

Introduction to Thompson's group F

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Chapter 1

Definition and first properties

1.1 Definition

Thompson's group F is defined as a group of piecewise linear maps of the interval $[0, 1]$. Piecewise linear maps will always be understood to be continuous, and those points where the map is not differentiable will be called *breakpoints*. Then, in our piecewise linear maps the set of breakpoints will always be discrete. In particular, if the piecewise linear map is defined in a compact interval, the set of breakpoints is finite.

Definition 1.1.1 *We define Thompson's group F as the group (under composition) of those homeomorphisms of the interval $[0, 1]$, which satisfy the following conditions:*

1. *they are piecewise linear and orientation-preserving,*
2. *in the pieces where the maps are linear, the slope is always a power of 2, and*
3. *the breakpoints are dyadic, i.e., they belong to the set $D \times D$, where $D = [0, 1] \cap \mathbb{Z}[\frac{1}{2}]$.*

An element of F has then finitely many breakpoints, which have dyadic coordinates. We can identify an element of F as a list of finitely many breakpoints: the list

$$[(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)]$$

will represent the element for which the interval (a_i, a_{i+1}) is mapped linearly to (b_i, b_{i+1}) , for $i = 0, 1, \dots, k$, if we consider $(a_0, b_0) = (0, 0)$ and $(a_{k+1}, b_{k+1}) = (1, 1)$. The conditions demand that the a_i and b_i are dyadic, and also that

$$\frac{b_{i+1} - b_i}{a_{i+1} - a_i}$$

is a power of 2 for all $i = 0, 1, \dots, k$.

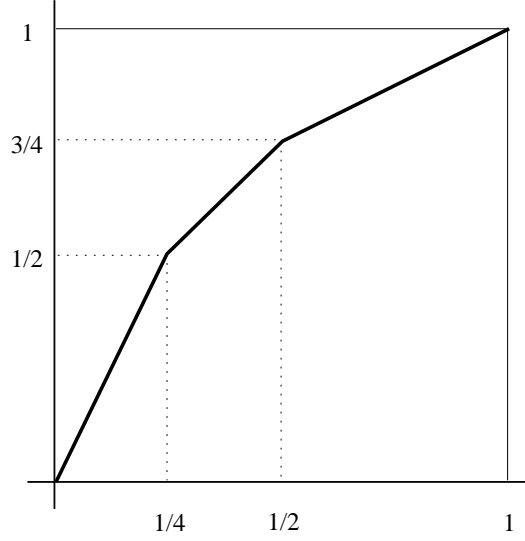


Figure 1.1: The graph of the element $[(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4})]$.

Example 1.1.2 The simplest element of F , with break-point list $[(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4})]$, is the map

$$f(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq \frac{1}{4} \\ t + \frac{1}{4} & \text{for } \frac{1}{4} \leq t \leq \frac{1}{2} \\ \frac{t+1}{2} & \text{for } \frac{1}{2} \leq t \leq 1 \end{cases}$$

and its graph is given in Figure 1.1.

Observe that pieces of the graph of the element may be on the diagonal, i.e., for a given element, there may be whole segments of fixed points. This motivates the following definition.

Definition 1.1.3 If $f \in F$, then its support is the closure of the set of points $t \in [0, 1]$ for which $f(t) \neq t$.

Example 1.1.4 The element $[(\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})]$ has support $[0, \frac{3}{4}]$.

Example 1.1.5 The support can obviously have more than one connected component. The element

$$[(\frac{1}{4}, \frac{1}{4}), (\frac{5}{16}, \frac{3}{8}), (\frac{3}{8}, \frac{7}{16}), (\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4}), (\frac{13}{16}, \frac{7}{8}), (\frac{15}{16}, \frac{15}{16})]$$

has support $[\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, \frac{15}{16}]$.

From this definition we can already prove the first group-theoretic property of F .

Theorem 1.1.6 F is torsion-free.

Proof. Take $f \in F$, f different from the identity, and let P be the smallest point in the support of f , which is either $(0, 0)$ or the first breakpoint, of the type (a, a) for $a \in [0, 1]$ a dyadic. The slope of f on the right of P is 2^k , for some $k \neq 0$. Then the element f^n has slope 2^{nk} on the right of P , and hence it is never equal to the identity. \square

1.2 Binary trees

As useful as maps are, the most visual realization of F is given by pairs of rooted binary trees. We will give a long list of definitions which will fix the language and the objects used in this realization.

Definition 1.2.1 A rooted binary tree is a tree which starts at some distinguished vertex (the root), which is the only vertex of degree two. All the other vertices have either degree three (the nodes) or degree one (the leaves). A caret is the subgraph containing a node (or the root), the two edges going down from it, and the two vertices at their ends (the node's children). These last two vertices may or may not be leaves.

See Figure 1.2 for a picture of a binary tree.

The significance of binary trees is that they encode subdivisions of the unit interval in successive halves. A binary tree is a subtree of the infinite binary tree, whose root correspond to the whole interval $[0, 1]$. The caret starting at the root has children $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Subdividing further we see that the interval $[a, b]$ has children $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. All endpoints of all intervals appearing in this infinite binary tree are dyadic rationals, and all dyadic rationals appear as endpoints of some interval. See Figure 1.3 for a picture of the infinite binary tree.

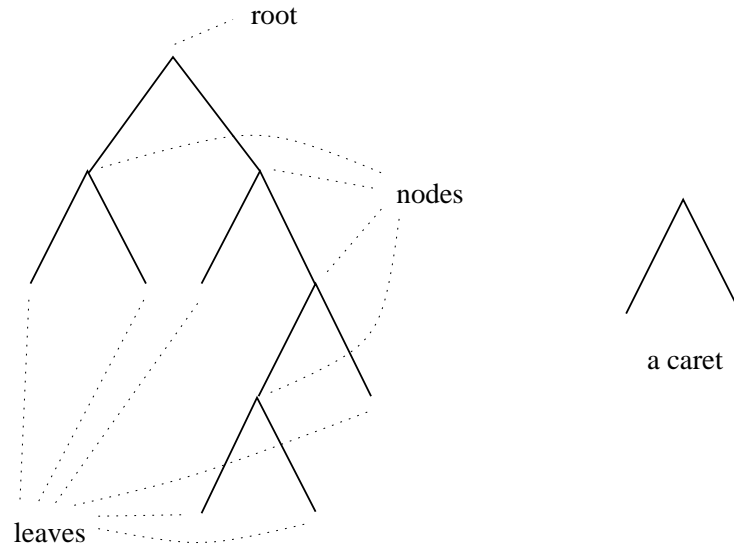


Figure 1.2: A rooted binary tree, and a caret.

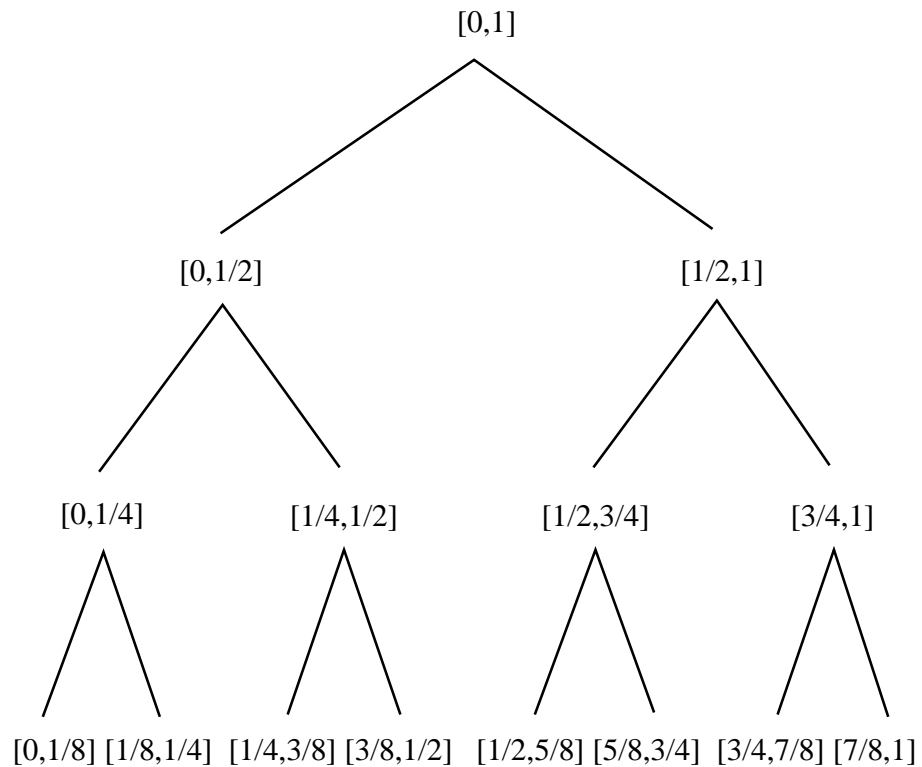


Figure 1.3: The intervals on the infinite binary tree

A finite binary tree is a finite subtree of this binary tree, and its leaves will spell out a subdivision of the unit interval in intervals with dyadic lengths and dyadic rationals as endpoints.

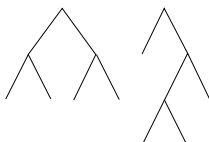
A *tree pair diagram* is a pair of binary trees with the same number of leaves, and it defines then an element of F . The element is found by mapping linearly the corresponding intervals to the leaves, in an order-preserving fashion. A few examples will clarify this construction.

Example 1.2.2 The two binary trees



correspond to the element $[(\frac{1}{4}, \frac{1}{2}), (\frac{1}{2}, \frac{3}{4})]$ described above.

Example 1.2.3 The two binary trees

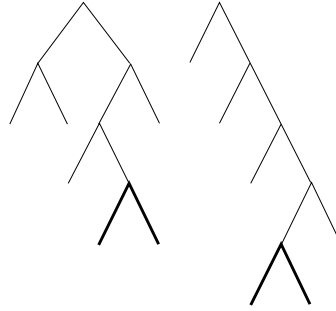


correspond to the element with list of breakpoints $[(\frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})]$.

If we observe this last element we see that the fact that we have a certain number of leaves does not mean that we have the same number of distinct intervals in the graph of the element. Two consecutive intervals may be disguised and merged into one if they have the same slope. In that case, there is no breakpoint between them, as it is shown by the point $(\frac{1}{2}, \frac{5}{8})$ in the second example above.

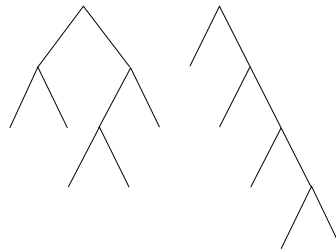
One last example to show another important feature.

Example 1.2.4 The pair of trees



corresponds to the element with breakpoints
 $[(\frac{1}{4}, \frac{1}{2}), (\frac{5}{8}, \frac{7}{8}), (\frac{3}{4}, \frac{15}{16})]$

In this tree pair diagram we observe the phenomenon of a *reducible caret*, indicated with thicker lines. The element could be written with the following trees:



Hence when we have a tree pair diagram representing an element of F , we could add extra carets and construct other pairs of trees which represent the same element, by adding meaningless subdivisions. Also, if we see a diagram where there is a reducible caret, we can reduce it. Hence we have an equivalence class of pairs of binary trees, where two elements are equivalent if there is a sequence of additions and reductions of carets which transforms one into the other. All pairs in the equivalence class represent the same element of F .

Conversely, we see that every element of F corresponds to an equivalence class of tree pair diagrams.

Proposition 1.2.5 *Given an element of F , there is a tree pair diagram representing it.*

Proof. Let $f \in F$, and let $[(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)]$ be its list of break-points. Let n be an integer such that all the a_i , for $i = 1, \dots, k$, can be expressed as a dyadic with a fraction with denominator 2^n . The same way, let m be an integer which works the same way for the b_i . This way we make sure that f is linear in each interval $[a_i, a_{i+1}]$, and the inverse map f^{-1} is also linear in $[b_i, b_{i+1}]$.

The division of the unit interval into the 2^n intervals is represented by a *balanced tree*, that is, a tree with 2^n leaves which are all at distance n from the root, i.e., a tree of depth n . All leaves in this tree represent intervals where the map f is linear. The same way, the subdivision of the target interval in 2^m pieces is represented by the balanced tree of depth m . The numbers of leaves in both trees may very well be different, so some leaves have to be subdivided further.

Consider f restricted to the interval $[a_i, a_{i+1}]$. We know that $a_{i+1} - a_i = l_i/2^n$ for some positive integer l_i . If the map f has slope 2^{r_i} in this linear piece, the image interval satisfies $b_{i+1} - b_i = l_i 2^{r_i}/2^n$. From the subdivision of the target interval, we know then that there exists a positive integer l'_i such that

$$b_{i+1} - b_i = l_i 2^{r_i}/2^n = l'_i/2^m.$$

The conclusion is that l'_i is equal to l_i multiplied by a power of 2, namely, 2^{s_i} , with $s_i = r_i + m - n$. Then, depending on which of the two numbers l_i and l'_i is larger, the leaves have to be subdivided in one tree or the other. And this depends on the sign of s_i . We have three cases:

- If $s_i > 0$, it means that the interval $[b_i, b_{i+1}]$ corresponds to l'_i leaves of the balanced tree of length m , but the interval $[a_i, a_{i+1}]$ corresponds to only l_i leaves. Hence these leaves in the source tree have to be subdivided each into 2^{s_i} leaves, again using a balanced tree of depth s_i for each leaf.
- If $s_i = 0$ then $l_i = l'_i$ and there is no need of subdividing anything.
- If $s_i < 0$, then $l_i > l'_i$, and by an analogous argument to the first case, each one of the l'_i leaves corresponding to $b_{i+1} - b_i$ can be subdivided into 2^{-s_i} leaves.

By this construction, we obtain two trees with the same number of leaves, and each leaf of the source tree is mapped to the corresponding one in the target tree. Hence the element we obtain from this tree pair diagram is f . \square

An example will clarify this construction.

Example 1.2.6 Take the element given by the breakpoints $[(\frac{1}{8}, \frac{1}{8}), (\frac{1}{4}, \frac{3}{8}), (\frac{1}{2}, \frac{7}{16}), (\frac{3}{4}, \frac{15}{16})]$. We observe that $n = 3$ and $m = 4$. Hence the source interval is divided in 8 pieces by a balanced tree T of depth 3 and 7 carets, and the target interval is divided in 16 pieces, represented by a balanced tree T' of depth 4. See Figure 1.4 for the steps of this construction.

The map has 5 linear pieces between breakpoints. The first piece corresponds to the interval $[0, \frac{1}{8}]$ mapping to itself, but it corresponds to one leaf in the tree T and to two leaves in T' . In the notation above, we have $l_0 = 1$, $l'_0 = 2$, $r_0 = 0$, and then $s_0 = 1$. Hence the leaf in T has to be subdivided in two, adding one caret.

The second piece corresponds to only one leaf in T but to 4 leaves in T' . We have $l_1 = 1$, $l'_1 = 4$, the slope is 2, so $r_1 = 1$ and then $s_1 = 2$. So the leaf in T is subdivided in 4 leaves, using a small balanced tree of depth 2 hanging from it.

The third piece corresponds to two leaves in T but to only one in T' . So this leaf is subdivided. The numbers are $l_2 = 2$, $l'_2 = 1$, the slope is $1/4$ and then $r_2 = -2$ and $s_2 = -1$.

The fourth piece is made up of two leaves of T and eight leaves of T' , so the two leaves of T are subdivided into four each. The numbers are $l_3 = 2$, $l'_3 = 8$, $r_3 = 1$ and $s_3 = 2$. The fifth piece makes the very last leaf of T' subdivide in two, and we let the reader compute the numbers for $i = 4$.

Observe that the diagram obtained is not reduced. See this diagram and its reduction in Figure 1.4.

Observe that for every element, in its equivalence class of tree pair diagrams there is at least one reduced diagram. Namely, take any tree pair diagram for the element and reduce it. Uniqueness for this reduced diagram (it is true that there is only one reduced diagram in each class) is much trickier to prove, and we will postpone the proof until the next chapter.

Finally, observe that the carets of a binary tree are totally ordered from left to right. The order is defined by the following rule: for a given caret, all the carets hanging from its left child precede it, and all the carets hanging from its right child follow it. This order is induced on the tree by the natural order of the dyadic numbers in the interval $[0, 1]$, because a caret represents a subdivision of an interval into two halves. According to the infinite tree of dyadic intervals defined above, each node corresponds to an interval, and one can identify the caret which subdivides it with the midpoint of the interval.

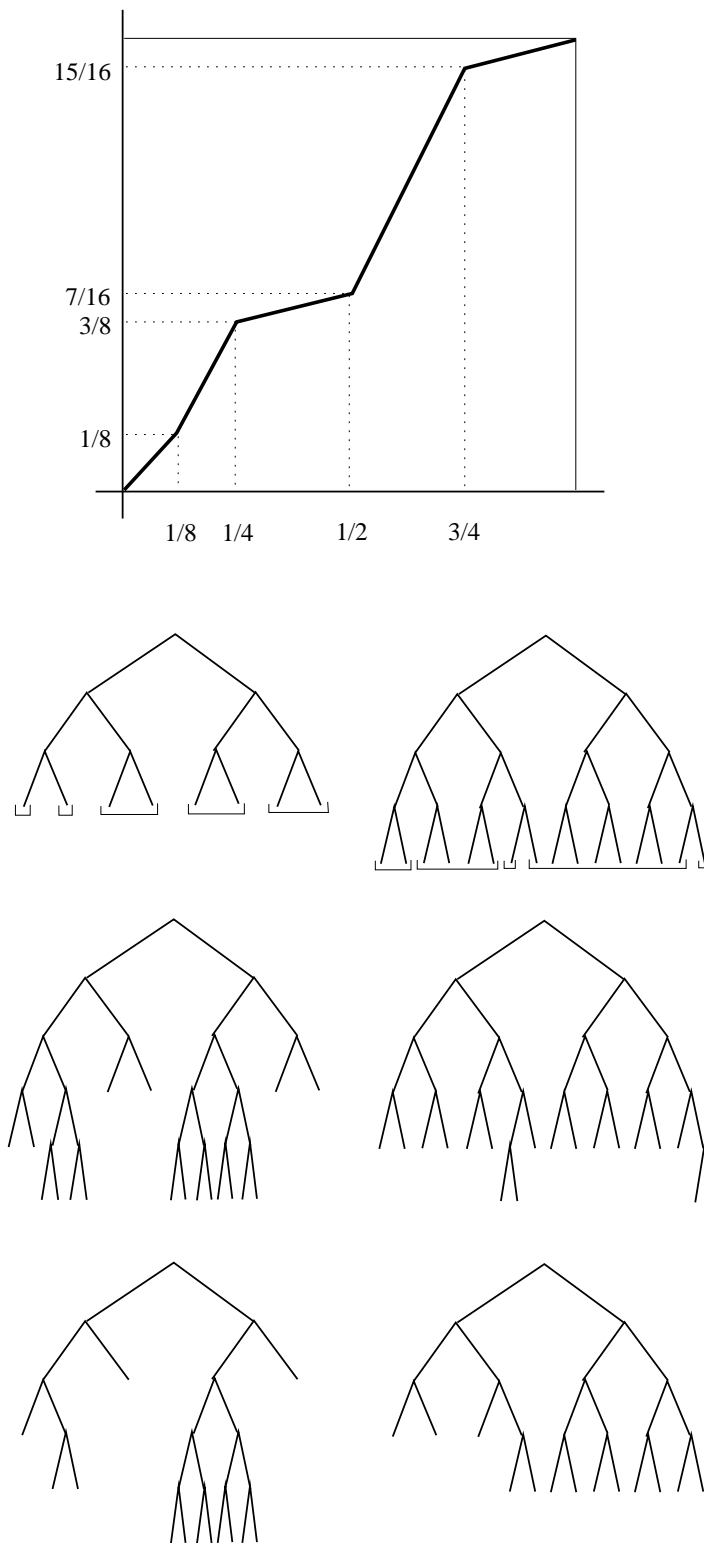


Figure 1.4: An example of construction of the tree diagram from the graph. From top to bottom, it shows: (1) the graph of the element; (2) the two balanced trees, with the leaves grouped for the corresponding intervals; (3) the diagram obtained for the element; and (4) the reduced diagram.

