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Symbolic dynamics in the restricted elliptic isosceles three body problem

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Abstract

The restricted elliptic isosceles three body problem (REI3BP) models the motion of a massless body under the influence of the Newtonian gravitational force caused by two other bodies called the primaries. The primaries of masses $m_1 = m_2$ move along a degenerate Keplerian elliptic collision orbit (on a line) under their gravitational attraction, whereas the third, massless particle, moves on the plane perpendicular to their line of motion and passing through the center of mass of the primaries. By symmetry, the component of the angular momentum G of the massless particle along the direction of the line of the primaries is conserved.

We show the existence of symbolic dynamics in the REI3BP for large G by building a Smale horseshoe on a certain subset of the phase space. As a consequence we deduce that the REI3BP possesses oscillatory motions, namely orbits which leave every bounded region but return infinitely often to some fixed bounded region. The proof relies on the existence of transversal homoclinic connections associated to an invariant manifold at infinity. Since the distance between the stable and unstable manifolds of infinity is exponentially small, Melnikov theory does not apply.

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1. Introduction

The restricted three body problem studies the motion of three bodies, one of them massless, under Newtonian gravitational force. The massless body does not exert any force on the other two, the primaries, and move therefore according to Kepler laws. As a particular case, in the restricted elliptic isosceles three body problem (REI3BP), the primaries move along a degenerate ellipse and the third (massless) body moves on the perpendicular plane to their line of motion passing through their center of mass, which is invariant. In this configuration the primaries collide, but since it is a Keplerian motion its collisions can be regularized. In a coordinate system with origin at the center of mass of the primaries, the position of the primaries is given by

$$q_1(t) = \frac{\rho(t)}{2}(0,0,1) \qquad q_2(t) = \frac{\rho(t)}{2}(0,0,-1), \tag{1.1}$$

where

$$\rho(t) = 1 - \cos E(t) \tag{1.2}$$

and the eccentric anomaly $E(t)$ satisfies

$$t = E - \sin E. \tag{1.3}$$

Introducing polar coordinates (r, y, α, G) in the plane of motion of the third body, where (y, G) denote the conjugated momenta to (r, α) the REI3BP is Hamiltonian with respect to

$$H(r, y, G, t) = \frac{y^2}{2} + \frac{G^2}{r^2} - \frac{1}{\sqrt{r^2 + \frac{\rho^2(t)}{4}}}. \tag{1.4}$$

It is immediate to check that G is a conserved quantity so the REI3BP is a system of $1 + 1/2$ degrees of freedom. We fix $G \neq 0$ in order to avoid triple collisions.

In [3] authors the study the existence of symmetric periodic solutions of the Hamiltonian system associated to (1.4). In the present paper we prove the existence of chaotic dynamics in the REI3BP for large values of the angular momentum G, by building a Smale horseshoe with infinitely many symbols on a certain subset of the phase space. To build this horseshoe we first prove that the stable and unstable manifold associated to a certain invariant manifold intersect transversally, giving rise to homoclinic connections to the invariant manifold.

As a consequence, from the way the horseshoe is built, we deduce the existence of different types of orbits of the REI3BP according to their behavior as $t \to \pm\infty$. In particular, the existence of infinitely many periodic orbits of arbitrary large period is obtained. A complete classification of the orbits of the three body problem according to their final motion was already established by Chazy in 1922 (see [1]). For the restricted three body problem (either planar or spatial, circular or elliptic) the possibilities reduce to four:

- $H^{\pm}$(hyperbolic) : $\|r(t)\| \to \infty$ and $\|\dot{r}(t)\| \to c > 0$ as $t \to \pm\infty$.

- $P^{\pm}$(parabolic) : $\|r(t)\| \to \infty$ and $\|\dot{r}(t)\| \to 0$ as $t \to \pm\infty$.
- $B^{\pm}$(bounded) : $\limsup_{t\to\pm\infty} \|r(t)\| < \infty$.
- $OS^{\pm}$(oscillatory) : $\limsup_{t\to\pm\infty} \|r(t)\| = \infty$ and $\liminf_{t\to\pm\infty} \|r(t)\| < \infty$.

Examples of hyperbolic, parabolic and bounded motions were already known by Chazy (in particular they are present in the two body problem). However, no examples of oscillatory motions were known until Sitnikov [28] proved their existence on a certain symmetric configuration of the spatial restricted three body problem, now called *the Sitnikov example*. We shall prove that any past-future combination of the four possible final motions exists in the REI3BP.

The connection between chaotic dynamics and the existence of different types of final motions was first devised by Moser [25], who gave a new proof of the existence of oscillatory motions in the Sitnikov model. Moser's approach relying on the connection between final motions, transversal homoclinic points and symbolic dynamics has been successfully extended to provide more examples of these motions [15,16,20,21,23,24]. When dealing with perturbations of integrable systems the classical strategy for showing the existence of transversal intersections between the invariant manifolds is to find non-degenerate zeros of the Melnikov function, which gives an asymptotic expression for the distance between them. However, when considering fast non-autonomous perturbations, the Melnikov function is exponentially small with respect to the perturbative parameter and the validity of Melnikov theory is not justified. This difficulty can be solved when the system in consideration has two perturbative parameters and an exponentially smallness condition between them is assumed. This was the approach in [20], where the existence of oscillatory motions in the restricted planar circular three body problem (RPC3BP) was shown for values of the mass ratio exponentially small compared to the value of the inverse of the Jacobi constant.

The study of the existence of intersections between invariant manifolds for fast non-autonomous perturbations without assuming smallness conditions on extra parameters requires showing that the distance between invariant manifolds is indeed exponentially small. This problem, now known as *exponentially small splitting of separatrices*, has drawn remarkable attention in the past decades, but, due to its difficulty most of the available results concern concrete models [9,12,14,17,19] or in general systems under very restrictive hypothesis to be applicable to problems in Celestial Mechanics [4,5,10,11,13,18,29]. Following these ideas, [16] proves the transversality of certain invariant manifolds of the RPC3BP for any mass ratio and large Jacobi constants, extending the result in [21] of existence of oscillatory motions to any mass ratio.

Following the same approach in [16], the present paper proves the exponentially small splitting of separatrices in a real problem arising from Celestial Mechanics, the aforementioned REI3BP, under the only assumption of large angular momentum G. It is worth pointing out that the Hamiltonian (1.4) is, in general, far from being integrable. However, we will see in Section 2 that for orbits with large angular momentum G, the Hamiltonian (1.4) can be considered as a fast non-autonomous perturbation of the two body problem, which is integrable.

From our result we deduce the existence of transverse homoclinic connections and we are able to build a Smale horseshoe on a certain subset which is close to the homoclinic points. This result completes the previous work [2], where the existence of symbolic dynamics in the EIR3BP was investigated for large values of G using numerical techniques for analyzing the exponentially small splitting of separatrices.

The main result of the present paper, which gives the existence of chaotic dynamics in the REI3BP, is the following.

Theorem 1.1. *Denote by ψ the Poincaré map induced by the flow of the Hamiltonian* (1.4) *on the section $\Sigma_+ = \{(r, y, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : y = 0,\ \dot{y} > 0\}$. Then, there exists $0 < G^* < \infty$ such that for $G > G^*$ there exists an invariant set $S \subset \Sigma_+$ such that the dynamics of $\psi : S \to S$ is topologically conjugated to the shift*

$$\sigma : \mathbb{N}^{\mathbb{Z}} \to \mathbb{N}^{\mathbb{Z}}$$
$$\{a_n\}_{n\in\mathbb{Z}} \mapsto \{a_{n-1}\}_{n\in\mathbb{Z}}$$

Namely ψ has a Smale horseshoe of infinite symbols.

An immediate consequence of Theorem 1.1 is the existence of infinitely many periodic orbits in the system associated to Hamiltonian (1.4). Moreover, from the way the Smale horseshoe of Theorem 1.1 is built, we obtain the second main result (see Section 2 for a detailed exposition of this connection).

Theorem 1.2. *Denote by X^+ (respectively Y^-) either H^+, P^+, B^+ or OS^+ (respectively H^-, P^-, B^- or OS^-). Then, there exists $G^* < \infty$ such that if $G > G^*$ we have*

$$X^+ \cap Y^- \neq \emptyset$$

for all possible combinations of X^+ and Y^-. In particular, the Hamiltonian system (1.4) *posses oscillatory orbits, that is, orbits such that*

$$\limsup_{t\to\pm\infty} |r(t)| = \infty \quad \text{and} \quad \liminf_{t\to\pm\infty} |r(t)| < \infty.$$

As commented above, the proof of Theorem 1.1 relies on two main ingredients: establishing the existence of transversal intersections between the invariant manifolds $\mathcal{W}_\infty^{u,s}$ associated to a periodic orbit at infinity and showing the existence of a Smale horseshoe on a certain subset close to the homoclinic points. The latter follows from the arguments presented in [25] without significant modifications. These arguments are sketched in Section 2 for the sake of self-completeness.

For the analysis of the splitting of the invariant manifolds, we use the fact that $\mathcal{W}_\infty^{u,s}$ are Lagrangian submanifolds so they can be parametrized as graphs which satisfy the Hamilton-Jacobi equation associated to H. Then, we study solutions to this equation in a suitable complex domain to get exponentially small asymptotics for the distance between $\mathcal{W}_\infty^s$ and $\mathcal{W}_\infty^u$. In order to obtain the appropriate exponent these parameterizations must be analyzed in a neighborhood $\mathcal{O}\left(G^{-3}\right)$ of the singularities of the unperturbed homoclinic ($G \to \infty$).

The document is organized as follows. In Section 2 we introduce the invariant manifolds at infinity and discuss the proofs and connection between Theorem 1.1 and Theorem 1.2. In particular, from Theorem 2.1, which claims the existence of transverse intersections of the infinity manifolds, we build a Smale horseshoe that is then used to show the existence of any past-future combination of final motions. The rest of the paper is devoted to the proof of Theorem 2.1. We discuss the integrable system ($G \to \infty$) and its homoclinic manifold in Section 3.1. Section 3.2 is devoted to rewrite the problem of existence of the infinity manifolds as a fixed point equation. We solve this equation and bound the solution in a suitable complex domain in Section 4. In Section 5 we show that the distance between the invariant manifolds is given, up to first order, by the Melnikov function and then we compute its asymptotic expansion for large G in Section 6.

2. Description of the proof of Theorems 1.1 and 1.2

We notice from the Hamiltonian (1.4) that the angular momentum G is a conserved quantity. Therefore, we apply the conformally symplectic change of variables

$$r = G^2\tilde{r}, \quad y = G^{-1}\tilde{y}, \quad t = G^3 s,$$

to the equations of motion associated to the Hamiltonian (1.4) to obtain a new system which is also Hamiltonian with respect to the scaled Hamiltonian.

$$\begin{aligned}\tilde{H}(\tilde{r},\tilde{y},s;G) &= G^2 H\left(G^2\tilde{r}, G^{-1}\tilde{y}, G^3 s\right)\\ &= \frac{\tilde{y}^2}{2} + \frac{1}{\tilde{r}^2} - \frac{1}{\tilde{r}} + U\left(\tilde{r}, G^3 s\right)\end{aligned} \tag{2.1}$$

where

$$U\left(\tilde{r}, G^3 s\right) = \frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2 + \rho^2(G^3 s)/4G^4}} = \frac{\rho^2\left(G^3 s\right)}{8G^4\tilde{r}^3}\left(1 + \mathcal{O}\left(\frac{1}{\tilde{r}^2 G^4}\right)\right). \tag{2.2}$$

Observe that, for G large, the system associated to the Hamiltonian (2.1) can be studied as a fast and small non-autonomous perturbation of the Kepler two-body problem. Adding time t as a phase variable, which we now denote by ξ, we see from the equations of motion associated to the Hamiltonian (2.1)

$$\begin{aligned}\frac{d\tilde{r}}{ds} &= \tilde{y}\\ \frac{d\tilde{y}}{ds} &= \frac{1}{\tilde{r}^3} - \frac{1}{\tilde{r}^2} - \partial_{\tilde{r}} U\\ \frac{d\xi}{ds} &= G^3,\end{aligned} \tag{2.3}$$

that $\Lambda = \{(\tilde{r},\tilde{y},\xi) = (\infty, 0, \xi) : \xi \in \mathbb{T}\}$ is a parabolic periodic orbit, which we will call infinity.

