Two Examples of Resurgence

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ABSTRACT. This paper gives an account of two articles which illustrate the use of Resurgence theory for estimating the difference between two complex invariant manifolds associated with the area-preserving Hénon map for the first one, and with a Hamiltonian system which stems from the rapidly forced pendulum for the second one.

1. Introduction

1.1. Motivation. It is usual that exponentially small phenomena appear in the study of near-integrable Hamiltonian systems with a weakly hyperbolic singular point. One of our goals is to estimate asymptotically the splitting of the separatrices associated with such a point, which is exponentially small with respect to the perturbation parameter, by using Resurgence theory.

A common approach in separatrix splitting problems is to look for good aproximations of the stable and unstable invariant manifolds. But in the cases we are interested in, which are singular in the sense that the hyperbolicity disappears when the perturbation parameter goes to zero (the fixed point is said to be weakly hyperbolic), these manifolds have different approximations in different regions of the complex plane. Using "matching techniques", one can obtain a so-called inner equation, which retains the dominant part of the invariant manifolds and of their splitting.

The papers [**GS01**] and [**OSS03**] are devoted to the resurgent study of the inner equations (1.6) and (1.7) below, obtained in two different settings: an area-preserving map and a Hamiltonian system. Here, we shall try to give an account of these two papers and to explain in a systematic way how Resurgence theory can be a used to estimate the exponentially small difference between special solutions of these inner equations. The passage from results for the inner equations (1.6) and (1.7) to the initial problems of singular separatrix splitting which have motivated them

(equations (1.1) and (1.3) below) would require further work which has not yet been done completely.

Although this paper is centred on two special equations, we want to present here a scheme that can be made suitable for the study of separatrix splitting in any functional equation: differential equations, difference equations, etc.

1.2. The two examples. We shall present the two examples quickly and then outline the approach followed in the papers [**GS01**] and [**OSS03**], to which the reader is referred for the details.

– The first example is the one-parameter family

$$(1.1) \qquad \mathcal{F}_\varepsilon : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x + \varepsilon y + \varepsilon^2 x(1-x) \\ y + \varepsilon x(1-x) \end{pmatrix}, \qquad \varepsilon > 0,$$

of quadratic area-preserving maps of $\mathbb{C}^2$. The origin is a weakly hyperbolic fixed point of $\mathcal{F}_\varepsilon$ and the discrete-time dynamical system defined by the iteration of $\mathcal{F}_\varepsilon$ has stable and unstable invariant curves W_ε^+ and W_ε^-, which are respectively attracted and repelled by the origin. They can be naturally parametrised by $(x^\pm(t), y^\pm(t))$ verifying

$$(1.2) \qquad \mathcal{F}_\varepsilon \begin{pmatrix} x^\pm(t) \\ y^\pm(t) \end{pmatrix} = \begin{pmatrix} x^\pm(t+h) \\ y^\pm(t+h) \end{pmatrix},$$

where $h = \ln \lambda$ and $\lambda = \lambda(\varepsilon)$ is the eigenvalue larger than one at the origin, and with asymptotic conditions

$$\lim_{t \to \pm\infty} (x^\pm(t), y^\pm(t)) = 0.$$

Near the origin, both curves look like the separatrix of the Hamitonian system generated by

$$h(x,y) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3,$$

which plays the role of first approximation for the map $\mathcal{F}_\varepsilon$ (this separatrix is given by $x_0(t) = \frac{3}{2\cosh^2(t/2)}$, $y_0(t) = \frac{dx_0}{dt}(t)$), but numerical experiments show that W_ε^+ and W_ε^- intersect transversally. The angle of intersection is necessarily exponentially small with respect to ε, according to the general results of [**FS90**], and an asymptotic formula was proposed in [**Gel91**] without a complete proof (see also [**Che98**]).

– The second example is a family of Hamiltonian systems with one and a half degrees of freedom, namely the rapidly forced pendulum generated by

$$(1.3) \qquad H_{\mu,\varepsilon}(q,p,t) = \frac{p^2}{2} - 1 + \cos q + \mu(\cos q - 1)\sin(t/\varepsilon),$$

where $\varepsilon > 0$ and μ are two parameters of which the first is assumed to be small but not necessarily the second.

The origin is a hyperbolic fixed point of the unperturbed pendulum, with coinciding stable and unstable curves (the separatrix is given by $q_0(t) = 4\arctan e^t$, $p_0(t) = \frac{dq_0}{dt}(t)$). It gives rise to a weakly hyperbolic $2\pi\varepsilon$-periodic orbit for $H_{\mu,\varepsilon}$ whose stable and unstable manifolds do not coincide any longer: the separatrix has "split" and the phenomenon is again exponentially small with respect to ε. This problem was already investigated by several authors ([**DS92**], [**Gel97**], [**Tre94**]).

Following Poincaré [**Poin93**], we can use the Lagrangian character of the two-dimensional stable and unstable manifolds and write them as graphs of differentials (see for example [**Sau95**] or [**LMS03**]): they admit equations $p = \partial_q S^{\pm}(q, t)$, where S^- (resp. S^+) is $2\pi\varepsilon$-periodic in t and analytic for $q > 0$ small enough (resp. for $q < 2\pi$ large enough) and satisfies the Hamilton-Jacobi equation

(1.4) $$\partial_t S + H_{\mu,\varepsilon}(q, \partial_q S, t) = 0,$$

with asymptotic conditions

$$\lim_{q \to 0} \partial_q S^-(q, t, \mu, \varepsilon) = 0 \quad (\text{resp. } \lim_{q \to 2\pi} \partial_q S^+(q, t, \mu, \varepsilon) = 0).$$

1.3. Derivation of the inner equation. Roughly speaking, the stable and unstable manifolds of (1.1) or (1.3) (described by solutions of (1.2) or (1.4)) are well approximated by the separatrix of the unperturbed problem in the real domain (outer domain), but to compute the exponentially small splitting we need to control these manifolds in some complex region (inner domain), where the separatrix has singularities ([**Gel97**], [**Che98**], [**DS92**], [**DR98**]). Approximations for these manifolds in the inner domain can be obtained as special solutions of the so-called inner equation, which is obtained as an approximation of the original equation in rescaled variables, and which does not depend on the singular parameter ε ([**OS99**]).

– In the first case, we use the change of coordinates $u = \varepsilon^2 x - \varepsilon^2/2$, $v = \varepsilon^3 y$, which turns $\mathcal{F}_\varepsilon$ into

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u + v - u^2 + \varepsilon^4/4 \\ v - u^2 + \varepsilon^4/4 \end{pmatrix}$$

and, for small positive values of ε, we can consider $\mathcal{F}_\varepsilon$ as a small perturbation of a particular case of the Hénon map

(1.5) $$\mathcal{F}: \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u + v - u^2 \\ v - u^2 \end{pmatrix}.$$

The origin is a parabolic fixed point for the map $\mathcal{F}$, which "inherits" from $\mathcal{F}_\varepsilon$ invariant curves which can be naturally parametrised by the complex variable $z = \frac{t - i\pi}{h}$ and written as $(u^{\pm}(z), v^{\pm}(z))$. These parametrisations must satisfy the equation $\mathcal{F}(u(z), v(z)) = (u(z+1), v(z+1))$, which can be reduced to a second-order difference equation for u:

(1.6) $$u(z + 1) - 2u(z) + u(z - 1) = -u^2(z).$$

Indeed, from any solution of (1.6), one obtains the parametrisation of an invariant curve of $\mathcal{F}$ by restoring the second component $v(z) = u(z) - u(z-1)$. Equation (1.6) is the **inner equation** of the problem. The two special curves we are interested in correspond to the asymptotic condition $\lim_{z \to \pm\infty} u^{\pm}(z) = 0$. The initial splitting problem gives rise to the question of estimating the difference $u^+(z) - u^-(z)$ for complex values of z of large negative imaginary part.