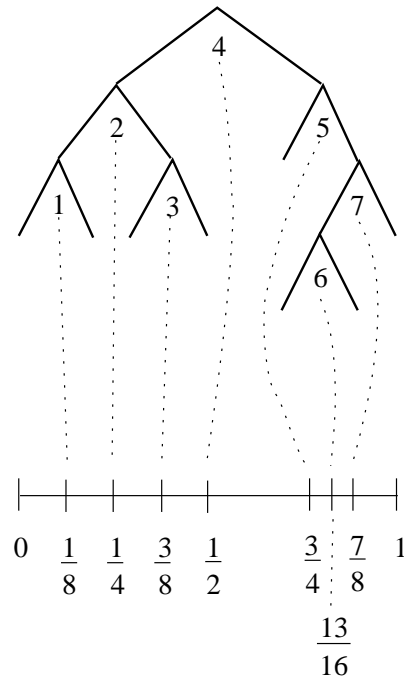


Figure 1.5: A tree where the carets have been given the natural order from left to right.

The example in Figure 1.5 will clarify this order.

1.3 Composition

The group law is the composition of maps. The correspondence between maps and pairs of binary trees can also be used to translate composition to binary trees. Observe that since properly speaking, elements are represented by equivalence classes of diagrams, we can choose the appropriate representatives to perform the composition.

So given two elements of F in terms of tree pair diagrams, one only needs to match composition of maps to compose them. Let (T, T') and (S, S') be the two tree pairs involved. We need an elementary lemma.

Lemma 1.3.1 *Two binary subdivisions of $[0, 1]$ always have a common subdivision. Equivalently, given two binary trees, there is a least common multiple, i.e., a unique minimal tree which contains both.*

Proof. The proof is elementary. Take the two trees and “superimpose” them, the tree obtained is the least common multiple, because it contains both, and if one deletes any caret it fails to be a multiple. \square

To compose the two elements, then, we only need to find the least common multiple R of T' and S , which will play the role of the middle interval in the composition. Find then two representatives for the two elements of the form (T'', R) and (R, S'') . The product element is then represented by the tree pair (T'', S'') .

The composition is well defined because of the correspondence with the maps. Two pairs representing the same element represent the same map, so since changing the representatives does not change the map, and the process of composition for diagrams translates that of maps, the result is the same independently of the representatives chosen.

One last remark is due, regarding left and right issues. If an element f of F is represented by (T, T') , and another element g is represented by (S, S') , then the product fg , or $(T, T')(S, S')$, represents the composition $g \circ f$. The reason for this is that it is more useful from the group-theoretic point of view to write the elements this way, because it will correspond easier with the generators and the normal forms. Equivalently, one can think of the maps acting on the interval on the right, with something like $(x)(fg) = ((x)f)g$ and written simply $(x)fg$, which is a more natural way of looking at maps as group elements. The reader should not be hampered by this notation, since composition will play a very small role in this work, since our interest in F will be mostly group-theoretic. Observe also that our convention is exactly the opposite that the one appearing in other places, most notably [8], but the issues are minor and should be easily figured out by the reader.

1.4 Maps of the real line

Sometimes, instead of considering maps of the unit interval, it is convenient to consider maps of the real line. In this section we will introduce a representation of F as a group of piecewise-linear maps of \mathbb{R} , which is essentially the same as the one in $[0, 1]$, but which is useful on its own to study some of the properties of F .

We can transport elements of F from $[0, 1]$ to \mathbb{R} by conjugating with some appropriate piecewise-linear map. We will construct this piecewise-linear map φ from \mathbb{R} to the unit interval by defining what are its breakpoints and then extending linearly between them. Define:

$$\varphi(k) = \begin{cases} 1 - \frac{1}{2^{k+1}} & \text{if } k \in \mathbb{Z} \text{ and } k \geq 0 \\ 2^{k-1} & \text{if } k \in \mathbb{Z} \text{ and } k < 0 \end{cases}$$

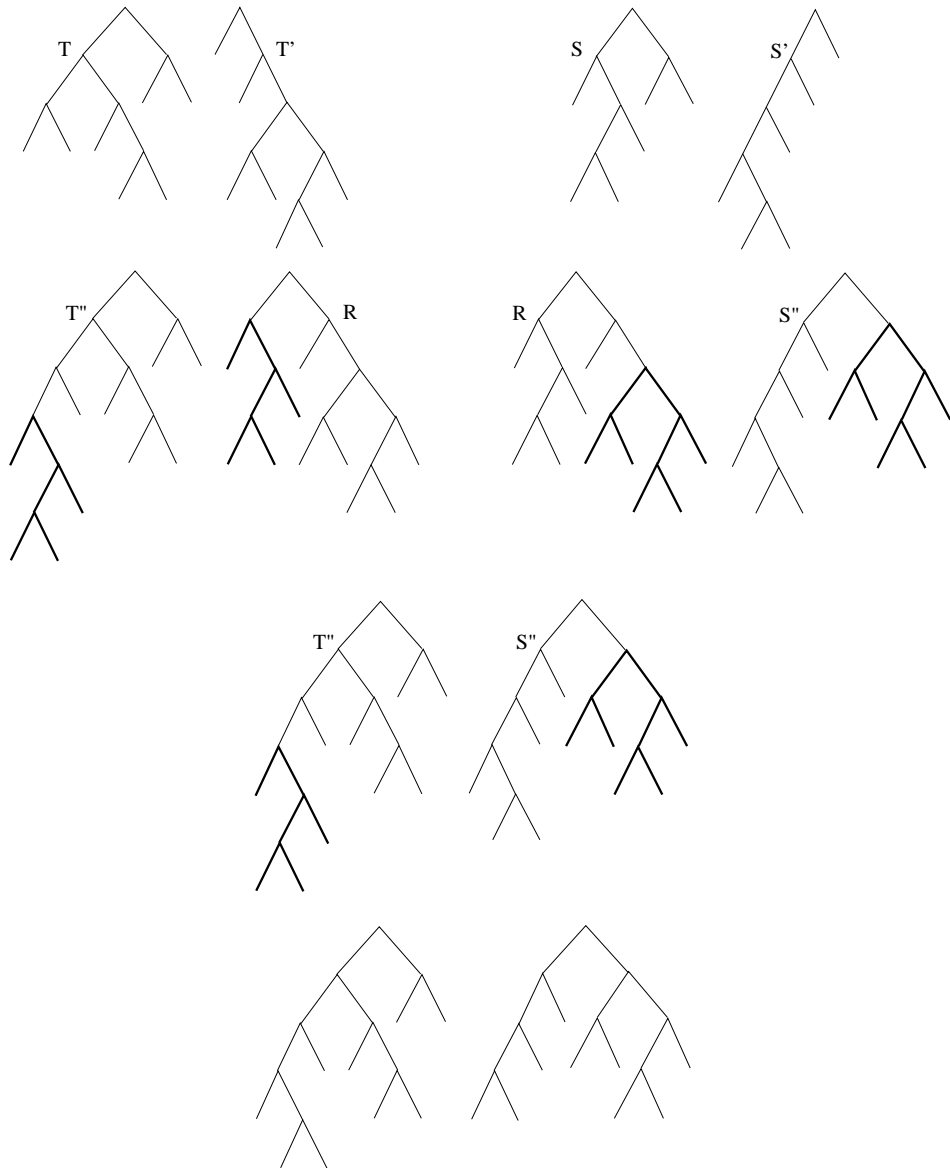
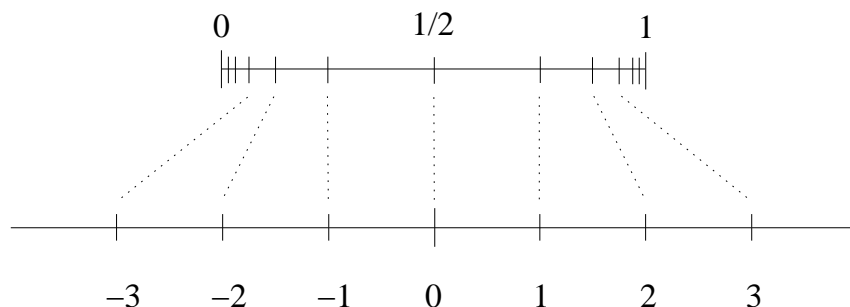


Figure 1.6: An example of the composition of two elements.

Figure 1.7: The conjugating map φ from \mathbb{R} to $[0, 1]$.

See Figure 1.7. And now consider, for each element $f \in F$, seen as a map in $[0, 1]$, its conjugate $\bar{f} = \varphi^{-1} \circ f \circ \varphi$, which is then a map from \mathbb{R} to \mathbb{R} . We obtain in this way a representation of the element f but now as a map of the whole real line. Also, since φ sends dyadics to dyadics, the breakpoints of the resulting map \bar{f} also have dyadic coordinates.

Observe that even though it looks like the map on the real line may have infinitely many breakpoints, it does not. When we look carefully at the neighborhood of 0 in the interval, we see that the map $y = 2^k$ sends an interval $[1/2^m, 1/2^{m-1}]$ to the interval $[1/2^{m-k}, 1/2^{m-k-1}]$, which is displaced k spots up or down. The outcome of this is that the corresponding map in \mathbb{R} is of the form $\bar{f}(t) = t + k$ in a neighborhood of $-\infty$, because here the intervals translated k spots are integer intervals. And a similar thing happens in the neighborhood of 1.

Hence we have found an interpretation of F as a group of maps of the real line:

Theorem 1.4.1 *F is isomorphic to the group of homeomorphisms α of \mathbb{R} which satisfy:*

1. *they are piecewise-linear and orientation preserving*
2. *they have a finite number of breakpoints, all with dyadic coordinates*
3. *there exists a real number M and two integers k and l such that if $t > M$, we have $\alpha(t) = t + k$, and that of $t < -M$, we have $\alpha(t) = t + l$.*

For instance, see the maps in \mathbb{R} which correspond to the generators x_i in Figure 1.8.

As it happens with the interpretations of F as maps of the unit interval, as pairs of binary trees, and algebraically as normal forms, this new realization

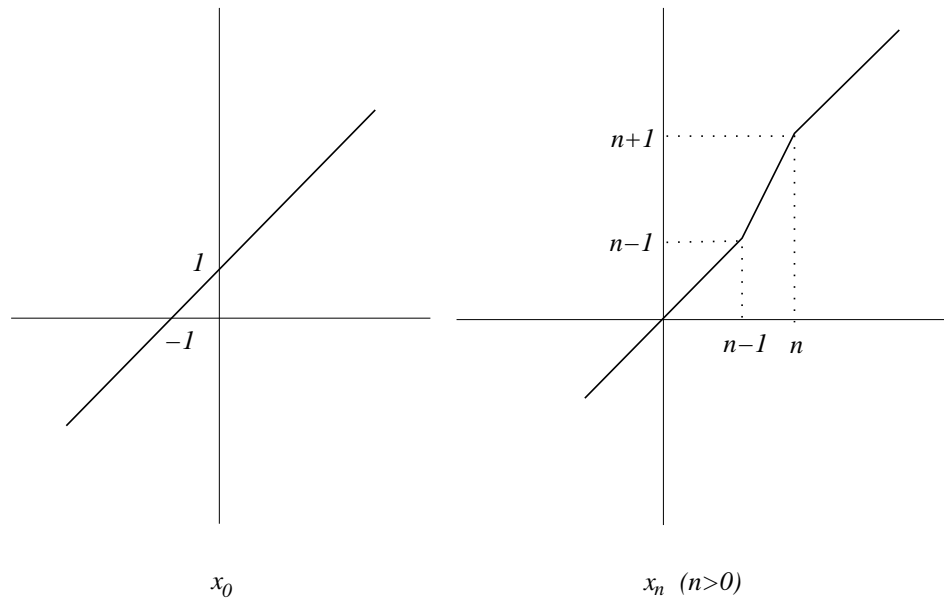


Figure 1.8: The generators of F when seen as maps on the real line.

as maps of the real line is useful on its own and can be used to understand some of the properties of F . Part of the richness of the structure of F is this fact that it can be seen from many multiple points of view, and they are each useful for some properties. This makes it quite easy and approachable to study its properties when one knows how to choose the right setting for F . We will use this interpretation several times later in this book.

Chapter 2

Presentations

2.1 The infinite presentation

Thompson's group F admits a presentation, which is infinite, but it is useful for its special simplicity, and the easiness and symmetry of its relators.

Theorem 2.1.1 *The following is a presentation for Thompson's group F .*

$$\langle x_0, x_1, x_2, \dots, x_n, \dots \mid x_i^{-1} x_j x_i = x_{j+1}, \text{ for } i < j \rangle.$$

The proof of this fact is long and will be spread out throughout the next sections. In the meantime, we will introduce some concepts which will be very useful in the future.

Let G be the group given by this presentation. To prove that G is isomorphic to F , we will define a homomorphism from G to F and then prove that it is one-to-one and onto. The homomorphism is defined by the image of the generators.

Definition 2.1.2 *There is a homomorphism*

$$\Theta : G \longrightarrow F$$

defined by the images of the generators. The image of the generator x_n is the element f_n with list of breakpoints

$$\left[\left(1 - \frac{1}{2^n}, 1 - \frac{1}{2^n} \right), \left(1 - \frac{3}{2^{n+2}}, 1 - \frac{1}{2^{n+1}} \right), \left(1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+2}} \right) \right]$$

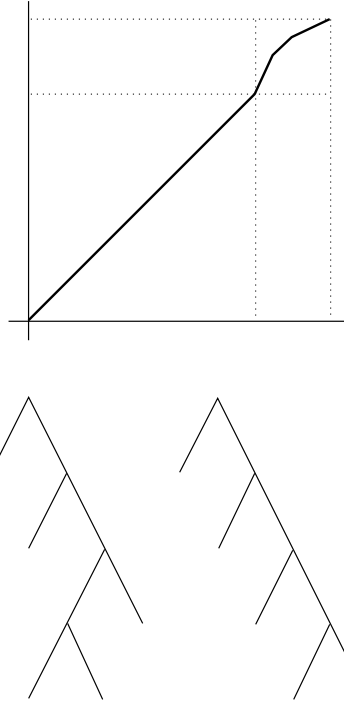


Figure 2.1: The graph and the tree pair diagram for the element f_2 , image of the generator x_2 , with breakpoints $[(\frac{3}{4}, \frac{3}{4}), (\frac{13}{16}, \frac{7}{8}), (\frac{7}{8}, \frac{15}{16})]$.

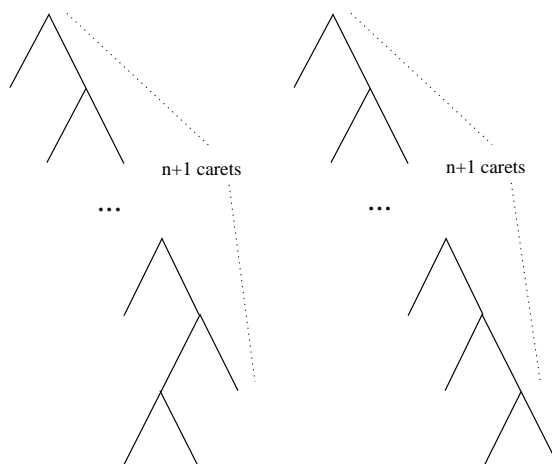


Figure 2.2: The tree pair diagram for the element f_n , image of the generator x_n .

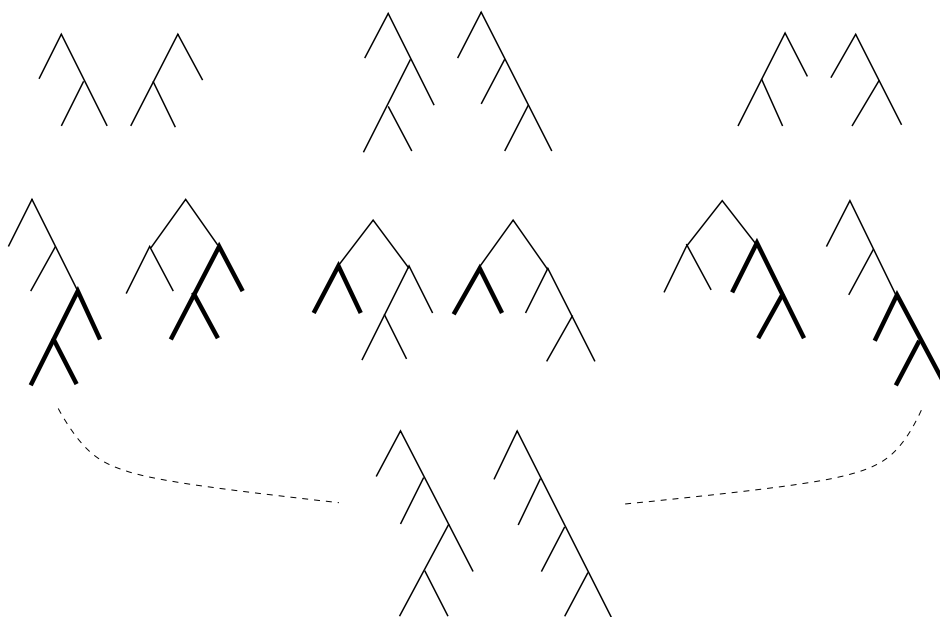


Figure 2.3: The verification of the relator $x_0^{-1}x_1x_0 = x_2$.

See Figure 2.1 and 2.2 for examples.

It is straightforward to see that the relations are satisfied. See Figure 2.3 for an example of the verification of a relator.

Hence, we have a well defined homomorphism Θ from G to F . We will work in the next sections towards proving that it is an isomorphism.

2.2 Positive words

Observe the presentation for G , and rewrite it the following way.

$$\langle x_0, x_1, x_2, \dots, x_n, \dots \mid x_j x_i = x_i x_{j+1}, \text{ for } i < j \rangle.$$

This presentation defines a monoid, called the *monoid of positive words*, and represented with the letter P . A *positive word* is a word on the generators x_i , for $i \geq 0$, without the involvement of any inverses. The monoid P is then a submonoid of G . The same way, the monoid $\Theta(P)$ is a submonoid of F , generated by the f_i .

An element of P is given by a word on the generators, but observe that the relators allow for a special writing of the elements. Note that if we have a generator with higher index located immediately before one with a lower index, we can switch them at the price of increasing the high index by 1.

Then, given a particular element we can perform the following process: first, all instances of x_0 can be moved all the way to the left, then all x_1 , and so on. We conclude the following result.

Theorem 2.2.1 *Each element of P can be written as a word of the following form*

$$x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}$$

for some n , and for some nonnegative integers a_n .

We will see in the next section that this expression is unique. Observe though, that some of the a_i may very well be zero.

We observe that in the monoid $\Theta(P)$ we have the analogous result, since the same relations are valid in it. We want to establish the shape of the tree pair diagram of an element in this monoid. For this we need the concepts of all-right tree and of leaf exponent.

Definition 2.2.2 *An all-right tree is a binary tree, where every caret hangs from the right child of the previous one. I.e., an all-right tree is made out of a string of carets, each hanging from the right child of the caret above it (except obviously the root).*

As an example, all the target trees of the generators f_n are all-right trees.

Definition 2.2.3 *Given a binary tree, number the leaves from left to right starting at zero. Then, for the i -th leaf, count the number of ascending left carets one can count from the leaf up in the tree, excluding, if the string arrives to it, a caret in the right side of the tree. This number is the i -th leaf exponent of the tree, and it is written a_i .*

See Figure 2.4 for a clarifying example.

The theorem is then that, for a given element of $\Theta(P)$, leaf exponents are exactly the exponents in the special expression given in Theorem 2.2.1.

Theorem 2.2.4 *An element in $\Theta(P)$, given by the expression*

$$f_0^{a_0} f_1^{a_1} \dots f_n^{a_n}$$

can be represented by a tree pair diagram (T, T') where T has leaf exponents a_0, a_1, \dots, a_n , and T' is an all-right tree.