Denoting by $\phi_s = (\phi_s^{\tilde{r}}, \phi_s^{\tilde{y}}, \phi_s^{\xi})$ the flow of the system (2.3), we define the stable and unstable manifolds of infinity as

$$\begin{aligned}\mathcal{W}^s_\infty &= \left\{(\tilde{r},\tilde{y},\xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \lim_{s\to+\infty} \phi_s^{\tilde{r}}(\tilde{r},\tilde{y},\xi) = \infty, \ \lim_{s\to+\infty} \phi_s^{\tilde{y}}(\tilde{r},\tilde{y},\xi) = 0\right\}\\ \mathcal{W}^u_\infty &= \left\{(\tilde{r},\tilde{y},\xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \lim_{s\to-\infty} \phi_s^{\tilde{r}}(\tilde{r},\tilde{y},\xi) = \infty, \ \lim_{s\to-\infty} \phi_s^{\tilde{y}}(\tilde{r},\tilde{y},\xi) = 0\right\}.\end{aligned} \tag{2.4}$$

The usual way to study the dynamics near infinity is to use McGehee coordinates $r = 2x^{-2}$ which map neighborhoods of infinity into bounded domains containing the origin. In particular, the periodic orbit Λ corresponds to the periodic orbit $\{(x, y, \xi) = (0, 0, \xi) : \xi \in \mathbb{T}\}$ in McGehee coordinates. This transformation was used in [22] to show that $\mathcal{W}^{u,s}_\infty$ exist and are analytic

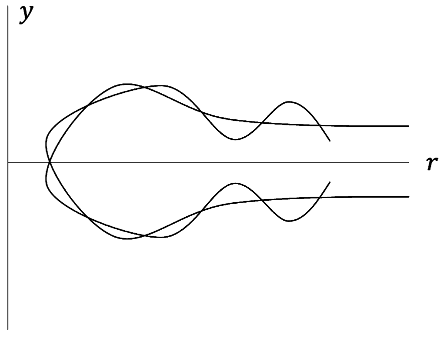

Fig. 2.1. Stable and unstable invariant manifolds of infinity for the Poincaré map $\mathcal{P}_{\xi_0}$ in (2.6).

submanifolds except at infinity, where only C^∞ regularity is proven (see [6,7] for more general results). However, in the present work we prefer to stick to the original variables since the symplectic form is non canonical in McGehee coordinates.

For $G \to \infty$ the system is integrable since $U \to 0$ and therefore $\mathcal{W}_\infty^s$ and $\mathcal{W}_\infty^u$ coincide along a two dimensional homoclinic manifold which is foliated by Keplerian parabolic orbits. Hence, it can be parametrized by the time section ξ and a suitable time parametrization $(\tilde{r}_h(v), \tilde{y}_h(v))$ of the parabolic orbit. We denote the parametrization of this invariant manifold as

$$\tilde{z}_h(v,\xi) = (\tilde{r}_h(v), \tilde{y}_h(v), \xi) \qquad \text{where} \qquad (v,\xi) \in \mathbb{R} \times \mathbb{T}, \tag{2.5}$$

and fix the origin of v such that $\tilde{y}_h(0) = 0$, which makes the homoclinic orbit symmetric under the map $v \to -v$. Some properties of this parametrization are discussed in Section 3.1.

We will prove that in the full problem (2.1), this two dimensional homoclinic manifold breaks down for $1 \ll G < \infty$, and $\mathcal{W}_\infty^s$, $\mathcal{W}_\infty^u$ do not longer coincide. In order to measure the distance between the invariant manifolds we introduce the Poincaré stroboscopic map

$$\begin{aligned} \mathcal{P}_{\xi_0} : \{\xi = \xi_0\} &\to \{\xi = \xi_0 + 2\pi\} \\ (\tilde{r}, \tilde{y}) &\mapsto \mathcal{P}_{\xi_0}(\tilde{r}, \tilde{y}) \end{aligned} \tag{2.6}$$

so $\mathcal{W}_\infty^{s,u} \cap \{\xi = \xi_0\}$ become invariant curves $\gamma^{s,u}$ (see Fig. 2.1).

Then, for $y > 0$, considering a parametrization of $\gamma^{s,u}$ of the form

$$\begin{aligned} \tilde{r} &= \tilde{r}_h(v) \\ \tilde{y} &= Y_{\xi_0}^{s,u}(v) \end{aligned} \tag{2.7}$$

where $\tilde{r}_h(v)$ is the parametrization of the unperturbed homoclinic (2.5), we observe that to measure the distance between the invariant manifolds along a suitable section $v = v^*$ it suffices to measure the difference between the functions $Y_{\xi_0}^{s,u}$. The following theorem is one of the two main ingredients needed for the proof of Theorem 1.1.

Theorem 2.1. *Let $\mathcal{W}_\infty^s$ and $\mathcal{W}_\infty^u$ be the infinity manifolds associated to the periodic orbit Λ and $\gamma^{s,u}$ the corresponding curves of the map $\mathcal{P}_{\xi_0}$. Then, for G large enough,*

(i) *The curves $\gamma^{s,u}$ exist and have a parametrization of the form* (2.7),
(ii) *If we fix a section $\tilde{r} = \tilde{r}(v^*)$ the distance d between these curves along this section is given by*

$$d = \frac{J_1(1)\sqrt{2\pi}}{\tilde{y}_h(v^*)} G^{1/2} e^{-\frac{G^3}{3}} \sin\left(\xi_0 - G^3 v^*\right) + E, \qquad |E| \le C G^{-1/2} e^{-\frac{G^3}{3}}, \tag{2.8}$$

where J_1 is the first Bessel function of first kind and $\tilde{y}_h$ correspond to the $\tilde{y}$ component of the unperturbed homoclinic and $C > 0$ is a constant independent of G.
(iii) *There exist (at least) two transverse homoclinic connections to the periodic orbit Λ.*

Item (iii) is a direct consequence of Item (ii). Indeed, since

$$J_1(1) \sim 0.44051 \neq 0$$

we observe that formula (2.8) in Theorem 2.1, implies that the zeros of the distance are given, up to first order, by the zeros of the function $\sin\left(\xi_0 - G^3 v^*\right)$. Therefore, transversal intersections of the invariant curves $\gamma^{s,u}$ will occur for values of $\xi_0 - G^3 v^*$ located in a neighborhood $\mathcal{O}\left(G^{-1}\right)$ of the points $\xi_0 - G^3 v^* = 0, \pi$. These transversal intersections give rise to two homoclinic connections to the invariant manifold Λ as stated in the third item of Theorem 2.1.

Observe that the distance between the invariant manifolds is exponentially small with respect to G. As usually happens in exponentially small splitting of separatrices phenomena, the smaller the period of the fast perturbation (in our case $2\pi/G^3$), the smaller the distance between the manifolds (see [27]).

2.1. Symbolic dynamics and oscillatory orbits

Once Theorem 2.1 is proven, the existence of chaotic dynamics is obtained following the techniques introduced in [25]. For that we define the section

$$\Sigma_+ = \left\{(\tilde{r}, \tilde{y}, \xi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{T} : \tilde{y} = 0, \dot{\tilde{y}} > 0\right\} \tag{2.9}$$

and use coordinates $(\tilde{r}_0, \xi_0)$ for this section. Then, we define the Poincaré map

$$\begin{aligned} \psi : \Sigma_+ &\to \Sigma_+ \\ (\tilde{r}_0, \xi_0) &\mapsto (\tilde{r}_1, \xi_1) \end{aligned} \tag{2.10}$$

where $\xi_1 = \xi_0 + G^3 s$, and $s > 0$ is the first time in which $\phi_s(\tilde{r}_0, 0, \xi_0)$ intersects Σ_+ again and $\tilde{r}_1$ is such that $\phi_s(\tilde{r}_0, 0, \xi_0) = (\tilde{r}_1, 0, \xi_1)$. We set $\xi_1 = \infty$ for points $(\tilde{r}_0, \xi_0)$ which do not intersect Σ_+ anymore in the future and define $D_0 \subset \Sigma_+$ as the set of points for which $\xi_1 < \infty$. In the unperturbed problem ($G \to \infty$) one easily deduces, using the conservation of energy, that Σ_+ is divided in two open sets, corresponding to initial conditions leading to hyperbolic and elliptic motions, whose common boundary is the curve in which the homoclinic manifold (2.5) intersects Σ_+. In this case, D_0 corresponds to the set of initial conditions leading to elliptic motions.

In order to characterize the set D_0 in the full problem (2.1) we make use of the following proposition, already proven in [2], which describes the intersection $\mathcal{W}^{s,u} \cap \Sigma_+$.

Proposition 2.2. *The stable manifold $\mathcal{W}^s$ intersects Σ_+ backwards for the first time in a simple curve*

$$\tilde{\gamma}^s = \left\{ \left(\tilde{r}_0^s(\xi_0), \xi_0\right) \in \Sigma_+ : \tilde{r}_0^s(\xi_0 + 2\pi) = \tilde{r}_0^s(\xi_0) \right\}. \tag{2.11}$$

Analogously, the unstable manifold $\mathcal{W}^u$ intersects Σ_+ forward for the first time in a simple curve

$$\tilde{\gamma}^u = \left\{ \left(\tilde{r}_0^u(\xi_0), \xi_0\right) \in \Sigma_+ : \tilde{r}_0^u(\xi_0 + 2\pi) = \tilde{r}_0^u(\xi_0) \right\}. \tag{2.12}$$

Remark 2.3. From Theorem 2.1 we deduce that the curves $\tilde{\gamma}^{s,u}$ described in Proposition 2.2 intersect transversally, a fact which is crucial for the proof of Theorem 2.4.

The curve $\tilde{\gamma}^s$ divides Σ_+ in two connected components. One of these components correspond to D_0 and the other component consists of initial conditions leading to orbits which do not intersect Σ_+ again and which escape to infinity with positive asymptotic radial velocity. We also define the set $D_1 \subset \Sigma_+$ of initial conditions $(\tilde{r}_0, \xi_0)$, in which the map ψ^{-1} is well defined. A similar argument to the one above using $\tilde{\gamma}^u$ instead of $\tilde{\gamma}^s$ can be used to identify this set.

Once we have identified D_0 and D_1, given a point $(\tilde{r}_0, \xi_0) \in D_0 \cap D_1$ we consider the sequence of consecutive times ξ_n given by $\psi^n(\tilde{r}_0, \xi_0) = (\tilde{r}_n, \xi_n)$ for $n \in \mathbb{Z}$ (whenever they exist) to define the sequence of integers

$$a_n = \left[\frac{\xi_n - \xi_{n-1}}{2\pi} \right],$$

where $[\cdot]$ defines the integer part. Thus, $a_n \in \mathbb{N}$ measures the number of binary collisions of the primaries between consecutive approaches of the third body. We introduce some technical concepts needed for stating the theorem that establishes the existence of symbolic dynamics on a subset of the closure $D_0 \cap D_1$ by conjugating ψ with the shift acting on a space of doubly infinite sequences.

Let A denote the set of all doubly infinite sequences

$$a = (\dots a_{-2}, a_{-1}, a_0; a_1, a_2 \dots)$$

of elements $a_n \in \mathbb{N}$. Equipping A with the product topology, the shift $\sigma : A \to A$ given by

$$\sigma\left(\{a_n\}_{n\in\mathbb{Z}}\right) = \{a_{n-1}\}_{n\in\mathbb{Z}} \tag{2.13}$$

is a homeomorphism.

We can define the compactification $\bar{A}$ of A admitting elements of the following type: For α, β integers satisfying $\alpha \leq 0$, $\beta \geq 1$, let

$$a = \left(\infty, a_{\alpha+1}, \dots, a_{\beta-1}, \infty\right) \quad a_n \in \mathbb{N}.$$

We also admit half infinite sequences with $\alpha = -\infty$, $\beta < \infty$ or $\alpha > -\infty$, $\beta = \infty$. It is possible to extend the topology defined on A to $\bar{A}$ in a way such that the shift (2.13) is a homeomorphism when restricted to

$$\bar{A}_0 = \left\{a \in \bar{A} \colon a_0 \neq \infty\right\}$$

(see [25] for details).