– For the second example, the method of complex matching, as described in [**OS99**], leads us to use the variables $\tau = t/\varepsilon$, $z = \frac{u - i\pi/2}{\varepsilon}$ where $u = \log \tan(q/4)$, and $\phi = \varepsilon S$. The Hamilton-Jacobi equation becomes

$$\partial_\tau \phi + \frac{\cosh^2(\varepsilon z + i\pi/2)}{8\varepsilon^2}(\partial_z \phi)^2 - \varepsilon^2 \frac{2}{\cosh^2(\varepsilon z + i\pi/2)}(1 - \mu \sin \tau) = 0$$

and can be considered as a perturbation of the **inner equation**

$$(1.7) \qquad \partial_\tau \phi - \frac{1}{8} z^2 (\partial_z \phi)^2 + 2z^{-2}(1 - \mu \sin \tau) = 0.$$

This equation, where the unknown function ϕ depends on $z \in \mathbb{C}$ and $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ (μ is a complex parameter), must be viewed as the Hamilton-Jacobi equation

$$\partial_\tau \phi + \mathcal{H}(z, \partial_z \phi, \tau) = 0$$

associated with the Hamiltonian function $\mathcal{H}(z, p, \tau) = -\frac{1}{8} z^2 p^2 + 2z^{-2}(1 - \mu \sin \tau)$. Observe that this first-order PDE comes with a periodicity requirement for the variable τ. The asymptotic condition $\lim_{z \to \pm\infty} \phi^\pm(z, \tau) = 0$ will select two particular solutions whose difference will be estimated for complex values of z of large negative imaginary part.

1.4. The rest of the present paper is devoted to the description of the results obtained for equation (1.6) in [**GS01**] and for equation (1.7) in [**OSS03**] and may serve as a guide for reading these articles, as well as a general method to deal with analogous problems.

The first step in the resurgent method is to look for formal solutions of the equation which verify the desired asymptotic condition. As we shall see (Section 2.1), in both examples the formal solution is essentially unique but divergent, and this divergence falls in the scope of Resurgence theory ([**Eca81**], [**Eca92a**], [**Eca92b**], [**CNP93a**], [**CNP93b**]). The idea is to analyse this divergence by means of formal Borel transform (i.e. formal inverse Laplace transform—see Section 2.2).

On the one hand, the study in the Borel plane will allow us to recover the different analytic solutions of the equation we are interested in, by applying Laplace transform in various directions (this is the Borel-Laplace summation process—see Section 3.3). On the other hand, it is in the Borel plane that precise information can be extracted, through the so-called alien calculus (Sections 3.1, 3.2 and 4), which yields for instance the desired estimation of the difference between the analytic solutions of (1.6) or (1.7).

2. Borel transforms of the formal solutions

2.1. The formal solutions $\tilde{u}_0(z)$ and $\tilde{\phi}_0(z, \tau)$. We shall denote by $\mathbb{C}[[z^{-1}]]$ the space of all power series in z^{-1} with complex coefficients and by $\mathbb{C}[z][[z^{-1}]]$ the space of sums of polynomials in z and power series in z^{-1} (i.e. formal Laurent series with finitely many positive powers of z).

For the second example, we shall also need the space $\mathcal{P} = \mathbb{C}[[e^{i\tau}, e^{-i\tau}]]$ of trigonometric polynomials of τ, and $\mathcal{P}[[z^{-1}]]$ will denote the space of formal series in z^{-1} whose coefficients are trigonometric polynomials (these coefficients will also depend on the parameter μ). The space $\mathcal{P}[z][[z^{-1}]]$ consists of formal Laurent series in z, with finitely many positive powers of z, all of whose coefficients also depend on τ as trigonometric polynomials. Thus the corresponding formal series may be expanded in two ways:

$$(2.1) \qquad \tilde{\varphi}(z, \tau) = \sum_{n \geq n_0} \varphi_n(\tau) z^{-n} = \sum_{k \in \mathbb{Z}} \tilde{\varphi}^{[k]}(z) e^{ik\tau},$$

where n_0 is some integer.

LEMMA 2.1. *All nonzero formal solutions of the difference equation (1.6) are of the form* $\tilde{u}_0(z+a)$, *where* $a \in \mathbb{C}$ *and*

$$(2.2) \qquad \tilde{u}_0(z) = -6z^{-2} + \frac{15}{2}z^{-4} - \frac{663}{40}z^{-6} + \cdots$$

is the unique nonzero even solution. The coefficients of $\tilde{u}_0$ are rationals of alternating sign.

For each $\mu \in \mathbb{C}$, the solutions in $\mathcal{P}[z][[z^{-1}]]$ of the Hamilton-Jacobi equation (1.7) are of the form

$$a + \tilde{\phi}_0(z,\tau) \quad or \quad a + \tilde{\phi}_0(-z,\tau),$$

where $a \in \mathbb{C}$ and

$$(2.3) \qquad \tilde{\phi}_0(z,\tau) = 4z^{-1} - (2\mu\cos\tau)z^{-2} - (4\mu\sin\tau + \frac{1}{3}\mu^2)z^{-3} + O(z^{-4})$$

is determined as the unique solution in $z^{-1}\mathcal{P}[[z^{-1}]]$ with leading term $4z^{-1}$.

This lemma is elementary; recursion formulae can be provided to determine the coefficients of the formal solutions inductively. As for most of the statements below, details are to be found in [**GS01**] and [**OSS03**].

2.2. Analyticity of the Borel transforms $\hat{u}_0(\zeta)$ and $\hat{\phi}_0(z,\tau)$.

The formal Borel transform is the linear operator $\mathcal{B}$ defined by

$$\tilde{\varphi}(z) = \sum_{n\geq 1} a_n z^{-n} \quad \mapsto \quad \mathcal{B}(\tilde{\varphi})(\zeta) = \hat{\varphi}(\zeta) = \sum_{n\geq 0} a_n \frac{\zeta^{n-1}}{(n-1)!}.$$

The coefficients a_n can be complex numbers or functions, i.e. $\mathcal{B}$ sends $z^{-1}\mathbb{C}[[z^{-1}]]$ into $\mathbb{C}[[\zeta]]$, and $z^{-1}\mathcal{P}[[z^{-1}]]$ into $\mathcal{P}[[\zeta]]$. Observe that, starting from (2.1), one gets

$$\mathcal{B}(\tilde{\varphi})(\zeta,\tau) = \hat{\varphi}(\zeta,\tau) = \sum_{n\in\mathbb{Z}} \hat{\varphi}^{[k]}(\zeta)e^{ik\tau}$$

with the notation $\hat{\varphi}^{[k]} = \mathcal{B}(\tilde{\varphi}^{[k]})$. Here we must assume $n_0 \geq 1$ in (2.1), but when necessary, the definition of $\mathcal{B}$ is extended to polynomials by use of the Dirac mass at 0 and its derivatives: $z^j \mapsto \delta^{(j)}$ if $j \geq 0$—see [**Eca81**] or [**CNP93a**].

When applied to a power series $\tilde{\varphi}(z)$ with nonzero radius of convergence, $\mathcal{B}$ yields a convergent power series $\hat{\varphi}(\zeta)$ which defines an entire function with at most exponential growth of order 1. We shall be dealing with formal series whose Borel transform converges near the origin but has finite radius of convergence. Such a series $\tilde{\varphi}(z)$ is thus divergent (and "Gevrey-1", i.e. its coefficients a_n are bounded by an expression of the form $CK^n n!$); the singularities of its Borel transform $\hat{\varphi}(\zeta)$ can be considered as "responsible" for the divergence, and it is important to study them.