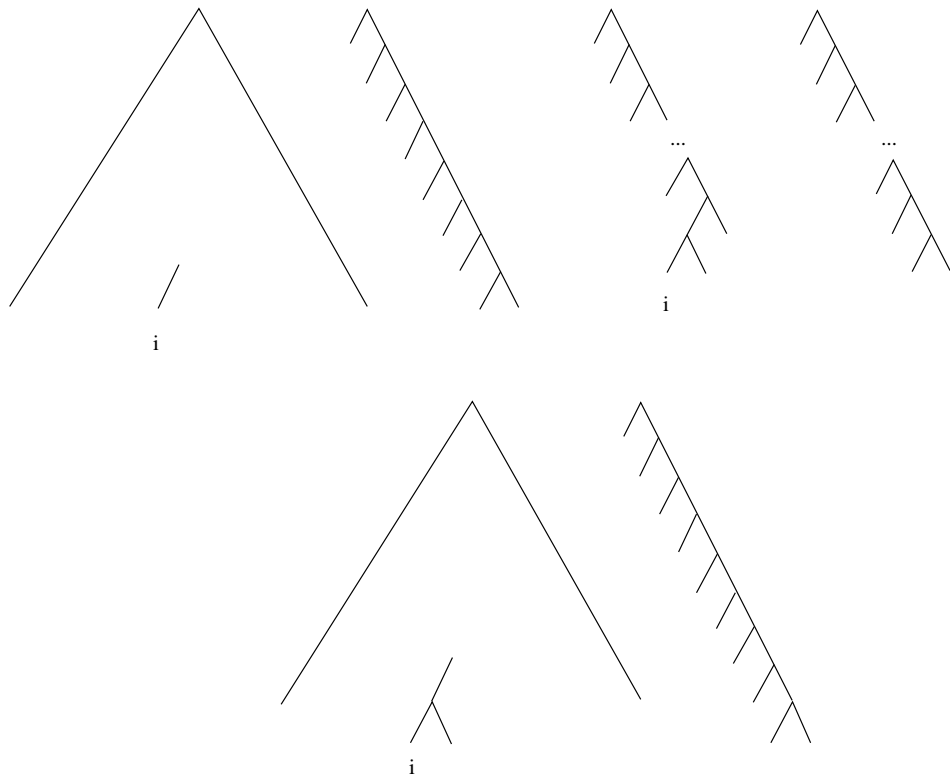


Figure 2.5: The proof of Theorem 2.2.4.

Proof. Observe that the tree diagrams for the generators f_n satisfy this theorem. The diagram for f_n has a tree with leaf exponent $a_n = 1$ (and the rest are zero), and then an all-right tree. Then we can continue by induction. Observe that if we multiply an element where the target tree is all-right by a generator f_i , the only thing we are doing is adding a caret to the i -th leaf. Continuing this process we observe that the result is satisfied. See Figure 2.5. \square

The tree of the Figure 2.4, together with an all-right tree with 11 carets, form a tree pair diagram of the element $f_0^2 f_1 f_3 f_5^3 f_6 f_9$.

What this proves, essentially, is that the f_n form a set of generators of F . An obvious corollary is now the surjectivity of Θ .

Proposition 2.2.5 *The map Θ is surjective.*

Proof. An element f of F is given by a tree pair diagram (T, T') . Observe that this tree pair diagram can be split into the product of two tree diagrams. If the trees T and T' have n carets, let R be an all-right tree with n carets. Then, we have that f is the product of the two elements given by (T, R) and

(R, T') , and according to Theorem 2.2.1, one of these elements is positive, while the other one is the inverse of another positive element. Using then leaf exponents for these two pairs of trees (one of them switched around) we see that f is a product of the form pq^{-1} , where p and q are positive elements. These can be given by

$$p = f_0^{a_0} f_1^{a_1} \cdots f_n^{a_n} \quad q = f_0^{b_0} f_1^{b_1} \cdots f_m^{b_m},$$

and then we see that f is the image of the element

$$x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} x_m^{-b_m} \cdots x_1^{-b_1} x_0^{-b_0}.$$

□

2.3 Normal forms

As we have just seen in the previous section, the relators can be used to find an expression for the elements of F (and also of G) which is of a very particular form, namely, first positive generators in ascending index, and then negative exponents in descending order of the indices. This is the first step in the construction of a normal form.

Proposition 2.3.1 *Every element of F admits an expression of the form*

$$f_0^{a_0} f_1^{a_1} \cdots f_n^{a_n} f_m^{-b_m} \cdots f_1^{-b_1} f_0^{-b_0}.$$

The proof has been spelled out in the previous section already going through the tree pair diagram, but we observe that the only thing needed to find this expression is the relators satisfied by the generators. Observe that the relators $f_i^{-1} f_j f_i = f_{j+1}$ are enough to find this expression, by just moving all the f_0 to the left and the f_0^{-1} to the right, continuing with the generator f_1 and its inverse, and continuing in order of the increasing index. Hence, the same result is true in the group G with the generators x_i .

This expression, sometimes called a *seminormal form*, is not unique, as we see by just using a relator in a slightly modified way, like $f_0 f_2 f_0^{-1} = f_1$. But the expression on the left is the example of how these expressions can be simplified to obtain the best possible one. See that the expressions where the generators f_i and f_i^{-1} both appear but neither one of the generators f_{i+1} or f_{i+1}^{-1} appear, can be reduced by applying relators to the generators in between, which are going to have an index $i+2$ or higher. Namely, if both a_i and b_i are different from zero and $a_{i+1} = b_{i+1} = 0$, the expression can be shortened by applying the relators.

Example 2.3.2 The element $f_0^2 f_1 f_2^3 f_4 f_6 f_5^{-1} f_2^{-1} f_1^{-1}$ is a reducible seminormal form, because a_2 and b_2 are nonzero but a_3 and b_3 are both zero. This element is the same as $f_0^2 f_1 f_2^2 f_3 f_5 f_4^{-1} f_1^{-1}$.

This phenomenon motivates the following result.

Theorem 2.3.3 *Every element in F admits an expression of the form*

$$f_0^{a_0} f_1^{a_1} \cdots f_n^{a_n} f_m^{-b_m} \cdots f_1^{-b_1} f_0^{-b_0},$$

where, for all i , if a_i and b_i are simultaneously nonzero, then either a_{i+1} or b_{i+1} (or both) is nonzero as well. Furthermore, this expression is unique, and it is the shortest expression for this element in the generators f_i .

This expression is called the *normal form* of the element. The condition on the exponents a_i and b_i is usually called the *extra condition* satisfied by normal forms.

Proof. The existence has been seen already. To prove uniqueness we borrow the proof appearing in [5]. Imagine that there are elements with more than one normal form, and take the two forms whose total length (adding both lengths of the two forms) is minimal among all pairs of normal forms representing the same element. Let

$$f_0^{a_0} f_1^{a_1} \cdots f_n^{a_n} f_m^{-b_m} \cdots f_1^{-b_1} f_0^{-b_0}$$

and

$$f_0^{c_0} f_1^{c_1} \cdots f_n^{c_n} f_m^{-d_m} \cdots f_1^{-d_1} f_0^{-d_0},$$

be these two normal forms. Let also be k the smallest index appearing in either one of these two forms, i.e., we have that $a_i = b_i = c_i = d_i = 0$ for $i < k$, and at least one of the a_k, b_k, c_k, d_k is nonzero. We first prove that the total exponent for f_k is the same in both forms, i.e. $a_k - b_k = c_k - d_k$. This is clear by appealing to the graph of this element as map in $[0, 1]$. Observe that since f_k is the smallest generator appearing, the support of this element is contained in $[1 - \frac{1}{2^k}, 1]$, and furthermore, the total exponent of this generator in the element is $\log_2 m$, where m is the right slope at the point $(1 - \frac{1}{2^k}, 1 - \frac{1}{2^k})$. A small analysis of the generators f_i for $i > k$ will convince the reader of this fact, since their supports are always smaller and they leave a neighborhood of the point $(1 - \frac{1}{2^k}, 1 - \frac{1}{2^k})$ unaffected. Hence both forms must have the same total exponent for f_k , since they both represent the same element and hence have the same graph.

The next step is to observe that, due to the minimality conditions, we must have that $a_k - b_k = c_k - d_k = 0$, that is, this common total exponent is zero. If it is positive, for instance, then both a_k and c_k are strictly positive, and a generator f_k could be cancelled from both forms to obtain shorter ones, contradicting minimality. If the total exponent is negative the same thing happens with b_k and d_k . So, the total exponent must be zero.

Finally, the fact just observed that a_k and c_k cannot be nonzero simultaneously (and neither can b_k and d_k), means that we must have that in one of the two expressions f_k must not appear at all. So we have that, for instance, $a_k = b_k = 0$ and c_k, d_k are both positive and equal. Hence, our two minimal expressions read $f_k z f_k^{-1}$ and w , where w has only indices strictly larger than k , and recall that they both must satisfy the conditions of normal forms. But then, moving the two generators f_k to the other side, we have that $z = f_k^{-1} w f_k$. Since all indices in w are larger than k , we can apply the relators to obtain $z = \bar{w}$, where \bar{w} is w with all indices increased by one. Since z and \bar{w} are shorter than the minimal ones, they must be exactly equal as forms (not only as elements of F). Hence, both z and \bar{w} have indices at least $k + 2$, and this contradicts the fact that $f_k z f_k^{-1}$ was a normal form, because it does not satisfy the extra condition. The uniqueness is thus proved.

Observe that given any word in the f_i , the process which is used to construct the normal form only decreases total length. Indeed, switching relators maintains length, and then we can cancel adjacent inverses or perform the reduction needed if the extra condition is not satisfied. Both these operations decrease length. Hence, the unique normal form must be the shortest, because if there were a shorter word, we could find a normal form from it which would be even shorter, contradicting the uniqueness.

The theorem is now proved in its entirety. □

The existence of a unique normal form for the elements of F has many interesting consequences, which will be exploited throughout this book. The first one is the injectivity of the map Θ and hence the isomorphism between the groups F and G .

Proposition 2.3.4 *The map Θ is one-to-one.*

Proof. Take two words in the generators x_i which map to the same element in F . Their image in F has a unique normal form which can be found from both images using the relators in F . Hence, using the same relators but now in G , we can arrive from both words to the same normal form too, and then, also using the relators, we could go from one word to the other. Hence the two words represent the same element. □

This finishes the proof of Theorem 2.1.1. From now on, we will eliminate from our notation the f_i and use only the x_i , and also the notation G for the group, using F at all times.

Our next goal is to see that the unique normal form is closely related to a reduced tree pair diagram.

Observe what are the leaf exponents of a nonreduced diagram. Consider an exposed caret (i.e. a reducible caret, with two leaves, not nodes), whose leaves are labelled i and $i + 1$ in the tree. Then, we have that as leaf exponents, a_i is nonzero but a_{i+1} is zero. If the two trees of the nonreduced diagram have the caret labelled $i, i + 1$ exposed, its leaf exponents satisfy that a_i and b_i are nonzero while $a_{i+1} = b_{i+1} = 0$. Then the extra condition is not satisfied and the form can be shortened, operation which corresponds exactly to reducing the diagram. Note too that if the exposed carets are on the right side of the tree, they do not affect the normal form or the leaf exponents, and can be reduced with no change.

Hence, we observe that using Theorem 2.2.4 about leaf exponents, there is an exact correspondence between tree pair diagrams and seminormal forms, which in particular matches normal forms with reduced diagrams. Uniqueness of these is now apparent.

Theorem 2.3.5 *Each element of F admits a unique reduced tree pair diagram.*

This correspondence is key in understanding both normal forms and tree pair diagrams. Since the latter correspond to maps in $[0, 1]$, we see that all three incarnations of F are totally interchangeable. From now on in this book, an element of F will be represented indistinctly as a map of $[0, 1]$, or by a reduced tree pair diagram, or by its unique normal form in the generators x_i , and we will interchange them at will. It is one of the most interesting features of F that there are these seemingly different ways of understanding it. We have already seen an example of this behavior when proving that Θ is an isomorphism. Surjectivity was best proved using diagrams, but injectivity is proved using maps. This phenomenon is constant throughout the study of F , depending on what one wants to prove, one can use one interpretation or another, according to the particular needs.

2.4 The finite presentation

When one observes closely the infinite presentation for F , it is easy to notice that, due to the relators $x_0^{-1}x_nx_0 = x_{n+1}$, only x_0 and x_1 are enough to generate the whole group. So clearly F is finitely generated. The purpose of this section is to prove that F is actually finitely presented, in particular, only two relations are necessary to present F .

Theorem 2.4.1 *Thompson's group F admits the following presentation:*

$$\langle a, b \mid [ab^{-1}, a^{-1}ba], [ab^{-1}, a^{-2}ba^2] \rangle.$$

Here $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y .

Proof. The proof of this fact is standard, although slightly long. One only needs to find relators from one presentation in terms of those of the other. Properly speaking, let H be the group given by this finite presentation. We will define two maps

$$\Phi : F \longrightarrow H \quad \Psi : H \longrightarrow F$$

which will be inverses from each other. As usual, the maps will be defined by the images of the generators. The proof will then be reduced to show that the maps are well defined, that is, to see that the relators are satisfied in the images. Define the map Ψ as

$$\Psi(a) = x_0 \quad \Psi(b) = x_1.$$

Then the relators on a and b , when rewritten in the x_i , are precisely conjugates of $x_1^{-1}x_2x_1 = x_3$ and $x_1^{-1}x_3x_1 = x_4$. Indeed,

$$\begin{aligned} \Psi([ab^{-1}, a^{-1}ba]) &= \Psi(ab^{-1}a^{-1}baba^{-1}a^{-1}b^{-1}a) \\ &= x_0x_1^{-1}x_0^{-1}x_1x_0x_1x_0^{-1}x_0^{-1}x_1^{-1}x_0 \\ &= x_0x_1^{-1}x_2x_1x_0^{-1}x_2^{-1}. \end{aligned}$$

This word is the identity because conjugating it by x_0 we obtain

$$x_1^{-1}x_2x_1x_0^{-1}x_2^{-1}x_0 = x_1^{-1}x_2x_1x_3^{-1}$$

as desired. These equalities are best seen in a picture, see Figure 2.6. The second relator is done in a very similar way.

For the inverse homomorphism, define $\Phi(x_0) = a$ and $\Phi(x_1) = b$, necessarily so if Φ has to be the inverse of Ψ . And also, if the relators have to be satisfied, we need to have

$$\Phi(x_n) = a^{-n+1}ba^{n-1}$$

for all $n \geq 2$.

We need to see that the relators $x_i^{-1}x_jx_i = x_{j+1}$ are satisfied for all $0 \leq i < j$ when one replaces the x_n by their images. If $i = 0$, then the definitions of the images of the x_n make sure that this relators are satisfied in an obvious manner. Also, if $i \geq 2$, we have that the relator is conjugate to one with $i = 1$, because

$$x_0^{i-1} (x_i^{-1}x_jx_ix_{j+1}^{-1}) x_0^{-i+1} = x_1^{-1}x_{j-i+1}x_1x_{j-i+2}^{-1}.$$

If this relator maps to the identity in terms of a and b , then conjugating it by a power of a we get the previous one.

So we only need to verify the relators $x_1^{-1}x_jx_1 = x_{j+1}$ for $j \geq 2$. And it is very easy to check that for $j = 2$ and $j = 3$ the two images are precisely the two relators in a and b , so these are satisfied. The remaining relators will be checked by induction in j . See Figure 2.7 for a picture which should make apparent the fact. Take $j \geq 4$ and rewrite it the following way

$$b^{-1}x_jbx_{j+1}^{-1} = b^{-1}a^{-1}x_{j-1}aba^{-1}x_j^{-1}a$$

Rewrite aba^{-1} using the first relation as $b^{-1}aba^{-1}a^{-1}b^{-1}a$. Then each one of these combined with x_{j-1} or lower, is a definition relator or one given by the induction hypothesis. Again, see Figure 2.7.

This concludes the proof. □

So the group F is finitely presented. This finite presentation is awkward since the relations are long and complicated, and it is difficult to see anything from them. But at least we know that the group admits a presentation with only two generators and two relators. For all practical purposes though, we will use predominantly the infinite presentation.

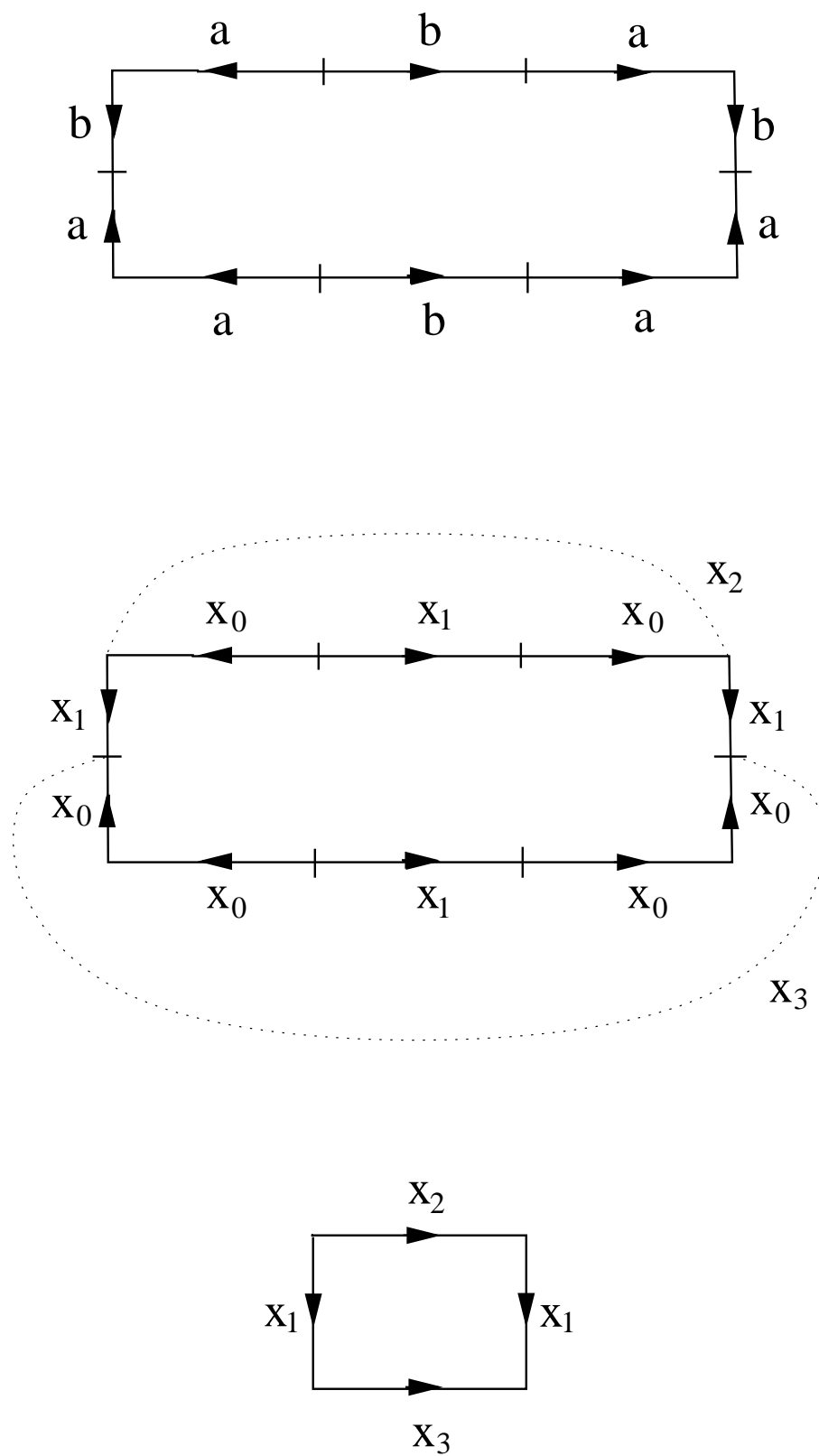


Figure 2.6: Checking the first relator in a and b when mapped into the x_i .