The proof of the following theorem, from which Theorems 1.1 and 1.2 are deduced, follows from direct adaptation of the ideas presented in [25] for the Sitnikov problem. The main ingredients are the transversal intersection of the curves $\gamma^{s,u}$ and a C^1 Lambda-Lemma for the parabolic invariant manifold Λ. This Lambda-Lemma follows from a careful analysis of the dynamics near Λ using McGehee coordinates which map neighborhoods of infinity into bounded neighborhoods of the origin.

Theorem 2.4. *There exists a set $S \subset (D_0 \cap D_1)$ which is invariant under the Poincaré map ψ defined in (2.10) and such that its restriction to S, is conjugated to the shift σ defined in (2.13). That is, there exists an homeomorphism $\chi : A \to S$ such that*

$$\psi\chi = \chi\sigma.$$

Moreover, χ can be extended to $\bar{\chi} : \bar{A} \to \bar{S}$ such that

$$\psi\bar{\chi} = \bar{\chi}\sigma$$

if both sides are restricted to $\bar{A}_0$.

In other words, to each point $p = (r_0, \xi_0) \in S$ we associate a sequence $a(p) \in A$ which codifies the time between successive intersections of the flow $\phi_s(r_0, 0, \xi_0)$ with Σ_+. In this setting, the connection between Theorem 1.1 and Theorem 1.2 becomes clear. The first part of Theorem 2.4 corresponds to sequences

- $a(p) = (\dots a_{-2}, a_{-1}, a_0, a_1, \dots)$ with $a_n \in \mathbb{N}$ for all $n \in \mathbb{Z}$. These represent orbits which perform an infinite number of "close" approaches to the line where the primaries move both in the past and in the future. From this result we deduce the existence of any past-future combination of bounded ($\sup_{n\in\mathbb{Z}} a_n < \infty$) and oscillatory ($\limsup_{n\to\pm\infty} a_n = \infty$) motions.

The second part of the theorem, concerns sequences of the following type

- $a(p) = (\infty, a_{-k}, a_{-k+1}, \dots)$ with $a_n \in \mathbb{N}$ for all $n > -k$, which represent capture orbits, i.e., orbits where the third body comes from infinity at $t \to -\infty$ and remains revolving around the line of primaries for all future times. In particular, we obtain orbits which are hyperbolic or parabolic in the past and bounded or oscillatory in the future.
- $a(p) = (\dots a_{l-1}, a_l, \infty)$ with $a_n \in \mathbb{N}$ for all $n < l$. In this case the third body performed an infinite number of oscillations in the past but escapes to infinity as $t \to \infty$. These sequences correspond to orbits which are bounded or oscillatory in the past and parabolic or hyperbolic in the future.
- $a(p) = (\infty, a_k, \dots, a_l, \infty)$ with $a_n \in \mathbb{N}$ for all $n \in \mathbb{Z}$ which corresponds to orbits coming from infinity, revolving around the primaries a finite number of times and escaping again to infinity as $t \to \infty$. They correspond to past-future combinations of parabolic and hyperbolic motions.

Finally, we point out that the existence of infinitely many periodic orbits in the REI3BP is deduced from Theorem 1.1 since fixed points for the shift correspond to periodic orbits of the Hamiltonian (2.1).

3. The invariant manifolds as graphs

3.1. The unperturbed homoclinic solution

For the unperturbed problem, $G \to \infty$ in (2.1), the equations of motion reduce to

$$\begin{aligned} \frac{\mathrm{d}\tilde{r}}{\mathrm{d}v} &= \tilde{y} \\ \frac{\mathrm{d}\tilde{y}}{\mathrm{d}v} &= \frac{1}{\tilde{r}^3} - \frac{1}{\tilde{r}^2}. \end{aligned} \tag{3.1}$$

In this case the infinity manifolds $\mathcal{W}_\infty^{s,u}$ associated to Λ coincide along the two dimensional homoclinic manifold $\tilde{z}_h$ introduced in (2.5). The (complex) singularities of $\tilde{z}_h(v, \xi)$ will be crucial for studying the existence of the invariant manifolds of the perturbed problem in certain complex domains. Thus, we state the following results, which were already obtained in [26].

1. The homoclinic solution (2.5) behaves as

$$\tilde{r}_h(v) \sim 3v^{2/3}, \qquad \tilde{y}_h(v) \sim 2v^{-1/3} \qquad \text{as} \qquad |v| \to \infty.$$

2. The homoclinic solution (2.5) is a real analytic function of v with singularities at $v = \pm i/3$.
3. Close to its singularities, the homoclinic solution (2.5) behaves as

$$\tilde{r}_h(v) \sim C\left(v \mp \frac{i}{3}\right)^{1/2}, \qquad \tilde{y}_h(v) \sim \frac{C}{2}\left(v \mp \frac{i}{3}\right)^{1/2}, \qquad \text{where} \qquad C^2 = \pm 2i.$$

3.2. The perturbed invariant manifolds and their difference

In this section we look for parametrizations of the infinity manifolds $\mathcal{W}_\infty^{u,s}$ in certain complex domains defined below. More concretely we look for graph parametrizations of $\mathcal{W}_\infty^{u,s}$ as solutions to a PDE. To do this we observe that the canonical form $\lambda = \tilde{r}\,\mathrm{d}\tilde{y} - \tilde{H}\,\mathrm{d}s$ is closed on the infinity manifolds (since the infinity manifolds are invariant by the flow it is enough to check that $\mathrm{d}\lambda$ is null on Λ). Then, one can see λ as the differential of a function $S(\tilde{r}, \xi)$ such that

$$\partial_{\tilde{r}} S = \tilde{y} \qquad G^3 \partial_\xi S = -\tilde{H}$$

or, putting this together, as a solution of the Hamilton-Jacobi equation

$$G^3 \partial_\xi S + H(\tilde{r}, \partial_{\tilde{r}} S, \xi) = 0.$$

We write $S = S_0 + S_1$ where S_0 is the solution to the unperturbed problem

$$G^3 \partial_\xi S_0 + \frac{(\partial_{\tilde{r}} S_0)^2}{2} + \frac{1}{2\tilde{r}^2} - \frac{1}{\tilde{r}} = 0$$

and perform the change of variables

$$(\tilde{r}, \xi) \mapsto (\tilde{r}_h(v), \xi). \tag{3.2}$$

Then, the equation for $T_1(v, \xi) = S_1(\tilde{r}_h(v), \xi)$ becomes

$$\partial_v T_1 + \frac{1}{2\tilde{y}_h{}^2} (\partial_v T_1)^2 + G^3 \partial_\xi T_1 + V(v, \xi) = 0, \tag{3.3}$$

where

$$V(v, \xi) = U(\tilde{r}_h(v), \xi). \tag{3.4}$$

Note that the change of variables (3.2) implies that we are looking for parametrizations of the stable and unstable manifolds of the form

$$\begin{aligned} \tilde{r} &= \tilde{r}_h(v) \\ \tilde{y} &= \tilde{y}_h(v) + \frac{1}{\tilde{y}_h(v)} \partial_v T_1^{u,s} \end{aligned} \tag{3.5}$$

where $\tilde{r}_h(v), \tilde{y}_h(v)$ correspond to the unperturbed homoclinic (2.5) and $T_1^{u,s}(v, \xi)$ are solutions of equation (3.3) with asymptotic boundary condition for the unstable manifold

$$\lim_{v \to -\infty} \frac{1}{\tilde{y}_h(v)} \partial_v T_1^u = 0 \tag{3.6}$$

and the analogous one for the stable manifold. Once we show the existence of the unstable manifold, the existence of the stable one is guaranteed by symmetry. Indeed, if $T_1(v, \xi)$ is a solution of (3.3), $-T_1(-v, -\xi)$ is also a solution satisfying the opposite boundary condition.

Before going into the analysis of the existence of the generating functions $T_1^{u,s}$ we recall that our goal is to have a first asymptotic approximation of the distance between the infinity manifolds which now boils down to obtain an asymptotic formula for $\partial_v \left(T_1^u - T_1^s\right)$. To this end, we introduce the Melnikov potential

$$L(v, \xi; G) = \int_{-\infty}^{\infty} V\left(\tilde{r}_h(v+s), \xi + G^3 s\right) \mathrm{d}s, \tag{3.7}$$

which, as we state in Theorem 3.2 below approximates to first order the difference $\Delta = T_1^s - T_1^u$.

We point out that the parametrization (3.5) becomes undefined at $v = 0$ since we have fixed v such that $\tilde{y}_h(0) = 0$. Since in order to measure $\partial_v \left(T_1^u - T_1^s\right)$ we need both functions to be defined in a common domain we will introduce a different parametrization to extend the unstable manifold across $v = 0$. This is discussed in full detail in Section 4.

The next proposition gives the first asymptotic term of the Melnikov potential and will be proved in Section 6.

Proposition 3.1. *The function $L(v,\xi;G)$ defined in (3.7) satisfies*

$$L(v,\xi;G) = L^{[0]}(G) + 2\sum_{l=1}^{\infty} L^{[l]}(G)\cos\left(l\left(\xi - G^3 v\right)\right),$$

where

$$L^{[1]}(G) = -J_1(1)\sqrt{2\pi}G^{-5/2}e^{\frac{-G^3}{3}}\left(1+\mathcal{O}\left(G^{-3/2}\right)\right)$$
$$\left|L^{[l]}(G)\right| \le KG^{-5/2}e^{l-1/2}e^{\frac{-|l|G^3}{3}}, \quad \text{for } l>1,$$

with J_1 the first Bessel function of the first kind and $K>0$ a constant independent of G.

Theorem 3.2. *Choose any $0<v_-<v_+<\infty$. Then, there exists $K>0$ such that for any $v\in\left[v_-,v_+\right]$ and for any G large enough, the generating functions $T_1^{u,s}(v,\xi)$ satisfy*

$$\left|T_1^s(v,\xi) - T_1^u(v,\xi) - L(v,\xi) - E\right| \le KG^{-7/2}e^{\frac{-G^3}{3}},$$

where $E\in\mathbb{R}$ is a constant and

$$\left|\partial_v\left(T_1^s(v,\xi) - T_1^u(v,\xi)\right) - \partial_v L(v,\xi)\right| \le KG^{-1/2}e^{\frac{-G^3}{3}}.$$

From Proposition 3.1, Theorem 3.2 and Equation (3.5) we deduce Theorem 2.1. We devote Sections 4 and 5 to the proof of Theorem 3.2.

4. The invariant manifolds in complex domains

The classical procedure when studying exponentially small splitting of separatrices is to look for the functions T_1^u and T_1^s in a complex common domain $D\times\mathbb{T}_\sigma$ where $D\subset\mathbb{C}$ is a connected domain which reaches a neighborhood of size $\mathcal{O}\left(G^{-3}\right)$ (recall that the period of the perturbation (2.2) is $2\pi/G^3$) of the singularities of the unperturbed separatrix, i.e., $v=\pm i/3$ (see Section 3.1) and

$$\mathbb{T}_\sigma = \{\xi\in\mathbb{C}/2\pi\mathbb{Z} : |\mathrm{Im}(\xi)|<\sigma\}.$$

The idea behind this approach is that for $v\in\mathbb{R}$ we will get exponentially small bounds on the distance $d(v,\xi)$ between the invariant manifolds if we show that d is a quasiperiodic function in some suitable coordinates and we manage to bound $|d|$ in a connected domain D which contains a subset of the real line and gets close to the singularities $v=\pm i/3$.

Since boundary conditions are imposed at infinity, we need to solve the equation (3.3) for T_1^u (resp. T_1^s) in a complex unbounded domain reaching $v\to-\infty$ (resp. $v\to\infty$). On the other hand, in order to measure their difference we need them to be defined in a common domain, we need to extend one of them across $v=0$. However, the equation (3.3) becomes singular at $v=0$ since $\tilde{y}_h(0)=0$. To overcome this problem we divide the process of extension of the invariant manifolds into three steps.