To study the formal Borel transforms $\hat{u}_0$ and $\hat{\phi}_0$ of the formal solutions $\tilde{u}_0$ and $\tilde{\phi}_0$, we write equations which they satisfy and which are obtained by applying $\mathcal{B}$ to equations (1.6) and (1.7). It is easily checked that $\mathcal{B}(z\partial_z\tilde{\varphi}) = -\partial_\zeta(\zeta\hat{\varphi})$, $\mathcal{B}(\tilde{\varphi}(z+c)) = e^{-c\zeta}\hat{\varphi}$ and $\mathcal{B}(\tilde{\varphi}\tilde{\psi}) = \mathcal{B}(\tilde{\varphi}) * \mathcal{B}(\tilde{\psi})$, where the convolution law is defined by

$$(2.4) \qquad (\hat{\varphi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\,d\zeta_1.$$

The above formula must be understood at a formal level or, if $\hat{\varphi}$ and $\hat{\psi}$ have positive radius of convergence, as defining an analytic germ at the origin, but in that case ζ

must be taken sufficiently close to the origin. In view of this, equation (1.6) is converted by $\mathcal{B}$ into

$$(2.5) \qquad \alpha(\zeta)\hat{u}(\zeta) = -\hat{u}^{*2}(\zeta), \qquad \alpha(\zeta) = 4\sinh^2\frac{\zeta}{2},$$

and equation (1.7) into

$$(2.6) \qquad \partial_\tau\hat{\phi} - \frac{1}{8}\left(\hat{\phi} + \zeta\partial_\zeta\hat{\phi}\right)^{*2} + 2\zeta(1 - \mu\sin\tau) = 0.$$

THEOREM 2.2. *The series $\hat{u}_0(\zeta)$, resp. $\hat{\phi}_0(\zeta,\tau)$, has positive radius of convergence with respect to ζ and defines an analytic germ whose analytic continuation can be followed along any path which starts from the origin and avoids the set $2\pi i\mathbb{Z}$, resp. the set $i\mathbb{Z}$.*

The resulting holomorphic functions $\hat{u}_0(\zeta)$ and $\hat{\phi}_0(\zeta,\tau)$ have at most exponential growth of order 1 with respect to ζ along any non-vertical half-line issuing from the origin.

Let us outline very briefly the technical work devoted to the proof of this result in [**GS01**] and [**OSS03**]. The idea is to devise an iterative scheme to solve (2.5) or (2.6) progessively, in which the desired property is easily checked at each step; all the work then consists in proving the convergence of the process, so as to express $\hat{u}_0$ or $\hat{\phi}_0$ as a uniform limit of analytic germs with this property.

In the first example, the origin of the singularities on $2\pi i\mathbb{Z}$ is the division by $\alpha(\zeta)$ when trying to solve equation (2.5). In the second example, we can write the equation that the new unknown function $\hat{\chi}$ defined by $\hat{\phi} = 4 + \mu\hat{\chi}$ satisfies:

$$(\zeta - \partial_\tau)\hat{\chi} + \mu\frac{1}{8}\left(\hat{\chi} + \zeta\partial_z\hat{\chi}\right)^{*2} + 2\zeta\sin\tau = 0.$$

Using Fourier expansions with respect to τ, we see that the inversion of the operator $\zeta - \partial_\tau$ involves division by $\zeta - ik$ for each Fourier number k, and thus produces singularities on $i\mathbb{Z}$.

From the analytical viewpoint, the arguments for the convergence are quite different from [**GS01**] to [**OSS03**], because a method of majorant series is used, which depends a lot on the precise form of the equation. There is also a geometric part, similar in both papers, devoted to the analytic continuation of convolution products.

Let $\mathcal{R}$ denote the Riemann surface consisting of all homotopy classes of paths issuing from the origin and lying in $\mathbb{C}\backslash 2\pi i\mathbb{Z}$, except for their origin (suppressing the factor 2π from this definition when the second example is considered). The main sheet $\mathcal{R}^{(0)}$ of $\mathcal{R}$ will be the holomorphic star for $\hat{u}_0$ or $\hat{\phi}_0$; it can be identified with the cut plane $\mathbb{C} \setminus \pm 2\pi i[1, +\infty)$. The convergence of the aforementioned iterative scheme is directly obtained in $\mathcal{R}^{(0)}$ with an ad hoc method of majorant series, using the fact that formula (2.4) holds true in $\mathcal{R}^{(0)}$ without any change when dealing with functions $\hat{\varphi}$ and $\hat{\psi}$ that are themselves analytic in $\mathcal{R}^{(0)}$.

The "contiguous half-sheets" are obtained as homotopy classes of paths issuing from the origin, lying in $\mathbb{C}\backslash 2\pi i\mathbb{Z}$ and crossing the imaginary axis at most once; we denote by $\mathcal{R}^{(1)}$ their union (the left part of Figure 1 shows an example of a path γ_ζ defining a point $\zeta \in \mathcal{R}^{(1)}$). Supposing that $\hat{\varphi}$ and $\hat{\psi}$ extend analytically to $\mathcal{R}^{(1)}$, one can check that the same is true for their convolution product thanks to the
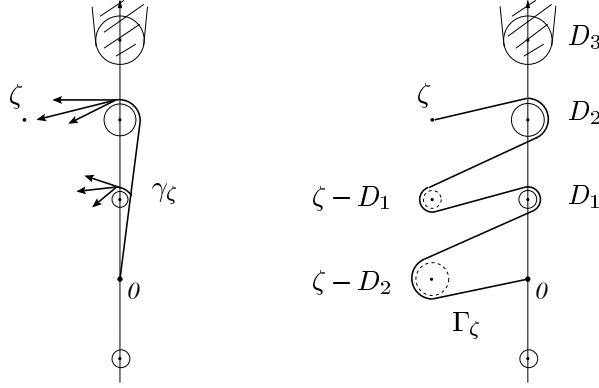
FIGURE 1. The paths γ_ζ and Γ_ζ define the same point $\zeta \in \mathcal{R}^{(1)} \subset \mathcal{R}$.

formula

$$(\hat{\varphi} * \hat{\psi})(\zeta) = \int_{\Gamma_\zeta} \hat{\varphi}(\zeta_1)\hat{\psi}(\zeta - \zeta_1)\, d\zeta_1, \qquad \zeta \in \mathcal{R}^{(1)},$$

where the "symmetrically contractile" path Γ_ζ is drawn on the right part of Figure 1. This formula yields also bounds of the convolution in terms of bounds of the factors, as far as one stays in $\mathcal{R}^{(1)}$. The convergence of the iterative scheme can thus be guaranteed in $\mathcal{R}^{(1)}$ by an appropriate adaptation of the majorant method.

But other ideas are needed to reach all the other sheets of the Riemann surface $\mathcal{R}$: "resurgence relations" are used for this—we shall return to that point in Section 4.

3. First alien derivatives and splitting for the inner equations

As already mentioned, we need to investigate the behaviour of the Borel transforms near their singular points (the points $2\pi i m$ or im, $m \in \mathbb{Z}$) in order to understand the divergence of $\tilde{u}_0$ and $\tilde{\phi}_0$. We shall begin with the singularity which corresponds to $m = 1$ (of course the singularity corresponding to $m = -1$ can be deduced by symmetry, since $\hat{u}_0$ and $\hat{\phi}_0$ are real-analytic) and introduce in this case some of the concepts defined by Écalle.

