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Exponentially Small Splitting for the Pendulum: A Classical Problem Revisited

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Abstract In this paper, we study the classical problem of the exponentially small splitting of separatrices of the rapidly forced pendulum. Firstly, we give an asymptotic formula for the distance between the perturbed invariant manifolds in the so-called singular case and we compare it with the prediction of Melnikov theory. Secondly, we give exponentially small upper bounds in some cases in which the perturbation is bigger than in the singular case and we give some heuristic ideas how to obtain an asymptotic formula for these cases. Finally, we study how the splitting of separatrices behaves when the parameters are close to a codimension-2 bifurcation point.

Keywords Exponentially small splitting of separatrices · Melnikov method · Resurgence theory · Averaging · Complex matching

Mathematics Subject Classification (2000) 34C29 · 34C37 · 37C29 · 34E10

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1 Introduction

In this paper, we consider the classical problem of the splitting of separatrices for the rapidly forced pendulum whose equation is

$$\ddot{x} = \sin x + \frac{\mu}{\varepsilon^2}\sin\frac{t}{\varepsilon}, \tag{1}$$

where μ is a real parameter and $\varepsilon > 0$ is a small parameter.

When $\mu\varepsilon^{-2}$ is small, this equation is a small perturbation of the classical pendulum

$$\ddot{x} = \sin x \tag{2}$$

and has been considered as a model of a two dimensional integrable system perturbed by a rapidly forcing term. Reparameterizing time $\tau = \frac{t}{\varepsilon}$, (1) can be considered as a nearly integrable system with slow dynamics

$$x'' = \varepsilon^2 \sin x + \mu \sin\tau \tag{3}$$

with $' = \frac{\mathrm{d}}{\mathrm{d}\tau}$. Rewriting this equation as a first order system,

$$\begin{cases} x' = \varepsilon y, \\ y' = \varepsilon \sin x + \mu\varepsilon^{-1}\sin\tau, \end{cases} \tag{4}$$

one can see that it is a Hamiltonian system of one and a half degrees of freedom, with Hamiltonian function

$$H(x, y, \tau, \mu, \varepsilon) = \varepsilon\left(\frac{y^2}{2} + \cos x - 1 - \mu\varepsilon^{-2}x\sin\tau\right). \tag{5}$$

Due to the $2\pi\varepsilon$-periodicity of the forcing in (1), its dynamical properties can be better visualized with the help of the Poincaré map P defined on a section $\Sigma_{t_0} = \{(x, y, t_0),\ (x, y) \in \mathbb{R}^2\}$.

If $\mu = 0$, the phase portrait of P is very simple. It is given by the level curves of the Hamiltonian $H_0(x, y) = (\frac{y^2}{2} + \cos x - 1)$. Our interest will be the stable and unstable manifolds of the hyperbolic fixed point $(0, 0)$, which in this case, coincide along two separatrices given by the homoclinic orbits of the pendulum: $(x_0(t), \pm y_0(t))$ where

$$x_0(t) = 4\arctan\left(e^t\right), \qquad y_0(t) = \dot{x}_0(t).$$

The phase space looks more complicated when $\mu \neq 0$. Roughly speaking, if $\mu\varepsilon^{-2}$ is small enough, there still exists a hyperbolic fixed point of P corresponding to a hyperbolic periodic orbit of (1), as well as the stable and unstable invariant curves $C^{\mathrm{s}}(t_0)$ and $C^{\mathrm{u}}(t_0)$, which lie near the unperturbed separatrix. Due to the symmetries of system (4), $C^{\mathrm{s}}(t_0)$ and $C^{\mathrm{u}}(t_0)$ intersect on a point z_{h}, which lies on the line $x = \pi$ in the case $t_0 = 0$. If this intersection is transversal at z_{h}, the curves enclose lobes whose area $\mathcal{A}$ does not depend on the homoclinic point we have chosen (see Fig. 1). The measure of this area in terms of ε and μ is the main purpose of this

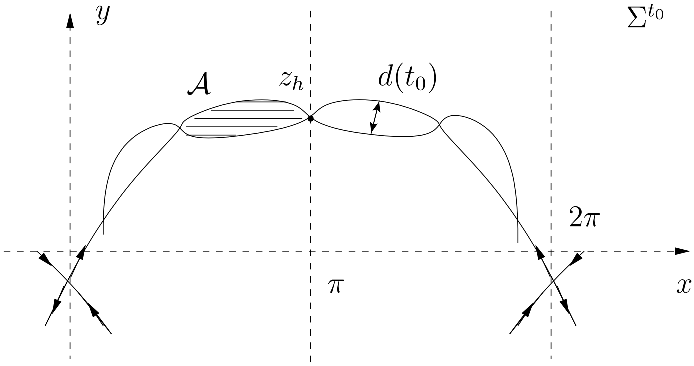


Fig. 1 Splitting of separatrices

paper. Another quantity that can be measured at the homoclinic points to check the transversality of the intersection, is the angle between the curves $C^{\mathrm{s}}(t_0)$ and $C^{\mathrm{u}}(t_0)$, but this quantity depends on the chosen homoclinic point. In fact, the corresponding invariant quantity is the so-called *Lazutkin invariant* (see, for instance Gelfreich et al. 1991).

As it is known in the dynamical systems community, the existence of transversal intersections between stable and unstable manifolds of one or more critical points in a dynamical system was described by Poincaré as the fundamental problem of mechanics (Poincaré 1890, Sect. 19). In fact, the transversal intersection of stable and unstable manifolds of fixed points of smooth planar diffeomorphisms is the simplest setting where this phenomenon gives rise to the existence of chaotic behavior (see Smale 1965). For this reason, it has been one of the most studied problems in the last century.

An easy way to produce planar diffeomorphisms is to consider the Poincaré map of a T-periodically perturbed planar vector field defined in a Poincaré section $t = t_0$. Furthermore, in this regular perturbative context, Poincaré, and later Melnikov (see Melnikov 1963), developed a method which measures the distance between the invariant manifolds of hyperbolic critical points which coincide in the unperturbed integrable system. The Poincaré–Melnikov method provides a function $L(t_0)$, called Poincaré function or Melnikov potential, whose nondegenerate critical points give rise to transversal intersections between the stable and unstable perturbed manifolds (see, for instance, Melnikov 1963; Guckenheimer and Holmes 1983). In fact, $\frac{\partial L}{\partial t_0}$ gives the main term of the distance $d(t_0)$ between the perturbed invariant manifolds in the Poincaré section $t = t_0$. Moreover, the main term of the area $\mathcal{A}$ is given by $L(t_0^1) - L(t_0^2)$, where t_0^1 and t_0^2 are two consecutive critical points of $L(t_0)$.

The generalization of this problem to higher dimensional systems has been achieved by several authors, mainly in the Hamiltonian case. See, for instance, Holmes and Marsden (1981, 1982, 1983), Eliasson (1994), Treschev (1994), Delshams and Gutiérrez (2000), Lochak et al. (2003) and references therein. A Melnikov theory for twist maps can be found in Delshams and Ramírez-Ros (1997).

In the case of rapidly forced systems, a difficult problem arises due to the fact that the Poincaré function depends on the perturbed parameter and, in fact, turns out to be exponentially small with respect to it. Sanders (1982) noticed this problem, previously stated by Poincaré, arising from the direct application of the Poincaré–Melnikov method in these cases. A similar problem occurs in families of area preserving diffeomorphisms close to the identity. In all these cases, even if the prediction of the Poincaré function for the splitting is exponentially small, it is not clear if this function gives the main term of this distance and even if the distance itself is exponentially small or not (see Scheurle et al. 1991; Fontich 1995).

Several authors have given partial answers to this problem. The first group of results is concerned about exponentially small upper bounds. Neĭshtadt (1984) gave exponentially small upper bounds for the splitting in two degrees of freedom Hamiltonians. For area preserving maps close to the identity, Fontich and Simó provided upper bounds in Fontich and Simó (1990). For second order equations with a rapidly forced periodic term, several authors gave sharp exponentially small upper bounds in Fontich (1993, 1995), Fiedler and Scheurle (1996) and, for the higher dimensional case, the papers Sauzin (2001), Simó (1994) gave (not sharp) exponentially small upper bounds.