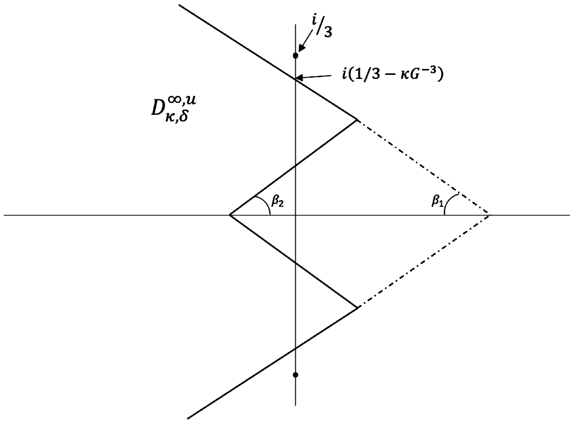

Fig. 4.1. The domain $D^{\infty,u}_{\kappa,\delta}$ defined in (4.1).

We first solve equation (3.3) together with the boundary condition (3.6) in the domain

$$D^{\infty,u}_{\kappa,\delta} = \left\{ v \in \mathbb{C} : \ |\mathrm{Im}(v)| < -\tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3}, |\mathrm{Im}(v)| > \tan\beta_2 \mathrm{Re}(v) + 1/6 - \delta \right\}, \tag{4.1}$$

which does not contain $v = 0$ and where κ, δ and $\beta_1, \beta_2 \in (0, \pi/2)$ are fixed independently of G (see Fig. 4.1). One can check that for $\delta \in (0, 1/12)$, $\kappa \sim \mathcal{O}(1)$, we can always find G big enough such that this domain is non empty. Once the existence of T_1^u in the domain $D^{\infty,u}_{\kappa,\delta}$ is proven, we exploit the symmetry of equation (3.3) under the map $(v, \xi) \to (-v, -\xi)$ to atutomatically deduce the existence of T_1^s in the domain

$$D^{\infty,s}_{\kappa,\delta} = \left\{ v \in \mathbb{C} : \ |\mathrm{Im}(v)| < \tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3}, |\mathrm{Im}(v)| > -\tan\beta_2 \mathrm{Re}(v) + 1/6 - \delta \right\}. \tag{4.2}$$

The next step is to perform the analytical continuation of T_1^u across the imaginary axis. Thus, we would have both invariant manifolds defined on a common domain (this domain will be contained in $D^{\infty,s}_{\kappa,\delta}$ where T_1^s is already defined). Since $y_h(0) = 0$, the equation (3.3) becomes singular at $v = 0$ so we change to a parametrization invariant by the flow in the bounded domain

$$D_{\rho,\kappa,\delta} = D^{\infty,u}_{\kappa,\delta} \cap (\mathrm{Re}(v) > -\rho) \tag{4.3}$$

for some finite $\rho > 0$. Then, we use the flow ϕ_s associated to the system (2.3) to extend the unstable manifold T_1^u to the domain

$$D^{\mathrm{flow}}_{\kappa,\delta} = \left\{ v \in \mathbb{C} : \ |\mathrm{Im}(v)| < -\tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3}, |\mathrm{Im} v| < \tan\beta_2 \mathrm{Re}(v) + 1/6 + \delta \right\} \tag{4.4}$$

which contains $v = 0$ (see Fig. 4.2). Then we go back to the original parametrization in a “boomerang domain”

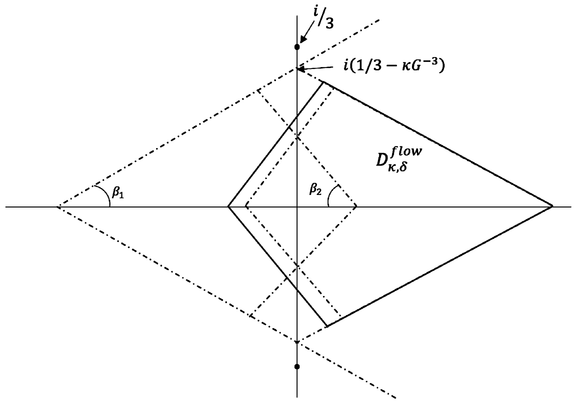

Fig. 4.2. The domain $D^{\rm flow}_{\kappa,\delta}$ defined in (4.4).

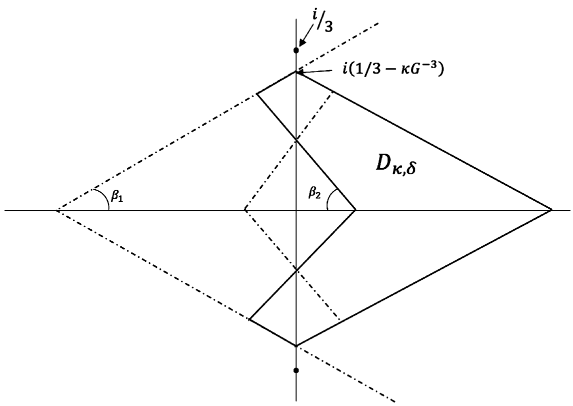

Fig. 4.3. The domain $D_{\kappa,\delta}$ defined in (4.5).

$$D_{\kappa,\delta} = \Big\{ v \in \mathbb{C} : |\mathrm{Im}(v)| < -\tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3}, |\mathrm{Im}(v)| < \tan\beta_1 \mathrm{Re}(v) + 1/3 - \kappa G^{-3}, \\ |\mathrm{Im}(v)| > -\tan\beta_2 \mathrm{Re}(v) + 1/6 - \delta \Big\}, \tag{4.5}$$

(which does not contain $v = 0$) in order to measure the distance between the stable and unstable manifold.

4.1. Existence of the invariant manifolds close to infinity

In order to prove existence of the invariant manifolds we rewrite equation (3.3) as a fixed point equation in a suitable Banach space. We start by defining the linear operator

$$\mathcal{L} = \partial_v + G^3 \partial_\xi \tag{4.6}$$

so equation (3.3) reads

$$\mathcal{L}\left(T_1^{u,s}\right) = \mathcal{F}\left(T_1^{u,s}\right) \qquad \text{where} \qquad \mathcal{F}\left(T_1^{u,s}\right) = -\frac{1}{2\tilde{y}_h^2}\left(\partial_v T_1^{u,s}\right)^2 - V(v,\xi). \tag{4.7}$$

We introduce the left inverse operators

$$\begin{aligned} \mathcal{G}^u(f)(v,\xi) &= \int_{-\infty}^{0} f\left(v+s,\xi+G^3 s\right)\mathrm{d}s \\ \mathcal{G}^s(f)(v,\xi) &= \int_{+\infty}^{0} f\left(v+s,\xi+G^3 s\right)\mathrm{d}s, \end{aligned} \tag{4.8}$$

so we can rewrite equation (4.7) as the fixed point equation

$$T_1^{u,s} = \mathcal{G}^{u,s} \circ \mathcal{F}\left(T_1^{u,s}\right). \tag{4.9}$$

Remark. Throughout this section we will only work with the unstable manifold so we will omit the superindex u and write $D^{\infty}_{\kappa,\delta}$, T_1 and $\mathcal{G}$ instead of $D^{\infty,u}_{\kappa,\delta}$, T_1^u and $\mathcal{G}^u$ if there is no possible confusion.

We look for solutions of this equation in the Banach spaces

$$\mathcal{Z}^{\infty}_{\nu,\mu} = \left\{h : D^{\infty}_{\kappa,\delta} \times \mathbb{T}_\sigma \to \mathbb{C} : h \text{ is real analytic, } \|h\|_{\nu,\mu} < \infty\right\}, \tag{4.10}$$

where

$$\|h\|_{\nu,\mu} = \sum_{l\in\mathbb{Z}} \left\|h^{[l]}\right\|_{\nu,\mu} e^{|l|\sigma}$$

and

$$\left\|h^{[l]}\right\|_{\nu,\mu} = \sup_{v\in D^{\infty}_{\kappa,\delta}\setminus D_{\rho,\kappa,\delta}} \left|v^\nu h^{[l]}(v)\right| + \sup_{v\in D_{\rho,\kappa,\delta}} \left|\left(v^2+1/9\right)^\mu h^{[l]}(v)\right|.$$

Notice that the first term takes account of the behavior at infinity and the second one of the behavior near the singularities since $v^2+1/9 = (v-i/3)(v+i/3)$. As we see from (4.9) we will also need to take control on the derivatives so we introduce

$$\tilde{\mathcal{Z}}^{\infty}_{\nu,\mu} = \left\{h : D^{\infty}_{\kappa,\delta} \times \mathbb{T}_\sigma \to \mathbb{C} : h \text{ is real analytic, } [\![h]\!]_{\nu,\mu} < \infty\right\}, \tag{4.11}$$

where

$$[\![h]\!]_{\nu,\mu} = \|h\|_{\nu,\mu} + \|\partial_v h\|_{\nu+1,\mu+1}.$$

The following lemma provides estimates for the norm of the perturbative potential.

Lemma 4.1. *Let V be the perturbative potential defined in (3.4). Then, for G large enough we have that*

$$\|V\|_{2,3/2} \leq K G^{-4}$$

for a constant $K > 0$ independent of G.

Proof. Since the domain $D^{\infty}_{\kappa,\delta}$ reaches a neighborhood of order $\mathcal{O}\left(G^{-3}\right)$ of $v = \pm i/3$ we have that for G sufficiently large

$$\left|\frac{1}{G^4 \tilde{r}_h^2(v)}\right| \leq K G^{-1},$$

for $K > 0$ independent of G. Therefore, from (2.2) we deduce that for all $(v, \xi) \in D^{\infty}_{\kappa,\delta} \times \mathbb{T}_\sigma$

$$|V(v, \xi)| \leq \frac{K}{G^4 |\tilde{r}_h(v)|^3}.$$

The conclusion follows now using the asymptotic expressions for $\tilde{r}_h(v)$ obtained in Section 3.1. □

We also state algebra-like properties for these spaces, which are straightforward from their definition and will be useful when dealing with the fixed point equation.

Lemma 4.2. *Let $\mathcal{Z}^{\infty}_{\nu,\mu}$ be the Banach spaces defined in (4.10). Then*

i) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ and $g \in \mathcal{Z}^{\infty}_{\nu',\mu'}$ then $hg \in \mathcal{Z}^{\infty}_{\nu+\nu',\mu+\mu'}$ with

$$\|hg\|_{\nu+\nu',\mu+\mu'} \leq \|h\|_{\nu,\mu} \|g\|_{\nu',\mu'}.$$

ii) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$, then $h \in \mathcal{Z}^{\infty}_{\nu-\alpha}$ for $\alpha > 0$ with

$$\|h\|_{\nu-\alpha,\mu} \leq K \|h\|_{\nu,\mu}.$$

iii) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ then, for $\alpha > 0$ we have that $h \in \mathcal{Z}^{\infty}_{\nu,\mu-\alpha}$ with

$$\|h\|_{\nu,\mu-\alpha} \leq K G^{3\alpha} \|h\|_{\nu,\mu}.$$

iv) If $h \in \mathcal{Z}^{\infty}_{\nu,\mu}$ then, for $\alpha > 0$ we have that $h \in \mathcal{Z}^{\infty}_{\nu,\mu+\alpha}$ with

$$\|h\|_{\nu,\mu+\alpha} \leq K \|h\|_{\nu,\mu}.$$

The following lemma provides estimates for the inverse operator. The proof follows the exact same lines as in Lemma 5.5. in [17] (see also [5]).

Lemma 4.3. *The operator $\mathcal{G}$ defined on (4.8) satisfies the following properties*

i) For any $\nu > 1$, μ 1, $\mathcal{G} : \mathcal{Z}_{\nu,\mu} \to \mathcal{Z}_{\nu-1,\mu-1}$ is well defined, linear and satisfies $\mathcal{L} \circ \mathcal{G} = \mathrm{Id}$.

ii) If $h \in \mathcal{Z}_{\nu,\mu}$ for some $\nu > 1$, $\mu > 1$, then

$$\|\mathcal{G}(h)\|_{\nu-1,\mu-1} \leq K \|h\|_{\nu,\mu}. \tag{4.12}$$

iii) If $h \in \mathcal{Z}_{\nu,\mu}$ for some $\nu \geq 1$, $\mu \geq 1$, then

$$\|\partial_v \mathcal{G}(h)\|_{\nu,\mu} \leq K \|h\|_{\nu,\mu}. \tag{4.13}$$

Now we are ready to solve the fixed point equation.