3.1. Majors of singularities. We can define analytic germs by the formulae

(3.1) $$\overset{\vee}{v}(\zeta) = \hat{u}_0(2\pi i + \zeta), \qquad \overset{\vee}{\psi}(\zeta, \tau) = \hat{\phi}_0(i + \zeta, \tau),$$

for $\zeta \notin i\mathbb{R}^+$ of small enough modulus, by requiring that the principal determination of $\hat{u}_0$ or $\hat{\phi}_0$ be used, i.e. that "$2\pi i + \zeta$" denote the point of $\mathcal{R}$ defined by the straight segment $[0, 2\pi i + \zeta]$ in the first case for instance. Germs like $\overset{\vee}{v}$ and $\overset{\vee}{\psi}$ are called "majors" because they are meant to determine a "singularity" at $\zeta = 0$; a singularity is nothing but a class of majors modulo regular germs.[1]

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[1] We do not give all the details here, but one has to specify a direction $\arg\zeta = \theta_0$ on the Riemann surface of the logarithm ($\theta_0 = \pi/2$ in our case) and to impose to majors to be holomorphic in sectors of the form $\{r\,e^{i\theta}; 0 < r < r_0, \theta_0 - 2\pi - \varpi < \theta < \theta_0 + \varpi\}$ for some $r_0, \varpi > 0$—see [**OSS03**, Section 2.4.1].

Our majors will turn out to belong to a space of germs on the Riemann surface of the logarithm which assume a special form ("simply ramified majors"):

$$\overset{\vee}{\varphi}(\zeta) = \sum_{n=0}^{n_0} \frac{(-1)^n n! A_n}{2\pi i \zeta^{n+1}} + \frac{1}{2\pi i} \hat{\varphi}(\zeta) \log \zeta + \hat{r}(\zeta),$$

where $n_0 \in \mathbb{N}$, $A_0, \ldots, A_{n_0} \in \mathbb{C}$, and $\hat{\varphi}, \hat{r} \in \mathbb{C}\{\zeta\}$. (We shall have $n_0 = 5$ in the case of $\overset{\vee}{v}$, while $n_0 = 1$ and the various ingredients of the formula depend also on τ for $\overset{\vee}{\psi}$.) The singularity $\overset{\triangledown}{\varphi} = \text{sing}(\overset{\vee}{\varphi})$ represented by such a major is thus determined by a finite number of complex numbers A_n and a regular germ $\hat{\varphi}$. But much more general singularities, represented by more general majors, are conceivable, and need in fact to be included in the theory of resurgent functions. For simply ramified singularities, we shall use the notation

$$\overset{\triangledown}{\varphi} = \sum_{n=0}^{n_0} A_n \delta^{(n)} + {}^b\hat{\varphi}, \qquad \delta^{(n)} = \text{sing}\left(\frac{(-1)^n n!}{2\pi i \zeta^{n+1}}\right), \qquad {}^b\hat{\varphi} = \text{sing}\left(\frac{1}{2\pi i} \hat{\varphi}(\zeta) \log \zeta\right),$$

and extend the formal Borel-Laplace isomorphism by considering

$$\tilde{\varphi} = \sum_{n=0}^{n_0} A_n z^n + \mathcal{B}^{-1} \hat{\varphi}$$

as the formal counterpart of the singularity $\overset{\triangledown}{\varphi}$. In the case of more general singularities, the formal Borel-Laplace isomorphism may lead to formal objects more general than power series (e.g. transseries of z).

A minor $\hat{\varphi}$ is associated to any singularity $\overset{\triangledown}{\varphi}$ by computing the monodromy around the origin of a representant $\overset{\vee}{\varphi}$ of $\overset{\triangledown}{\varphi}$:

$$\hat{\varphi}(\zeta) = \overset{\vee}{\varphi}(\zeta) - \overset{\vee}{\varphi}(\zeta e^{-2\pi i})$$

(obviously, the result does not depend on the choice of the major $\overset{\vee}{\varphi}$). The simply ramified case thus corresponds to minors $\hat{\varphi}$ that are regular at the origin.

Let us sum up the objects we have at our disposal starting from $\tilde{u}_0$ or $\tilde{\phi}_0$. We may consider $\hat{u}_0 = \mathcal{B}\tilde{u}_0$ and $\hat{\phi}_0 = \mathcal{B}\tilde{\phi}_0$ as the minors of some simply ramified singularities at the origin, $\overset{\triangledown}{u}_0 = {}^b\hat{u}_0$ and $\overset{\triangledown}{\phi}_0 = {}^b\hat{\phi}_0$. Following the analytic continuation of the minors, we encounter a first singularity at $2\pi i$ or i, encoded in $\overset{\triangledown}{v}$ or $\overset{\triangledown}{\psi}$. This is the definition of the first **alien derivatives**:

$$\Delta_{2\pi i}\overset{\triangledown}{u}_0 = \overset{\triangledown}{v} = \text{sing}\big(\hat{u}_0(2\pi i + \zeta)\big), \quad \Delta_i\overset{\triangledown}{\phi}_0 = \overset{\triangledown}{\psi} = \text{sing}\left(\hat{\phi}_0(i + \zeta)\right).$$

The operators $\Delta_{2\pi i}$ and Δ_i, whose definition we have just sketched out, are a particular case of the alien derivations Δ_ω defined by Écalle for all $\omega \in \mathbb{C}$ (or even for all ω on the Riemann surface of the logarithm in the most general case). The operator Δ_ω measures the singularity at ω of the analytic continuation of the minors, but in the general case its definition must take into account the possible multivaluedness of the minors: one has to consider a well-balanced average of the various determinations obtained by following $]0, \omega[$ and avoiding the possible intermediary singular points.

In our case, only the operators $\Delta_{2\pi i m}$ or Δ_{im} ($m \in \mathbb{Z}^*$) act non-trivially on $\overset{\triangledown}{u}_0$ or $\overset{\triangledown}{\phi}_0$, and the singular behaviour of the analytic continuation of $\hat{u}_0$ (resp. $\hat{\phi}_0$) is

encoded by all the successive alien derivatives $\Delta_{\omega_r} \ldots \Delta_{\omega_1} \overset{\vee}{u}_0$, where $r \geq 1$ and $\omega_1, \ldots, \omega_r \in 2\pi i \mathbb{Z}$ (resp. $\Delta_{\omega_r} \ldots \Delta_{\omega_1} \overset{\vee}{\phi}_0$, $\omega_1, \ldots, \omega_r \in i\mathbb{Z}$).

The space of **resurgent functions** consists of the singularities for which the action of all the successive alien derivations $\Delta_{\omega_r} \ldots \Delta_{\omega_1}$ is well defined; this amounts to a condition of discreteness of the set of singular points obtained when following the analytic continuation of their minors. Theorem 2.2 claims that $\overset{\vee}{u}_0$ and $\overset{\vee}{\phi}_0$ are such singularities, but observe that the part of the proof explained in Section 2.2 (analyticity of the minors $\hat{u}_0$ and $\hat{\phi}_0$ in $\mathcal{R}^{(1)}$) allows us to consider only the first alien derivatives $\overset{\vee}{v}$ and $\overset{\vee}{\psi}$, since we do not know enough on $\hat{v}$ and $\hat{\psi}$ at that stage to be sure that further alien derivatives are defined (the definition of $\Delta_{2\pi i}\Delta_{2\pi i}\overset{\vee}{u}_0$ for instance involves the analytic continuation of $\hat{u}_0$ along paths which cross $i\mathbb{R}^{*+}$ twice).

All this formalism is adapted to nonlinear contexts—and this essential feature may serve as its main justification—because one can define a convolution law on the space of singularities under which the subspace of resurgent functions is stable and for which the alien derivations satisfy the Leibniz rule (thus they deserve their name of derivation). This law is a suitable extension of the convolution of minors (2.4), so that $\overset{b}{\hat{\varphi}} * \overset{b}{\hat{\psi}} = \overset{b}{(\hat{\varphi} * \hat{\psi})}$ for instance, and more generally it corresponds by Borel-Laplace to the multiplication of formal series.

The differentiation with respect to z corresponds to the operator

$$\partial : \operatorname{sing}\big(\overset{\vee}{\varphi}(\zeta)\big) \mapsto \operatorname{sing}\big(-\zeta \overset{\vee}{\varphi}(\zeta)\big),$$

which is the "natural derivation". One can check that $[\Delta_\omega, \partial] = -\omega \Delta_\omega$.