The second group of results is concerned about the validity of the Melnikov potential in the exponentially small case. This problem needs more information about the system under consideration and, for this reason, the results existing in this direction deal mostly with specific examples. The first result was obtained by Holmes et al. (1988) (followed by Scheurle 1989; Angenent 1993), where they studied the rapidly perturbed pendulum (1). Taking $\mu = \mathcal{O}(\varepsilon^p)$, they confirmed the prediction of the Melnikov potential establishing exponentially small upper and lower bounds for the area $\mathcal{A}$ provided $p \geq 10$,

$$c_2 \varepsilon^{p-1} \mathrm{e}^{-\frac{\pi}{2\varepsilon}} \leq \mathcal{A} \leq c_1 \varepsilon^{p-1} \mathrm{e}^{-\frac{\pi}{2\varepsilon}}.$$

Let us observe that for (1), the Melnikov potential is given by

$$L(t_0) = -\frac{2\pi\varepsilon}{\cosh\frac{\pi}{2\varepsilon}} \cos t_0$$

and, therefore, the prediction for the area is

$$\mathcal{A} \sim \frac{4\pi\mu}{\varepsilon\cosh\frac{\pi}{2\varepsilon}} \sim 8\pi\varepsilon^{p-1}\mathrm{e}^{-\frac{\pi}{2\varepsilon}}. \tag{6}$$

In Ellison et al. (1993), the range for p was extended to $p \geq 5$ using the same approach. Following the ideas in Gelfreich et al. (1991), Gelfreich (1994) established an asymptotic expression for the splitting provided $p > 7$ and Delshams and Seara established rigorously the result in Delshams and Seara (1992) for $p > 2$. An alternative proof, using parametric resurgence, was done in Sauzin (1995) for a simplified perturbation of the pendulum equation. Later, in Delshams and Seara (1997), Gelfreich (1997a), the authors gave a proof for the validity of the Melnikov method for general rapidly periodic Hamiltonian perturbations of a class of second order equations. The case of a perturbed second order equation with a parabolic point was studied in Baldomá and Fontich (2004), Baldomá and Fontich (2005). Some results about the validity of the prediction given by the Poincaré function for area preserving maps were given in Delshams and Ramírez-Ros (1998) and, for higher dimensional Hamiltonian systems, in Gallavotti (1994), Chierchia and Gallavotti (1994), Delshams et al. (1997, 2004), Gallavotti et al. (1999), Sauzin (2001). Finally, in a non-Hamiltonian setting, in Baldomá and Seara (2006), the splitting of a heteroclinic orbit for some degenerate unfoldings of the Hopf-zero singularity of vector fields in $\mathbb{R}^3$ was found.

The third group of results deal with what is called the *singular case*. In these cases, the splitting of separatrices is still exponentially small but its asymptotic value is no longer given by the Melnikov potential. The first paper that dealt with this kind of problem was the paper by Lazutkin (1984, 2003). There, the author studied the splitting of separatrices in the classical Chirikov standard map, and gave the main tools to obtain an asymptotic formula for it. The problem, in this case, can be approximated by an integrable flow, but there is not a good Melnikov potential which gives the asymptotic value of the area between the stable and unstable invariant curves.

Even if Lazutkin (1984) was not complete (the complete proof was achieved by Gelfreich 1999), the ideas in this paper have inspired most of the work in this area. As it is known by the experts in the subject, the detection of an exponentially small splitting relies on suitable complex parameterizations of the invariant manifolds. These parameterizations are analytic in a complex strip, whose size is limited by the singularities of the unperturbed homoclinic orbit. In the regular case, these manifolds are well approximated by the unperturbed homoclinic orbit even for complex values of the parameter. Hence, the prediction of the Melnikov potential, which is mainly an integral over the unperturbed homoclinic orbit, gives the main term of the distance.

However, in the singular case, one has to deal with different approximations of the invariant manifolds in different zones of the complex plane. Close to the singularities of the homoclinic orbit, an equation for the leading term is obtained and it is called the *inner equation*. It is a nonintegrable equation without parameters, which needs a deep study itself. Once this equation is solved, matching techniques are required to match the different approximations obtained for the invariant manifolds. Roughly speaking, the difference between the solutions of the inner equation replaces the Melnikov potential in the asymptotic formula for the splitting.

Following these ideas, some authors have obtained partial results for some specific equations. The study of the inner equation of the Hénon map, using the resurgence theory, can be found in Gelfreich and Sauzin (2001) and the inner system associated to the Hopf-zero singularity was studied in Baldomá and Seara (2008) using functional analysis techniques. For several periodically perturbed second order equations, Gelfreich stated in Gelfreich (1997b) the corresponding inner equation which he called the reference system. The analysis of the inner equation for the Hamilton–Jacobi equation associated to a system analogous to (1) was done in Olivé et al. (2003) using the resurgence theory, and for a wider class of second order equations in Baldomá (2006). Using the results of the inner equation, Gelfreich (2000) gave a detailed sketch of the proof for the splitting of separatrices for (1). Numerical results about the splitting for this problem can be found in Benseny and Olivé (1993), Gelfreich (1997b). The complete proof for the pendulum with a different perturbation was achieved in Olivé (2006). Following the same approach, Lombardi proved in Lombardi (2000) the splitting of separatrix connections for a certain class of reversible systems in $\mathbb{R}^4$. Finally, Treschev gave in Treschev (1997) an asymptotic formula for the splitting in the case of a pendulum with a moving suspension point using a different method called continuous averaging and a related problem about adiabatic invariants in the harmonic oscillator using matching techniques and the resurgence theory was done in Bonet et al. (1998).

As we have said before, most of the previous works adapted the ideas in Lazutkin (1984, 2003) to rigorously prove the asymptotic formula for the splitting in different settings. A fundamental tool in Lazutkin's work is the use of "flow box coordinates," called "straightening the flow" in Gelfreich (2000), around one of the manifolds. In this way, one obtains a periodic function whose values are related with the distance between the manifolds and whose zeros correspond to the intersections between them. Consequently, the result about exponentially small splitting is derived from some properties of analytic periodic functions bounded in complex strips (see, for instance, Proposition 2.7 in Delshams and Seara 1997).

A significantly different method was presented by Lochak, Marco, and Sauzin in the papers Sauzin (2001), Lochak et al. (2003). There, the authors were able to measure the distance between the manifolds (and all the related quantities like the angle, etc.) in the original variables of the problem without using "flow box coordinates." The authors used a very simple but clever idea: since the graphs of both manifolds are solutions of the same equation, their difference satisfies a linear equation and is bounded in some complex strip. Studying the properties of bounded solutions of this linear equation, where periodicity also plays a role, one obtains exponentially small results.

In the present paper, following the method of Sauzin (2001), Lochak et al. (2003), we obtain three main results. The main part of the paper is devoted to rigorously prove the asymptotic formula of the splitting of separatrices for (1) in the singular case ($|\mu| \leq \mathcal{O}(1)$) following the ideas given in Sauzin (2001), Olivé et al. (2003), Olivé (2006). The key idea is to look for the invariant manifolds as graphs of the differential of certain functions $S^{\pm}$. These functions, called generating functions, are solutions of the Hamilton–Jacobi equation associated to (1), which is a first order partial differential equation. In this way, the method to establish an asymptotic formula for the area of the lobes relies on computing the difference $S^+ - S^-$ in their common domain.

As we have already pointed out, the main difference of this approach from the previous ones, is the fact that we do not use "complex flow box variables" to obtain a good "splitting function" which measures the distance between both manifolds. Instead, we give a formula for the distance in the original parameterizations of the manifolds, working in the original variables of the problem. The main idea that was already used in Sauzin (2001), Olivé et al. (2003), Olivé (2006) is the fact that the difference $S^+ - S^-$ satisfies a linear partial differential equation whose solutions can be characterized and bounded by exponentially small terms in $\mathbb{R}$, provided they are bounded in a complex strip. The results obtained in this singular case coincide with Gelfreich (2000).