Theorem 4.4. *Fix $\kappa > 0$, $\delta > 0$ and $\sigma > 0$. Then, for G large enough the fixed point equation (4.9) has a unique solution T_1^u on $D^\infty_{\kappa,\delta} \times \mathbb{T}_\sigma$ which satisfies*

$$[\![T_1^u]\!]_{1,1/2} \leq b_0 G^{-4}$$

with $b_0 > 0$ independent of G. Moreover, if we define the function

$$L_1^u(v,\xi) = \mathcal{G}^u(V)(v,\xi)$$

we have

$$\left\|T_1^u - L_1^u\right\|_{1,1/2} \leq K G^{-13/2} \tag{4.14}$$

where $K > 0$ is independent of G.

Proof. We show that T_1 is the unique solution the fixed point equation (4.9). For that we first check that the operator $\mathcal{G} \circ \mathcal{F}$ is well defined from $\tilde{\mathcal{Z}}_{1,1/2}$ to itself. Indeed, from Lemma 4.1 we have that

$$\|V\|_{2,3/2} \leq K G^{-4}.$$

Then, the result follows from direct application of the properties of the homoclinic solution stated in Section 3.1, the algebra properties of the norm stated in Lemma 4.2 and Lemma 4.3 since we obtain that for $h \in \tilde{\mathcal{Z}}_{1,1/2}$

$$[\![\mathcal{G} \circ \mathcal{F}(h)]\!]_{1,1/2} \leq K \min\left([\![h]\!]_{1,1/2}, G^{-4}\right) \tag{4.15}$$

for some $K > 0$ independent of G. In particular we deduce that there exists $b_0 > 0$ independent of G such that

$$[\![\mathcal{G} \circ \mathcal{F}(0)]\!]_{1,1/2} \leq \frac{b_0}{2} G^{-4}.$$

Then in order to show existence and uniqueness of solutions it is enough to show that the map $\mathcal{G} \circ \mathcal{F}$ is contractive on the ball $B\left(b_0 G^{-4}\right) \subset \tilde{\mathcal{Z}}_{1,1/2}$ centered at 0. For that purpose we write

$$\mathcal{F}(h_2) - \mathcal{F}(h_1) = \frac{1}{2y_h^2}(\partial_v h_1 + \partial_v h_2)(\partial_v h_1 - \partial_v h_2)$$

so using that $h_1, h_2 \in B\left(b_0 G^{-4}\right) \subset \tilde{\mathcal{Z}}_{1,1/2,\kappa,\delta,\sigma}$ we have

$$\begin{aligned}\|\mathcal{F}(h_2) - \mathcal{F}(h_1)\|_{2,3/2} &\leq \left\| \frac{1}{2y_h^2}(\partial_v h_1 + \partial_v h_2) \right\|_{0,0} \|\partial_v h_1 - \partial_v h_2\|_{2,3/2} \\ &\leq K G^{3/2} \left\| \frac{1}{2y_h^2}(\partial_v h_1 + \partial_v h_2) \right\|_{0,1/2} [\![h_1 - h_2]\!]_{1,1/2} \\ &\leq K G^{-5/2} [\![h_1 - h_2]\!]_{1,1/2},\end{aligned}$$

and contractivity follows from Lemma 4.3 (enlarging G if necessary).

To obtain (4.14) we notice that

$$\begin{aligned}\left\| T_1 - L_1^u \right\|_{1,1/2} &= \|\mathcal{G} \circ (\mathcal{F}(T_1) - \mathcal{F}(0))\|_{1,1/2} \\ &\leq [\![\mathcal{G} \circ (\mathcal{F}(T_1) - \mathcal{F}(0))]\!]_{1,1/2} \\ &\leq K G^{-5/2} [\![T_1]\!]_{1,1/2} \leq K G^{-13/2}. \quad \square\end{aligned}$$

Since the parametrization (3.5) becomes singular at $v = 0$, in the next section we look for a new parametrization of the unstable manifold which is regular at $v = 0$ and therefore allows us to extend it across $v = 0$.

4.2. Analytic continuation of the solution to the domain $D_{\kappa,\delta}^{flow}$

In order to measure the distance between the stable and unstable manifolds we need them to be defined in a common domain. However, a parametrization of the form

$$\Gamma(v, \xi) = \begin{pmatrix} \tilde{r}(v, \xi) \\ \tilde{y}(v, \xi) \end{pmatrix} = \begin{pmatrix} \tilde{r}_h(v) \\ \frac{1}{\tilde{y}_h(v)} \partial_v T^u \end{pmatrix}$$

becomes undefined at $v = 0$. To avoid this difficulty we look for a different parametrization of the unstable manifold in the domain $D_{\rho,\kappa,\delta}$ (4.3) which does not contain $v = 0$ and then extend it by the flow. In order to proceed, we introduce the Banach spaces

$$\mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma} = \left\{ h : D_{\rho,\kappa,\delta} \times \mathbb{T}_\sigma \to \mathbb{C} : h \text{ is real analytic}, \ \|h\|_\mu < \infty \right\} \tag{4.16}$$

where

$$\|h\|_\mu = \sum_{l \in \mathbb{Z}} \left\| h^{[l]} \right\|_\mu \tag{4.17}$$

and

$$\left\| h^{[l]} \right\|_{\mu} = \sup_{v \in D_{\rho,\kappa,\delta}} \left| \left(v^2 + 1/9 \right)^{\mu} h^{[l]}(v) \right| \tag{4.18}$$

and the analogues of (4.11)

$$\tilde{\mathcal{Y}}_{\mu,\rho,\kappa,\delta} = \left\{ h : D_{\rho,\kappa,\delta} \times \mathbb{T}_{\sigma} \to \mathbb{C} : h \text{ is real analytic, } \llbracket h \rrbracket_{\mu} < \infty \right\}$$

with

$$\llbracket h \rrbracket_{\mu} = \| h \|_{\mu} + \| \partial_v h \|_{\mu+1} .$$

Remark 4.5. Throughout this section we will work on different domains $D_{\rho,\kappa,\delta}$, $D^{\text{flow}}_{\kappa,\delta}$ and $\tilde{D}_{\kappa,\delta}$ (the latter is defined in (4.32)). We will denote by $\mathcal{Y}_{\mu,\kappa,\delta}$ the analogue to the Banach spaces (4.16) associated to the domain $\tilde{D}_{\kappa,\delta}$, and by $\mathcal{Y}^{\text{flow}}_{\mu,\kappa,\delta}$ the analogues for domain $D^{\text{flow}}_{\kappa,\delta}$ (4.4) (in this case for vectorial functions since we will work with vector fields on the plane).

4.2.1. From Hamilton-Jacobi parametrizations to parametrizations invariant by the flow

We look for a change of variables of the form $\text{Id} + g : (v, \xi) \mapsto (v + g(v, \xi), \xi)$ such that

$$\hat{\Gamma}(v, \xi) = \Gamma \circ (\text{Id} + g)(v, \xi) \tag{4.19}$$

satisfies

$$\phi_s \left(\hat{\Gamma}(v, \xi) \right) = \hat{\Gamma} \left(v + s, \xi + G^3 s \right).$$

Denoting by X the vector field generated by the Hamiltonian (2.1), this equation is equivalent to

$$X \circ \hat{\Gamma} = \mathcal{L}\left(\hat{\Gamma} \right), \tag{4.20}$$

which we can rewrite as

$$\mathcal{L}(g)(v, \xi) = \mathcal{F} \circ (\text{Id} + g)(v, \xi) \qquad \text{where} \qquad \mathcal{F} = \frac{1}{y_h^2} \partial_v T_1 \tag{4.21}$$

and $\mathcal{L}$ stands for the differential operator (4.6). As before we transform (4.21) into a fixed point equation. Thus, we introduce the inverse operator

$$\tilde{\mathcal{G}}(h) = \sum_{l \in \mathbb{Z}} \tilde{\mathcal{G}}(h)^{[l]} e^{il\xi}$$

where

$$\tilde{\mathcal{G}}(h)^{[l]} = \int_{v_1}^{v} e^{ilG^3(t-v)} h^{[l]}(t)\,\mathrm{d}t$$

$$\tilde{\mathcal{G}}(h)^{[0]} = \int_{-\rho}^{v} h^{[l]}(t)\,\mathrm{d}t \tag{4.22}$$

$$\tilde{\mathcal{G}}(h)^{[l]} = \int_{\bar{v}_1}^{v} e^{ilG^3(t-v)} h^{[l]}(t)\,\mathrm{d}t,$$

and $v_1, \bar{v}_1$ are the top and bottom points of the domain $D_{\rho,\kappa,\delta}$ defined in equation (4.3). The following lemma is proved as Lemma 5.5 in [17].

Lemma 4.6. *The operator $\tilde{\mathcal{G}}$ defined on (4.22) satisfies the following properties.*

i) For any $\mu \geq 0$, $\tilde{\mathcal{G}} : \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma} \to \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma}$ is well defined, linear and satisfies $\mathcal{L} \circ \tilde{\mathcal{G}} = Id$.

ii) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma}$ for some $\mu > 1$, then

$$\left\| \tilde{\mathcal{G}}(h) \right\|_{\mu-1} \leq K \|h\|_{\mu}. \tag{4.23}$$

iii) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma}$ for some $\mu \geq 1$, then

$$\left\| \partial_v \tilde{\mathcal{G}}(h) \right\|_{\mu} \leq K \|h\|_{\mu}. \tag{4.24}$$

Therefore, solutions of (4.21) are also fixed points of

$$g = \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g). \tag{4.25}$$

We state two technical lemmas which will be useful for dealing with compositions of functions and are deduced from the proofs of Lemmas 5.14 and 5.15 in [16].

Lemma 4.7. *Fix constants $\delta' < \delta$, $\rho' < \rho$, $\kappa' > \kappa$ and take $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma}$. Then, $\partial_v h \in \mathcal{Y}_{\mu,\rho',\kappa',\delta',\sigma}$ and satisfy*

$$\|\partial_v h\|_{\mu} \leq \frac{G^3}{(\kappa' - \kappa)} \left(\frac{\kappa'}{\kappa}\right)^{\mu} \|h\|_{\mu}.$$

Lemma 4.8. *Fix constants $\rho' < \rho$, $\delta' < \delta$, $\kappa' > \kappa + 1$ and $\sigma' < \sigma$. Then,*

i) If $h \in \mathcal{Y}_{\mu,\rho,\kappa,\delta,\sigma}$ and $g \in B\left(G^{-3}\right) \subset \mathcal{Y}_{\mu,\rho',\kappa',\delta',\sigma'}$ we have that $\tilde{h} = h \circ (\mathrm{Id} + g) \in \mathcal{Y}_{\mu,\rho',\kappa',\delta',\sigma'}$ and

$$\left\| \tilde{h} \right\|_{\mu} \leq \left(\frac{\kappa'}{\kappa}\right)^{\mu} \|h\|_{\mu}.$$

ii) Moreover if $g_1, g_2 \in B\left(G^{-3}\right) \subset \mathcal{Y}_{\mu,\rho',\kappa',\delta',\sigma'}$, then $f = h \circ (\mathrm{Id} + g_1) - h \circ (\mathrm{Id} + g_2)$ satisfies

$$\|f\|_\mu \le \frac{G^3}{(\kappa'-\kappa)}\left(\frac{\kappa'}{\kappa}\right)^\mu \|h\|_\mu \|g_1-g_2\|_{0,0}.$$

Theorem 4.9. *Let δ, κ and σ be the constants given by Theorem 4.4. Let $\rho_1<\rho$, $\delta_1<\delta$, $\sigma_1<\sigma$ and $\kappa_1>\kappa$. Then, for G big enough, there exist a function $g\in\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}$ satisfying*

$$\|g\|_0 \le b_1 G^{-7/2}$$

for $b_1>0$ independent of G and such that

$$\hat{\Gamma} = \Gamma\circ(\mathrm{Id}+g)$$

satisfies (4.20).