It is in fact convenient to pull-back everything in the formal model whenever possible. We continue to call "resurgent functions" the formal series whose formal Borel transform are resurgent functions in the above sense, and "alien derivations" the operators which act in the formal model (still denoted by Δ_ω). With that convention, we may say that $[\Delta_\omega, \frac{d}{dz}] = -\omega \Delta_\omega$, or that $\frac{d}{dz}$ commutes with the "dotted alien derivations" $\overset{\bullet}{\Delta}_\omega = e^{-\omega z}\Delta_\omega$. But, as a rule, the interpretation and the justification of resurgent formulae is best seen in the convolutive model. We refer particularly to Sections 2.4 and 3.1.3 of [**OSS03**] for more details.

3.2. First resurgent relations. Let us illustrate the efficiency of the formalism outlined in the previous section by the computation of $\overset{\vee}{v} = \Delta_{2\pi i}\overset{\vee}{u}_0$ and $\overset{\vee}{\psi} = \Delta_i\overset{\vee}{\phi}_0$. These two singularities are easily seen to satisfy linear equations derived from (2.5) and (2.6) by rephrasing these equations as

$$\alpha(\zeta)\overset{\vee}{u}_0 = -(\overset{\vee}{u}_0)^{*2}, \quad \partial_\tau \overset{\vee}{\phi}_0 - \frac{1}{8}\delta^{(2)} * \big(\partial \overset{\vee}{\phi}_0\big)^{*2} + 2\delta^{(-2)}(1 - \mu\sin\tau) = 0.$$

The multiplication of a singularity by a regular germ like $\alpha(\zeta)$ is defined by considering the product of α with any representant of the singularity; in the second equation we have used the notation $\delta^{(-n-1)} = \overset{b}{(\zeta^n/n!)}$ for $n \in \mathbb{N}$. Since Δ_ω is a derivation whose commutators with the operators involved in these equations are known (for instance $\Delta_{2\pi i}$ commutes with the multiplication by $\alpha(\zeta)$ because α is $2\pi i$-periodic), applying the operator Δ_ω with $\omega = 2\pi i$ or i, we find

$$(3.2) \qquad \alpha(\zeta)\overset{\vee}{v} = -2\overset{\vee}{u}_0 * \overset{\vee}{v}, \qquad \partial_\tau\overset{\vee}{\psi} + \overset{\vee}{D}_0 * \partial\overset{\vee}{\psi} = i\overset{\vee}{D}_0 * \overset{\vee}{\psi},$$

where $\overset{\vee}{D}_0 = -\frac{1}{4}\delta^{(2)} * \big(\partial\overset{\vee}{\phi}_0\big)$ (the right-hand side in (3.2b) stems from $[\partial, \Delta_i]$; of course Δ_i annihilates all singularities whose minors have no singularity at i, in particular every $\delta^{(n)}$, $n \in \mathbb{Z}$).

– Let us first consider the case of equation (3.2a) satisfied by $\overset{\vee}{v} = \Delta_{2\pi i}\overset{\vee}{u}_0$. This equation is obviously satisfied by $\partial\overset{\vee}{u}_0$ too: in the formal model, this amounts simply to saying that the linearized equation $\tilde{v}(z+1) - 2\tilde{v}(z) + \tilde{v}(z-1) = -2\tilde{u}_0(z)\tilde{v}(z)$ is satisfied by $\frac{d\tilde{u}_0}{dz}$.

LEMMA 3.1. *The set of solutions of equation (3.2a) in the space of singularities consists of all linear combinations (with constant coefficients) of two particular solutions $\overset{\vee}{v}_1 = \partial\overset{\vee}{u}_0$ and $\overset{\vee}{v}_2$ which are simply ramified singularities and whose coefficients can be determined inductively. The second fundamental solution is uniquely determined by the condition that its leading term be $\frac{1}{84}\delta^{(4)}$.*

The formal counterparts of $\overset{\vee}{v}_1$, $\overset{\vee}{v}_2$ are of the form

$$\tilde{v}_1 = \frac{d\tilde{u}_0}{dz} = \sum_{k=1}^{\infty} \frac{b_k}{z^{2k+1}}, \quad \tilde{v}_2 = \sum_{k=-2}^{\infty} \frac{d_k}{z^{2k}} = \frac{1}{84}z^4 + \frac{17}{840}z^2 - \frac{17}{2240} + O(z^{-2}).$$

This lemma is not stated in that form in [**GS01**] which indicates rather the space of solutions of the formal counterpart of the equation in $\mathbb{C}[z][[z^{-1}]]$. But the above lemma (which can be proved essentially by the same arguments, transposed in the convolutive model) is stronger, because it shows that any singularity $\overset{\vee}{v}$ solving (3.2a) is necessarily simply ramified (we need not assume a priori it to be the Borel transform of an element of $\mathbb{C}[z][[z^{-1}]]$), and this allows us to shorten all the chain of reasoning, as noticed in [**OSS03**]. The point we wish to make here is that some technical verifications like Section 4.1 in [**GS01**] can be dispensed with by the use of the formalism of majors and singularities.

The consequence for the first singularity of $\hat{u}_0$ is immediate:

COROLLARY 3.2. *There exist complex numbers Λ and Θ such that*

(3.3) $$\Delta_{2\pi i}\tilde{u}_0 = \Lambda\tilde{v}_1 + \Theta\tilde{v}_2.$$

In other words, the principal determination of $\hat{u}_0$ is of the form

(3.4) $$\hat{u}_0(2\pi i + \zeta) = \frac{\Theta}{2\pi i}\left(\frac{4!d_{-2}}{\zeta^5} + \frac{2!d_{-1}}{\zeta^3} + \frac{d_0}{\zeta}\right) + \frac{1}{2\pi i}\hat{h}(\zeta)\log\zeta + \hat{r}(\zeta),$$

for $\zeta \notin i\mathbb{R}^+$ of small enough modulus, where $\hat{h} = \Lambda\hat{v}_1 + \Theta\hat{v}_2 \in \mathbb{C}\{\zeta\}$ and $\hat{r}$ is some other regular germ.

One can check that $\Lambda \in \mathbb{R}$ and $\Theta \in i\mathbb{R}$; the second constant is the most important since it governs the singular behaviour at $2\pi i$. A simple argument is given in [**GS01**, Section 2] to prove that $\Im m\,\Theta < 0$ (the idea is to reach a contradiction when assuming $\Theta = 0$ by observing that this would imply the boundedness of $\hat{u}_0$ near $2\pi i$).

– As for equation (3.2b), we can view it as a linear PDE in the formal model, easy to solve by the method of characteristics applied to the operator $\partial_\tau + \tilde{D}_0(z,\tau)\partial_z$, where $\tilde{D}_0 = -\frac{1}{4}z^2\partial_z\tilde{\phi}_0(z,\tau)$ corresponds to $\overset{\vee}{D}_0$. We refer the reader to [**OSS03**, Section 2.1.4] for

LEMMA 3.3. *The equation*

(3.5) $$\partial_\tau Y + \tilde{D}_0(z,\tau)\partial_z Y = 0$$

admits a unique formal solution of the form

$$Y = z - \tau + \tilde{S}, \qquad \tilde{S} \in z^{-1}\mathcal{P}[[z^{-1}]].$$

The coefficients of the formal series $\tilde{S}$ can be determined inductively, and

$$\tilde{S}(z,\tau) = (-\frac{1}{4}\mu^2 + \mu\sin\tau)z^{-1} - 4\mu(\cos\tau)z^{-2} + O(z^{-3}).$$

Now, from (3.2b), one can check that $e^{-iz}\tilde{\psi}(z,\tau) = \overset{\bullet}{\Delta}_i\tilde{\phi}_0$ solves equation (3.5) (or directly from (1.7) using the fact that $\overset{\bullet}{\Delta}_i$ commutes with ∂_z), and thus must be a "function" of $z - \tau + \tilde{S}(z,\tau)$; in fact, the requirement of periodicity with respect to τ and the nature of the dependence of $e^{-iz}\tilde{\psi}(z,\tau)$ upon z forces it to be proportional to $e^{-iz+i\tau-i\tilde{S}}$.