On the other hand, studying the manifolds in the original variables, one can see that they exist even if the parameter μ in (1) is big with respect to ε or when μ is finite but approaches some values of bifurcation μ_i, which correspond to the zeros of the zero order Bessel function of first type (see (10)). So, there is still the question of the size of the splitting of these manifolds in these cases that we call *below the singular case* and *close to a bifurcation case*. The rest of the paper is devoted to study the splitting of separatrices in these cases, that is, when $\mu = \varepsilon^p$ with $p < 0$ or $\mu = \mu_0 - c\varepsilon^r$ with $r > 0$ and $c > 0$, which as far as the authors know, had not been studied before.

As a first result, we state the existence of the periodic orbit for (1) for any μ, including the case $\mu = \varepsilon^p$ with $p < 0$, even if in this case it is no longer close to the hyperbolic critical point of the pendulum, but to the periodic solution of (3) for $\varepsilon = 0$. Later, for $\mu = \varepsilon^p$ with $p \in (-4, 0)$ or $\mu = \mu_0 - c\varepsilon^r$ with $r \in (0, 2)$ and $c > 0$, we state that this periodic orbit is still hyperbolic and we obtain an exponentially small upper bound for the distance between its global stable and unstable manifolds. Finally, we give some conjectures about the size of the splitting of separatrices when $\mu = \mu_0 - c\varepsilon^r$ with $r \geq 2$ and $c > 0$.

2 Main Results

The system associated to (2) has a hyperbolic fixed point at the origin. The first result in this paper is Theorem 2.1 where we prove that, if ε is small enough, (1) has a $2\pi\varepsilon$-periodic orbit for any value of μ, even if $\mu = \varepsilon^p$ for $p < 0$.

Theorem 2.1 *Let us consider* (1), *then there exists* $\varepsilon_0 > 0$ *such that*:

(i) *There exists a constant* $C_1 > 0$, *such that for any* $\mu > 0$ *and* $\varepsilon \in (0, \varepsilon_0)$, (1) *has a* $2\pi\varepsilon$*-periodic orbit* $x_{\mathrm{p}}(\frac{t}{\varepsilon})$, *which satisfies for* $t \in \mathbb{R}$:

$$\left| x_{\mathrm{p}}\left(\frac{t}{\varepsilon}\right) + \mu \sin \frac{t}{\varepsilon} \right| \leq C_1 \varepsilon^2.$$

(ii) *Moreover, for any fixed* $\bar{\mu} > 0$, *there exists a constant* $C_2 > 0$ *such that if* $\mu \in (0, \bar{\mu})$, $\varepsilon \in (0, \varepsilon_0)$ *and* $t \in \mathbb{R}$, x_{p} *satisfies the sharper bound*:

$$\left| x_{\mathrm{p}}\left(\frac{t}{\varepsilon}\right) + \mu \sin \frac{t}{\varepsilon} \right| \leq C_2 \mu \varepsilon^2.$$

This theorem is proved in Sect. 4.

Once we know the existence of the periodic orbit x_{p} for (1), the next step is to establish for which values of the parameter μ, taking ε small enough, x_{p} is hyperbolic as a solution of the corresponding system. When it is hyperbolic, we will study the existence of the stable and unstable manifolds associated to it and their possible intersection.

Classical perturbation theory applied to (1), gives a positive answer for these questions provided $\mu\varepsilon^{-2}$ is small enough because (1) is close to the classical pendulum. Nevertheless, to obtain the widest range of values of μ where the periodic orbit is still hyperbolic, we use that (1) is periodic in time with period $2\pi\varepsilon$. This is due to the fact that, in the fast periodic case, there is a natural method to obtain good autonomous approximations for this equation even if μ is not small: the averaging method. Performing several steps of the averaging procedure and looking for the critical point of the averaged system corresponding to our periodic orbit, we will be able to deduce the "true limit" of the splitting problem. This limit will occur when the averaged system is not the pendulum equation anymore and so its critical point can lose its hyperbolicity. We will see that it occurs when $\mu = \mathcal{O}(\varepsilon^p)$, with $p = -4$.

On the other hand, studying the averaged system, even when $\mu = \mathcal{O}(1)$, we will find values where the corresponding averaged system encounters bifurcations and then the critical point also loses its hyperbolicity. This phenomenon will give rise, as we explain in Sect. 2.4, to a new splitting problem.

2.1 Averaging Method: A Tool to Obtain the "the True Limit" of the Splitting Problem

In this section, we use averaging theory to obtain a good approximation, when ε is small but not μ, of the system associated to (1) (see Simó 1994; Gelfreich 1997b). To obtain this approximation, one can average several times. In fact, two steps of averaging give the change

$$x = \tilde{x} - \mu \sin\frac{t}{\varepsilon}, \qquad y = \tilde{y} - \frac{\mu}{\varepsilon}\cos\frac{t}{\varepsilon}, \tag{7}$$

which leads to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \tilde{y}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{y} = \sin\left(\tilde{x} - \mu\sin\dfrac{t}{\varepsilon}\right), \end{cases} \tag{8}$$

whose averaged system is

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \tilde{y}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{y} = J_0(\mu)\sin\tilde{x}, \end{cases} \tag{9}$$

where

$$J_0(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\mu\sin\tau)\,\mathrm{d}\tau \tag{10}$$

is the zero order Bessel function of first type (see Abramowitz and Stegun 1992).

In order to perform another step of averaging, we write system (8) as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x} = \tilde{y}, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{y} = J_0(\mu)\sin\tilde{x} + \sin\tilde{x}\left(\cos\left(\mu\sin\dfrac{t}{\varepsilon}\right) - J_0(\mu)\right) \\ \qquad\qquad - \cos\tilde{x}\sin\left(\mu\sin\dfrac{t}{\varepsilon}\right). \end{cases} \tag{11}$$

Later, in Sect. 2.5, we will study the behavior of this system for values of μ such that $J_0(\mu)=0$. We assume now that $J_0(\mu)>0$ (the case $J_0(\mu)<0$ can be done analogously), and we consider the scaling of time and variables

$$(\hat{x},\hat{y})=\big(\tilde{x},\tilde{y}/\sqrt{J_0(\mu)}\big),\qquad s=t\sqrt{J_0(\mu)}, \tag{12}$$

obtaining

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\hat{x}=\hat{y},\\ \dfrac{\mathrm{d}}{\mathrm{d}s}\hat{y}=\sin\hat{x}+\dfrac{1}{J_0(\mu)}\sin\hat{x}\left(\cos\left(\mu\sin\dfrac{s}{\delta}\right)-J_0(\mu)\right)\\ \qquad\quad -\dfrac{1}{J_0(\mu)}\cos\hat{x}\sin\left(\mu\sin\dfrac{s}{\delta}\right), \end{cases} \tag{13}$$

where $\delta=\varepsilon\sqrt{J_0(\mu)}$.

This system is a $2\pi\delta$-periodic perturbation of size $1/J_0(\mu)$ of the system associated to pendulum equation (2). We perform one step more of averaging in this new time, which corresponds to the change of variables

$$\begin{cases} \hat{x}=\bar{x},\\ \hat{y}=\bar{y}+\dfrac{\delta}{J_0(\mu)}h_1\left(\dfrac{s}{\delta}\right)\sin\bar{x}-\dfrac{\delta}{J_0(\mu)}h_2\left(\dfrac{s}{\delta}\right)\cos\bar{x}, \end{cases} \tag{14}$$

where $h_1(\tau)$ and $h_2(\tau)$ are the primitives of $f_1(\tau)=\cos(\mu\sin\tau)-J_0(\mu)$ and $f_2(\tau)=\sin(\mu\sin\tau)$ with zero average, and thus h_1 and h_2 are respectively an odd and an even function. Moreover, they depend on $\mu=\varepsilon^p$, but are of order 1 with respect to ε for $\tau\in\mathbb{R}$ even when $p<0$.