Proof. To find g we solve the fixed point equation (4.25). For that, we take $g\in B\left(KG^{-7/2}\right)\subset\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}$, with K a constant independent of G. Then by Lemma 4.8 and using the estimate for $\partial_v T_1$ obtained in Theorem 4.4 we have

$$\begin{aligned}\|\mathcal{F}\circ(\mathrm{Id}+g)\|_{1/2} &\le \left(\frac{\kappa_1}{\kappa}\right)^{1/2}\|\mathcal{F}\|_{1/2}\\ &\le \left(\frac{\kappa_1}{\kappa}\right)^{1/2} KG^{-4}\\ &\le KG^{-4}\end{aligned}$$

where K is a constant depending only on the reduction of the domain. From here it is clear using Lemma 4.6 that the map $\tilde{\mathcal{G}}\circ\mathcal{F}\circ(\mathrm{Id}+g) : B\left(KG^{-7/2}\right)\subset\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}\to\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}$ is well defined. Moreover, we obtain that

$$\begin{aligned}\left\|\tilde{\mathcal{G}}\circ\mathcal{F}\circ(\mathrm{Id}+g)_{|g=0}\right\|_0 &\le KG^{1/2}\left\|\tilde{\mathcal{G}}\circ\mathcal{F}\circ(\mathrm{Id}+g)_{|g=0}\right\|_{1/6}\\ &\le KG^{1/2}\left\|\mathcal{F}\circ(\mathrm{Id}+g)_{|g=0}\right\|_{7/6}\\ &\le KG^{1/2}\left\|\mathcal{F}\circ(\mathrm{Id}+g)_{|g=0}\right\|_{1/2}\\ &\le b_1G^{-7/2}\end{aligned} \tag{4.26}$$

for some b_1 independent of G. It only remains to show that the map $\tilde{\mathcal{G}}\circ\mathcal{F}\circ(\mathrm{Id}+g)$ is contractive in a neighborhood of the origin. Take $g_1, g_2\in B\left(b_1G^{-7/2}\right)\subset\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}$, using again Lemma 4.8 we have that

$$\|\mathcal{F}\circ(\mathrm{Id}+g_1)-\mathcal{F}\circ(\mathrm{Id}+g_2)\|_{1/2} \le \tilde{K}G^{-1/2}\|g_1-g_2\|_0.$$

Direct application of Lemma 4.3 yields

$$\left\|\tilde{\mathcal{G}}\left(\mathcal{F}\circ(\mathrm{Id}+g_1)-\mathcal{F}\circ(\mathrm{Id}+g_2)\right)\right\|_0 \le \tilde{K}G^{-1/2}\|g_1-g_2\|_0,$$

so for G big enough the map $g \mapsto \tilde{\mathcal{G}} \circ \mathcal{F} \circ (\mathrm{Id} + g)$ is contractive on $B\left(b_1 G^{-7/2}\right) \subset \mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1,\sigma_1}$ and the proof is completed. □

4.2.2. Analytic extension of the unstable manifold by the flow parametrization

Now we perform the analytic continuation of the parametrization (4.19) given by Theorem 4.4 to the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ defined in (4.4) using the flow of the Hamiltonian (2.1). Notice that since the domain $D^{\mathrm{flow}}_{\kappa,\delta}$ is bounded and at distance of order $\mathcal{O}(1)$ with respect to the singularities all norms $\|h\|_\mu$ are equivalent, therefore it will suffice to get estimates on the norm $\|h\|_0$.

Write $\hat{\Gamma} = \hat{\Gamma}_0 + \hat{\Gamma}_1$, where

$$\hat{\Gamma}_0(v,\xi) = \Gamma_0 \circ (\mathrm{Id} + g)(v,\xi) \qquad \Gamma_0(v) = (\tilde{r}_h(v), \tilde{y}_h(v)). \tag{4.27}$$

Then, the equation (4.20) that defines this extension is rewritten as

$$\hat{\mathcal{L}}\left(\hat{\Gamma}_1\right) = \hat{\mathcal{F}}\left(\hat{\Gamma}_1\right) \tag{4.28}$$

where

$$\begin{aligned}\hat{\mathcal{L}}(h) =& \mathcal{L}(h) - DX_0\left(\hat{\Gamma}_0\right) h \\ \hat{\mathcal{F}}(h) =& X_0\left(\hat{\Gamma}_0 + h\right) - X_0\left(\hat{\Gamma}_0\right) - DX_0\left(\hat{\Gamma}_0\right) h + X_1\left(\hat{\Gamma}_0 + h\right).\end{aligned}$$

Denote by $\Psi(v)$ the fundamental matrix of the linear system

$$\dot{z}(v) = DX_0(\Gamma_0(v,\xi)) z(v), \qquad v \in D^{\mathrm{flow}}_{\kappa,\delta}.$$

Then, equation (4.28), together with a suitable initial condition $\hat{\Gamma}_h$, can be reformulated as the fixed point equation

$$\hat{\Gamma}_1 = \hat{\Gamma}_h + \hat{\mathcal{G}} \circ \hat{\mathcal{F}}\left(\hat{\Gamma}_1\right), \tag{4.29}$$

where

$$\begin{aligned}\hat{\Gamma}_h =& \sum_{l>0} \Psi(v) \Psi^{-1}(v_1) \hat{\Gamma}_1^{[l]}(v_1) e^{ilG^3(v_1 - v)} e^{il\xi} \\ &+ \sum_{l<0} \Psi(v) \Psi^{-1}(\bar{v}_1) \hat{\Gamma}_1^{[l]}(\bar{v}_1) e^{ilG^3(\bar{v}_1 - v)} e^{il\xi} \\ &+ \Psi(v) \Psi^{-1}(-\rho_1) \hat{\Gamma}_1^{[0]}(-\rho_1)\end{aligned}$$

is the solution of the homogeneous equation $\hat{\mathcal{L}}(h) = 0$ (observe that since $v_1, \bar{v}_1, -\rho_1$ are contained in $D_{\rho,\kappa,\delta}$, the terms $\hat{\Gamma}_1^{[l]}(v_1), \hat{\Gamma}_1^{[l]}(\bar{v}_1)$ and $\hat{\Gamma}_1^{[l]}(-\rho_1)$ are already defined) and

$$\hat{\mathcal{G}}(h) = \Psi \tilde{\mathcal{G}}\left(\Psi^{-1} h\right)$$

is a right inverse operator. Notice that since $DX\left(\hat{\Gamma}_0(v,\xi)\right)$ is continuous and $D^{\text{flow}}_{\kappa,\delta}$ is a compact domain at distance $\mathcal{O}(1)$ from the singularities, we have that there exists $K>0$ such that

$$\sup_{v\in D^{\text{flow}}_{\kappa,\delta}} \max\left\{\|\Psi\|_0, \left\|\Psi^{-1}\right\|_0\right\} \leq K, \tag{4.30}$$

in the matrix norm associated to the usual vector norm in $\mathbb{C}^2$.

Lemma 4.10. *Assume $h,\tilde{h}\in B\left(KG^{-4}\right)\subset \mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$ for some $K>0$. Then there exists $K'>0$ such that*

i) Defining $Y(h)=X_0\left(\hat{\Gamma}_0+h\right)-X_0\left(\hat{\Gamma}_0\right)-DX_0\left(\hat{\Gamma}_0\right)h$ we have that $Y(h)\in\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$, and

$$\|Y(h)\|_0 \leq K'G^{-4},$$

ii) $X_1\left(\hat{\Gamma}_0+h\right)\in\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$. with $\left\|X_1\left(\hat{\Gamma}_0+h\right)\right\|_0\leq K'G^{-4}$,

iii) $\left\|Y(h)-Y\left(\tilde{h}\right)\right\|_0\leq K'G^{-4}\left\|h-\tilde{h}\right\|_0$,

iv) $\left\|X_1\left(\hat{\Gamma}_0+h\right)-X_1\left(\hat{\Gamma}_0+\tilde{h}\right)\right\|_0\leq K'G^{-4}\left\|h-\tilde{h}\right\|_0$.

Proof. The proof follows from the mean value theorem together with the straightforward bounds

$$\left\|DX_0\left(\hat{\Gamma}_0\right)\right\|_0\leq K' \quad \left\|X_1\left(\hat{\Gamma}_0\right)\right\|_0\leq K'G^{-4} \quad \left\|DX_1\left(\hat{\Gamma}_0\right)\right\|_0\leq K'G^{-4}. \quad \square$$

Proposition 4.11. *Let κ_1, δ_1 and σ_1 be the constants considered in Theorem 4.9. Then, there exists $b_2>0$ such that if G is large enough, the fixed point equation (4.29) has a unique solution $\hat{\Gamma}_1\in B\left(b_2G^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$.*

Proof. As $v_1,\bar{v}_1,\rho_1\in D_{\rho_1,\kappa_1,\delta_1}$ we have that $\hat{\Gamma}_h\in\mathcal{Y}_{0,\rho_1,\kappa_1,\delta_1}$ with

$$\left\|\hat{\Gamma}_h\right\|_0\leq KG^{-4}.$$

We claim using Lemma 4.10 that the map $\hat{\mathcal{K}}: h\mapsto \Gamma_h+\hat{\mathcal{G}}\circ\hat{\mathcal{F}}(h)$ is well defined from $B\left(KG^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$ to $\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$. Moreover, we see from the estimate (4.30) for the fundamental matrix $\Psi(v)$ that there exists b_2 such that

$$\left\|\hat{\mathcal{K}}(0)\right\|_0=\left\|\Gamma_h+\hat{\mathcal{G}}(X_1\circ\Gamma_0)\right\|_0\leq\frac{b_2}{2}G^{-4}.$$

Finally, from Lemma 4.10, we conclude that for G big enough $\hat{\mathcal{K}}$ is Lipschitz in $B\left(b_2G^{-4}\right)\subset\mathcal{Y}^{\text{flow}}_{0,\kappa_1,\delta_1,\sigma_1}$ with Lipschitz constant KG^{-4}. $\square$

4.2.3. From flow parametrization to Hamilton-Jacobi parametrization

Now that we have extended the parametrization (4.19) across $v = 0$, the next step is to come back to the Hamilton-Jacobi parametrization (3.5) so we have both stable and unstable manifolds parametrized as graphs of the form $(\tilde{r}_h(v), \tilde{y}^{u,s}(v,\xi))$ and we can easily measure the distance between them.

We look for a change of variables of the form $\mathrm{Id} + f$ such that

$$\pi_1 \circ \hat{\Gamma} \circ (\mathrm{Id} + f)(v,\xi) = \tilde{r}_h(v) \tag{4.31}$$

in the domain

$$\tilde{D}_{\kappa_1,\delta_1} = D^{flow}_{\kappa_1,\delta_1} \cap D_{\kappa_1,\delta_1}, \tag{4.32}$$

where $D^{flow}_{\kappa_1,\delta_1}$, D_{κ_1,δ_1} are the domains defined in (4.4) and (4.5). Therefore, in $D^u_{\rho_1,\kappa_1,\delta_1} \cap \tilde{D}_{\kappa_1,\delta_1}$ the change $\mathrm{Id} + f$ is the inverse of the change $\mathrm{Id} + g$ obtained in Theorem 4.9. We will see that this change of variables is unique under certain conditions, therefore, once we have f, the second component of the unstable manifold is given by

$$\pi_2 \circ \hat{\Gamma}_1 \circ (\mathrm{Id} + f)(v,\xi) = \frac{1}{y_h(v)} \partial_v T_1. \tag{4.33}$$

Using the properties of the unperturbed solution, i.e. $\pi_1 \circ \Gamma_0(v,\xi) = \tilde{r}_h(v)$, we can write equation (4.31) as

$$f = \mathcal{P}(f)$$

where

$$\mathcal{P}(f) = \frac{-1}{y_h(v)} \left(\tilde{r}_h(v + f(v,\xi)) - \tilde{r}_h(v) - \tilde{y}_h(v) f(v,\xi) - \pi_1 \circ \Gamma_1 \circ (\mathrm{Id} + f)(v,\xi)\right).$$

Proposition 4.12. *Consider the constants κ_1, δ_1 and σ_1 given by Proposition 4.11 and any $\kappa_2 > \kappa_1$, $\delta_2 < \delta_1$, $\sigma_2 < \sigma_1$. Then,*

i) There exists $b_3 > 0$ such that for G large enough, the operator $\mathcal{P}$ has a unique fixed point $f \in \mathcal{Y}_{0,\kappa_2,\delta_2,\sigma_2}$ with

$$\|f\|_0 \le b_3 G^{-4}.$$

ii) Equation (4.33) defines the graph of the unstable manifold which can be written as $T^u = T_0 + T_1^u$ where T_1^u satisfies

$$\left\|\partial_v T_1^u\right\|_0 \le K G^{-4}.$$

Proof. For the first part we observe that, for $f_2, f_1 \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2,\sigma_2}$,

$$
\begin{aligned}
\left|\tilde{r}_h\left(v+f_2\right)-\tilde{r}_h\left(v+f_1\right)-\tilde{y}_h\left(f_2-f_1\right)\right| &\leq K\left|f_2^2-f_1^2\right| \\
&\leq KG^{-4}\left|f_2-f_1\right|.
\end{aligned}
$$

Then, from Lemma 4.8 and the fact and $\left\|\hat{\Gamma}_1^u\right\|_0 \leq KG^{-4}$ we deduce that

$$
\left|\mathcal{P}\left(f_2\right)-\mathcal{P}\left(f_1\right)\right| \leq KG^{-4}\left|f_2-f_1\right|,
$$

i.e. $\mathcal{P}(f)$ is a contractive mapping on $B\left(b_3G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2,\sigma_2}$ for some $b_3>0$ so there exists a unique $f \in B\left(b_3G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_2,\delta_2,\sigma_2}$ solving $f=\mathcal{P}(f)$.