The previous reasoning somewhat lacks rigour because it takes for granted that $\overset{\triangledown}{\psi} = \Delta_i\overset{\triangledown}{\phi}_0$ is the Borel transform of a member of $\mathcal{P}[z][[z^{-1}]]$ (i.e. that it is a simply ramified singularity). But there is no difficulty in making it rigorous by transposing it in the convolutive model, as shown in [**OSS03**, Section 2.4.3]. We thus state

COROLLARY 3.4. *For each $\mu \in \mathbb{C}$, there exists $\beta \in \mathbb{C}$ such that*

$$(3.6) \qquad \Delta_i\tilde{\phi}_0 = \beta\,e^{i\tau-i\tilde{S}}.$$

In other words, the principal determination of $\hat{\phi}_0$ is of the form

$$(3.7) \qquad \hat{\phi}_0(i+\zeta) = \frac{\beta\,e^{i\tau}}{2\pi i\zeta} + \frac{1}{2\pi i}\hat{\psi}(\zeta,\tau)\log\zeta + \hat{r}(\zeta,\tau),$$

for $\zeta \notin i\mathbb{R}^+$ of small enough modulus, where $\hat{\psi}$ and $\hat{r}$ are regular with respect to ζ, and $\hat{\psi}$ is determined as

$$(3.8) \qquad \hat{\psi}(z,\tau) = \beta\,e^{i\tau}\sum_{n\geq 1}\frac{(-i)^n}{n!}\hat{S}^{*n}(\zeta,\tau).$$

One can check that β depends on the parameter μ as an odd entire function of the form $-2\pi i\mu + O(\mu^3)$ for small $|\mu|$.

3.3. Splitting associated with the inner equations. We now show how to recover analytic solutions of (1.6) and (1.4) from our formal solutions, and how to use the description of the first singularity in the Borel plane to study the corresponding splitting. Theorem 2.2 admits the following

COROLLARY 3.5. *The formulae*

$$u^\pm(z) = \int_0^{\pm\infty} e^{-z\zeta}\,\hat{u}_0(\zeta)\,d\zeta, \qquad \phi^\pm(z,\tau) = \int_0^{\pm\infty} e^{-z\zeta}\,\hat{\phi}_0(\zeta,\tau)\,d\zeta$$

define analytic functions u^+, u^- or ϕ^+, ϕ^-, which solve equations (1.6) or (1.7) and satisfy the asymptotic conditions

$$\lim_{\Re e\,z\to\pm\infty} u^\pm(z) = 0, \qquad \lim_{\Re e\,z\to\pm\infty} \phi^\pm(z,\tau) = 0$$

(uniformly in $\tau \in \mathbb{R}/2\pi\mathbb{Z}$ in the case of $\phi^\pm$).

Observe that exponential bounds of the type $|\hat{u}(\zeta)|, |\hat{\phi}_0(\zeta,\tau)| \leq c_1\,e^{c_2|\zeta|}$ (whose existence is claimed in Theorem 2.2) are necessary to define the Laplace transforms $u^\pm$ or $\phi^\pm$. These functions are analytic in the half-planes $\{\pm\Re e\,z > c_2\}$. But, by Cauchy Theorem, we can move the half-line of integration: the use of $\arg\zeta = \theta$
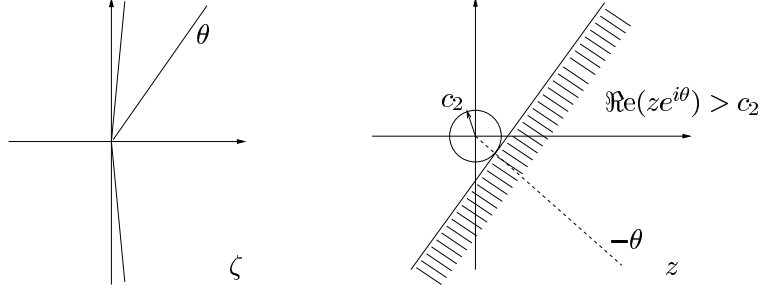
FIGURE 2.   Laplace transforms in directions $\arg \zeta = \theta$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ yield a domain $\mathcal{D}^+$ of validity of the asymptotic expansion for u^+ or ϕ^+.

with $-\frac{\pi}{2} + \varpi < \theta < \frac{\pi}{2} - \varpi$ (for some small positive ϖ), or with $\frac{\pi}{2} + \varpi < \theta < \frac{3\pi}{2} - \varpi$, yields analytic continuation in a domain $\mathcal{D}^+$ or $\mathcal{D}^-$, obtained as the union of the corresponding half-planes $\{\Re e(z\, e^{i\theta}) > c_2\}$—see Figure 2 for $\mathcal{D}^+$; the domain $\mathcal{D}^-$ can be deduced from $\mathcal{D}^+$ by symmetry with respect to the imaginary axis. These domains are sectorial neighbourhoods of infinity, of aperture $2\pi - 2\varpi$, bissected by $\mathbb{R}^{\pm}$, in which asymptotic expansions hold:

$$u^{\pm}(z) \sim \tilde{u}_0(z), \qquad \phi^{\pm}(z, \tau) \sim \tilde{\phi}_0(z, \tau), \qquad |z| \to \infty, \ z \in \mathcal{D}^{\pm}.$$

These are standard facts of the theory of the Borel-Laplace transform; the divergence of $\tilde{u}_0$ or $\tilde{\phi}_0$ is manifested by the existence of singularities in the Borel plane for $\hat{u}_0$ or $\hat{\phi}_0$, and by ambiguities of resummation: two different sums are attributed to each divergent series.

We are now ready to see that the "splitting" functions, i.e. the differences between the two different sums, admit exponentially small asymptotics which can be derived from our study of the first singularities in the Borel plane.

THEOREM 3.6. *We have*

$$u^+ - u^- \sim e^{-2\pi i z} \Delta_{2\pi i} \tilde{u}_0, \qquad \phi^+ - \phi^- \sim e^{-iz} \Delta_i \tilde{\phi}_0,$$

in the part of $\mathcal{D}^+ \cap \mathcal{D}^-$ contained in the half-plane $\{\Im m\, z < 0\}$. In other words, we have in this domain the asymptotic expansions

$$e^{2\pi i z}\left(u^+ - u^-\right) \sim \Theta(d_{-2} z^4 + d_{-1} z^2 + d_0) + \tilde{h}(z) = \Lambda \tilde{v}_1(z) + \Theta \tilde{v}_2(z)$$

and

$$e^{iz}\left(\phi^+ - \phi^-\right) \sim \beta\, e^{i\tau} + \tilde{\psi}(z, \tau) = \beta\, e^{i\tau - i\tilde{S}(z,\tau)},$$

where Λ, Θ and β are defined by Corollaries 3.2 and 3.4.

Of course $\mathcal{D}^+ \cap \mathcal{D}^-$ has two connected components (each one being a sectorial neighbourhood of infinity, of aperture $\pi - 2\varpi$, bissected by $i\mathbb{R}^{\pm}$), and a similar result holds in the component which is contained in the upper half-plane (but it involves the alien derivative at $-2\pi i$ or $-i$ instead of $2\pi i$ or i).