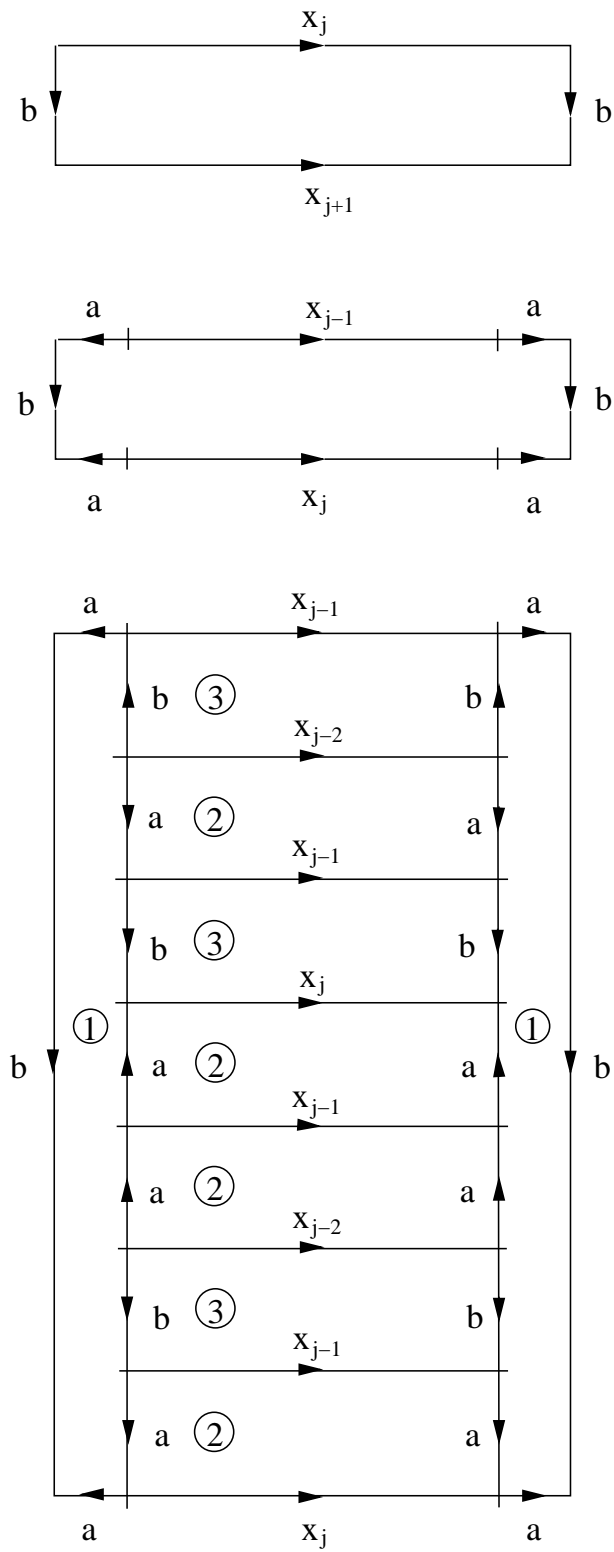


Figure 2.7: The relator $x_1^{-1}x_jx_1 = x_{j+1}$ in terms of the relators in a and b . The cells in the bottom diagram are marked the following way: (1) the first relator on a and b , (2) the definition of the x_i , and (3) the induction hypothesis.

Chapter 3

Further properties of F

From the different ways we have to study the group F (as maps of the interval, as pairs of binary trees, or algebraically from its presentations) we can now deduce some more properties of F . Many of these are proved using the incarnation of F as maps of the interval, taking full advantage of the dynamics of the maps. Combining these interpretation with the algebraic expressions gives fruitful results. The interpretation as binary trees will be most useful in the next chapter, when we study the metric properties of F .

3.1 Transitivity and F -subgroups

An elementary but extremely useful property is the n -transitivity of maps in $[0, 1]$, namely, that maps in F can send any given sequence of elements to another. This property will be used later to study subgroups with a given support, for instance, or to show that maps in F show enough versatility to send a given interval to another.

This study of transitivity begins with an elementary but useful lemma.

Lemma 3.1.1 *Let $a/2^n$ be a dyadic number, with $a > 0$, and let b be another integer with $a \leq b$. Then, the interval $[0, a/2^n]$ can be divided into b intervals, all of them having as length a power of 2 (although the power may be different for each interval).*

Proof. First divide the interval in a subintervals, each of them of length $1/2^n$. Since $b \geq a$, we only need to subdivide some of these intervals (maybe more than once) into two intervals of equal length. Each subdivision increases the

number of intervals by one, so subdividing $b - a$ intervals will provide the required number of subintervals. Each of them has as length a power of 2, being subdivisions of intervals of length $1/2^n$. \square

This lemma is useful to map any dyadic interval into another, since they both can be subdivided into the same number of subintervals which can then be mapped linearly. An iterated use of this fact yields the n -transitivity.

Theorem 3.1.2 *Let $x_0 < x_1 < \dots < x_n$ and $y_0 < y_1 < \dots < y_n$ be dyadic numbers in the interval $[0, 1]$, and assume $x_0 = y_0 = 0$ and $x_n = y_n = 1$. Then, there exists a map $f \in F$ such that $f(x_i) = y_i$ for all $i = 0, 1, \dots, n$.*

Proof. We will construct the map f piece by piece in each of the intervals $[x_i, x_{i+1}]$. Assume this interval has length $a_i/2^{n_i}$, and also let $b_i/2^{m_i}$ be the length of the interval $[y_i, y_{i+1}]$. Assume without loss of generality that $a_i \leq b_i$. Then subdivide $[y_i, y_{i+1}]$ into b_i intervals of length $1/2^{m_i}$, and, using the previous lemma, also subdivide $[x_i, x_{i+1}]$ into b_i subintervals of length some power of 2. Construct now the map linearly in each subinterval. This produces a piecewise-linear map, whose slopes are all powers of two, and the breaks are all dyadic. Doing this for all intervals $[x_i, x_{i+1}]$, for $i = 0, 1, \dots, n - 1$ will produce the map f . \square

Hence, the group F is n -transitive. In particular, given a pair of dyadics a and b , there is always a map sending a to b , and also, given two intervals $[a, a']$ and $[b, b']$, there is always a map in F which sends the interval $[a, a']$ exactly into $[b, b']$. What this means is that each element plays the same role (except obviously 0 and 1), and also that each dyadic subinterval is exactly like any other.

As a consequence of the transitivity we can prove that F has many subgroups isomorphic to F itself. This somewhat surprising fact will also be very useful in the future, when we study the commutator subgroup and its incidence with subgroups of F .

Theorem 3.1.3 *Let a and b two dyadic numbers in $[0, 1]$. Then, the subgroup of elements of F which have support included in the interval $[a, b]$ is isomorphic to F .*

From now on, due to the fact that they will be used extensively, we will name $F[a, b]$ the subgroup of those elements of F with support included in $[a, b]$. Recall that this means that these elements are the identity outside this interval.

To prove this theorem we will use transitivity to reduce it to the special case of some particular intervals. We start with this fact:

Proposition 3.1.4 *The subgroups $F[0, \frac{1}{2}]$, $F[\frac{1}{4}, \frac{3}{4}]$ and $F[\frac{1}{2}, 1]$ of F are all isomorphic to F .*

Proof. It is very easy to construct linear maps which send each of these subintervals to the whole $[0, 1]$ and which preserve dyadics. Then, conjugating with this map gives the desired result. For instance, for $F[0, \frac{1}{2}]$, take any element $f \in F$ and construct the following map

$$\bar{f}(t) = \begin{cases} \frac{f(2t)}{2} & \text{for } 0 \leq t \leq \frac{1}{2} \\ t & \text{for } \frac{1}{2} < t \leq 1 \end{cases}.$$

It is straightforward to show that this correspondence is a homomorphism which sends bijectively F to $F[0, \frac{1}{2}]$. Observe that the crucial part of this proof is that the conjugating map $y = 2x$ preserves dyadics.

A similar correspondence given by

$$\bar{f}(t) = \begin{cases} \frac{2f\left(2t - \frac{1}{2}\right) + 1}{4} & \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \\ t & \text{for } t \notin [\frac{1}{4}, \frac{3}{4}] \end{cases}$$

gives the same result for $F[\frac{1}{4}, \frac{3}{4}]$. The case $[\frac{1}{2}, 1]$ is left to the reader. \square

These particular subgroups of F and the fact that they are isomorphic to F itself can be easily understood using binary tree diagrams. See Figure 3.1 to see the correspondence written out in diagrams.

Using this result we can now prove Theorem 3.1.3.

Proof of Theorem 3.1.3. We have to distinguish three cases for the interval $[a, b]$, depending on whether the endpoints coincide with 0 or 1 or not. Each of these cases can be modeled on one of the three particular cases above using the transitivity. For instance, if $0 < a < b < 1$, we only need to find an element $\alpha \in F$ such that $\alpha(\frac{1}{4}) = a$ and $\alpha(\frac{3}{4}) = b$. Conjugating $F[\frac{1}{4}, \frac{3}{4}]$ by α we obtain $F[a, b]$, and this conjugation is an isomorphism of those groups. Do a similar thing for the cases $[0, b]$ and $[a, 1]$. \square

This transitivity is an elementary feature for F , but it will have quite interesting consequences later.

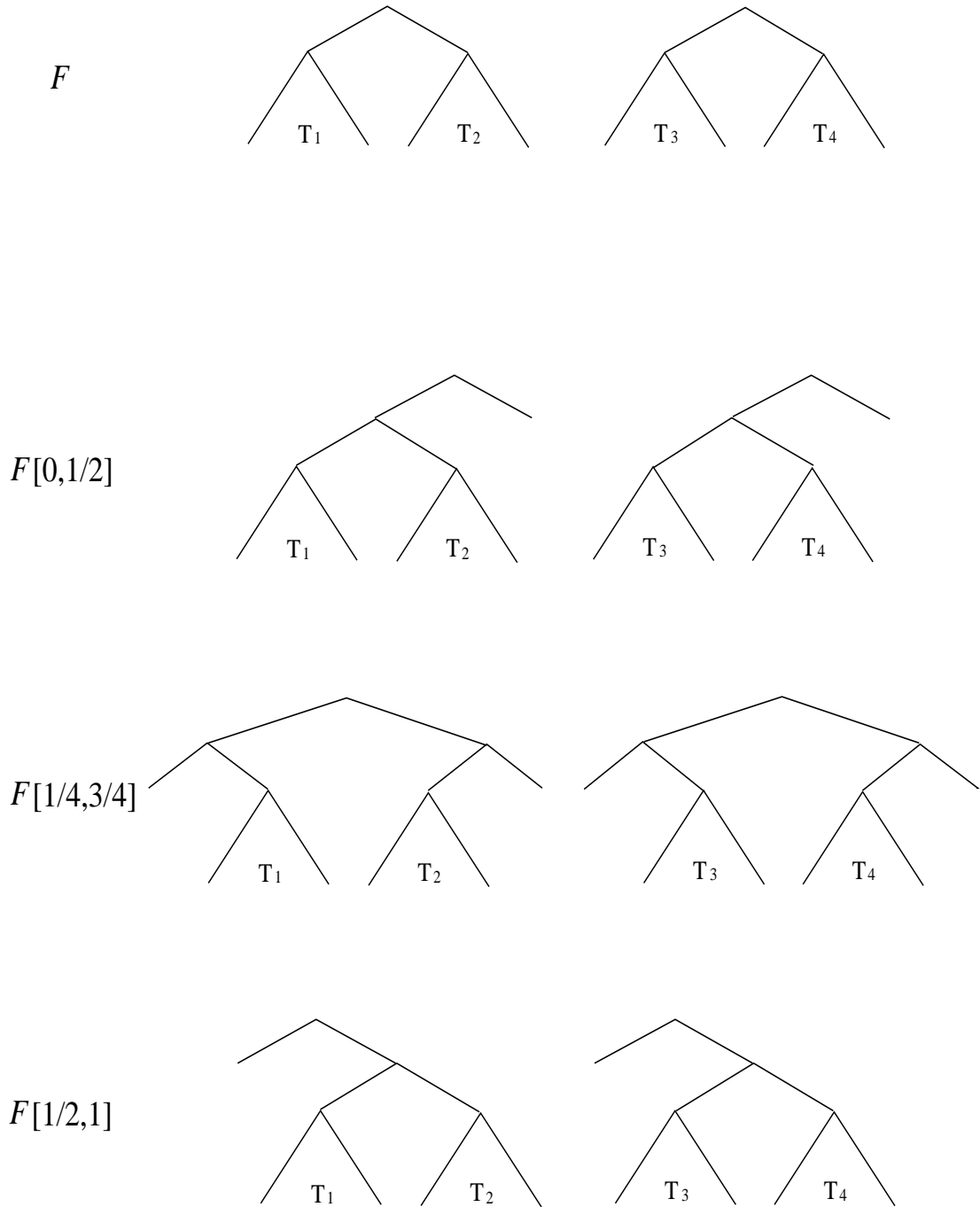


Figure 3.1: An element in F and its corresponding elements in $F[0, \frac{1}{2}]$, $F[\frac{1}{4}, \frac{3}{4}]$ and $F[\frac{1}{2}, 1]$. The trees T_i can be any trees, even empty. Observe that if the diagram of the element in F is reduced, then the other three diagrams are reduced as well, which reaffirms the fact that this correspondences are isomorphisms.

3.2 Abelianization

A clear feature of an element of F , seen as a map in $[0, 1]$, is the initial slope at 0 as well as the final slope at 1. An element of F is always of the form $f(t) = 2^k t$ for some integer k , and for t in a neighborhood of 0. Note also that when we compose two elements of F , the slopes at zero get multiplied directly, in some neighborhood of the origin. Hence we can define a map

$$\pi^{\text{ab}} : F \longrightarrow \mathbb{Z}^2$$

which sends an element $f \in F$ to $(k, l) \in \mathbb{Z}^2$ if the right-slope at 0 is 2^k and the left-slope at 1 is 2^l . The fact that these slopes are multiplied when two elements are composed means exactly that π^{ab} is a homomorphism.

But observe that F can be generated with two elements, so its abelianization can also be generated by two elements. Any abelian group generated by two elements is a quotient of \mathbb{Z}^2 , so since we already have a map onto it, this map has to be the abelianization map.

Theorem 3.2.1 *The map π^{ab} is the abelianization map.*

As a consequence of this fact, we can easily describe the commutator of F , being the kernel of the abelianization. We will denote the commutator subgroup $[F, F]$ by F' .

Theorem 3.2.2 *The commutator F' of F is exactly the subgroup of all those elements of F which are the identity in a neighborhood of 0 and in a neighborhood of 1.*

Namely, an element f of F is in the commutator subgroup if there exists an $\varepsilon > 0$ such that the support of f is included in $[\varepsilon, 1 - \varepsilon]$.

It is also interesting to observe that as a consequence of the transitivity of F , we can prove a similar transitivity property for F' .

Theorem 3.2.3 *Let $x_0 < x_1 < \dots < x_n$ and $y_0 < y_1 < \dots < y_n$ be dyadic numbers in the interval $[0, 1]$, and assume $x_0 = y_0 = 0$ and $x_n = y_n = 1$. Then, there exists a map $f \in F'$ such that $f(x_i) = y_i$ for all $i = 0, 1, \dots, n$.*

Proof. We only need to apply the transitivity theorem for F in a smaller interval. Choose two dyadic numbers a and b such that

$$0 < a < \min\{x_1, y_1\} \quad \text{and} \quad \max\{x_{n-1}, y_{n-1}\} < b < 1.$$

We apply now the transitivity theorem to $F[a, b]$, having the points x_1, \dots, x_{n-1} and y_1, \dots, y_{n-1} inside $[a, b]$, and find an element in $F[a, b]$ which maps these points the appropriate way. Extending the map with the identity outside $[a, b]$ gives the desired map in F' . \square

Hence, n -transitivity is also true for F' .

3.3 The commutator subgroup and quotients

The most important fact about the commutator subgroup is the fact that it is a simple group. Recall that a group is simple if it does not have any normal subgroups except the trivial ones (the identity and the whole group).

Theorem 3.3.1 *The group F' is simple.*

The proof of this fact will be given by the combination of the next two results.

Let F'' be the double commutator group, namely, $F'' = [F', F']$.

Proposition 3.3.2 *We have $F' = F''$.*

Proof. It is clear that $F'' \subset F'$. So let $f \in F'$, and let $[a, b]$ be its support. Choose two dyadics c and d such that $0 < c < a$ and $b < d < 1$. Clearly, we have that $f \in F[a, b] \subset F[c, d] \subset F'$. But observe that $F[a, b]$ is actually inside $F'[c, d]$, where $F'[c, d]$ denotes the commutator subgroup of $F[c, d]$. Since $F[c, d] \subset F'$, we obtain that $f \in F''$. \square

As an obvious corollary of this result, we have that F is not nilpotent or solvable.

The key to the proof of the fact that F' is simple, is its transitivity, and the fact that one can send any interval inside $[0, 1]$ to another. Since the elements of F' have a support which is strictly included in $[0, 1]$, all supports are similar for all elements of F' . The result is the following result, proved by Higman in 1954. The proof is taken from Higman's paper [18].

Let G be a group of bijections of some set E . For $g \in G$ define its support $\text{supp}(g)$ as the set of points in $x \in E$ such that $g(x) \neq x$.

Theorem 3.3.3 *Let α and β be two elements of such a group G . Call $S = \text{supp}(\alpha) \cup \text{supp}(\beta)$. And let $\gamma \in G$ be an element, with $\gamma \neq 1$. If, for these elements α, β, γ we can find $\rho \in G$ such that $\gamma(\rho(S))$ and $\rho(S)$ are disjoint, then G' is simple.*

Despite its apparently obscure statement, what Higman's result indicates is that if α and β have a combined support which can be sent inside the support of ρ and is moved completely by γ , then what the result says is that the commutator of α and β is inside any normal subgroup of G' . Hence, if a normal subgroup contains a nontrivial γ , then it contains every commutator and hence G' is simple.

The proof will need the following lemma.

Lemma 3.3.4 *Let G be a group satisfying the conditions of Theorem 3.3.3, and let N be a normal subgroup of G , with $N \neq 1$. Then, $G' \subset N$.*

Proof of Lemma 3.3.4. By hypothesis there exists $\gamma \neq 1$ in N . Let α and β be any two elements of G . And let ρ be the element which exists by the hypothesis of the Theorem. Consider $\delta = \rho^{-1}\gamma\rho$, which is in N since N is normal in G . As a consequence of our assumption, we have that $\gamma(\rho(\text{supp}(\alpha)))$ and $\rho(\text{supp}(\beta))$ are disjoint, and then applying ρ^{-1} we have that $\delta(\text{supp}(\alpha))$ and $\text{supp}(\beta)$ are disjoint as well. But since $\delta(\text{supp}(\alpha)) = \text{supp}(\delta^{-1}\alpha\delta)$, we have that $\delta^{-1}\alpha\delta$ and β commute.

In that case, we have

$$\begin{aligned}\alpha^{-1}\beta^{-1}\alpha\beta &= \alpha^{-1}(\delta^{-1}\alpha\delta)\beta(\delta^{-1}\alpha^{-1}\delta)\alpha\beta \\ &= (\alpha^{-1}\delta^{-1}\alpha)\delta(\beta\delta^{-1}\beta)(\beta^{-1}\alpha^{-1}\delta\alpha\beta)\end{aligned}$$

and we see that the commutator $[\alpha, \beta]$ is product of four conjugates of δ , which is a conjugate of $\gamma \in N$. Hence, for any α and β , we have that $[\alpha, \beta] \in N$. So $G' \subset N$. \square

Proof of Theorem 3.3.3. For our group G satisfying the hypothesis, the first thing we will prove is that $G' = G''$. Apply Lemma 3.3.4 to $N = G''$. Since G'' is normal in G , we only need to find an element of G'' which is not the identity.

Since the case $G' = 1$ is trivial, take $\alpha = \beta = \gamma \neq 1$ in G' and apply the hypothesis, so by replicating the proof of Lemma 3.3.4, we obtain a conjugate δ of γ such that $\text{supp}(\delta^{-1}\gamma\delta)$ and $\text{supp}(\gamma)$ are disjoint. Hence, γ and $\delta^{-1}\gamma\delta$ are actually different, and then $\gamma^{-1}\delta^{-1}\gamma\delta$ is nontrivial. Since both γ and δ are in G' , we have found a nontrivial element of G'' . The conclusion, using the lemma, is that $G' = G''$.