For the second part we have from equation (4.33) that

$$
\pi_2 \circ \hat{\Gamma}_1 \circ(\mathrm{Id}+f)(v, \xi)=\frac{1}{y_h(v)} \partial_v T_1 .
$$

Therefore,

$$
\begin{aligned}
\left\|\partial_v T_1\right\|_{0,0} &\leq K\left\|\frac{1}{y_h(v)} \partial_v T_1\right\|_0 \\
&=K\left\|\pi_2 \circ \hat{\Gamma}_1 \circ(\mathrm{Id}+f)\right\|_0 \\
&\leq K\left\|\hat{\Gamma}_1 \circ(\mathrm{Id}+f)\right\|_0 \\
&\leq K\left\|\hat{\Gamma}_1\right\|_0 \leq KG^{-4},
\end{aligned}
$$

where we have used Lemma 4.8 and the estimate for $\left\|\hat{\Gamma}_1\right\|_0$ obtained in Proposition 4.11. □

We sum up the results obtained in this section in the following theorem.

Theorem 4.13. *Let κ_2, δ_2 and σ_2 the constants given by Proposition 4.12. Then, for G big enough there exist real analytic functions $T_1^{u,s}$ defined in D_{κ_2,δ_2} which are solutions of equation (3.3) and satisfy*

$$
\left\|\partial_v T_1^{u,s}\right\|_{3/2} \leq b_4 G^{-4}
$$

for a certain $b_4>0$ independent of G.

Proof. For the stable manifold, the result was obtained in Theorem 4.4 since $D_{\kappa_2,\delta_2} \subset D_{\kappa,\delta}^{\infty,s}$. For the unstable manifold, using that $D_{\kappa_2,\delta_2} \subset D_{\kappa,\delta}^{\infty,u} \cup \tilde{D}_{\kappa_2,\delta_2}$ the result follows from the combination of Theorem 4.4 and Proposition 4.12. □

5. The difference between the manifolds

Once we have obtained the parametrization of the invariant manifolds in the common domain $D_{\kappa,\delta}$ defined in (4.3), the next step is to study their difference. To this end we define

$$\tilde{\Delta}(v,\xi) = T^s(v,\xi) - T^u(v,\xi). \tag{5.1}$$

Substracting equation (3.3) for T_1^s and T_1^u one obtains that

$$\tilde{\Delta} \in \mathrm{Ker}\tilde{\mathcal{L}}$$

where $\tilde{\mathcal{L}}$ is the differential operator

$$\tilde{\mathcal{L}} = (1 + A(v,\xi))\,\partial_v - G^3\partial_\xi$$

with

$$A(v,\xi) = \frac{1}{2\tilde{y}_h^2}\left(\partial_v T_1^s - \partial_v T_1^u\right). \tag{5.2}$$

To obtain exponentially small bounds on the difference between the invariant manifolds we will look for a close to identity change of variables $(v,\xi) = (w + C(w,\xi), \xi)$ such that the function

$$\Delta(w,\xi) = \tilde{\Delta}(w + C(w,\xi), \xi) \qquad (w,\xi) \in D_{\kappa,\delta} \times \mathbb{T}_\sigma, \tag{5.3}$$

satisfies

$$\Delta \in \mathrm{Ker}\mathcal{L}$$

where $\mathcal{L}$ is the differential operator defined in (4.6). The condition $\Delta \in \mathrm{Ker}\mathcal{L}$ implies that $\Delta = f\left(\xi - G^3 w\right)$. Therefore, since Δ is periodic in ξ it must be periodic in w. Since Δ is real analytic and bounded in a strip that reaches up to points $\mathcal{O}\left(G^{-3}\right)$ close to the singularities the exponentially small bound for $|\Delta(w,\xi)|$ where $w \in \mathbb{R}$ comes straightforward by a classical argument (see Lemma 5.2 below). We devote the rest of the section to make this rigorous.

5.1. *Straightening the operator $\tilde{\mathcal{L}}$*

As we did in the previous sections we introduce the Banach spaces

$$\mathcal{Q}_{\mu,\rho,\kappa,\delta,\sigma} = \left\{h : D_{\kappa,\delta} \times \mathbb{T}_\sigma \to \mathbb{C} : h \text{ is real analytic, } \|h\|_\mu < \infty\right\}$$

where

$$\|h\|_\mu = \sup_{v \in D_{\kappa,\delta}} \left|\left(v^2 + 1/9\right)^\mu h(v)\right|.$$

Theorem 5.1. *Let κ_2 and δ_2 the constants defined in Theorem 4.13. Let $\kappa_3 > \kappa_2$, $\delta_3 < \delta_2$ and $\sigma_3 < \sigma_2$ be fixed. Then, for G big enough, there exists $C \in \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3}$ such that the function*

$$\Delta(w,\xi) = \tilde{\Delta}(w + C(w,\xi), \xi)$$

satisfies that $\Delta \in \mathrm{Ker}\mathcal{L}$. Moreover, we have that

$$\|C\|_0 \le b_5 G^{-7/2}$$

for a certain $b_5 > 0$ independent of G.

Proof. Using the chain rule we obtain that the implication $\Delta \in \mathrm{Ker}\mathcal{L}$ if and only if $\Delta \in \mathrm{Ker}\tilde{\mathcal{L}}$, is equivalent to finding C satisfying

$$\begin{aligned}\mathcal{L}(C) &= A_{|v=w+C(w)} \\ &= A \circ (\mathrm{Id} + C),\end{aligned}$$

where $A(v,\xi)$ was defined in (5.2). We can rewrite this equation as a fixed point equation

$$C = \tilde{\mathcal{G}}(A \circ (\mathrm{Id} + C)),$$

where $\tilde{\mathcal{G}}$ is the inverse operator defined in (4.22). Using the bounds for $\partial_v T_1^{u,s}$ in Theorem 4.13, the properties of the homoclinic orbit stated in Section 3.1, and Lemma 4.8 for the composition, we obtain that, for $C \in B\left(KG^{-4}\right) \subset \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3}$,

$$\|A \circ (\mathrm{Id} + C)\|_{1/2} \le K' G^{-4}$$

for some $K' > 0$ independent of G. Hence, from Lemma 4.8 we observe that the map $C \mapsto \tilde{\mathcal{G}}(A \circ (\mathrm{Id} + C))$ is well defined from $C \in B\left(KG^{-7/2}\right) \subset \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3} \to \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3}$. Moreover, we also get

$$\left\| \tilde{\mathcal{G}}\left(A \circ (\mathrm{Id} + C)_{|C=0}\right) \right\|_0 \le \frac{b_5}{2} G^{-7/2},$$

for some b_5 independent of G. Hence, it only remains to prove that the map $C \mapsto \tilde{\mathcal{G}}(A \circ (\mathrm{Id} + C))$ is contractive on the ball $B\left(b_5 G^{-7/2}\right) \subset \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3}$. Again by Lemma 4.8 we have that if $C_1, C_2 \in B\left(b_5 G^{-7/2}\right) \subset \mathcal{Q}_{0,\kappa_3,\delta_3,\sigma_3}$, then

$$\begin{aligned}\|A \circ (\mathrm{Id} + C_2) - A \circ (\mathrm{Id} + C_1)\|_{1/2} &\le KG^3 \|A\|_{1/2} \|C_2 - C_1\|_0 \\ &\le KG^{-1} \|C_2 - C_1\|_0,\end{aligned}$$

and contractivity follows from Lemma 4.6 for G big enough. □

5.2. Estimates for the difference between the invariant manifolds

Now we exploit the fact that the function $\Delta(w,\xi)$ defined in (5.3) satisfies

$$\Delta \in \mathrm{Ker}\mathcal{L}$$

to get exponentially small bounds on the real line.

Lemma 5.2. *Let $h : D_{\kappa,\delta} \times \mathbb{T}_\sigma \to \mathbb{C}$ be a real-analytic function such that $h \in \mathcal{Q}_{0,\kappa,\delta,\sigma}$ and $h \in \mathrm{Ker}\mathcal{L}$. Then,*

i) h is of the form

$$h(w,\xi) = \sum_{l\in\mathbb{Z}} h^{[l]}(w)\, e^{il\xi} = \sum_{l\in\mathbb{Z}} \beta^{[l]} e^{il\left(\xi - G^3 w\right)}.$$

ii) the coefficients $\beta^{[l]}$ satisfy the bounds

$$\left|\beta^{[l]}\right| \le \|h\|_0\, K^{|l|} e^{\frac{-|l|G^3}{3}}.$$

Proof. Since $h \in \mathrm{Ker}\mathcal{L}$ and is periodic in ξ, we have that each Fourier coefficient $h^{[l]}$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}w} h^{[l]} + ilG^3 h^{[l]} = 0$$

so it has to be

$$h^{[l]}(w) = \beta^{[l]} e^{-ilG^3 w}$$

for certain constants $\beta^{[l]}$. Moreover, evaluating this equality at the top vertex $w_2 = i\left(1/3 - \kappa G^{-3}\right)$ of the domain $D_{\kappa,\delta}$ for $l < 0$ and at the bottom vertex $\bar{w}_2 = i\left(1/3 - \kappa G^{-3}\right)$ for $l > 0$ we obtain that

$$\begin{aligned}
\left|\beta^{[l]}\right| &\le \max\left\{h^{[l]}(w_2), h^{[l]}(\bar{w}_2)\right\} e^{\frac{-|l|G^3}{3}} e^{|l|\kappa_3} \\
&\le \|h\|_0\, e^{|l|\kappa_3} e^{\frac{-|l|G^3}{3}} \\
&\le \|h\|_0\, K^{|l|} e^{\frac{-|l|G^3}{3}},
\end{aligned}$$

for a constant K independent of G and l. Therefore, for $u \in \mathbb{R} \cap D_{\kappa,\delta}$

$$\left|h^{[l]}(u)\right| = \left|\beta^{[l]}\right| \le \|h\|_0\, K^{|l|} e^{\frac{-|l|G^3}{3}}. \quad \square$$

Using this lemma we already have exponentially small bounds for $\Delta(w,\xi)$. Nevertheless, our goal is to prove that the function L defined in (3.7) is the main term in Δ. Thus we study the function

$$\mathcal{E}(w,\xi) = \Delta(w,\xi) - L(w,\xi).$$

Lemma 5.3. *Consider the constants κ_3 δ_3 and σ_3 defined in Theorem 5.1. Then, for $(w,\xi) \in \left(D_{\kappa_3,\delta_3} \cap \mathbb{R}\right) \times \mathbb{T}$ we get*

$$|\mathcal{E}(w,\xi) - E| \leq KG^{-7/2} e^{\frac{-G^3}{3}},$$

where E is a constant and

$$|\partial_w \mathcal{E}| \leq KG^{-1/2} e^{\frac{-G^3}{3}}.$$

Proof. Notice that $L = L^s - L^u$ where $L^* = \mathcal{G}^*(V)$, with $\mathcal{G}^{u,s}$ are the left inverse operators introduced in (4.8). Then, it is clear that $\mathcal{L}(L) = 0$ and we have that $\mathcal{E} \in \mathrm{Ker}\mathcal{L}$. We bound $\mathcal{E}$ in the domain $D_{\kappa,\delta}$ so that we can apply Lemma 5.2. We decompose $\mathcal{E} = \mathcal{E}_1^s - \mathcal{E}_1^u + \mathcal{E}_2$ where

$$\begin{aligned}
\mathcal{E}_1^* &= T_1^* - L^* \\
\mathcal{E}_2 &= \Delta - \tilde{\Delta}.
\end{aligned}$$