Let us explain briefly the proof of this theorem in the case of $\phi^+ - \phi^-$ (the other one is similar), and with $z \in i\mathbb{R}^-$ of large modulus. The splitting function can be written

$$\phi^+(z, \tau) - \phi^-(z, \tau) = \int_{e^{i(\pi - \theta)}\infty}^{e^{i\theta}\infty} e^{-z\zeta}\, \hat{\phi}_0(\zeta, \tau)\, d\zeta,$$
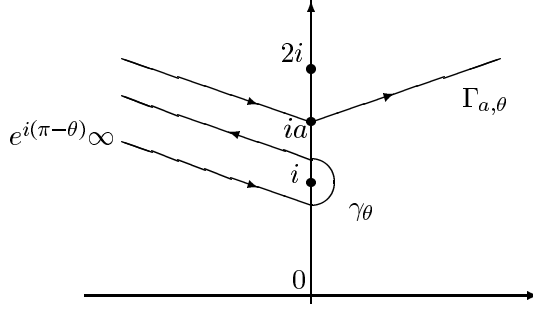
FIGURE 3. Deformation of contour for the study of $\phi^+ - \phi^-$

with any $\theta \in \,]0, \frac{\pi}{2}[$. By Cauchy Theorem, the path of integration can be deformed as indicated on Figure 3, into a path γ_θ which crosses the positive imaginary axis between the first two singularities and a path $\Gamma_{a,\theta}$ for which $\Im m\, \zeta \geq a$ with $a \in \,]1, 2[$. Correspondingly, we can decompose the integral into the contribution of the first singularity and an exponentially smaller term:

$$\phi^+ - \phi^- = \int_{\gamma_\theta} \hat{\phi}_0(\zeta, \tau)\, e^{-z\zeta}\, d\zeta + R(z, \tau), \qquad R(z, \tau) = O(e^{-a|\Im m\, z|}).$$

By virtue of equation (3.7) in Corollary 3.4, the first integral can be written as

$$e^{-iz} \Big(\beta\, e^{i\tau} + \int_0^{e^{i(\pi - \theta)} \infty} \hat{\psi}(\zeta, \tau)\, e^{-z\zeta}\, d\zeta \Big)$$

which admits the announced asymptotic expansion.

4. Formal integral and Bridge equation

We shall now try to describe the whole resurgent structure of $\tilde{u}_0$ and $\tilde{\phi}_0$, i.e. the relations satisfied by all the singularities encountered when following the analytic continuation of their Borel transforms on the Riemann surface $\mathcal{R}$. There will appear new "resurgent relations", similar to equations (3.3) or (3.6), involving new constants, similar to Λ and Θ, or β. This important phenomenon is at the origin of the name "resurgence" and will be encoded into a single equation, the so-called Bridge equation.

Again, the exposition will be sketchy, but we shall indicate how the proof of all the resurgent relations is intertwined with the part of the proof of Theorem 2.2 which was lacking at the end of Section 2.2.

In order to proceed, we must extend the notion of formal solution and embed $\tilde{u}_0$ or $\tilde{\phi}_0$ into a one-parameter formal object which will be called the **formal integral** of equation (1.6) or (1.7). In the first example, this new formal object will be

$$\tilde{u}(z, c) = \sum_{n \geq 0} c^n \tilde{u}_n(z),$$

where the formal solution $\tilde{u}_0$ already studied is supplemented with new formal series $\tilde{u}_n \in \mathbb{C}[z][[z^{-1}]]$, $n \geq 1$. In the second example, it will be

$$\tilde{\phi}(z, \tau, c) = \sum_{n \geq 0} c^n \tilde{\phi}_n(z, \tau),$$

with new formal series $\tilde{\phi}_n \in \mathcal{P}[z][[z^{-1}]]$, $n \geq 1$.

Plugging these expressions into the inner equations, we obtain a system of linear difference equations or linear partial differential equations to be satisfied by the components of the formal integral. Indeed, using the notation

$$P : u(z) \longmapsto u(z+1) - 2u(z) + u(z-1)$$

and expanding equation (1.6) with respect to c, we find the system

(4.1) $$P\tilde{u}_1 + 2\tilde{u}_0\tilde{u}_1 = 0,$$

(4.2) $$P\tilde{u}_n + 2\tilde{u}_0\tilde{u}_1 = \sum_{k=1}^{n-1} \tilde{u}_k\tilde{u}_{n-k}, \qquad n \geq 2,$$

while expanding (1.7) we find

(4.3) $$\left(\partial_\tau + \tilde{D}_0\partial_z\right)\tilde{\phi}_1 = 0$$

(4.4) $$\left(\partial_\tau + \tilde{D}_0\partial_z\right)\tilde{\phi}_n = \frac{1}{8}z^2 \sum_{k=1}^{n-1} \partial_z\tilde{\phi}_k\partial_z\tilde{\phi}_{n-k}, \qquad n \geq 2.$$

Equation (4.1) is nothing but the formal counterpart of (3.2a); in view of Lemma 3.1, we can thus choose the solution $\tilde{u}_1 = 84\tilde{v}_2$, so as to have $\tilde{u}_1(z) = z^4 + O(z^2)$. As for equation (4.3), Lemma 3.3 provides us with the solution $\tilde{\phi}_1 = z - \tau + \tilde{S}$.

PROPOSITION 4.1. *There is a unique sequence of nonzero even series $(\tilde{u}_n)_{n \geq 2}$ in $\mathbb{C}[z][[z^{-1}]]$ such that the coefficient of z^4 vanishes in each of them and equation (4.2) is fulfilled for each n.*

For each $\mu \in \mathbb{C}$, there is a unique sequence $(\tilde{\phi}_n)_{n \geq 2}$ in $\mathcal{P}[z][[z^{-1}]]$ such that the constant term (the coefficient of $e^{ik\tau}z^{-j}$ when $k = j = 0$) vanishes in each of them and equation (4.4) is fulfilled for each n.

We can thus define the formal integrals of equations (1.6) and (1.7)

$$\tilde{u}(z,c) = \sum_{n \geq 0} c^n\tilde{u}_n(z), \qquad \tilde{\phi}(z,\tau,c) = \sum_{n \geq 0} c^n\tilde{\phi}_n(z,\tau),$$

using the above formal series together with $\tilde{u}_1 = 84\tilde{v}_2$, $\tilde{\phi}_1 = z - \tau + \tilde{S}$ from Lemmas 3.1 and 3.3, and the formal solutions $\tilde{u}_0$ and $\tilde{\phi}_0$ from Lemma 2.1.

The proof is elementary, but involves some technical verifications. In the case of $\tilde{u}_n$, tools of the theory of second-order linear difference equations are used, which are analogous to well-known facts in the theory of differential equations (a discrete Wronskian for instance). And the idea for finding $\tilde{\phi}_n$ is to consider $x = z + \tilde{S}(z, \tau)$ as a formal change of variable which conjugates the vector field $\partial_\tau + \tilde{D}_0(z, \tau)\partial_z$ with $\partial_\tau + \partial_x$.

We shall skip the induction which supplies all these formal series, but it must be mentioned that *the way $\tilde{u}_1$ and the next $\tilde{u}_n$'s relate to $\tilde{u}_0$ directly shows that their Borel transforms inherit the properties of convergence and analytic continuation of $\hat{u}_0$, and similarly for the $\tilde{\phi}_n$'s.* As a result, the arguments given in Section 2.2 are sufficient to establish the analyticity of the minors $\hat{u}_n$ and $\hat{\phi}_n$ in $\mathcal{R}^{(1)}$.