We want to apply the lemma to G' and a normal subgroup in it, but then we need to verify that G' also satisfies the hypotheses of the theorem. Clearly G' is in G , so it also is a group of bijections of E . Take α, β in G' , and also

$\gamma \in G'$ which is not 1. We can choose ρ , but the hypothesis only assures us that $\rho \in G$. To solve this, apply the hypothesis of the theorem now to γ and ρ (playing the roles of α and β) and also to the same $\gamma \neq 1$. Again, we can choose σ such that $\gamma\sigma(\text{supp}(\gamma) \cup \text{supp}(\rho))$ and $\sigma(\text{supp}(\gamma) \cup \text{supp}(\rho))$ are disjoint. Again, following the proof of the lemma, we can consider $\eta = \sigma\gamma\sigma^{-1}$ for which $\text{supp}(\gamma)$ and $\text{supp}(\eta^{-1}\rho\eta)$ are disjoint. In particular, $\eta^{-1}\rho\eta$ is the identity on $\rho(\text{supp}(\alpha) \cup \text{supp}(\beta))$, which is contained in $\text{supp}(\gamma)$. Then,

$$\rho(\text{supp}(\alpha) \cup \text{supp}(\beta)) = \eta^{-1}\rho^{-1}\eta\rho(\text{supp}(\alpha) \cup \text{supp}(\beta))$$

and the hypothesis of the theorem can be applied to G' , because we have found an element of G' which plays the role of ρ .

Hence, the lemma can be applied to G' . Let N be any normal subgroup of G' , different from 1. By Lemma 3.3.4, $G'' \subset N$, and since $G' = G''$, we have $N = G'$ and then G' is simple. \square

Observe that the condition of the theorem is not applied to F , but to F' . The condition is not true for F because if α or β has as support the whole $[0, 1]$, this support cannot be moved inside a smaller interval. But it is satisfied in F' , because supports are smaller. Taking any α and β , we have that their combined support is an interval included inside $[\varepsilon, 1 - \varepsilon]$ for some ε . Since $\gamma \neq 1$, there is an interval I such that $\gamma(I)$ and I are disjoint. Using the transitivity for F' , choose a ρ which sends $\text{supp}(\alpha) \cup \text{supp}(\beta)$ inside I . This guarantees Higman's condition, and the conclusion would be that F'' is simple. Having proved before that $F'' = F'$, we can finally conclude that F' is simple and Theorem 3.3.1 is proved.

This interesting fact has some useful consequences for the structure of F . The most important one is that all nontrivial quotients of F are abelian. To prove this result we need the following group-theoretic fact. Recall that the center of a group contains those elements which commute with every element of the group.

Proposition 3.3.5 *The center of F is trivial.*

Proof. Let x be a nontrivial element of F , and assume for now that x is not a power of x_0 . Then, x does not commute with x_0 . Let

$$x_0^{a_0} x_1^{a_1} \dots x_n^{a_n} x_m^{-b_m} \dots x_1^{-b_1} x_0^{-b_0}$$

be its normal form. Observe that because of our assumption at least one of the a_i or b_i is nonzero for some $i > 0$. If x were to commute with x_0 , the element $x_0^{-1}xx_0$ would have the exact same normal form as x , and this is never the case. The proof depends on three cases.

- Assume $a_0 = b_0 = 0$. Then the normal form for $x_0^{-1}xx_0$ is

$$x_2^{a_1} \dots x_{n+1}^{a_n} x_{m+1}^{-b_m} \dots x_2^{-b_1}.$$

This situation is obtained looking at the process used to get this normal form. The conjugating x_0 at the end needs to move to the beginning of the word, where it cancels with its inverse, increasing all indices in the meantime.

- Assume $a_0 > 0$ and $b_0 = 0$. The process is very similar. The x_0^{-1} at the beginning cancels with one of the $x_0^{a_0}$, but the x_0 at the end will again move to the beginning, increasing all the other indices. Hence the new normal form is now

$$x_0^{a_0} x_2^{a_1} \dots x_{n+1}^{a_n} x_{m+1}^{-b_m} \dots x_2^{-b_1}.$$

Observe that in this case it is crucial that x is not a power of x_0 , hence this normal form is actually different from the one for x . The case where $a_0 = 0$ and $b_0 > 0$ is symmetric and is solved the same way.

- Finally, if $a_0 > 0$ and $b_0 > 0$ then the conjugating element just cancels one x_0 at each end and we obtain

$$x_0^{a_0-1} x_1^{a_1} \dots x_n^{a_n} x_m^{-b_m} \dots x_1^{-b_1} x_0^{-b_0+1}.$$

In any case, the normal form of the conjugate is different. Hence, by uniqueness of the normal form, we have that x does not commute with x_0 .

To end the proof, we just need to observe that x_0^a , does not commute with x_1 (for $a \neq 0$ obviously), as it is elementary to verify, again using normal forms. Hence, there is no element which commutes with every element of F . \square

As an example of the versatility of F , note that there is a proof of this result in [8] using the dynamics of the maps in $[0, 1]$.

Using this fact for the center of F we can prove the result we wanted.

Theorem 3.3.6 *Let*

$$\varphi : F \longrightarrow Q$$

be a surjective homomorphism. Then, either φ is an isomorphism, or Q is abelian.

Proof. If φ is not an isomorphism, then there exist $x \in \ker \varphi$, where $x \neq 1$. Since there is no center, there exists an element y such that $[x, y] =$

$xyx^{-1}y^{-1} \neq 1$. But observe that $[x, y] \in \ker \varphi \cap F'$, and this subgroup is normal in F' . Since F' is simple, and $[x, y]$ is not trivial, then we must have $F' \subset \ker \varphi$, and then Q is abelian. \square

Hence, every nontrivial quotient of F is abelian. The structure of normal subgroups of F is actually quite restricted, since each nontrivial normal subgroup must contain F' . The nontrivial normal subgroups of F are in one-to-one correspondence with subgroups of \mathbb{Z}^2 , via the preimage by the abelianization map.

A corollary for this fact is that F is not residually finite, because any nontrivial element $x \in F'$ can never be separated of 1 by a subgroup of finite index. If we have a map onto a finite group, this finite group is always abelian, and x is always in its kernel, since the kernel must contain F' .

Chapter 4

Metric properties

4.1 Estimates of the word metric

The Cayley graph of F , with respect to x_0 and x_1 , is an awkward and complicated object, which is practically impossible to visualize, and from which very few properties can be deduced. This is because the two relators are actually quite long (lengths 10 and 14 respectively). However, there is a general principle which is true for many groups which holds also for F : the more complicated an element is, the farther away it is from the identity in the Cayley graph.

Hence, we can try to estimate the word length of an element in terms of its complexity, namely, in terms of the normal form or the number of carets.

Recall that the word metric of an element with respect for a finite generating set is the smallest number of generators needed to write it, i.e., the length of a shortest word representing it. The word metric of an element $x \in F$ will be denoted by $\|x\|$.

Definition 4.1.1 *Let x be an element of F with normal form*

$$x_0^{a_0} x_1^{a_1} \dots x_n^{a_n} x_m^{-b_m} \dots x_1^{-b_1} x_0^{-b_0}$$

and assume here that both a_n and b_m are not zero. We define the estimate of the word metric D as

$$D(x) = a_0 + a_1 + \dots + a_n + b_0 + b_1 + \dots + b_m + n + m.$$

Also, denote by $N(x)$ the number of carets that appear in either tree of the reduced tree diagram for x .

Both these quantities $D(x)$ and $N(x)$ give estimates of the word metric, up to multiplicative constants. We must emphasize that even though we work (for sake of regularity and easiness) with the infinite presentation for F , the word metric is always the one for the generating set $\{x_0, x_1\}$.

Theorem 4.1.2 *There exists a constant $C > 0$ such that the inequalities*

$$\frac{D(x)}{C} \leq \|x\| \leq C D(x) \quad \text{and} \quad \frac{N(x)}{C} \leq \|x\| \leq C N(x)$$

hold for every element $x \in F$.

Proof. Using Theorem 2.2.4, about the correspondence of normal forms with leaf exponents in tree diagrams, we obtain that all four inequalities

$$\begin{aligned} N(x) &\geq a_0 + a_1 + \dots + a_n & N(x) &\geq b_0 + b_1 + \dots + b_m \\ N(x) &\geq n & N(x) &\geq m \end{aligned}$$

hold. Hence, from here we have that $D(x) \leq 4N(x)$. To obtain the upper bound for the metric, just rewrite the normal form in terms of x_0 and x_1 , to see a word in x_0 and x_1 representing x . The key fact is that the term

$$x_i^{a_i} x_{i+1}^{a_{i+1}},$$

for $i \geq 2$, can be rewritten as

$$x_0^{-i+1} x_1^{a_i} x_0^{i-1} x_0^{-i} x_1^{a_{i+1}} x_0^i = x_0^{-i+1} x_1^{a_i} x_0^{-1} x_1^{a_{i+1}} x_0^i$$

and observe the cancellation of the generators x_0 between two instances of x_1 to leave only one of them. Hence the normal form becomes

$$\begin{aligned} &x_0^{a_0} x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} x_m^{-b_m} \dots x_1^{-b_1} x_0^{-b_0} = \\ &x_0^{a_0} x_1^{a_1} x_0^{-1} x_1^{a_2} x_0^{-1} \dots x_0^{-1} x_1^{a_n} x_0^{n-1} x_0^{-m+1} x_1^{-b_m} x_0 \dots x_0 x_1^{-b_2} x_0 x_1^{b_1} x_0^{b_0} \end{aligned}$$

and this word has length exactly

$$a_0 + a_1 + \dots + a_n + 2n - 2 + b_0 + b_1 + \dots + b_m + 2m - 2$$

which is bounded above by $2D(x)$. Hence we obtain the upper bounds for the word metric, namely, we have proved that

$$\|x\| \leq 2D(x) \leq 8N(x).$$

To obtain the lower bound, we only need to observe that the reduced diagram for x_0 has just two carets, and the one for x_1 has three. If we have a diagram

representing an element and we multiply it by a generator, we see by the way the composition is carried out in diagrams, that the number of carets can increase by, at most, two carets. The reader will be convinced by the fact that to match any tree with a tree for x_1 , the tree only needs to have at most two carets added, because the root is always in any tree.

Hence, if an element has length $\|x\|$, starting with (at most) three carets for the first generator and increasing it by two each multiplication by a generator, the largest number of carets it can have is $2\|x\| + 1$, or bounding it crudely, at most $3\|x\|$. Hence

$$N(x) \leq 3\|x\|,$$

which combines with the relation between N and D to obtain

$$\frac{D(x)}{12} \leq \frac{N(x)}{3} \leq \|x\|$$

to finish the lower bounds and the proof. \square

These estimates of the word metric, even being accurate only up to a multiplicative constant, are already useful to prove some results for F . These estimates are most useful by their simplicity: one of them can be read directly from the normal form, and the other one from the tree diagram. Hence, if one only needs metrics accurate up to a constant, they provide very simple estimates.

An example of this fact is the following result. This fact is already proved in CFP by using direct algebraic methods.

Theorem 4.1.3 *Thompson's group F has exponential growth.*

Proof. Having exponential growth does not depend on a multiplicative constant. If we are trying to bound from below the number of elements x which satisfy $\|x\| \leq n$, by the inequalities above we have that among these elements, we have all those that satisfy $N(x) \leq n/C$. So it is enough to see that the number of reduced tree pair diagrams which have n carets is exponential in n , and this is a well-known fact of Catalan numbers. Indeed, consider, to avoid nonreduced diagrams, only positive elements, which recall that have a diagram with an all-right tree as a target tree. Also, to ensure that the diagram is reduced with the all-right target tree, assume that the root caret of the source tree has an empty right child, see Figure 4.1. Then there are as many elements of that form with n carets as there are binary trees with $n - 1$ carets, which is known to be exponential in n . According to this simplification, the number of elements with n carets is at least

$$\frac{1}{n} \binom{2n-2}{n-1}$$

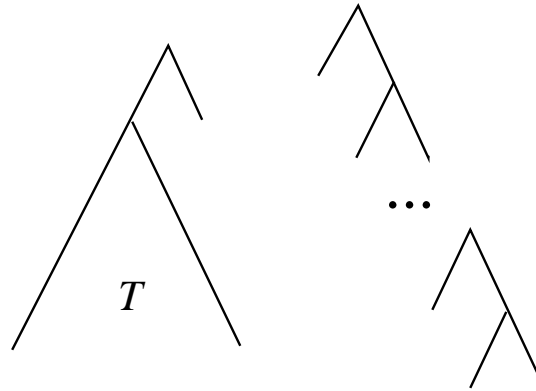


Figure 4.1: A positive element of the type counted in Theorem 4.1.3. Observe that no matter what the tree T is, this diagram is always reduced. So the number of elements with n carets is bounded below by the number of binary trees of $n - 1$ carets. This is a Catalan number.

which grows as 4^n .

□

4.2 Fordham's algorithm

In his PhD thesis from 1995 at Brigham Young University, S. Blake Fordham described an algorithm to compute the word length of an element in F , with respect to the generators x_0 and x_1 . The base for this method is the description of an element as a tree pair diagram. Fordham classifies each caret in a binary tree in one of seven types, according to its relative position in the tree. Once each caret is classified, carets are paired using the total order on them, and each pair of carets is assigned a weight, an integer between zero and four. The total weight for the diagram is the length of the element.

This remarkable result has been used extensively by researchers since then, to obtain several results where the exact length is needed (and not only an estimate up to a multiplicative constant). Examples of these works are [6], [10] or [11].

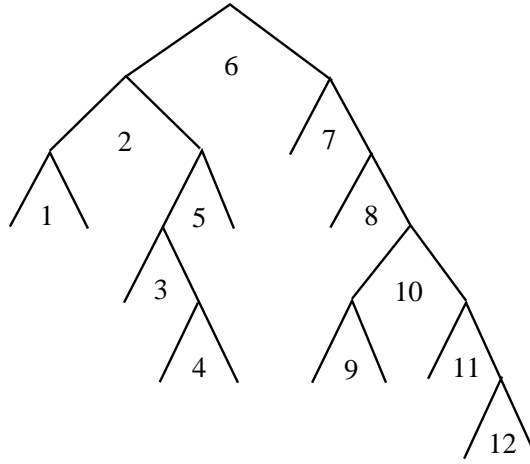
We will describe the method to find the length without proof, because the proof is long and highly technical. The proof goes over the effects on the carets of a tree when multiplying by all generators, and studies which carets get affected. It is quite tedious and the reader can understand and use the method without reading the proof. However, it is elementary, and can be read without any extra difficulty. A reader interested in the proof is directed to Fordham's paper [16] for a short, concise version, or to the original PhD thesis for a longer and more elaborate version.

Recall that a caret in a binary tree can be described as right, left or interior in an obvious manner. This is one of those notions that is easier understood than explained. A left caret is on the left side of the tree, and the path that connects it to the root involves only left carets. Analogously right carets can be defined, and then an interior caret is one which is neither right or left. The root caret is special, being both left and right (or maybe neither), but for the purpose of Fordham's method it will be considered always as a left caret.

Here is the description of all seven types of carets, roughly from left to right in their situation in the tree. See Figure 4.2 for an example of a tree with carets of all types. Recall that the carets of a binary tree are totally ordered from left to right, so when we speak of the following caret here it means the caret that follows in the total order, and which may or may not be a child.

Definition 4.2.1 *The carets of a binary rooted tree can be divided in the following seven types:*

- *Type \mathcal{L}_\emptyset . A caret is of type \mathcal{L}_\emptyset if it is a left caret and it does not have a left child. Equivalently, it is the leftmost left caret. It can be the root caret if it does not have a left child. In that case, the root caret is the only left caret and it is of type \mathcal{L}_\emptyset .*
- *Type \mathcal{L}_L . A caret is of type \mathcal{L}_L if it is a left caret and it has a left child. These are all the left carets except the leftmost one.*
- *Type \mathcal{I}_\emptyset . These are interior carets which do not have right children. A caret of type \mathcal{I}_\emptyset can have a left child but not a right child.*
- *Type \mathcal{I}_R . A caret is of type \mathcal{I}_R is an interior caret which has a right child.*
- *Type \mathcal{R}_i . A caret is of type \mathcal{R}_i if it is a right caret, and the following caret is not a right caret. This means that a caret is of type \mathcal{R}_i if it is right, it has a right child, and this right child has a left child.*
- *Type \mathcal{R}_{ni} . These is a right caret which has a right child, this right child has no left child, but further along the right side there is a right caret with left children.*
- *Type \mathcal{R}_\emptyset . A caret is of type \mathcal{R}_\emptyset if it is a right caret, it may have right children, but all the right carets following it cannot have any left children. Namely, if a caret is of type \mathcal{R}_\emptyset , all the following carets are of the type \mathcal{R}_\emptyset .*



1	2	3	4	5	6	7	8	9	10	11	12
\mathcal{L}_\emptyset	\mathcal{L}_L	\mathcal{I}_R	\mathcal{I}_\emptyset	\mathcal{I}_\emptyset	\mathcal{L}_L	\mathcal{R}_{ni}	\mathcal{R}_i	\mathcal{I}_\emptyset	\mathcal{R}_\emptyset	\mathcal{R}_\emptyset	\mathcal{R}_\emptyset

Figure 4.2: An example of a tree with all its carets numbered in order. The table spells out the type for each caret.

There is a built-in non-symmetric flavor to this classification, namely, why interior carets are singled out if they have right children and not left ones, or the complicated classification of right carets. This is due to the biased definition of the generators for F , which tend to favor the right side. Observe that the generators x_n are heavily directed to the right side, with a row of right carets defining the interval where the map actually has its support. This is the reason for the complicated definition of the types of right carets.

Once all carets in both trees have been assigned a type, each caret of the source tree is paired with its corresponding caret in the target tree, in an order-preserving fashion. Hence each pair of carets has a pair of types assigned. We will assign to each pair a weight according to the following definition.

Definition 4.2.2 *Given an element of Thompson's group F , its weight is defined by the following procedure. Take the reduced tree pair diagram (S, T) for it, and number the carets on each tree from left to right according to the natural order. Give each caret its type according to the definition of types above. Pair the carets which matching number, that is, the first caret of S with the first caret of T , the second with the second, and so on, to obtain a series of pairs of types. Give each pair of types its weight according to the*

following table:

	\mathcal{R}_\emptyset	\mathcal{R}_{ni}	\mathcal{R}_i	\mathcal{L}_L	\mathcal{I}_\emptyset	\mathcal{I}_R
\mathcal{R}_\emptyset	0	2	2	1	1	3
\mathcal{R}_{ni}	2	2	2	1	1	3
\mathcal{R}_i	2	2	2	1	3	3
\mathcal{L}_L	1	1	1	2	2	2
\mathcal{I}_\emptyset	1	1	3	2	2	4
\mathcal{I}_R	3	3	3	2	4	4

In the element, there is always a pair of the type $(\mathcal{L}_\emptyset, \mathcal{L}_\emptyset)$, which carries no weight. Then, the sum of all the weights of all pairs is the weight of the element.

We can state now the remarkable Fordham's theorem.

Theorem 4.2.3 *The weight of an element in F is exactly equal to its length in the word metric with respect to the generating set $\{x_0, x_1\}$.*

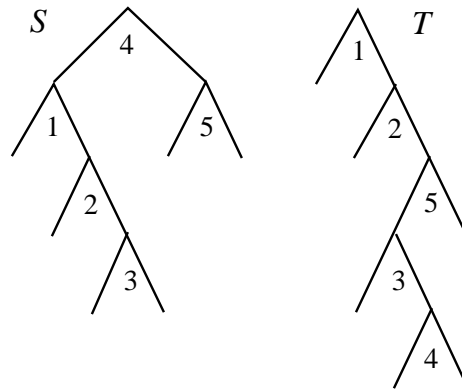
An example is carried out in Figure 4.3. Fordham's method is an outstanding result, giving the exact value of the length of an element. Groups where the exact length of elements is easily computable are few and far between, and they tend to be easy (abelian, free, maybe solvable). For groups as complicated as F , it is extremely rare that an easy algorithm is known to compute the word metric.

Observe that some of the estimates obtained in Section 4.1 are immediate corollaries of Fordham's method. Namely we have the following corollary.