This change leads to

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s}\bar{x}=\bar{y}+\dfrac{\delta}{J_0(\mu)}\left(h_1\left(\dfrac{s}{\delta}\right)\sin\bar{x}-h_2\left(\dfrac{s}{\delta}\right)\cos\bar{x}\right),\\ \dfrac{\mathrm{d}}{\mathrm{d}s}\bar{y}=\sin\bar{x}-\dfrac{\delta}{J_0(\mu)}\bar{y}\left(h_1\left(\dfrac{s}{\delta}\right)\cos\bar{x}+h_2\left(\dfrac{s}{\delta}\right)\sin\bar{x}\right)\\ \qquad\quad +\left(\dfrac{\delta}{J_0(\mu)}\right)^2 m\left(\bar{x},\dfrac{s}{\delta}\right), \end{cases} \tag{15}$$

where

$$m(x,\tau)=-\big(h_1(\tau)\cos x+h_2(\tau)\sin x\big)\big(h_1(\tau)\sin x-h_2(\tau)\cos x\big). \tag{16}$$

Let us observe that for values of μ such that $J_0(\mu)<0$, an analogous procedure leads to a system whose first order is the pendulum with the x coordinate shifted by π (see Gelfreich 1997b).

Once we have averaged our system three times, we are going to discuss the true limit of the splitting problem, simply by studying the hyperbolicity of the periodic orbit of the different approximations of system (15), depending on the value of μ. It is clear that the main point will be the size of the function $J_0(\mu)$ which appears in systems (11) and (15), obtained after two or three steps of averaging, and the scaling (12), respectively.

The function $J_0(\mu)$ (see Abramowitz and Stegun 1992) has isolated zeros $\mu_0 < \mu_1 < \mu_2 < \cdots$ with $\mu_0 \simeq 2.404825558$ and tends to zero as $\mu \to +\infty$ as

$$J_0(\mu) \sim \sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4}\right) \quad \text{as } \mu \to +\infty. \tag{17}$$

For fixed $\mu \in (0, \mu_0)$, $J_0(\mu) > 0$ and system (15) is a small and fast perturbation of period $2\pi\delta = 2\pi\varepsilon\sqrt{J_0(\mu)}$ of the classical pendulum. Going back to the original variables of the problem and due to the scaling (12), we expect that, for fixed $\mu \in (0, \mu_0)$, the periodic orbit of system (4) will be hyperbolic and that its invariant manifolds will be close to the separatrix of system (9), which is the pendulum slightly modified by this coefficient $\sqrt{J_0(\mu)}$.

The same argument applies to any fixed μ belonging to any compact subset of (μ_{2i+1}, μ_{2i+2}).

Once we have understood the behavior of system (15) for any fixed value of μ for which $J_0(\mu) \neq 0$, next step is to study the case $J_0(\mu) \to 0$. This occurs when μ is close to a zero μ_i of the Bessel function $J_0(\mu)$ or when $\mu \to \infty$.

We first study this last case, and thus we take $\mu = \varepsilon^p$ with $p < 0$. Moreover, we restrict ourselves to values of ε such that μ belongs to compact intervals $I_i \subset (\mu_{2i+1}, \mu_{2i+2})$, in such a way that $\cos(\mu - \pi/4)$ has a positive lower bound independent of ε. In particular, $J_0(\mu) > 0$ and $J_0(\mu) = \mathcal{O}(\varepsilon^{-p/2})$.

Using (17), we have that $\delta/J_0(\mu) = \varepsilon/\sqrt{J_0(\mu)} \sim \varepsilon^{1+p/4}$. Therefore, if $\varepsilon^{1+p/4} \ll 1$ and $\mu = \varepsilon^p \in I_i$ for some $i \in \mathbb{N}$, system (15) is still a small and fast perturbation of period $2\pi\delta = \mathcal{O}(\varepsilon^{1-p/4})$ of the classical pendulum. Thus, considering $\mu = \varepsilon^p$ with $\mu \in I_i$ and $p \in (-4, 0)$, the periodic orbit is still hyperbolic and consequently it has local stable and unstable manifolds.

The question about the global behavior of these manifolds and their splitting is more involved. In Theorems 2.2 and 2.7, we state their existence and we give a bound of their distance to the separatrix of system (9). We also provide, in these theorems, an asymptotic formula for the splitting for $p \geq 0$ and an exponentially small upper bound for $p \in (-4, 0)$.

In Sect. 2.3, we conjecture the possible size of the splitting for $p \in (-4, 0)$ and give some ideas of how to prove it. Nevertheless, the authors think that the "true limit" of the splitting problem is given by $p = -4$, in the sense that for $p \leq -4$, system (15) is not a perturbation of the pendulum. In fact, if we look at the averaged system of system (15):

$$\left\{\begin{aligned} \frac{d\bar{x}}{ds} &= \bar{y}, \\ \frac{d\bar{y}}{ds} &= \sin\bar{x} + \left(\frac{\delta}{J_0(\mu)}\right)^2 \langle m\rangle(\bar{x}), \end{aligned}\right. \tag{18}$$

where $\langle m\rangle(x) = 1/(2\pi)\int_0^{2\pi} m(x,\tau)\,\mathrm{d}\tau$ is the average of m in (16) and is given by

$$\langle m\rangle(x) = \frac{1}{2}\big(\langle h_2^2\rangle - \langle h_1^2\rangle\big)\sin 2x,$$

and $p < -4$, the leading term of the averaged system will be given by $\langle m\rangle(\bar{x})$ instead of $\sin\bar{x}$.

Another case in which $J_0(\mu) \to 0$ is when $\mu \to \mu_i$ (being μ_i any zero of $J_0(\mu)$). Let us observe that μ_i are simple zeros, and then $J_0(\mu) \sim \mu - \mu_i$. If we focus on the first zero and we reach it from below, that is taking $\mu = \mu_0 - c\varepsilon^r$ with $c > 0$, we have that $J_0(\mu) > 0$. Now system (15) is a $2\pi\delta = \mathcal{O}(\varepsilon^{1+r/2})$-periodic perturbation of size $\delta/J_0(\mu) \sim \mathcal{O}(\varepsilon^{1-r/2})$ of the classical pendulum. Analogously to the results obtained for the *below the singular case* in Theorem 2.7, in Theorem 2.9 of Sect. 2.4 we state the existence of the invariant manifolds and we give an exponentially small upper bound of the size of their splitting provided $r < 2$. In Sect. 2.4, we also give some insight about the case μ closer to μ_0, that is, $\mu = \mu_0 - c\varepsilon^r$ with $r \geq 2$ and $c > 0$, which is analogous to the case $\mu = \varepsilon^p$ with $p \leq -4$. In this case, the leading term of the averaged system (18) is not the pendulum anymore, and thus the hyperbolic structure, and so the splitting changes drastically. The same argument applies to any other fixed zero μ_{2i} approached from below. To approach them from above, one has to take into account that $J_0(\mu) < 0$ and, therefore, the first order is the pendulum with the x coordinate shifted by π. For any fixed zero μ_{2i+1}, the arguments go the other way round.

For the rest of this section, we consider $\mu\varepsilon^4 \ll 1$ and $|\mu - \mu_i| \gg \varepsilon^2$ and we study the splitting between its stable and unstable manifolds. The results are split in three cases depending on μ. The results for μ belonging to any compact subset of $(0, \mu_0)$ are stated in Sect. 2.2. The asymptotic formula for the splitting of separatrices is given in the Analytic Theorem 2.2. Later, in the Geometric Theorem 2.6, we give an asymptotic formula for the splitting in terms of the area of the lobes between the invariant curves $C^{\mathrm{s}}(t_0)$ and $C^{\mathrm{u}}(t_0)$ in the Poincaré section (see Fig. 1).