But, according to Section 3.1, this makes it possible to define the singularities $\Delta_{\pm 2\pi i}\tilde{u}_n$ or $\Delta_{\pm i}\tilde{\phi}_n$. And the same kind of arguments as in Section 3.2 will apply to them, yielding relations between them and the components of the formal

integrals. In the same time, since

$$\Delta_{2\pi i}\tilde{u}_0 = a\partial_z\tilde{u}_0 + b\tilde{u}_1, \qquad a = \Lambda, \ b = \Theta/84$$

(by (3.3)), we see that the minor of $\Delta_{2\pi i}\tilde{u}_0$ itself extends analytically to $\mathcal{R}^{(1)}$. This amounts to a property of analytic continuation of $\hat{u}_0$ in half-sheets of $\mathcal{R}$ which are part of a subset $\mathcal{R}^{(2)}$ (accessed from $\mathcal{R}^{(1)}$ by crossing once more the imaginary axis), and allows us to define $\Delta_{\pm 2\pi i}\Delta_{2\pi i}\tilde{u}_0$ and $\Delta_{4\pi i}\tilde{u}_0$, and similarly $\Delta_{\pm 2\pi i}\Delta_{-2\pi i}\tilde{u}_0$ and $\Delta_{-4\pi i}\tilde{u}_0$ (using a counterpart of Corollary 3.2 for the alien derivative at $-2\pi i$). And still by the same kind of arguments, these new singularities will turn out to be combinations of components of the formal integral, as will be the case of the successive alien derivatives of all the $\tilde{u}_n$'s.

Of course, the chain of reasoning that we have just sketched for the $\tilde{u}_n$'s works the same for the $\tilde{\phi}_n$'s, starting with $\overset{\bullet}{\Delta}_i\tilde{\phi}_0 = \beta\, e^{-i\tilde{\phi}_1}$.

This explains why in [**GS01**] or [**OSS03**] the proof of the resurgent relations and that of Theorem 2.2 are done simultaneously: once that the possibility of following the analytic continuation of $\hat{u}_0$ or $\hat{\phi}_0$ in $\mathcal{R}^{(1)}$ has been checked (by the method indicated in Section 2.2) and that the first resurgent relations (3.3) or (3.6) have been derived, analyticity is, so to say, automatically propagated in the further sheets of $\mathcal{R}$.

Summing up, and taking for granted the analyticity of all minors in $\mathcal{R}$, the shape of the resurgent relations can be derived very simply, at the level of the formal integrals, by the following computations (whose justifications must however be provided component-wise):

– Since $\tilde{u}(z,c)$ solves the equation $P\tilde{u} = -\tilde{u}^2$ and Δ_ω is a derivation which commutes with P for $\omega \in 2\pi i\mathbb{Z}^*$, the generating series $\Delta_\omega\tilde{u} = \sum_{n\geq 0} c^n\Delta_\omega\tilde{u}_n(z)$ is solution of the linearized equation $P\tilde{v} = 2\tilde{u}\tilde{v}$; but $(\partial_c\tilde{u}, \partial_z\tilde{u})$ is a system of fundamental solutions of the linearized equation, hence $\Delta_\omega\tilde{u}$ must be a linear combination of these two series.

– Since $\tilde{\phi}(z,\tau,c)$ solves equation (1.7) and $\overset{\bullet}{\Delta}_\omega$ is a derivation which commutes with ∂_z and ∂_τ for $\omega \in i\mathbb{Z}^*$, the generating series $\overset{\bullet}{\Delta}_\omega\tilde{\phi} = \sum_{n\geq 0} c^n\, \overset{\bullet}{\Delta}_\omega\tilde{\phi}_n(z,\tau)$ is solution of the linearized equation $(\partial_\tau + \tilde{D}\partial_z)Y = 0$, where $\tilde{D} = -\frac{1}{4}z^2\partial_z\tilde{\phi}(z,\tau,c)$; but $\partial_c\tilde{\phi}(z,\tau,c) = z - \tau + O(c,z^{-1})$ is solution of the linearized equation, hence $\overset{\bullet}{\Delta}_\omega\tilde{\phi}$ must be a function of $\partial_c\tilde{\phi}$ and, in fact, proportional to $\exp(-\omega\partial_c\tilde{\phi})$ (because of the periodicity requirement with respect to τ and for homogeneity reasons with respect to e^{iz}).

The reader is once more referred to [**GS01**] and [**OSS03**] for more details. Let us synthetize our conclusions:

THEOREM 4.2. *Each minor $\hat{u}_n(\zeta)$ or $\hat{\phi}_n(\zeta,\tau)$ extends analytically to $\mathcal{R}$, with at most exponential growth of order 1 with respect to ζ along non-vertical rays; the series $\tilde{u}_n$ and $\tilde{\phi}_n$ are thus resurgent functions. There exist three families of formal series*

$$A_\omega(c) = \sum_{n\geq 0} A_{\omega,n}c^n, \qquad B_\omega(c) = \sum_{n\geq 0} B_{\omega,n}c^n, \qquad \omega \in 2\pi i\mathbb{Z}^*,$$

and

$$C_\omega(c) = \sum_{n\geq 0} C_{\omega,n}c^n, \qquad \omega \in i\mathbb{Z}^*,$$

such that

$$(4.5) \qquad \Delta_\omega \tilde{u}(z,c) = \big(A_\omega(c)\partial_c + B_\omega(c)\partial_z\big)\tilde{u}, \qquad \dot{\Delta}_\omega \tilde{\phi}(z,\tau,c) = C_\omega(c)\, e^{-\omega\partial_c \tilde{\phi}}.$$

These equations must be understood as a compact way of writing the systems of resurgence relations

$$\Delta_\omega \tilde{u}_n = \sum_{n_1+n_2=n} \big[(n_2+1)A_{\omega,n_1}\tilde{u}_{n_2+1} + B_{\omega,n_1}\partial_z\tilde{u}_{n_2}\big], \qquad n \geq 0,$$

and $\Delta_\omega \tilde{\phi}_0 = C_{\omega,0}\, e^{\omega\tau-\omega\tilde{S}}$,

$$\Delta_\omega \tilde{\phi}_n = \Big[C_{\omega,n} + \sum_{r=1}^{n} \frac{(-1)^r \omega^r}{r!} \sum_{\substack{n_0+\ldots+n_r=n+r \\ n_0 \geq 0 \\ n_1,\ldots,n_r \geq 2}} n_1 \ldots n_r\, C_{\omega,n_0}\, \tilde{\phi}_{n_1} \ldots \tilde{\phi}_{n_r} \Big] e^{\omega\tau-\omega\tilde{S}}$$

for $n \geq 1$, *which allow one to compute all the successive alien derivatives of the resurgent functions* $\tilde{u}_n$ *or* $\tilde{\phi}_n$.

Both equations in (4.5) are examples of what Écalle called the **Bridge equation:** they throw a bridge between alien calculus (the alien derivatives in the left-hand sides) and usual differential calculus (derivatives with respect to c or z in the right-hand sides).

The coefficients $C_{\omega,n}$ depend on the parameter μ of equation (1.7) as entire functions; we have $C_{i,0} = \beta$ and $A_{2\pi i,0} = \Theta$, $B_{2\pi i,0} = \Lambda$. The numbers $A_{\omega,n}, B_{\omega,n}, C_{\omega,n}$ represent the transcendental part of the information, in contrast with the coefficients of the series $\tilde{u}_n$ or $\tilde{\phi}_n$ which are computable by induction. They can be compared with Stokes constants, inasmuch as the Bridge equation can be viewed as the infinitesimal version of a nonlinear Stokes phenomenon. One is indeed tempted to consider the formal automorphisms obtained by exponentiating the operators involved in the right-hand sides of (4.5), in view of connecting the two different sums of each formal integral

$$u^\pm(z,c) = \sum_{n\geq 0} c^n \mathcal{L}^\pm \overset{\triangledown}{\tilde{u}}_n, \qquad \phi^\pm(z,\tau,c) = \sum_{n\geq 0} c^n \mathcal{L}^\pm \overset{\triangledown}{\tilde{\phi}}_n$$

obtained by extended Borel-Laplace summation in the direction of $\mathbb{R}^\pm$. But there is here an analytical difficulty: in [**GS01**, Section 5.5b], the convergence of the above $u^\pm(z,c)$ with respect to c was conjectured, whereas numerical experiments by R. Schäfke and coworkers seem to indicate divergence for any fixed value of z. The situation might be more favourable for the convergence of $\phi^\pm(z,\tau,c)$. Both questions are currently under investigation.

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