Corollary 4.2.4 *We have the following inequalities:*

$$N(x) - 2 \leq ||x|| \leq 4N(x)$$

Proof. Since, according to the table, each pair of carets has weight at most 4, the second inequality is obvious. For the lower bound, observe that the only pair types that have weight zero are $(\mathcal{L}_\emptyset, \mathcal{L}_\emptyset)$ and $(\mathcal{R}_\emptyset, \mathcal{R}_\emptyset)$, and there can be at most one of each type. For $(\mathcal{L}_\emptyset, \mathcal{L}_\emptyset)$ this is clear, since there can be only one caret of type \mathcal{L}_\emptyset , and also note that if there are two (or more) pairs of carets of the type $(\mathcal{R}_\emptyset, \mathcal{R}_\emptyset)$, then the diagram is not reduced. \square



Caret	Type S	Type T	Weight
1	\mathcal{L}_\emptyset	\mathcal{L}_\emptyset	0
2	\mathcal{I}_R	\mathcal{R}_i	3
3	\mathcal{I}_\emptyset	\mathcal{I}_R	4
4	\mathcal{L}_L	\mathcal{I}_\emptyset	2
5	\mathcal{R}_\emptyset	\mathcal{R}_\emptyset	0
Total			9

Figure 4.3: An example of the weight of an element computed using Fordham's method. The element is $x_0x_1x_2x_3^{-1}x_2^{-1}$, which can be written as $x_0x_1x_0^{-1}x_1x_0^{-1}x_1x_0x_1x_0$. This word has length 9, so, according to Fordham's theorem, it is a shortest word for this element.

Chapter 5

Subgroups of F

The purpose of this chapter is to introduce and study interesting subgroups of F . The subgroup structure of F is quite rich, and it contains several subgroups interesting on its own, given by some features of the group. We will study them, their properties, and their distortion. The metric estimates obtained in the previous chapter are extremely useful to establish whether subgroups of F are distorted, and we will introduce all these interesting subgroups and study their algebraic, combinatorial, and geometric properties.

Recall that distortion is a concept which is devised to compare the word metric of a group with that for one of its finitely generated subgroups. So let G be a finitely generated group, with system of generators X , and let H be a finitely generated subgroup, whose generating set will be denoted by Y . The distortion function relates the two metrics on H , namely, the word metric with respect to Y and the restriction to H of the word metric of G .

Definition 5.0.5 *The distortion function of H in G is defined by*

$$\Delta(r) = \sup \{ \|x\|_Y : \|x\|_X \leq r \}$$

Observe that the distance in Y is larger than that of X , because Y is a subset of X , so the amount of paths available to find the shortest distance is smaller. Observe that if we change generating sets, the metric changes only linearly, so the distortion is also defined up to a linear factor. Hence, if the distortion function is linear, we say that the subgroup is non-distorted. In other cases, we say that the group has quadratic or exponential distortion, for instance. [Need a quote here for properties of distortion].

A subgroup H on G in the situation above is said to be quasi-isometrically

embedded if there exists a constant C such that

$$\frac{\|x\|_Y}{C} \leq \|x\|_X \leq C\|x\|_Y.$$

Observe that if the subgroup is quasi-isometrically embedded, then its distortion function is bounded above and below by a linear function. So being quasi-isometrically embedded is the same as being non-distorted.

A fact that will be useful in the following sections is that if we conjugate an undistorted subgroup, the conjugate is also undistorted.

Theorem 5.0.6 *Let H_1 and H_2 be subgroups of G , so that $H_2 = gH_1g^{-1}$, for some $g \in G$. Then H_1 is undistorted if and only if H_2 is.*

Proof. Let $Y_1 = \{y_1, \dots, y_k\}$ be a generating set for H_1 . Clearly, then $Y_2 = \{gy_1g^{-1}, \dots, gy_kg^{-1}\}$ is a generating set for H_2 . It is clear then that, for $x \in H_2$, we have that its length in H_1 is at most its length in H_2 plus twice the length of g (in H_1). Hence, we have

$$\|x\|_{Y_2} \leq \|x\|_{Y_1} + 2\|g\|_{Y_1}$$

The distance changes at most by an additive constant, so distortion is linear. \square

Throughout the chapter, we will use the estimates of the word metric we constructed in the previous chapter to prove that some subgroups are undistorted. Observe that since both D and N are similar to the metric up to multiplicative constants, they can be used for this purpose.

5.1 The subgroups $F[a, b]$ and $F \times F$

For the first examples of interesting nondistorted subgroups, we will consider the subgroups $F[a, b]$ we already know (see Theorem 3.1.3). This will give us an idea of how many of these proofs related to distortion work.

Theorem 5.1.1 *Given two dyadics a and b in $[0, 1]$, then the subgroup $F[a, b]$ in F is undistorted.*

Proof. According to the estimate of the word metric given by the number of carets, and observing Figure 3.1, we see that the model subgroups $F[0, \frac{1}{2}]$,

$F[\frac{1}{4}, \frac{3}{4}]$ and $F[\frac{1}{2}, 1]$ are all undistorted. Observe that each of them is isomorphic to F , and the isomorphism is obtained by just adding at most two carets to the reduced diagram. Hence, the distance of the element when counted in the subgroup and when counted in the ambient F is similar.

If we have now a general $F[a, b]$, we observe that by transitivity (Theorem 3.1.2), it can be conjugated into one of these three model subgroups. \square

We want to consider now direct products of copies of F with disjoint supports. It is clear, and it has been used already earlier in this book (see Section 3.3), that if two elements have disjoint supports, or moreover, if the two supports just have an endpoint in common, that the two elements commute. Hence, if we consider two intervals $[a, b]$ and $[c, d]$ inside $[0, 1]$ such that $a < b \leq c < d$, then all elements of $F[a, b]$ commute with all elements of $F[c, d]$, and we can consider the subgroup $F[a, b] \times F[c, d]$ inside F . Our next goal is to prove that these subgroups are also undistorted.

The proof of this fact will again go, as before, through conjugating them into model subgroups which correspond to simple subintervals of $[0, 1]$. To achieve this goal, we establish it for the easiest pair of subgroups and then extend it to the rest.

Proposition 5.1.2 *The subgroup $F[0, \frac{1}{2}] \times F[\frac{1}{2}, 1]$ is undistorted.*

Proof. The proof is elementary when we see the tree pair diagram which corresponds to elements of this subgroup. See Figure 5.1. Taking an element in $F[0, \frac{1}{2}] \times F[\frac{1}{2}, 1]$, we see that both its norm in F and its norm in $F[0, \frac{1}{2}] \times F[\frac{1}{2}, 1]$ correspond almost exactly to the sum of the two norms of the two pieces in the subgroups $F[0, \frac{1}{2}]$ and $F[\frac{1}{2}, 1]$. The number of carets of the tree diagram of the combined element is the sum of the number of carets of the two elements which combine into it. \square

This fact gives us a possibly surprising fact: a quasi-isometrically embedded copy of $F \times F$ inside F . In fact, nothing prevents us to continue this fact indefinitely, and obtain copies of F^n inside F , all of them quasi-isometrically embedded.

Using transitivity (Theorem 3.1.2) and conjugation (Theorem 5.0.6), we can now extend this fact to any subgroups $F[a, b]$ inside.

Theorem 5.1.3 *Let a, b, c, d be dyadic numbers in $[0, 1]$, such that $a < b \leq c < d$. Then, the subgroup $F[a, b] \times F[c, d]$ is undistorted.*

Proof. The proof is straightforward. By iterating the situation in Proposition 5.1.2, we can find a pair of intervals that model the relative position of $[a, b]$

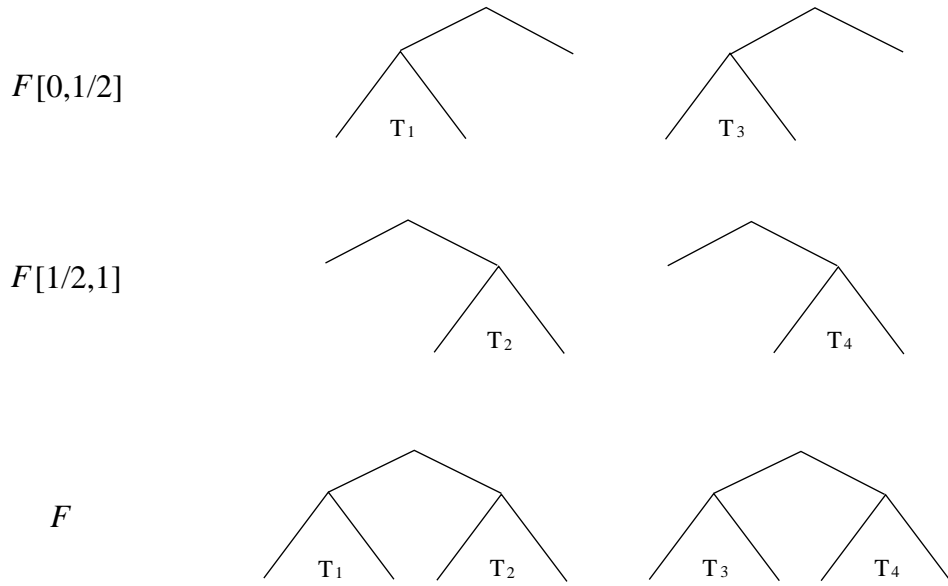


Figure 5.1: The proof of the fact that $F[0, \frac{1}{2}] \times F[\frac{1}{2}, 1]$ is undistorted. For an element of the product, the number of carets is almost exactly the sum of the number of carets of the two elements which originate it.

and c, d . There are several possible cases (whether $a = 0$ or not, whether $d = 1$ or not, and whether $b = c$ or not). In each case it is easy to match the intervals with some intervals of the type $[i/2^n, (i+1)/2^n]$ which have the same relative position. For instance, the case $0 < a < b < c < d = 1$ can be modeled on the intervals $[\frac{1}{4}, \frac{1}{2}]$ and $[\frac{3}{4}, 1]$. It is clear that $F[\frac{1}{4}, \frac{1}{2}] \times F[\frac{3}{4}, 1]$ is quasi-isometrically embedded by iterated applications of Proposition 5.1.2, and by transitivity we can conjugate it into the case of the general a, b, c, d . Proceed in a similar way for all possible combinations for the endpoints. \square

Obviously, nothing prevents us to extend this result to any number of intervals. Hence n intervals with disjoint interiors will yield a quasi-isometrically embedded copy of F^n .

5.2 Cyclic and abelian subgroups

We continue with interesting types of subgroups of F . In this section we consider cyclic groups and free abelian groups of arbitrary (finite) rank.

Let x be an element of F , different from 1. Recall that F is torsion-free, so the subgroup generated by x is infinite cyclic. Then this subgroup is non-distorted.

Theorem 5.2.1 *In F , cyclic subgroups are non-distorted.*

Proof. Let

$$x_0^{a_0} x_1^{a_1} \dots x_n^{a_n} x_m^{-b_m} \dots x_1^{-b_1} x_0^{-b_0}.$$

be the normal form for x . Let k be the minimal index such that a_k or b_k is nonzero, namely,

$$a_0 = a_1 = \dots = a_{k-1} = b_0 = b_1 = \dots = b_{k-1} = 0$$

We know from Theorem 5.0.6 we can replace x by a conjugate. So conjugate x until we can assume that $a_k \neq 0$ and $b_k = 0$, maybe taking the inverse of x , and changing k if necessary. Then, we can compute the normal form for x^n , but we observe that in the product x^n there are na_k appearances of x_k , which are never going to cancel because all the other indices are larger. In the process of constructing the normal form of x^n , all the appearances of x_k will be moved to the beginning of the word, and the normal form will start by $x_k^{na_k}$, and then we have $D(x^n) \geq na_k$, where D is the estimate from Theorem 4.1.2. Hence, using this theorem, we have

$$\|x^n\| \geq \frac{D(x^n)}{C} \geq \frac{na_k}{C}$$

And then

$$\frac{a_k}{C}n \leq \|x^n\| \leq \|x\|n$$

and since n is the length of x^n inside the cyclic subgroup, this inequality finishes the proof. \square

Corollary 5.2.2 *For Thompson's group F , all translation numbers are positive.*

This fact has some interesting consequences, for instance, according to Eckmann [13], F satisfies the Kaplansky conjecture, i.e., the group ring $\mathbb{C}F$ has no idempotents besides 0 and 1.

We can combine cyclic subgroups to obtain free abelian subgroups. One only needs to take two elements of F with disjoint supports to see that they commute, and hence the group they generate is isomorphic to \mathbb{Z}^2 . There can be no torsion, so all abelian subgroups of F are free. When we observe the previous section, looking at copies of F^n inside F , by taking one element in each F component, we obtain a free abelian subgroup of rank n .

Proposition 5.2.3 *F contains free abelian subgroups of rank n for all n , and it also contains free abelian subgroups of countable rank.*

Moreover, when we consider one of these free abelian subgroups of rank n , obtained by picking an element in each of the copies of F in F^n , and such that all these copies of F have disjoint supports, we see immediately that all these free abelian subgroups are again undistorted. Clearly, each cyclic subgroup is undistorted inside its copy of F , and, according to the previous section, the product of all these copies of F is undistorted in the large ambient F . Hence, we see that the direct product of all the cyclic groups (our abelian group) is also undistorted. We have proved the following result.

Theorem 5.2.4 *Let g_1, \dots, g_n be nontrivial elements of F , such that their supports satisfy that*

$$\text{supp}(g_i) \subset [a_i, b_i],$$

where we have that

$$(a_i, b_i) \cap (a_{i+1}, b_{i+1}) = \emptyset.$$

Then, the free abelian subgroup generated by the g_i is undistorted.

All these results combine into the following.

Theorem 5.2.5 *For any integers $n, m \geq 0$, we have infinitely many undistorted copies of $F^n \times \mathbb{Z}^m$ inside F .*

Using these facts, and considering the arbitrarily large (finitely generated) free abelian subgroups inside F , we obtain corollaries referring to the fact that F behaves in certain ways in an infinite-dimensional manner.

Corollary 5.2.6 1. *F has infinite cohomological dimension.*

2. *All asymptotic cones for F are infinite-dimensional.*

In fact, F was the first example of a group which is of type FP_∞ (and hence has infinite cohomological dimension) and is torsion-free (see [5]). Infinite cohomological dimension is usually obtained with finite groups or with groups with nontrivial torsion, so this is one more of the strange properties of F .

5.3 Wreath products

The goal for this section is to show that the wreath product $\mathbb{Z} \wr \mathbb{Z}$ can be realized as a subgroup of F . Recall that $\mathbb{Z} \wr \mathbb{Z}$ admits the presentation

$$\langle a, t \mid [t^i a t^{-i}, t^j a t^{-j}], i, j \in \mathbb{Z} \rangle$$

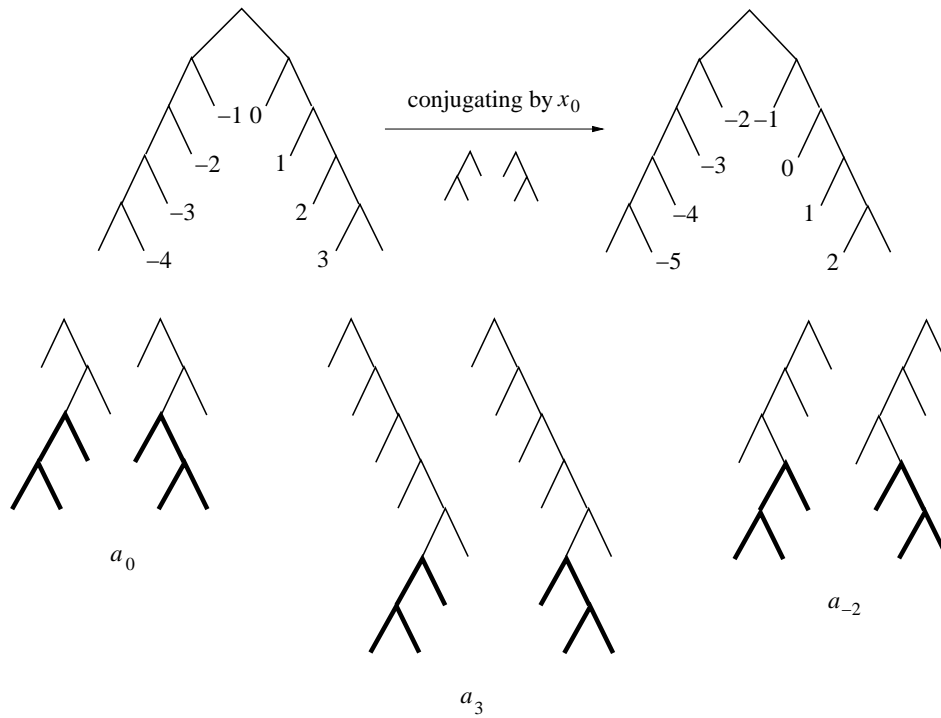


Figure 5.2: The diagram needed for the embedding of $\mathbb{Z} \wr \mathbb{Z}$ in F . Conjugating by x_0 moves the diagram left and right on the tree, creating commuting elements. In the bottom part of the picture we see three of the generators of the direct sum, which commute pairwise. The thicker carets are moved around by conjugation by x_0 .

and fits into a short exact sequence

$$1 \longrightarrow \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} \longrightarrow \mathbb{Z} \wr \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 1$$

where the quotient \mathbb{Z} is generated by t and the infinite direct sum by all the $t^i a t^{-i}$. As is common, we will write a_i instead of $t^i a t^{-i}$. This short exact sequence splits, hence, the wreath product is, in particular, a semidirect product.

To construct a copy of $\mathbb{Z} \wr \mathbb{Z}$ inside F , consider the string made of all right and left carets in an infinite tree, and observe that conjugating by x_0 acts basically by left and right translation on the leaves of this tree (see Figure 5.2). So the idea is to take x_0 as t , and the corresponding a will be an element which has carets only on one of the leaves of the infinite string. The element we will take is $x_1^{-2} x_2^{-1} x_1^{-1}$, which has support on the interval $[\frac{1}{2}, \frac{3}{4}]$. Conjugating by x_0 moves the graph up and down the diagonal, and all supports are disjoint. See Figure 5.2 for a picture of three of the a_i .

Let W be the subgroup of F generated by x_0 and $x_1^2 x_2^{-1} x_1^{-1}$.

Theorem 5.3.1 *The map*

$$\begin{aligned} f : \mathbb{Z} \wr \mathbb{Z} &\longrightarrow W \\ t &\mapsto x_0 \\ a &\mapsto x_1^2 x_2^{-1} x_1^{-1} \end{aligned}$$

is an isomorphism.

Proof. It is straightforward to see that the map is well-defined, since the relations are satisfied, and also the map is surjective, because x_0 and $x_1^2 x_2^{-1} x_1^{-1}$ generated W . To see that the map is one-to-one, we observe that an element of $\mathbb{Z} \wr \mathbb{Z}$, being a semidirect product, admits a unique expression of the type

$$a_{i_1}^{r_1} a_{i_2}^{r_2} \dots a_{i_n}^{r_n} t^m$$

where the $m, i_1, i_2, \dots, i_n, r_1, r_2, \dots, r_n \in \mathbb{Z}$, with $i_1 < i_2 < \dots < i_n$. Each one of the a_i will map to its corresponding pair of trees in the i -th position, and the remaining t^m will move the root, so the only way that this element maps to the identity is if the element is the identity itself. \square

Hence, W is a copy of $\mathbb{Z} \wr \mathbb{Z}$ inside F . It is an interesting fact that this subgroup is also undistorted, which has been proved in [9]. It falls a bit outside of the purpose of this book to prove this fact, since it requires a somewhat long computation of the norm of an element in the wreath product and the comparison with its norm in F , but the interested reader can obtain the proof in the given reference.

5.4 A distorted subgroup

By observing the previous sections, one could think that a reasonable conjecture would be that every finitely generated subgroup of F is undistorted, as it happens in free groups and free abelian groups, for example. In fact, it is quite hard to find distorted examples. But we can take advantage of the fact that the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is a subgroup of F , as we will see. This example was constructed by Guba and Sapir in [17, Theorem 38].

Consider $F \times F$ inside F as we saw in Section 5.1, and consider $W \times W$ inside $F \times F$, where W is the wreath product defined in the previous section. We will define a subgroup of $W \times W$ which is distorted in it. The construction

is somewhat reminiscent of the Mihailova construction [21] for subgroups of $F_2 \times F_2$, using the same kind of elements.

Let

$$K = \langle (a, a), (t, t), ([a, t], 1) \rangle$$

be a subgroup of $W \times W$. We claim that this subgroup is quadratically distorted in $W \times W$, and hence also in F . We need to construct a family of elements w_n , for $n = 1, 2, \dots$, such that the norm of w_n in $W \times W$ is linear in n and its norm in K is quadratic in n .

