n \log_{2d} n$ out of elements that have tree-pair diagrams of size n is less than

$$\frac{n^n}{\sqrt{\frac{2}{\pi}} \frac{16^n}{e^n} n^{(n-2)} (1 + o(1))} \sim c \frac{n^2 e^n}{16^n}$$

which converges to 0 exponentially fast, as desired. So by complementing we have the result.

5. Distortion

Elements of $2V$ can be represented with pairs of binary trees, where the carets have been subdivided in two types. This fact makes it more difficult to multiply elements whose caret types disagree, illustrated already in Theorem 3.2, and the number of carets can

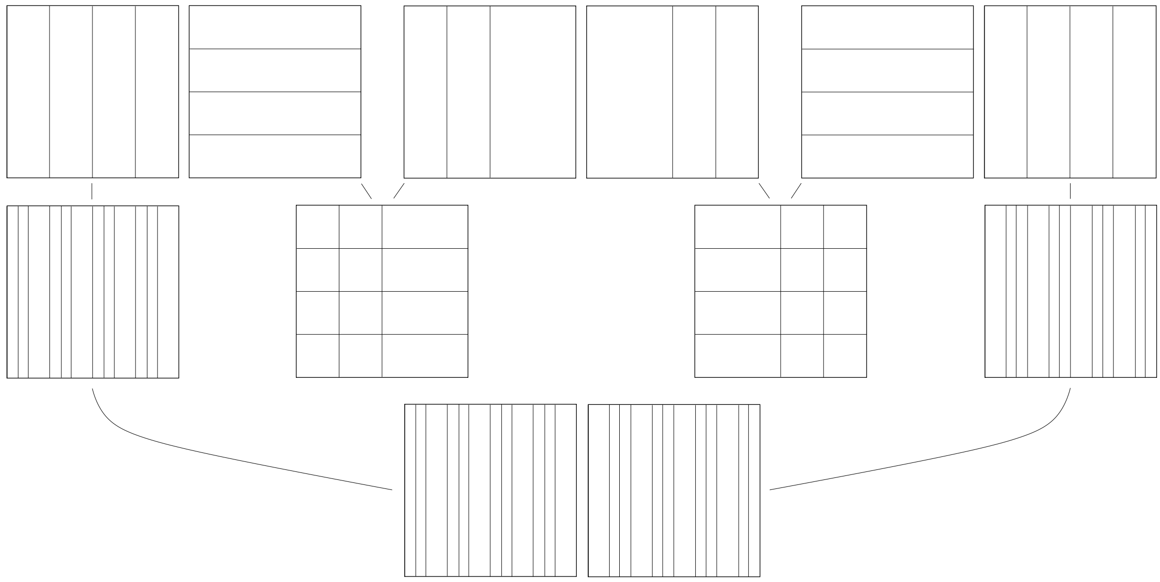

FIGURE 9. The process of multiplication for $C_0^{-2}X_0C_0^2$, illustrating the exponential number of subdivisions.

grow faster because of this situation. This phenomenon has been described already by Wladis in [7] for the group $F(2,3)$, which has also carets of two types (binary and ternary). This feature of $2V$ implies now that the groups F, T and V, when seen as subgroups of $2V$ (by doing only vertical subdivisions, for instance), are exponentially distorted.

Theorem 5.1. *The groups F, T and V are at least exponentially distorted in $2V$.*

Proof. We consider the specific element $C_0^{-n}X_0C_0^n$, illustrated in Figure 9. This element lies in a copy of F in $2V$ obtained by putting F into $2V$ using only vertical subdivisions.

The element C_0^n has 2^n carets, as it is seen with an easy induction. Its two trees are balanced trees of depth n, one with only vertical subdivisions, and one with only horizontal subdivisions. By matching the horizontal subdivisions with the vertical ones in X_0 we obtain that the element $C_0^{-n}X_0C_0^n$ has a number of carets of the order of 2^n, and all of them of vertical type, so the element is in F. The element $C_0^{-n}X_0C_0^n$ has length in V no more than $2n+1$, but the number of carets is exponential in n and thus its word length as an element of the vertical copy of F in $2V$ is also exponential.

Thus F is at least exponentially distorted in $2V$. Since F is undistorted in T and V, then T and V (as subgroups using only vertical subdivisions) are also at least exponentially distorted in $2V$. □

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