Later, in Sect. 2.3, we consider the *below the singular case*, that is, $\mu = \varepsilon^p$ with $p \in (-4, 0)$ and, in Theorem 2.7, we provide exponentially small upper bounds for the distance between the invariant manifolds.

Finally, in Theorem 2.9 of Sect. 2.4, we give the result for $\mu = \mu_0 - c\varepsilon^r$ with $r \in (0, 2)$ and $c > 0$. Even if we focus in the first zero of $J_0(\mu)$, the results are analogous for all the zeros. The asymptotic size of the distance between the invariant manifolds in these two last cases is still an open problem.

2.2 The Classical Singular Case

From now on, we work with the fast time $\tau = t/\varepsilon$ and, therefore, we deal with system (4). This system has a periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$, where $x_{\mathrm{p}}'(\tau) = \varepsilon y_{\mathrm{p}}(\tau)$ and $x_{\mathrm{p}}(t/\varepsilon)$ is given in Theorem 2.1.

In order to study the invariant manifolds of this periodic orbit, as a first step, we consider the symplectic time-dependent change of variables,

$$\begin{cases} q = x - x_{\mathrm{p}}(\tau), \\ p = y - y_{\mathrm{p}}(\tau). \end{cases} \tag{19}$$

With this change, system (4) becomes a new Hamiltonian system with a (weakly) hyperbolic periodic orbit at the origin which has stable and unstable 2-dimensional invariant manifolds. Unlike the unperturbed system, these invariant manifolds do not coincide. On the other hand, since they are Lagrangian (see Lochak et al. 2003), they can be written locally as the graph of the differential of certain functions as $p = \partial_q S^{\pm}(q, \tau)$, where $S^{\pm}$ are usually called generating functions and satisfy the so-called Hamilton–Jacobi equation (see (32)). However, since for the unperturbed system they can be written as a graph globally, it is expected that the same happens for the perturbed case.

Moreover, the difference $\partial_q S^+(q, \tau) - \partial_q S^-(q, \tau)$ gives us the difference between the manifolds in the original coordinates.

Theorem 2.2 (Main theorem: analytic version) *Let μ_0 be the first zero of the Bessel function $J_0(\mu)$. Then for fixed $Q_0 \in (0, \pi/4)$ and $\bar{\mu} \in (0, \mu_0)$, there exists $\varepsilon_0 > 0$ (depending on $\bar{\mu}$) such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \bar{\mu})$:*

(i) *System (4) has a hyperbolic 2π-periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$. Moreover, it has stable and unstable invariant manifolds which can be parameterized as graphs*

$$y = y_{\mathrm{p}}(\tau) + \partial_q S^{\pm}\big(x - x_{\mathrm{p}}(\tau), \tau\big).$$

(ii) *Let $S_0(q, \tau) = S_0(q) = 4(1 - \cos(q/2))$ be the separatrix of system (4) for $\mu = 0$, which corresponds to the separatrix of the pendulum equation (2). Then writing $q = x - x_{\mathrm{p}}(\tau)$, the functions $S^{\pm}$ satisfy*

$$\partial_q S^{\pm}(q, \tau) = y_{\mathrm{p}}(\tau) + \sqrt{J_0(\mu)}\partial_q S_0(q, \tau) + \mathcal{O}\big(\mu\varepsilon^2\big),$$

for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$.

(iii) *For $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$,*

$$\begin{aligned} &\partial_q S^+(q, \tau) - \partial_q S^-(q, \tau) \\ &\quad = \frac{\mu}{\sqrt{J_0(\mu)}\varepsilon^2} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(f(\mu) \frac{\sin h(q, \tau)}{\sin(q/2)} + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)} \right) \right), \end{aligned} \tag{20}$$

where

$$h(q, \tau) = \tau - \big(\varepsilon\sqrt{J_0(\mu)}\big)^{-1} \ln\big(\tan(q/4)\big)$$

and $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$ is a real-analytic even function.

(iv) *There exists a constant $\overline{\alpha}(\mu, \varepsilon) = \mathcal{O}(\mu\varepsilon^2)$, such that*

$$\begin{aligned} &S^+(q, \tau) - S^-(q, \tau) - \overline{\alpha}(\mu, \varepsilon) \\ &\quad = \frac{2\mu}{\varepsilon} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(f(\mu) \cos h(q, \tau) + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)} \right) \right). \end{aligned} \tag{21}$$

Remark 2.3 The result presented in this theorem is valid for μ belonging to any compact set in which $J_0(\mu) > 0$. However, in order to simplify the notation and to make this result comparable to the previous ones (see Delshams and Seara 1992;

Gelfreich 1994) we restrict ourselves to $\mu \in (0, \bar{\mu}) \subset (0, \mu_0)$. When μ belongs to a compact set in which $J_0(\mu) < 0$, an analogous result can be obtained making little changes (see Gelfreich 1997b, 2000).

Remark 2.4 The exponent $\pi/(2\varepsilon\sqrt{J_0(\mu)})$ in the exponentially small term of the formulae (20) and (21) is the natural one if we take into account that the invariant manifolds of system (4) are also, up to the change (7) and the rescalings (12), the invariant manifolds of system (13). Indeed, applying Melnikov theory to this system, we would obtain the same exponent since the perturbation in system (13) is $2\pi\varepsilon\sqrt{J_0(\mu)}$-periodic in time.

Remark 2.5 Let us observe that formulas (20) and (21) of Theorem 2.2 give the main order of the difference between the functions $S^{\pm}$ and their derivatives provided $|f(\mu)|$ has a positive lower bound independent of ε. Since $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$, we know that $f(\mu) > 0$ for μ small enough.

From Theorem 2.2, one can easily derive asymptotic formulae for several geometric quantities related with the splitting of the invariant manifolds of system (4).

In next theorem, which is indeed a corollary of Theorem 2.2, we compute the area of the lobes delimited by the stable and unstable curves of the hyperbolic fixed point of the Poincaré map between to consecutive homoclinic points. The computation of the maximal distance between the stable and unstable manifolds, the splitting angle, or the Lazutkin invariant at any homoclinic point can be done in an analogous way.

Theorem 2.6 (Main theorem: geometric version) *Consider the hyperbolic fixed point of the Poincaré map of system* (4) *associated to the periodic orbit* $(x_p(\tau), y_p(\tau))$ *of system* (4). *Let* μ_0 *and* $\tilde{\mu}_0$ *be respectively the first zero of the Bessel function* $J_0(\mu)$ *in* (10) *and the function* $f(\mu)$ *given in Theorem* 2.2. *Then for any* $\bar{\mu} \in (0, \mu_0) \cap (0, \tilde{\mu}_0)$, *there exists* $\varepsilon_0 > 0$ (*depending on* $\bar{\mu}$) *such that for* $\varepsilon \in (0, \varepsilon_0)$ *and* $\mu \in (0, \bar{\mu})$, *the area of the lobes between the invariant stable and unstable curves associated to this point is given by the asymptotic formula*

$$\mathcal{A} = \frac{\mu}{\varepsilon} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(4\left|f(\mu)\right| + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right), \tag{22}$$

where $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$ *is the function given in Theorem* 2.2.

Proof The result follows almost directly from equality (21) of Theorem 2.2. The intersections of the invariant manifolds with a transversal section $\Sigma^{\tau_0} = \{(x, y, \tau_0),\ (x, y) \in \mathbb{R}^2\}$ are two different curves $C^{\mathrm{s,u}}(\tau_0)$, which can be parameterized by $\gamma^{\pm}(x) = (x, y_p(\tau_0) + \partial_q S^{\pm}(x - x_p(\tau_0), \tau_0))$. Then taking $q = x - x_p(\tau_0)$, the area is given by

$$\mathcal{A} = \left| \int_{q_0}^{q_1} \partial_q S^+(q, \tau_0) - \partial_q S^-(q, \tau_0)\,\mathrm{d}q \right| = \left| S^+(q, \tau_0) - S^-(q, \tau_0)\big|_{q_0}^{q_1} \right|, \tag{23}$$

where q_0 and q_1 correspond to two consecutive points of $C^{\mathrm{s}}(\tau_0) \cap C^{\mathrm{u}}(\tau_0) \subset \Sigma^{\tau_0}$, that is, $\partial_q S^+(q_i, \tau_0) = \partial_q S^-(q_i, \tau_0)$ for $i = 1, 2$.