Let $w_n = ([a^n, t^n], 1)$. Obviously this word has length $4n$ in $W \times W$. We see first that this word is in K . Note that $w_n = (a_0^n a_n^{-n}, 1)$. Name $c_i = a_i a_{i+1}^{-1}$, and notice that $(c_i, 1) \in K$, because $c_i = t^i [a, t] t^{-i}$ and then obviously $(c_i, 1) = (t, t)([a, t], 1)(t, t)^{-1}$. So we have that $w_n = (c_0, 1)^n (c_1, 1)^n \dots (c_{n-1}, 1)^n$, and hence $w_n \in K$.

This last expression is the key to see that w_n has quadratic length in K , because this word in the c_i involves n^2 elements. The idea of this proof is that creating the n instances of a_0 will require introducing a_1^n . To eliminate those, we need to introduce n copies of a_2 , so to arrive to a_n^{-n} we will require at least n^2 generators.

Imagine w_n has length L in K . Hence, it admits an expression of length L in its generators. This expression would be something like

$$w_n = (u_0, u_0)([a, t], 1)^{m_1}(u_1, u_1)([a, t], 1)^{m_2}(u_2, u_2) \dots \\ (u_{n-1}, u_{n-1})([a, t], 1)^{m_n}(u_n, u_n)$$

where the appearances of (a, a) and (t, t) have been grouped into the pairs (u_i, u_i) , i.e., u_i are elements in W , and which also satisfy $u_0 u_1 u_2 \dots u_n = 1$, since the second component of w_n is trivial. And also observe that $m_1 + m_2 + \dots + m_n \leq L$. Observe that the word can be seen as a product of conjugates of $([a, t], 1)$ by the words $(u_0, u_0), (u_0 u_1, u_0 u_1), \dots$. The conclusion of all this digression is that if w_n has length L in K , then it can be written as a product of fewer than L conjugates of $([a, t], 1)$.

But we had that $w_n = (c_0, 1)^n (c_1, 1)^n \dots (c_{n-1}, 1)^n$, and each c_i is a conjugate of $c_0 = ([a, t], 1)$. Note that $[a^n, t^n]$ belongs to the free abelian subgroup generated by a_0, a_1, \dots, a_n , and in particular, inside this one, it also belongs to the free abelian subgroup generated by c_0, c_1, \dots, c_{n-1} . Moreover, in this free abelian group, $[a^n, t^n]$ cannot be written with fewer of n^2 generators c_i . Then, it is clear that w_n has quadratic length, because w_n cannot be written with fewer instances of the $(c_i, 1)$. The word of length L above produces a word for w_n in the c_i of length less than L , so we conclude that L is at least n^2 .

This example given by Guba and Sapir can be generalized to obtain subgroups with polynomial distortion n^d for any degree d . It is not known whether F has subgroups with higher distortion, for example, exponential.

Chapter 6

The amenability question

Amenability is a fascinating subject which dates back to the early 20th century, when Banach and Tarski formulated their celebrated Banach–Tarski paradox about subsets of \mathbb{R}^3 . It was von Neumann [25] who, a few years later, realized that the paradox was more properly a property which should be bestowed on the group more than in the space, and the Banach–Tarski paradox should be attributed to the fact that the group of isometries of \mathbb{R}^3 is not amenable. And this is the reason why the paradox is not possible with subsets of \mathbb{R}^2 , since its isometry group is indeed amenable. Readers who are interested in amenability and in general in the Banach–Tarski paradox should read the wonderful book [26].

The nonabelian free group on two generators is the standard model against which nonamenable groups are measured. For many years the only nonamenable groups which were known were those with free subgroups, and it was conjectured that nonamenability was equivalent to not having free subgroups. Day was the first to mention this conjecture in [12], and he attributed it to von Neumann. Olshanskii [23] found the first counterexample, and later Olshanskii–Sapir [24] the first finitely presented example, so the conjecture is now known to be false in all instances. We will see in a later chapter a particular counterexample due to Lodha and Moore.

But at the time (around 1979) researchers started paying attention to Thompson’s group F , the conjecture was still standing, and Geoghegan first realized that F was a prime candidate for a counterexample. Geoghegan stated the following conjecture about F :

Conjecture 6.0.1 *Thompson’s group F satisfies the following properties:*

- (1) *It is of type F_∞ .*

- (2) *It has trivial homotopy at infinity.*
- (3) *It has no nonabelian free subgroups.*
- (4) *It is nonamenable.*

Geoghegan himself, together with Brown, in their celebrated paper [5] proved statements (1) and (2) of the conjecture. Statements (3) and (4) combined form the contents of the von Neumann conjecture, and at the time F was considered the most likely counterexample. Even more when Brin and Squier [4] settled property (3), as we will see later. Hence, the only thing needed was to decide the amenability of F . All the way until now. It is still not known whether F is amenable or not, and this problem is the most tantalizing question which nowadays involves F , and a great deal of research has been developed in trying to answer this question.

In this chapter we aim to show a proof of the Brin–Squier theorem and to give an idea of the amenability question, with some approaches and partial results. The goal is to have a picture of where the problem stands at the time of writing.

6.1 No free subgroups

The goal of this section is to prove the following theorem, due to Brin and Squier in [4].

Theorem 6.1.1 *F does not admit free nonabelian subgroups.*

This theorem is actually quite general, and it can be stated and proved in a more general setting, since it is the whole group of piecewise linear maps the one that already has no free subgroups. Since the proof is exactly the same, we will state the theorem in high generality, so that it can be applied to other situations later. In this chapter, we will consider F as a group of maps on the real line, as seen in section 1.4.

Let $PLF(\mathbb{R})$ the group of piecewise linear maps of \mathbb{R} with finitely many breakpoints. Observe that slopes can be any real number, and also, breaks can be anywhere. The only important thing is to have finitely many breaks. The proof is purely dynamic, appealing only to the shapes of the maps, hence slopes and breaks are arbitrary. Of course, the same proof would apply directly to F . The proof here is adapted directly from the original paper [4].

Theorem 6.1.2 *The group $PLF(\mathbb{R})$ does not admit free nonabelian subgroups.*

The proof will be divided in several propositions, and all of them combined will yield the theorem.

Definition 6.1.3 *Let f be an element of $PLF(\mathbb{R})$. We define its support by*

$$\text{supp } f = \{t \in \mathbb{R} \mid f(t) \neq t\}$$

Observe that for any f , its support is the disjoint union of finitely many open intervals, which are all bounded except at most two, which can extend to infinity on each end of the real line. For instance, $\text{supp } x_0 = (-\infty, \infty)$ and $\text{supp } x_n = (n, \infty)$ if $n > 0$.

Two facts that are elementary and which will be used throughout are first, that the group $PLF(\mathbb{R})$ is torsion-free, and second, that if two elements have disjoint support always commute, and hence they generate a copy of \mathbb{Z}^2 . Proofs of these two facts are elementary and are left to the reader.

The proof uses the commutator subgroup of $PLF(\mathbb{R})$, which will be denoted by $PLF'(\mathbb{R})$. Here are some elementary facts about it.

Lemma 6.1.4 *Let f, g be two elements of $PLF(\mathbb{R})$.*

1. *The commutator $[f, g]$ has slope 1 at ∞ and $-\infty$.*
2. *If f and g have slope 1 at ∞ and $-\infty$ (in particular if $f, g \in PLF'(\mathbb{R})$) then $[f, g]$ has compact support.*
3. *If f and g have a common fixed point t , then $[f, g]$ is the identity in a neighborhood of t .*

Proof. The proof is elementary. For 1, observe that the slope is multiplicative in $PLF(\mathbb{R})$, so in a commutator element the slopes will cancel out and the slope will be 1. For 2, observe that if the slope is 1, elements are $t + a$ and $t + b$ near ∞ (or near $-\infty$), so the commutator is just t . For 3, observe that the common fixed point will also be a fixed point of the commutator $[f, g]$, and in a neighborhood the slopes (which are also multiplicative) will be 1. \square

The following two lemmas are also elementary, and illustrate very well how the dynamics of the maps can be used to understand group-theoretic properties. We emphasize that it is a key (and beautiful) fact about F that different

situations involve different interpretations (maps, trees, presentations) and one may choose the one that suits better to the current purposes.

Lemma 6.1.5 *Let $f \in PLF(\mathbb{R})$, and let $a, b \in \mathbb{R}$ such that $[a, b] \subset \text{supp } f$. Then, there exists $n \in \mathbb{Z}$ such that $f^n(a) > b$.*

Note that $\text{supp } f$ is open and $[a, b]$ is a closed interval, so it must be included in an open interval in \mathbb{R} . So it makes sense to have room in $\text{supp } f$ outside $[a, b]$.

Proof. The proof is based on the well-known fact that if the interval (c, d) is a component of the support of a map in \mathbb{R} , and $t \in (c, d)$, then:

1. If $f(t) > t$, we have $f^n(t) \rightarrow d$ and $f^{-n}(t) \rightarrow c$ when $n \rightarrow \infty$.
2. If $f(t) < t$, we have $f^n(t) \rightarrow c$ and $f^{-n}(t) \rightarrow d$ when $n \rightarrow \infty$.

and it is best understood with a picture, see figure 6.1. The result is now clear, because taking either positive or negative powers the images of a will approach d , so they will go beyond b . \square

The last lemma is similar, but with two maps and complicated supports.

Lemma 6.1.6 *Let $f, g \in PLF(\mathbb{R})$, and let $a, b \in \mathbb{R}$ such that $[a, b] \subset \text{supp } f \cup \text{supp } g$. Then, there exists a word w in f, g such that $w(f, g)(a) > b$.*

Proof. The proof is easier seen in a picture than explained, due to complicated notation. See figure 6.2.

Let (c_i, d_i) , for $i = 1, \dots, k$ be intervals in \mathbb{R} such that each c_i, d_i is a component of either f or g . Assume that

$$[a, b] \subset \bigcup_{i=1}^k (c_i, d_i).$$

and also that $c_1 < c_2 < \dots < c_k$ and $d_1 < d_2 < \dots < d_k$. Suppose that $a \in (c_1, d_1)$ and that (c_1, d_1) is a component of $\text{supp } f$. Applying the previous lemma, we can find a power of f such that $f^n(a) > c_2$. Since now (c_2, d_2) is in the support of g and $f^n(a)$ is in it, we can find a power of g such that $g^m(f^n(a)) > c_3$.

Inductively then, we can translate the image of a from one interval to the next, and since $[a, b]$ is in the union of the intervals, taking the appropriate

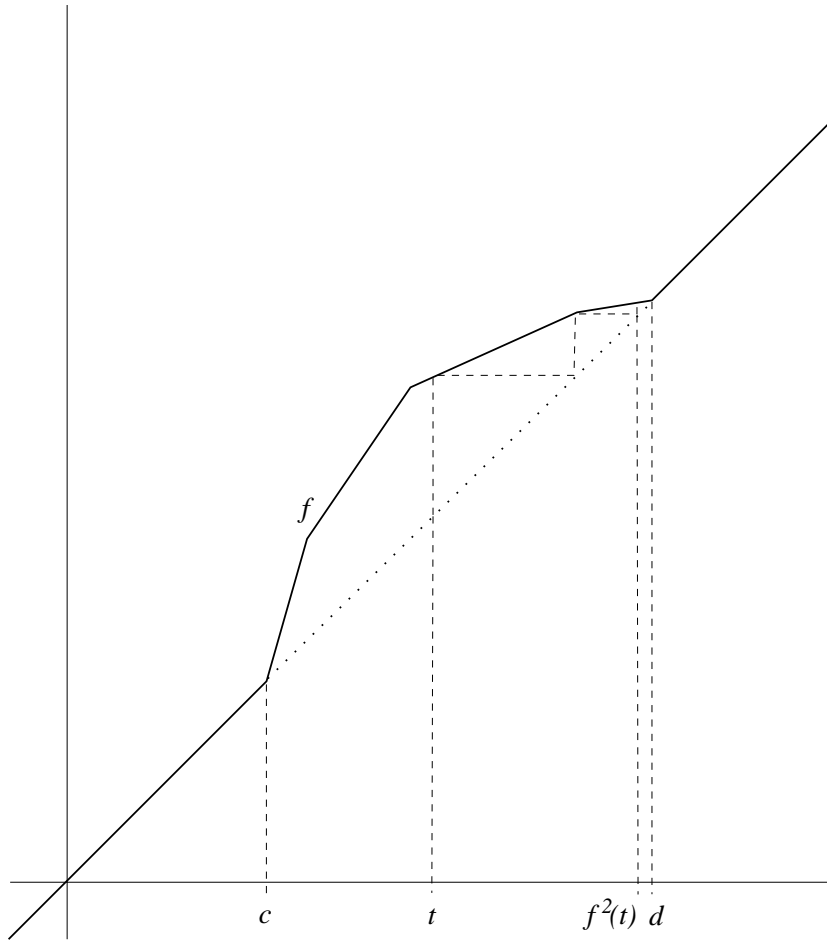


Figure 6.1: Dynamics of a map, where iterations converge to fixed points

word (composition of powers of f and g) the point will approach d_k , and hence it will go beyond b . \square

We are finally ready to finish the proof, with the following result.

Theorem 6.1.7 *Let G be a subgroup of the commutator subgroup $PLF'(\mathbb{R})$. Then, either G is abelian or G contains a copy of \mathbb{Z}^2 .*

Proof. If G is not abelian, there exist $f, g \in G$ such that $z = [f, g] \neq 1$. Observe that $\text{supp } z \subset \text{supp } f \cup \text{supp } g$, because if an element is in the support of z , it must be moved by either f or g . Suppose that the components of $\text{supp } f \cup \text{supp } g$ are the intervals $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$, where a_1 could be $-\infty$ and b_k could be ∞ . Observe that then all the a_i and b_i are fixed points of both f and g , and hence z is the identity in a neighborhood of all these points. This means that if for some i we have that $\text{supp } z \cap (a_i, b_i) \neq \emptyset$,

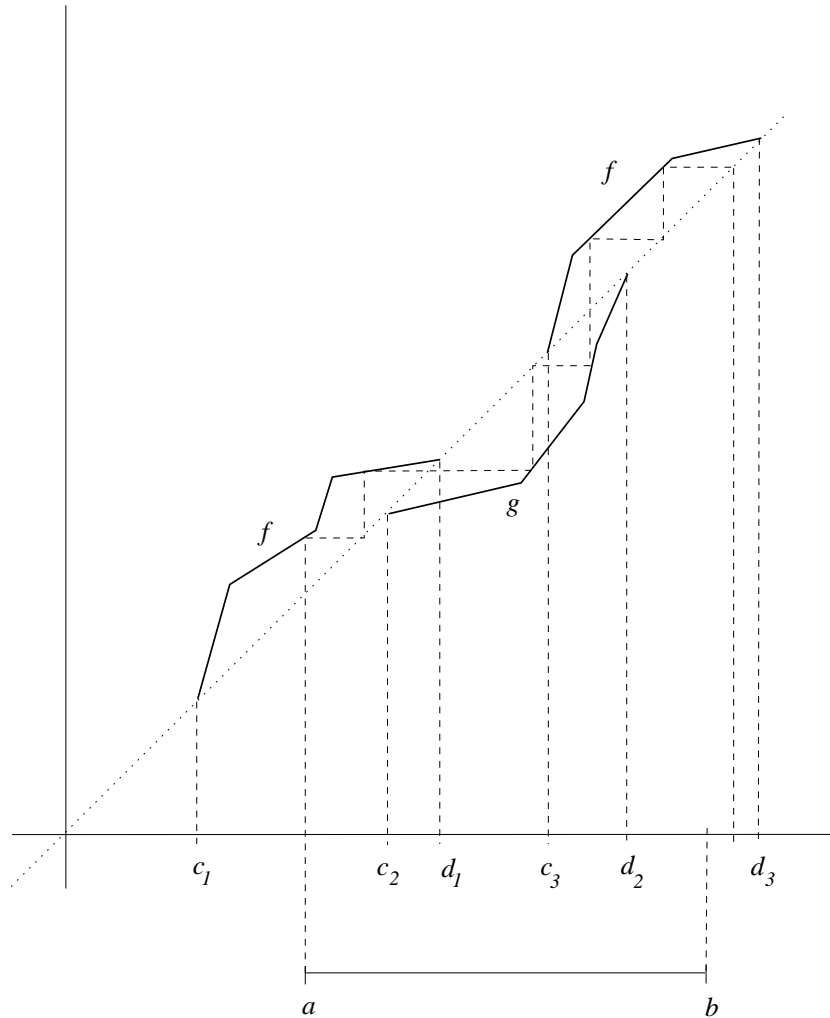


Figure 6.2: Combining two maps to move a point. Observe that $f^2 g^{-2} f^2(a) > b$.

then there exist c, d such that

$$\text{supp } z \cap (a_i, b_i) \subset [c, d] \subset (a_i, b_i).$$

We would like to apply lemma 6.1.6 to this situation, but it may be impossible to apply this to all components, ones would mess up the others, there may not be one word which works for all of them.

So we will resort to a slightly more complicated argument. Let W be the set of all non-identity homeomorphisms in the subgroup generated by f and g whose support is included in a compact subset of $\text{supp } f \cup \text{supp } g$. We have that $z \in W$, so W is nonempty. Let w be a word whose support has nonempty intersection with a minimal number of components of $\text{supp } f \cup \text{supp } g$. Choose

i such that we have

$$\text{supp } w \cap (a_i, b_i) \subset [c, d] \subset (a_i, b_i).$$

and apply lemma 6.1.6 to it, to conclude that there exists a word u in f and g such that $u(c) > d$. Now, observe that w and $u^{-1}wu$ have disjoint supports inside (a_i, b_i) , that is:

$$\text{supp } w \cap \text{supp } u^{-1}wu \cap (a_i, b_i) = \emptyset.$$

This means that the commutator of w and $u^{-1}wu$ has a support which does not intersect (a_i, b_i) and hence has intersection with fewer components than w , which contradicts the minimality. So it must be the identity, and finally w and $u^{-1}wu$ commute. Then they generate a copy of \mathbb{Z}^2 . \square

Now the theorem follows.

Corollary 6.1.8 *If G is a subgroup of $PLF(\mathbb{R})$ then either G is metabelian (i.e. $G'' = 1$) or G contains a copy of \mathbb{Z}^2 .*

One only needs to apply the theorem to the subgroup G' .

Corollary 6.1.9 *The group $PLF(\mathbb{R})$ (and hence F) has no free nonabelian subgroups.*

This is because free nonabelian groups are not metabelian and have no free abelian subgroups of rank 2.

6.2 Følner sets

The definition of amenability of a group G usually given in the literature is one that involves the existence of a left-invariant, finitely additive measure on $\mathcal{P}(G)$ of total measure 1. From the group theory point of view, though, this definition is not very illuminating. Even the proof of the fact that \mathbb{Z} is amenable is nonconstructive and already involves the compactness of the space of measures deduced from Tychonoff's theorem. See [26, Chapter 10] for details.

Fortunately enough, amenability enjoys several equivalent definitions, which at the first glance seem to have nothing to do with each other. For group-theoretical purposes, the one that is more useful and suits groups better is

Følner's condition. The condition states that a group is amenable if it admits a sequence of Følner sets, which are sets where the boundary is small with respect to the interior. These concepts involve the geometric realization of the group in terms of its Cayley graph, and are very apparent geometrically and easy to understand for a group theorist with a bit of familiarity with geometric methods.

Finally, before we start with the definitions, just remark that we will restrict ourselves to the case of finitely generated groups, since all groups involved here will be finitely generated. But Følner theory can be defined as well for arbitrary groups, and many of the theorems work similarly.

Let G be a finitely generated group, and let $\{x_1, x_2, \dots, x_n\}$ be a finite symmetric generating set (that is to say, S is finite and if $x \in S$, then $x^{-1} \in S$).

Definition 6.2.1 *Let A be a finite subset of G . The boundary of A , denoted by ∂A , is the set of edges in the Cayley graph of G which join a vertex in A with a vertex in $G \setminus A$.*

Basically, an edge starting at $g \in A$ and labelled x_i is in the boundary if gx_i is not in A . This boundary is sometimes called the *Cheeger boundary* or the *edge boundary*, to distinguish it from the *interior boundary*, which just considers elements of A which have a neighbor outside A .

Definition 6.2.2 *A family of subsets $A_n \subset G, n \geq 0$ is called a family of Følner sets if it satisfies that*

$$\lim_{n \rightarrow \infty} \frac{\#\partial A_n}{\#A_n} = 0.$$

There is no such thing as a single Følner set, there must be a family of them. Observe that by the structure of the Cayley graph and the Følner sets, if A_n is a family of Følner sets, and $g_n, n \geq 0$ is a sequence of elements in G , then $g_n A_n$ is also a family of Følner set. That means that Følner sets can be translated anywhere in the Cayley graph by left multiplication.