From Lemma 4.2 and equation (4.14) we have

$$\left\|\mathcal{E}_1^*\right\|_0 = \left\|T_1^* - L^*\right\|_0 \leq KG^{3/2} \left\|T_1^* - L^*\right\|_{1/2} \leq KG^{-5}.$$

For the second term we use Lemmas 4.2, 4.8 and the bounds for $\tilde{\Delta}$ and C from Theorems 4.13 and 5.1 to obtain

$$\begin{aligned}
\|\mathcal{E}_2\|_0 = \left\|\tilde{\Delta} \circ (\mathrm{Id} + C) - \tilde{\Delta}\right\|_0 &\leq KG^3 \left\|\tilde{\Delta}\right\|_0 \|C\|_0 \\
&\leq KG^{9/2} \left\|\tilde{\Delta}\right\|_{1/2} \|C\|_0 \leq KG^{-7/2},
\end{aligned}$$

Combining these results

$$\|\mathcal{E}\|_0 \leq KG^{-7/2}.$$

Hence, by direct application of Lemma 5.2 we obtain that for $u \in D_{\kappa_3,\delta_3} \cap \mathbb{R}$

$$\left|\mathcal{E}^{[l]}(w)\right| \leq G^{-7/2} K^{|l|} e^{\frac{-|l|G^3}{3}}.$$

Now, defining $E = \mathcal{E}^{[0]}$ (notice that by Lemma 5.2, $\mathcal{E}^0$ is constant) we have that for $(w,\xi) \in \left(D_{\kappa_3,\delta_3} \cap \mathbb{R}\right) \times \mathbb{T}_{\sigma_3}$

$$
\begin{aligned}
|\mathcal{E}-E| &\leq \sum_{|l|>1} \left|\mathcal{E}^{[l]}(w)\right| \\
&\leq G^{-7/2} e^{\frac{-G^3}{3}} \sum_{|l|>2} \left(K e^{\frac{-G^3}{3}}\right)^{|l|} \\
&\leq K G^{-7/2} e^{\frac{-G^3}{3}}.
\end{aligned}
$$

Finally, it is a straightforward computation to check that

$$\left|\frac{\mathrm{d}}{\mathrm{d}w}\mathcal{E}^{[l]}(w)\right| \leq G^{-1/2} K^{|l|} e^{\frac{-|l|G^3}{3}}$$

so we conclude that

$$|\partial_w \mathcal{E}| \leq K G^{-1/2} e^{\frac{-G^3}{3}}. \quad \square$$

There is only one step left for achieving our goal, going back to the original variables (v,ξ). This is done in the next lemma.

Lemma 5.4. *Consider the function*

$$\tilde{\mathcal{E}}(v,\xi) = \tilde{\Delta}(v,\xi) - L(v,\xi)$$

where $\tilde{\Delta}(v,\xi)$ *is defined in* (5.1) *and* $L(v,\xi)$ *is defined in* (3.7). *Fix* $\kappa_4 > \kappa_3$, $\delta_4 < \delta_3$ *and* $\sigma_4 > \sigma_3$. *Then, for* $(v,\xi) \in \left(D_{\kappa_4,\delta_4} \cap \mathbb{R}\right) \times \mathbb{T}_{\sigma_4}$,

$$\left|\tilde{\mathcal{E}}(v,\xi) - E\right| \leq K G^{-7/2} e^{\frac{-G^3}{3}} \tag{5.4}$$

where E *is a constant and*

$$\left|\partial_v \tilde{\mathcal{E}}(v,\xi)\right| \leq K G^{-1/2} e^{\frac{-G^3}{3}}. \tag{5.5}$$

Proof. We look for a function $\varphi(v,\xi)$ such that $(\mathrm{Id}+C)\circ(\mathrm{Id}+\varphi)(v,\xi) = (v,\xi)$, i.e., φ must satisfy

$$v = v + \varphi(v,\xi) + C(v+\varphi(v,\xi),\xi)$$

or what is the same

$$\varphi(v,\xi) = -C(v+\varphi(v,\xi),\xi). \tag{5.6}$$

In order to solve this fixed point equation we first use Lemma 4.8 to obtain that for $\varphi \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4,\sigma_4}$

$$\|C\circ(\mathrm{Id}+\varphi)\|_0 \leq \|C\|_0 \leq KG^{-4}$$

so the map $\varphi \mapsto C \circ (\mathrm{Id} + \varphi)$ is well defined from $B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4,\sigma_4} \to \mathcal{Y}_{0,\kappa_4,\delta_4,\sigma_4}$. Moreover we get that there exists b_6 such that

$$\left\| C \circ (\mathrm{Id} + \varphi)_{|\varphi=0} \right\|_0 \leq \frac{b_6}{2} G^{-4}.$$

Since for $\varphi_1, \varphi_2 \in B\left(KG^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4,\sigma_4}$ we have

$$\| C \circ (\mathrm{Id} + \varphi_2) - C \circ (\mathrm{Id} + \varphi_1) \|_0 \leq KG^{-4} \, \| \varphi_2 - \varphi_1 \|_0$$

we have shown the existence of a unique $\varphi \in B\left(b_6 G^{-4}\right) \subset \mathcal{Y}_{0,\kappa_4,\delta_4,\sigma_4}$ solving (5.6).

Now that we have obtained the inverse change of variables, the bounds (5.4) and (5.5) follow from direct application of Lemma 4.8 if we notice that

$$\begin{aligned} \mathcal{E}\left(w\left(v,\xi\right),\xi\right) &= \left(\Delta - L\right) \circ \left(\mathrm{Id} + \varphi\right)\left(v,\xi\right) \\ &= \left(\tilde{\Delta} \circ \left(\mathrm{Id} + C\right) - L\right) \circ \left(\mathrm{Id} + \varphi\right)\left(v,\xi\right) \\ &= \tilde{\Delta}\left(v,\xi\right) - L \circ \left(\mathrm{Id} + \varphi\right)\left(v,\xi\right) \end{aligned}$$

so

$$\tilde{\mathcal{E}}\left(v,\xi\right) = \mathcal{E}\left(v,\xi\right) + L \circ \left(\mathrm{Id} + \varphi\right)\left(v,\xi\right) - L\left(v,\xi\right).$$

Then, the result follows from Lemma 4.8 and the estimates on Proposition 3.1. $\square$

6. Computation of the Melnikov potential

We devote this section to the computation of the Melnikov potential $L\left(v,\xi\right)$ whose partial derivative with respect to v gives us the first order term of the distance between the infinity manifolds. From its definition (3.7) we have

$$\begin{aligned} L\left(v,\xi\right) &= \int_{-\infty}^{\infty} V\left(\tilde{r}_h\left(v+s\right), \xi + G^3 s\right) \mathrm{d}s \\ &= \int_{-\infty}^{\infty} V\left(\tilde{r}_h\left(s\right), \xi + G^3\left(s - v\right)\right) \mathrm{d}s. \end{aligned}$$

Expanding in Taylor series the square root in (2.2) we obtain that

$$V\left(\tilde{r}_h\left(s\right), \xi + G^3\left(s-v\right)\right) = -\sum_{k=1}^{\infty} \binom{\frac{-1}{2}}{k} \left(4G^4\right)^{-k} \int_{-\infty}^{\infty} \frac{\rho^{2k}\left(\xi + G^3\left(s-v\right)\right) \mathrm{d}s}{\tilde{r}_h^{2k+1}\left(s\right)}.$$

Hence, expanding now the terms ρ^{2k} in Fourier series we get

$$L(v,\xi) = -\sum_{l\in\mathbb{Z}} e^{il\left(\xi - G^3 v\right)} \sum_{k=1}^{\infty} \binom{\frac{-1}{2}}{k} a_{l,k} \left(4G^4\right)^{-k} \int_{-\infty}^{\infty} \frac{e^{ilG^3 s}\mathrm{d}s}{\tilde{r}_h^{2k+1}(s)},$$

where

$$a_{l,k} = \frac{1}{2\pi}\int_0^{2\pi} \rho^{2k}(\sigma)\, e^{-il\sigma}\mathrm{d}\sigma.$$

Since for all $\sigma \in [0, 2\pi]$ we have $|\rho| < 2$ we easily bound

$$\left|a_{l,k}\right| \le 4^k. \tag{6.1}$$

Moreover, changing the integration variable to the eccentric anomaly E defined by $t = E - \sin E$

$$\rho(E) = 1 - \cos E,$$

we obtain that

$$a_{1,1} = -2J_1(1) \neq 0 \tag{6.2}$$

where J_1 is the Bessel function of first kind.

Under the time reparametrization

$$s = \frac{1}{2}\left(\tau + \frac{\tau^3}{3}\right),$$

we can write

$$\begin{aligned} L(v,\xi) &= -2\sum_{l\in\mathbb{Z}} e^{il\left(\xi - G^3 v\right)} \sum_{k=1}^{\infty} \binom{\frac{-1}{2}}{k} a_{l,k} G^{4k} \int_{-\infty}^{\infty} \frac{e^{ilG^3\left(\tau + \frac{\tau^3}{3}\right)/2}\mathrm{d}\tau}{(\tau - i)^{2k}(\tau + i)^{2k}} \\ &= -2\sum_{l\in\mathbb{Z}} e^{il\left(\xi - G^3 v\right)} \sum_{k=1}^{\infty} \binom{\frac{-1}{2}}{k} a_{l,k} G^{4k} I(l,k) \\ &= \sum_{l\in\mathbb{Z}} L^{[l]} e^{il\left(\xi - G^3 v\right)}. \end{aligned} \tag{6.3}$$

The harmonic with $l = 0$ is readily bounded using that

$$I(0,k) = \sqrt{\pi}\,\frac{\Gamma(2k - 1/2)}{\Gamma(2k)},$$

where Γ stands for the Gamma function.

A standard computation shows that $L^{[l]} = L^{[-l]}$ so we focus only on the case $l > 0$. The next proposition, which can be deduced from Propositions 19 and 22 in [8] gives estimates for $|I(l,k)|$ and the asymptotic first order term for $I(1,1)$ which we use to identify the main term in $L^{[1]}(v,\xi)$.

Proposition 6.1. *Let G be large enough, then the estimate*

$$|I(l,k)| \leq 8e^{l} G^{3k-3/2} e^{\frac{-lG^3}{3}},$$

holds for $l \geq 1$, $k \geq 1$. Moreover we have that

$$I(1,1) = \sqrt{\pi}\left(\frac{G}{2}\right)^{3/2} e^{\frac{-G^3}{3}}\left(1+\mathcal{O}\left(G^{-3/2}\right)\right).$$

For $l = 1$ we have

$$L^{[1]} = -2\left(-\frac{1}{2}a_{1,1}G^{-4}I_{1,1} + \sum_{k=2}^{\infty}\binom{\frac{-1}{2}}{k} a_{1,k}G^{4k}I(1,k)\right).$$

Using Proposition 6.1 and the estimate in (6.1) we have that

$$\begin{aligned}\left|\sum_{k=2}^{\infty}\binom{\frac{-1}{2}}{k} a_{1,k}G^{4k}I(1,k)\right| &\leq 8e^{1/2}e^{\frac{-G^3}{3}}G^{-3/2}\sum_{k=2}^{\infty}G^{-k}\\ &\leq 16e^{1/2}e^{\frac{-G^3}{3}}G^{-7/2}.\end{aligned}$$

Therefore

$$L^{[1]} = a_{1,1}\sqrt{\pi}2^{-3/2}G^{-5/2}e^{\frac{-G^3}{3}}\left(1+\mathcal{O}\left(G^{-1}\right)\right).$$

For $l \geq 2$ we have

$$L^{[l]} = -2\sum_{k=1}^{\infty}\binom{\frac{-1}{2}}{k} a_{l,k}G^{-4k}I_{l,k}$$

and again from Proposition 6.1 and the estimate in (6.1) we obtain

$$\left|L^{[l]}\right| \leq 32e^{l-1/2}G^{-5/2}e^{\frac{-lG^3}{3}}.$$

From the estimates we have obtained for $\left|L^{[l]}\right|$ the double series is absolutely convergent, which justify the expansions in Taylor and Fourier series and the proof of Proposition 3.1 is completed.

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