To simplify the notation, we define

$$F_0(q,\tau) = \frac{2}{\varepsilon} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} f(\mu) \cos h(q,\tau). \tag{24}$$

Then by (21) and (23),

$$\mathcal{A} = \mu \left| F_0(q_1,\tau_0) - F_0(q_0,\tau_0) + (q_0 - q_1)\mathcal{O}\left(\frac{1}{\varepsilon \ln(1/\varepsilon)} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}}\right)\right|.$$

Using (20), we have that q_0 and q_1 are given by $h(q_i,\tau_0) = r_i$, where r_i satisfies

$$f(\mu)\sin r_i + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right) = 0,$$

and, therefore,

$$r_1 = 0 + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right) \quad \text{and} \quad r_2 = \pi + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right).$$

Their existence for ε small enough is obtained applying the implicit function theorem using that $1/f(\mu)$ is bounded for $\mu \in [0, \bar{\mu}]$.

Now,

$$\begin{aligned} \mathcal{A} &= \mu \left| \frac{2}{\varepsilon} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} f(\mu)(\cos r_1 - \cos r_0) + \mathcal{O}\left(\frac{1}{\varepsilon \ln(1/\varepsilon)} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}}\right)\right| \\ &= \frac{\mu}{\varepsilon} \mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(4\left|f(\mu)\right| + \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right). \end{aligned}$$

□

Comparing the result of Theorem 2.6 with the prediction of the Melnikov formula (6) and using that $J_0(\mu) = 1 + \mathcal{O}(\mu^2)$ for μ small (see Abramowitz and Stegun 1992), it is clear that both coincide provided $\mu = \varepsilon^p$ with $p > 1/2$.

For $\mu = \varepsilon^p$ with $p \in (0, 1/2]$, the result of Theorem 2.6 does not coincide with the Melnikov formula applied to system (4) but to system (13) modulo the rescaling (12).

Finally, for the classical singular case $\mu = \mathcal{O}(1)$, Melnikov theory fails to predict correctly the exponentially small splitting as is showed in Theorem 2.6.

2.3 Below the Singular Case

In the *below the singular case*, that is, $\mu = \varepsilon^p$ with $p \in (-4, 0)$, the next theorem states that the invariant manifolds of the periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$ of the Hamiltonian system (4) given in Theorem 2.1 are still close to the separatrix of the pendulum (9). Furthermore, it gives an exponentially small upper bound for their splitting.

Theorem 2.7 (Below the singular case) *Let μ_k be the kth zero of the Bessel function in* (10). *Then for fixed $Q_0 \in (0, \pi/4)$, $p \in (-4, 0)$, $\bar{a} > 0$ and $\bar{\gamma} \in [0, 1 + p/4)$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu = \varepsilon^p \in I_i \subsetneq (\mu_{2i+1}, \mu_{2i+2})$:*

(i) *The 2π-periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$ of system (4) is hyperbolic and its stable and unstable invariant manifolds can be parameterized as graphs*

$$y = y_{\mathrm{p}}(\tau) + \partial_q S^{\pm}\big(x - x_{\mathrm{p}}(\tau), \tau\big).$$

(ii) *Writing $x = x_{\mathrm{p}}(\tau) + q$, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$, the functions $S^{\pm}$ satisfy*

$$\partial_q S^{\pm}(q, \tau) = y_{\mathrm{p}}(\tau) + \sqrt{J_0(\mu)}\partial_q S_0(q, \tau) + \mathcal{O}\big(\varepsilon^{1+\frac{p}{4}} \ln(1/\varepsilon)\big),$$

where $S_0(q, \tau) = S_0(q) = 4(1 - \cos(q/2))$ is the separatrix of system (4) for $\mu = 0$, which corresponds to the separatrix of the pendulum equation (2).

(iii) *There exists a constant $\bar{C} > 0$ independent of ε such that, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$,*

$$\big|\partial_q S^{+}(q, \tau) - \partial_q S^{-}(q, \tau)\big| \leq \bar{C}\varepsilon^{2-4\bar{\gamma}} \mathrm{e}^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}(\frac{\pi}{2} - \bar{a}\varepsilon^{\bar{\gamma}})}.$$

Theorem 2.7 gives an exponentially small upper bound for the splitting of separatrices for system (4) when $\mu = \varepsilon^p$ and $p \in (-4, 0)$. Nevertheless, in this theorem, the exponent $\bar{\gamma}$, which appears in the term $\bar{a}\varepsilon^{\bar{\gamma}}$ that corrects the exponent $\pi/2$, can not be taken equal to $1 - p/4$, which corresponds to the size of the denominator $\varepsilon\sqrt{J_0(\mu)} = \mathcal{O}(\varepsilon^{1-p/4})$.

A sharper estimate for the splitting could be obtained working with system (15) instead of system (9) and looking for the stable and unstable manifolds as perturbations of the separatrix of its averaged system, which is given in (18). This system has a separatrix contained in the 0 level set of the Hamiltonian

$$\bar{H}(x, y) = \frac{y^2}{2} + \cos x - 1 + \frac{(\langle h_2^2\rangle - \langle h_1^2\rangle)\delta^2}{4(J_0(\mu))^2}(\cos 2x - 1),$$

which is given by $(x_{\mathrm{h}}(t), y_{\mathrm{h}}(t))$ with $\dot{x}_{\mathrm{h}}(t) = y_{\mathrm{h}}(t)$ and

$$1 - \cos x_{\mathrm{h}}(t) = \frac{8(1 + 2A)\mathrm{e}^{2t\sqrt{1+2A}}}{\mathrm{e}^{4t\sqrt{1+2A}} + 2\mathrm{e}^{2t\sqrt{1+2A}}(1 + 4A) + 1}, \tag{25}$$

where

$$A = \frac{(\langle h_1^2\rangle - \langle h_2^2\rangle)\delta^2}{2(J_0(\mu))^2}, \tag{26}$$

so that $A = \mathcal{O}(\varepsilon^{2+p/2})$.

One can see that the singularities of this separatrix closest to the real axis are given by

$$t = \frac{1}{\sqrt{1 + 2A}} \ln\big|1 + 4A \pm \sqrt{8A + 16A^2}\big| \pm \mathrm{i}\frac{\pi}{2\sqrt{1 + 2A}}.$$

Therefore, their imaginary parts are given by

$$\Im t = \pm\mathrm{i}\frac{\pi}{2\sqrt{1 + 2A}} = \pm\mathrm{i}\frac{\pi}{2} + \mathcal{O}\big(\varepsilon^{2+p/2}\big).$$

We recall that system (15) is a $2\pi\varepsilon\sqrt{J_0(\mu)} = O(\varepsilon^{1-p/4})$-periodic perturbation of system (18). Therefore, one can expect an upper bound of the distance between the perturbed invariant manifolds of order

$$\mathcal{O}\big(\varepsilon^{\beta}\mathrm{e}^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)(1+2A)}}}\big),$$

for some $\beta\in\mathbb{R}$. Consequently, a necessary condition to have bounds of the form $\mathcal{O}(\varepsilon^{\beta}\mathrm{e}^{-\pi/(2\varepsilon\sqrt{J_0})})$, as we had in the singular case, is that $2+p/2>1-p/4$. Then we expect that a similar formula to the one given in Theorem 2.2 can be obtained for $p>-4/3$. Nevertheless, for lower values of p the exponent in the asymptotic formula for the difference between the functions $S^{\pm}$ will have the exponent $-\pi/(2\varepsilon\sqrt{J_0(\mu)(1+2A)})$, so that the size of the splitting changes dramatically.