Theorem 6.2.3 (Følner [15]) *A finitely generated group G is amenable if and only if it admits a family of Følner sets.*

Again, see [26] for details. Also, see the pictures in figure 6.3 for illustration.

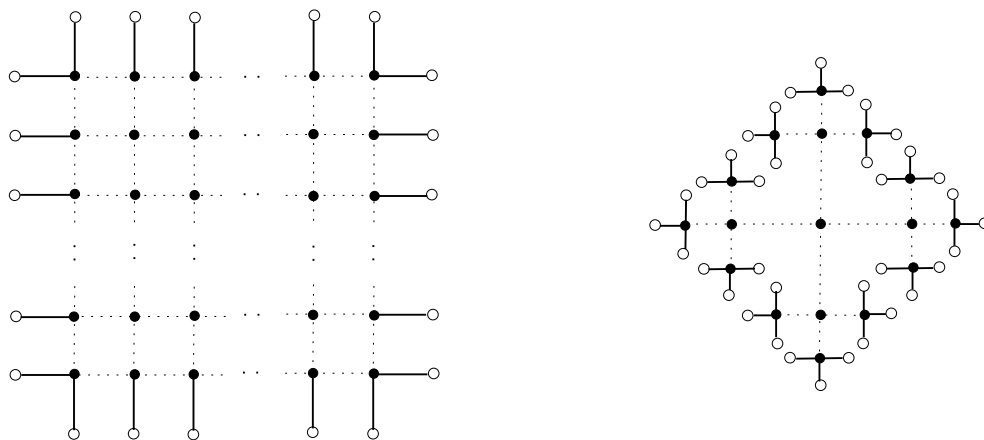


Figure 6.3: On the left, Følner sets for \mathbb{Z}^2 are squares $n \times n$, where A_n has n^2 elements and its boundary has $4n$. On the right, the Cayley graph of F_2 , where there are no Følner sets. The ball of radius n , for instance, has approximately 3^n elements while its boundary also has about 3^n elements.

Finding Følner sets for F would be a daunting task. The reader is encouraged to try, for instance, sets like

$$A_n = \{x_0^{a_0} x_1^{a_1} \dots x_n^{a_n} \mid 0 \leq a_i \leq n\}$$

to be convinced that this is not going to be easy. A reason for this can be the following result by Moore [22].

Theorem 6.2.4 *For every finite symmetric generating set S of Thompson's group F , there is a constant $C > 0$ such that if $A \subset F$ is a C^{-n} -Følner set, i.e.*

$$\frac{\#\partial A}{\#A} \leq C^{-n}$$

with respect to S , then A contains at least an n -iterated exponential

$$\exp_n(0) = 2^{2^{2^{\dots^2}}}$$

of elements.

Hence someone looking for Følner sets for F must be prepared to deal with huge sets.

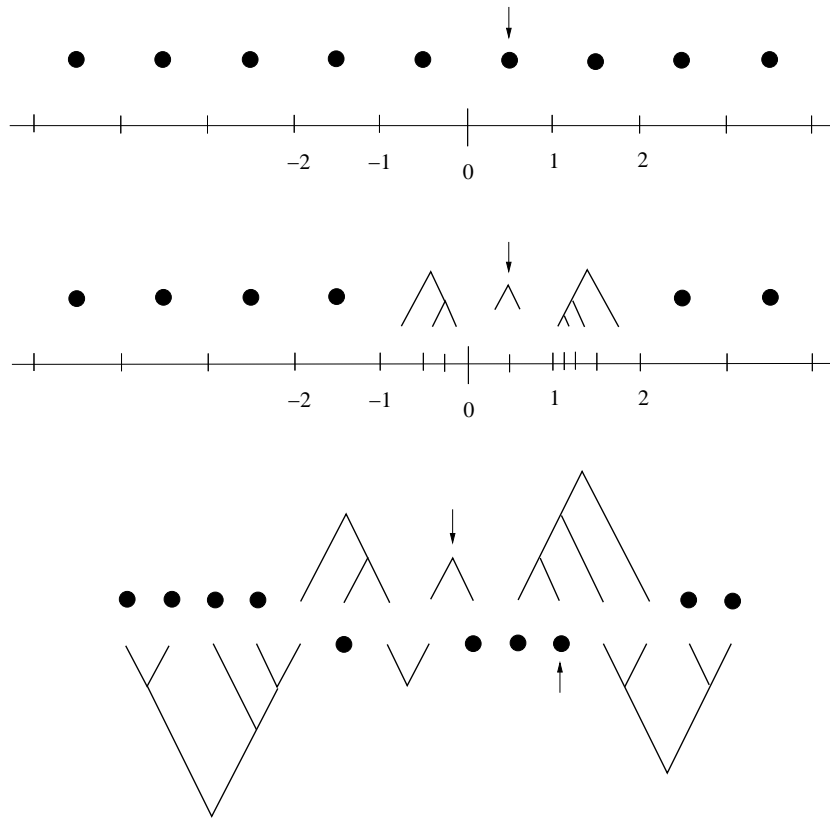


Figure 6.4: Belk–Brown forest diagrams. The first row is the subdivision of the line into length-one intervals. The second row has three of the intervals subdivided. The third row is what an element of F looks like, meaning that the top subdivision is mapped to the bottom subdivision. The top subdivision is the one in the second row.

6.3 Forest diagrams

If we cannot find Følner sets, we can try to find finite sets which have the smallest possible boundary. The currently known examples which have smallest boundary are the Belk–Brown sets, constructed in [3]. To understand these sets we need the concept of forest diagram. Forest diagrams are yet another way to understand elements of F , which is essentially equivalent to three pair diagrams, but which is particularly good for the purpose of computing this boundary.

To define forest diagrams, first consider the real line, and associate a dot for each interval of the type $[n, n + 1]$, where n is an integer. Since the line is doubly infinite, mark with an arrow the dot corresponding to the interval $[0, 1]$. Then, add a caret from its dot if the interval is subdivided, and add carets to the leaves if the intervals are subdivided further. See figure 6.4.

The forest diagram representation of an element can be obtained by taking the tree pair diagram and removing all the left and right carets. See Figure 6.5.

See the figure 6.6 to see what the generators x_0 and x_1 look like as forest diagrams. It is interesting also to see how a diagram gets affected when multiplying by a generator: multiplying by x_0 or x_0^{-1} just moves the pointer from the forest right or left. Multiplying by x_1 just erases the caret immediately below the pointer and moves the pointer to its left child. Multiplying by x_1^{-1} does the opposite: join with a caret the pointer tree with the one to its right, and puts the pointer on top of this new caret. These actions will be useful in the next section.

Observe too that if we have a positive element, then we have that the second subdivision (the one at the bottom) has only dots, and the marking is on the dot below the first leaf of the first nondot tree of the top forest. Hence, positive elements can be identified with only a single forest, because the bottom forest is always understood. In the next section we will deal only with positive elements, with a single forest.

6.4 Belk–Brown sets

Belk–Brown sets are a family of sets which are defined using forest diagrams, and which up to now are the known sets with smallest boundary. For a two-generated group such as F , the boundary of a set can go from 0 up to 4, and the Belk–Brown sets have a boundary of 0.5. The purpose of this section is to describe these sets and to give a (sketch of a) proof of the boundary fact.

The first thing to notice is that the sets will include only positive elements. This is for simplicity, since then only one forest is necessary, as pointed out in the previous section. Furthermore, since the diagrams have infinite tails with only dots, we will consider only finite diagrams. A finite diagram will be understood to have infinitely many dots at each side, which will not be depicted. Start with some definitions:

Definition 6.4.1 *Given a finite forest diagram, we say it has width n if the number of spaces at the bottom of the diagram is n .*

For instance, the forest diagrams in figure 6.4 have width 14.

Definition 6.4.2 *Given a forest diagram, we will call its height to the maximal distance from the root to one of its leaves in any of the trees in it.*

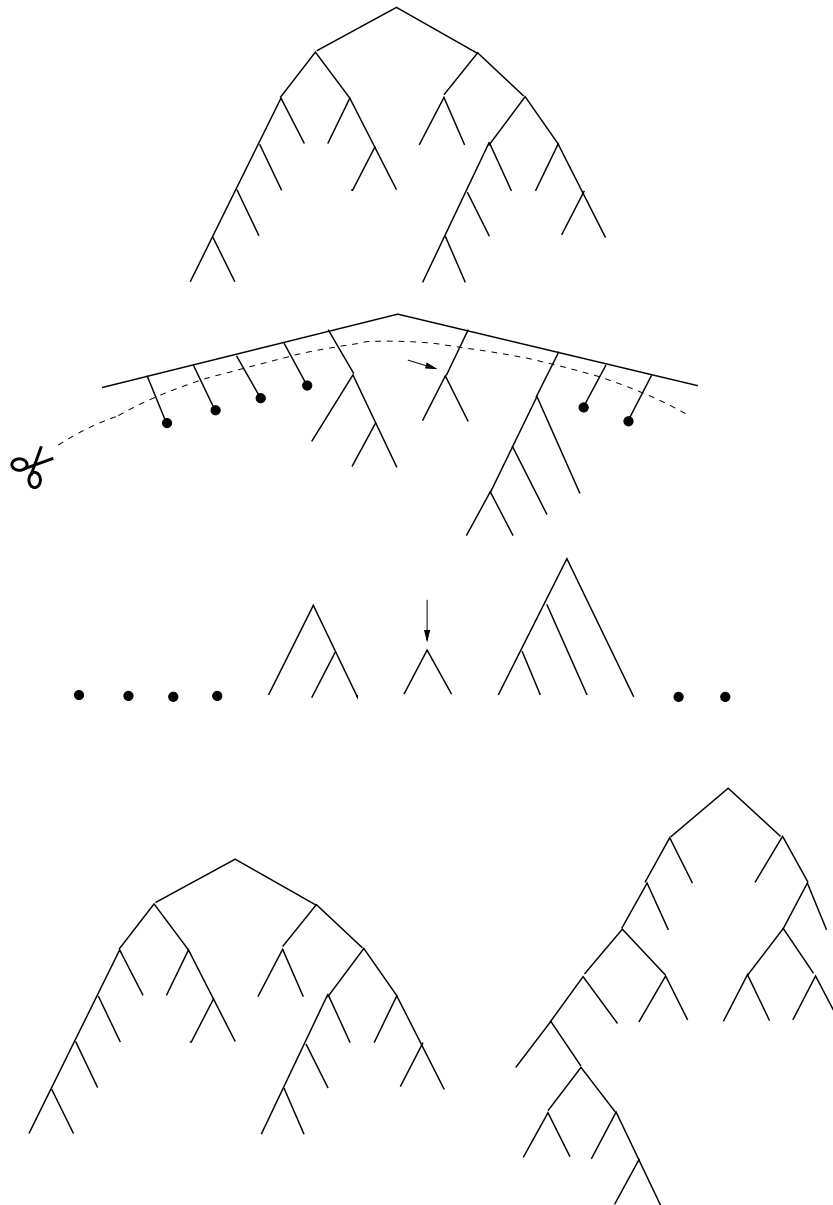
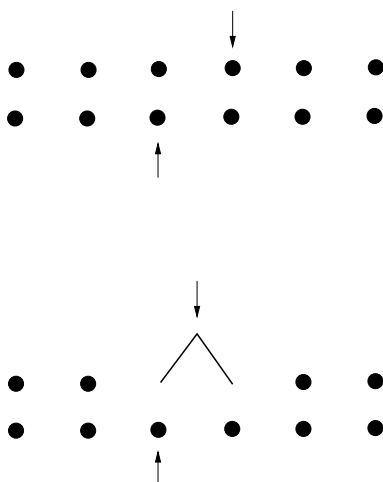


Figure 6.5: How a binary tree corresponds to a forest diagram. The tree pair diagram at the bottom corresponds to the element represented by the forest diagram at the bottom of the figure 6.4.

Figure 6.6: The generators x_0 and x_1 as forest diagrams.

Height is sometimes called *depth*. The meaning of this definition is clear. For instance, both diagrams in figure 6.4 have height 3.

Height and width are the two key ingredients in the definition of the Belk–Brown sets.

Definition 6.4.3 *The Belk–Brown set $S_{k,n}$ is the set of pointed binary diagrams which have width at most n and height at most k .*

The significance of these sets is the following theorem, due to Belk and Brown in [3].

Theorem 6.4.4 *We have that*

$$\lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\partial S_{k,n}}{S_{k,n}} = \frac{1}{2}.$$

Proof. The proof of this fact is not included in the paper [3], but the interested reader can go to the original source, which is Belk’s PhD thesis [2]. We will give a sketch here.

Observe that the quotient can be expressed in terms of probabilities, that is, that given an element $f \in S_{n,k}$, we are interested in the probability that a generator multiplied by f is not in the set. So we will compute these probabilities.

Recall that the boundary is the number of directed edges which have one vertex in the set and one vertex in the complement. The first thing to notice

is that the number of edges x_0 pointing out is exactly the same as the number of edges x_0 pointing in (and the same is true for x_1). This is a general fact about finite sets in Cayley graphs, and the reader just needs to think about this and draw some pictures. For instance, see figure 6.3.

So start fixing k . Observe the following fact: given $f \in S_{k,n}$, the probability that $x_0 f$ is outside the set is small, and will approach zero when n goes to infinity. Why is this? Notice that if k is fixed, a tree of height k can have at most 2^k leaves. Hence, as n grows and grows, the number of trees must grow as well. For a diagram of width n , the number of trees is at least $n/2^k$. Now, the only elements whose multiplication by x_0 will fall outside are those where the pointer is in the very first tree of the forest. So the probability of the pointer falling in the exact first tree is the inverse of the number of trees, so bounded above by $2^k/n$, and hence approaching zero when $n \rightarrow \infty$.

So we have the following equality:

$$\lim_{n \rightarrow \infty} \frac{\partial S_{k,n}}{S_{k,n}} = 2p[x_1^{-1} f \notin S_{k,n} : f \in S_{k,n}]$$

where $p[\]$ indicates the probability. When does an element fall outside the set when multiplied by x_1^{-1} ? Exactly when the pointer is on a trivial tree, i.e., a dot. Because the dot is the only tree which cannot be split, and if that is the case, the multiplication by x_1^{-1} will not be positive (and will have carets on the bottom forest of the diagram). If we were dealing with x_1 , the pointer would have to fall on a tree with height exactly h , which can also be done.

So what is left to prove is that

$$p[x_1^{-1} f \notin S_{k,n} : f \in S_{k,n}] = \frac{1}{4}.$$

So how many forest diagrams do we have with height k and width n ? Let $t_{k,\ell}$ be the number of trees of height k and ℓ leaves. Observe that this number exists for $1 \leq \ell \leq 2^k$. If we have a forest diagram in $S_{k,n}$ with $n \gg k$, we can consider the first tree, which will have some number of leaves ℓ , and observe that then the remaining forest has width $n - \ell$. Hence, f_n satisfies a recurrence

$$f_n = \sum_{i=1}^{2^k} t_{k,i} f_{n-i}.$$

To understand this recurrence we need to consider the polynomial

$$T_k(x) = \sum_{i=1}^{2^k} t_{k,i} x^i.$$

It is a well-known fact that if this is the case, then

$$\lim_{n \rightarrow \infty} \frac{f_{n-1}}{f_n} = p_k$$

where p_k is the positive solution of $T_k(x) = 1$ (check that there is only one, noticing that all $t_{k,i} > 0$).

The number f_n is for nonpointed forests. Let f_n^* be the number of pointed forests (of width n and height k , which, remember, is fixed). The step to pointed forests is not hard, take a pointed binary forest and observe that having a pointer is equivalent to having a separator splitting the forest into one with width i and another with width $n-i$ (separate on the space immediately to the left of the pointer). It is not hard to see that also

$$\lim_{n \rightarrow \infty} \frac{f_{n-1}^*}{f_n^*} = p_k.$$

So we are only left to compute



$$\lim_{k \rightarrow \infty} p_k.$$

And this is a calculus exercise. To compute this limit, observe that the polynomials $T_k(x)$ satisfy

$$T_k(x) = T_{k-1}(x)^2 + x$$

due to the fact that if a tree has height k , split at the root and the two remaining trees have height at most $k-1$. The x comes from the trivial tree (the dot). As indicated in [2], computing the zeros of these polynomials is an exercise on studying the parabolas $y = x^2 + c$ for different values of c . Full details can be found in [2]. \square

To illustrate this recurrence and the sequence, the first terms are the following. The trees with height at most zero is just the dot, so $t_{0,1} = 1$ and $T_0(x) = x$. For height at most one, we have the dot and a single caret, so $t_{1,1} = t_{1,2} = 1$, and $T_1(x) = x^2 + x$. For height at most two, we have:

1 leaf	$t_{2,1}=1$	dot
2 leaves	$t_{2,2} = 1$	single caret
3 leaves	$t_{2,3} = 2$	
4 leaves	$t_{2,4} = 1$	

and the polynomial is

$$T_2(x) = x^4 + 2x^3 + x^2 + x.$$

The sequence of p_k starts by

$$\begin{aligned} p_1 &= 0.618034 & p_2 &= 0.484028 & p_3 &= 0.416318 \\ p_4 &= 0.375927 & p_5 &= 0.349387 & p_6 &= 0.330798. \end{aligned}$$

These are the sets that have the smallest boundary known to date. In fact, in [2], Belk proves that this boundary is in some sense optimal. This fact has led to believe many researchers that these sets offer the best possible boundary (and then the group would not be amenable), but this is still not known.

6.5 Computational approaches

The fact that amenability has so many equivalent definitions and characterizations makes it possible to estimate whether a group is amenable by computational methods. The main tool used to implement amenability computationally is cogrowth and Kesten's criterion.

Definition 6.5.1 *Let G be a finitely generated group and let*

$$1 \rightarrow K \rightarrow F_m \rightarrow G \rightarrow 1$$

be a presentation for G . The cogrowth of G is the growth of the subgroup K inside F_m . In particular, the cogrowth function of G is

$$g(n) = \#(B(n) \cap K),$$

where $B(n)$ is the ball of radius n in F_m and the cogrowth rate of G is

$$\gamma = \lim_{n \rightarrow \infty} g(n)^{1/n}.$$

Kesten's cogrowth criterion for amenability states basically that a group is amenable when it has a large proportion of freely reduced words, for every length n , representing the trivial element; that is, when the cogrowth is large.

Theorem 6.5.2 (Kesten [19, 20]) *Let G be a finitely generated group, and let X be a finite set of generators, with cardinal m . Let γ be its cogrowth rate. Then G is amenable if and only if $\gamma = 2m - 1$.*

Hence a group is amenable if almost all elements in the ball of radius n in the free group are trivial when seen in the group. This can be seen in terms of random words. Take a random word of length L . The term $g(L)$ above divided by the cardinal of the ball is exactly the probability that a random word of length L is in K . Let $p(L)$ be this probability. Hence $p(L)$ can be estimated by taking many random words of length L , compute the fraction of which are trivial, and take this number as an estimate for the probability. Hence, when L approaches ∞ , this number should approach 1 if and only if the group is amenable, according to Kesten's criterion.

This was implemented in [7] using words of lengths up to 320, but the results are inconclusive, since the numbers obtained appear as follows:

$$\sqrt[100]{p(100)} = 0.8806 \quad \sqrt[200]{p(200)} = 0.8918 \quad \sqrt[320]{p(320)} = 0.9003$$

It is not clear whether these numbers approach 1 or they have an asymptotic value strictly less than 1.

In the paper [14], the authors implement a different algorithm to sample random words in the ball, and they compute the cogrowth directly. The estimate for the cogrowth rate they obtain is 2.53 ± 0.03 , which seems to suggest F would not be amenable.

Several more works have been implemented to estimate numerically the amenability of F , such as [1]. Most of them are not decisive, but if they suggest something, it is that F is not amenable. The difficulty of finding Følner sets, together with Moore's theorem, also seems to point in that direction. Even more is given by the possibility that Belk–Brown sets cannot be improved and have the best boundary. For all these reasons, by now most researchers lean towards F being not amenable. But the problem is as open as ever and it is one of the most exciting open problems in group theory today.

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