The method to prove these conjectures, could be to study the corresponding Hamilton–Jacobi equation associated to the system (15), and thus reparameterizing through the new homoclinic orbit (25) of the averaged system (18), and then looking for a suitable inner equation which gave the first order approximation of the invariant manifolds in the inner domain. Nevertheless, the main purpose of Theorem 2.7 is to illustrate that the splitting problem of the pendulum still has meaning for $\mu=\varepsilon^{p}$ with $p\in(-4,0)$, which is below the usually considered "limiting case" $p=0$.

2.4 Close to a Codimension-2 Bifurcation

Finally, we deal with the case in which μ is close to μ_0 where μ_0 is the first zero of the Bessel function, so that

$$J_0(\mu)\sim\mu-\mu_0.$$

We restrict ourselves to the case $\mu<\mu_0$ since the other one can be done analogously. Therefore, we take $\mu=\mu_0-c\varepsilon^{r}$ with $c>0$, and thus we have that $J_0(\mu)=\mathcal{O}(\varepsilon^{r})$. We can deal with this case as we have done with the *below the singular case* in Theorem 2.7: provided $r<2$, the system (15) is still close to the pendulum and, therefore, the invariant manifolds of the periodic orbit are close to the separatrix of the pendulum (9). Let us observe that now r plays the role of $-p/2$ in Sect. 2.3.

Remark 2.8 We want to point out that if we focus on system (11) instead of system (4), the study of the splitting of separatrices has interest in itself, since system (11) can be seen as a very simple toy model for two degrees of freedom Hamiltonian system close to a second order resonance. In fact, when one studies perturbations of a completely integrable Hamiltonian system close to a simple resonance and averages with respect to the fast frequency, at first order one obtains a pendulum-like structure of size square root of the size of the perturbation. Nevertheless, in some degenerate cases this first order vanishes in the resonance and, therefore, there is no impediment to perform another step of non-resonant averaging. In that case, we have a higher order resonance in which the hyperbolic structure is smaller (at least, same order as the size of the perturbation) as it is happening in this problem. Even if a more accurate model of a Hamiltonian system close to second order resonances should depend on the actions in a more complicated form, it would be interesting, as a first step, to study system (11) with μ close to μ_0.

Theorem 2.9 *Let μ_0 be the first zero of the Bessel function $J_0(\mu)$, then for fixed $Q_0 \in (0, \pi/4)$, $r \in (0, 2)$, $c > 0$, $\bar{a} > 0$ and $\bar{\gamma} \in [0, 1 - r/2)$, there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu = \mu_0 - c\varepsilon^r$:*

(i) *The 2π-periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$ of system (4) is hyperbolic and its stable and unstable invariant manifolds can be parameterized as graphs*

$$y = y_{\mathrm{p}}(\tau) + \partial_q S^{\pm}\big(x - x_{\mathrm{p}}(\tau), \tau\big).$$

(ii) *Writing $x = x_{\mathrm{p}}(\tau) + q$, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$, the invariant manifolds satisfy*

$$\partial_q S^{\pm}(q, \tau) = y_{\mathrm{p}}(\tau) + \sqrt{J_0(\mu)}\partial_q S_0(q, \tau) + \mathcal{O}\big(\varepsilon^{1-\frac{r}{2}} \ln(1/\varepsilon)\big),$$

where $S_0(q, \tau) = S_0(q) = 4(1 - \cos(q/2))$ is the separatrix of system (4) for $\mu = 0$, which corresponds to the separatrix of the pendulum equation (2).

(iii) *There exists a constant $\bar{C} > 0$ independent of ε such that, for $q \in (\pi - Q_0, \pi + Q_0)$ and $\tau \in \mathbb{R}$,*

$$\big|\partial_q S^{+}(q, \tau) - \partial_q S^{-}(q, \tau)\big| \leq \bar{C}\varepsilon^{2-4\bar{\gamma}} \mathrm{e}^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}(\frac{\pi}{2} - \bar{a}\varepsilon^{\bar{\gamma}})}.$$

Remark 2.10 A similar observation to the one made after Theorem 2.7 about the possible asymptotic formula of the splitting of separatrices applies also in this case. The exponent in the asymptotic formula for the difference between the functions $S^{\pm}$ should be $-\pi/(2\varepsilon\sqrt{J_0(\mu)(1 + 2A)})$ where A is the constant in (26) and, therefore, $A = \mathcal{O}(\varepsilon^{2-r})$. So, a similar asymptotic formula to the one in Theorem 2.2 can be expected provided $r \in (0, 2/3)$. Nevertheless, for $r \in (2/3, 2)$, the size of the splitting could change dramatically.

2.5 The Codimension-2 Bifurcation

The problem of considering μ closer to μ_0, namely $\mu = \mu_0 - c\varepsilon^r$ with $r \geq 2$ and $c > 0$, has to be studied in a different way. Indeed, the averaging procedure performed in Sect. 2.1 cannot be completed since the rescaling (12) is close to singular.

Now system (11) is better understood by considering it as being close to the corresponding one at the bifurcating value μ_0. For $\mu = \mu_0$, this system has zero average and period $2\pi\varepsilon$, and thus for $\mu = \mu_0 - c\varepsilon^r$ with $r \geq 2$ and $c > 0$, it is worth performing another step of averaging to understand its behavior. One step of averaging gives the change

$$\begin{cases} \tilde{x} = \hat{x}, \\ \tilde{y} = \hat{y} + \varepsilon h_1\left(\dfrac{t}{\varepsilon}\right)\sin\hat{x} - \varepsilon h_2\left(\dfrac{t}{\varepsilon}\right)\cos\hat{x}, \end{cases}$$

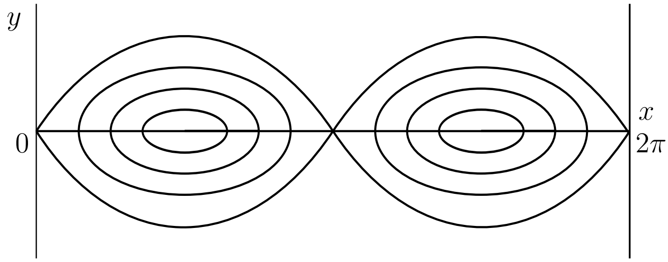


Fig. 2 Phase space of the averaged system (29)

which after the rescaling $\hat{y}=\varepsilon\bar{y}$ and $t=s/\varepsilon$, transforms system (11) into

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}s}=\bar{y}+h_1\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x}-h_2\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x},\\ \dfrac{\mathrm{d}\bar{y}}{\mathrm{d}s}=\langle m\rangle(\bar{x})+\dfrac{J_0(\mu)}{\varepsilon^2}\sin\bar{x}-\bar{y}\left(h_1\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x}+h_2\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x}\right)\\ \qquad +\left(m\left(\bar{x},\dfrac{s}{\varepsilon^2}\right)-\langle m\rangle(\bar{x})\right) \end{cases} \tag{27}$$

with h_1, h_2, and m the functions which appear in system (15). The averaged system of (27) is given by

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}s}=\bar{y},\\ \dfrac{\mathrm{d}\bar{y}}{\mathrm{d}s}=\langle m\rangle(\bar{x})+\dfrac{J_0(\mu)}{\varepsilon^2}\sin\bar{x}, \end{cases} \tag{28}$$

where $J_0(\mu)/\varepsilon^2=\mathcal{O}(\varepsilon^{r-2})$ and $\langle m\rangle(\bar{x})=\frac{1}{2}(\langle h_2^2\rangle-\langle h_1^2\rangle)\sin 2\bar{x}$, which is a Hamiltonian system with Hamiltonian

$$K(\bar{x},\bar{y})=\frac{\bar{y}^2}{2}+\frac{J_0(\mu)}{\varepsilon^2}(\cos\bar{x}-1)+\frac{1}{4}\big(\langle h_2^2\rangle-\langle h_1^2\rangle\big)(\cos 2\bar{x}-1).$$

Since $J_0(\mu_0)=0$, for $\mu=\mu_0$, the averaged system is

$$\begin{cases} \dfrac{\mathrm{d}\bar{x}}{\mathrm{d}s}=\bar{y},\\ \dfrac{\mathrm{d}\bar{y}}{\mathrm{d}s}=\langle m\rangle(\bar{x}), \end{cases} \tag{29}$$

which has a double-well potential. Moreover, it can be checked that $\langle h_2^2\rangle-\langle h_1^2\rangle\simeq 0.4298451>0$ for $\mu=\mu_0$ and, therefore, it has two hyperbolic critical points at $(0,0)$ and $(\pi,0)$ which are in the same energy level and are connected by four heteroclinic orbits and it has also two elliptic points at $(\pi/2,0)$ and $(3\pi/2,0)$ (see Fig. 2).

Taking now $\mu=\mu_0-c\varepsilon^r$ with $r>2$, $c>0$ and ε small enough, the averaged system (28) is a small perturbation of system (29). Therefore, this system has also two elliptic points close to $(\pi/2,0)$ and $(3\pi/2,0)$ and the two hyperbolic critical

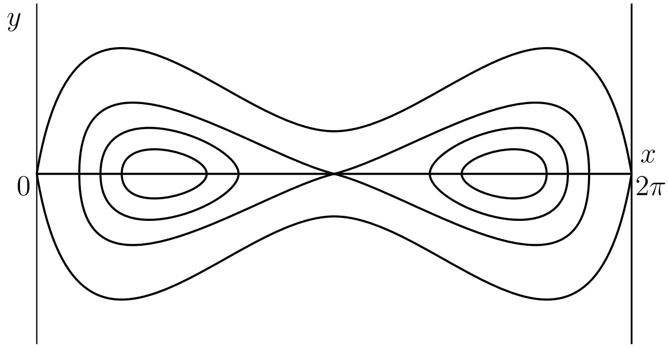


Fig. 3 Phase space of the averaged system (28) for ε small enough

points remain the same. Nevertheless, they belong to different levels of energy and, therefore, the heteroclinic connections bifurcate into four homoclinic orbits, as can be seen in Fig. 3: two of them forming an "eye" as in the classical pendulum given by $K(\bar{x},\bar{y})=0$ and the other two forming a figure eight around the elliptic points given by $K(\bar{x},\bar{y})=-2J_0(\mu)/\varepsilon^2$.

Between the dynamics observed for $\mu=\mu_0-c\varepsilon^r$ with $r>2$ and the one studied in Theorem 2.9, and thus for $r\in(0,2)$, there exists a codimension-1 bifurcation of the averaged system (28) which takes place at a curve in the (μ,ε) parameter space which is given by

$$\mu(\varepsilon)=\mu_0+\frac{\langle h_2^2\rangle-\langle h_1^2\rangle}{J_0'(\mu_0)}\varepsilon^2+\mathrm{o}\big(\varepsilon^2\big).$$

Since system (28) is reversible with respect to the involution $R(\bar{x},\bar{y},s)=(2\pi-\bar{x},\bar{y},-s)$, this is a pitchfork bifurcation, namely the two elliptic critical points and the hyperbolic critical point at $(\pi,0)$ merge together and become a parabolic point. At the same time, the figure eight homoclinic orbits shrink, also merging with the parabolic point. After the bifurcation, this point becomes elliptic, and thus we obtain again the classical picture of the pendulum equation.

Once we have studied the dynamics of the averaged integrable system (28), one could continue the study of the splitting of separatrices of $(0,0)$ for system (27) for $\mu=\mu_0-c\varepsilon^r$ with $r\geq 2$ and $c>0$. As in the case $r\in(2/3,2)$, the main point would be to consider the separatrix of system (28) as a first order approximation for the perturbed invariant manifolds. Moreover, for $r>2$, a new splitting of separatrices problem arises since in system (28) there exist also two separatrices of $(\pi,0)$. So, one can expect that system (27) has two exponentially small chaotic layers in the phase space around the four homoclinics of system (28).

Remark 2.11 Looking at the averaged system (28), it is straightforward to see that the existence of heteroclinic orbits is only possible on the line $\mu=\mu_0$. Nevertheless, the transversal splitting of the homoclinic orbits which exist away from $\mu=\mu_0$ for system (27) might imply the transversal intersection between the stable and the unstable invariant manifolds of the two hyperbolic points. However, since this splitting is exponentially small, it is expected that the region in the parameter space where these heteroclinic orbits could exist would be delimited by two curves emanating from $(\varepsilon, \mu) = (0, \mu_0)$ which are exponentially close (see Gelfreich and Naudot 2009 for a related problem). Away from this region, the two exponentially small chaotic zones seem to be enough separated to allow the existence of some invariant curves, as is indicated in Treschev (1998).

Remark 2.12 Let us observe that to study system (27) for $\mu \sim \mu_0$ with $\mu > \mu_0$, that is, taking for instance $\mu = \mu_0 + c\varepsilon^r$ with $c > 0$, one has to take into account that $J_0(\mu) < 0$ for $\mu \in (\mu_0, \mu_1)$ (see Gelfreich 1997b, 2000). Then, in particular, the pitchfork bifurcation takes place at the point $(0, 0)$ (instead of $(\pi, 0)$) along the curve

$$\mu(\varepsilon) = \mu_0 - \frac{\langle h_2^2 \rangle - \langle h_1^2 \rangle}{J_0'(\mu_0)}\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big).$$

2.6 Structure of the Paper

The rest of the paper is organized as follows. In Sect. 3, we write the invariant manifolds as graphs of functions $S^{\pm}(q, \tau)$ which are solutions of the Hamilton–Jacobi equation (32) associated to (4). We also give the complete description of the proof of Theorem 2.2. However, to make the paper more readable, we postpone the proofs of the theorems stated in Sect. 3 to Sects. 4, 5, 6, 7, and 8.

Section 4 is devoted to the proof of the first part of Theorem 2.1. The existence of stable and unstable manifolds and their first order approximations for $|\mu| \le \mathcal{O}(1)$ are established, in different domains, in Sects. 5 and 6. Firstly, in Sect. 5, we present results in the *outer domain*, a region of the complex plane up to a distance $\mathcal{O}(\varepsilon^\gamma)$, with $0 < \gamma < 1$, from $\pm i\pi/2$. These points are the singularities of the unperturbed homoclinic orbit closest to $\mathbb{R}$. In this domain, the manifolds are well approximated by the unperturbed homoclinic orbit. Secondly, in Sect. 6, we present the results in the *inner domain*, a region of the complex plane up to a distance $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ from the singularities. As it is shown in Sect. 3, in the inner region the manifolds are well approximated by the solutions of the so-called inner equation, which was studied in Olivé et al. (2003) and Baldomá (2006) and corresponds to the reference system in Gelfreich (1997a). This is done using complex matching techniques in the Hamilton–Jacobi equation.

In Sects. 7 and 8, we obtain a change of variables which allows us to have the difference between manifolds as a solution of a homogeneous linear partial differential equation with constant coefficients. Proceeding as in the preceding sections, first we obtain this change of variables in the outer domain and afterwards, using complex matching techniques, we extend it to the inner one. The final result follows from straightforward properties of linear partial differential equations with constant coefficients. This idea was already used in Sauzin (2001), Olivé et al. (2003), Baldomá (2006).

Finally, the results about the *below the singular case* given in Theorem 2.7 are proved in Sect. 9. In this case the manifolds, as well as the change of variables, are just studied in the outer domain where they are well approximated by the unperturbed homoclinic orbit. Consequently, we only obtain exponentially small upper bounds which are not sharp.

The proof of Theorem 2.9 will not be given since it is completely analogous to the one of Theorem 2.7 considering $p = -2r$ with $r < 2$.

3 Description of the Proof of Theorem 2.2

3.1 The Hamilton–Jacobi Equation

Even if Hamiltonian (5) is entire as a function of τ, we do not take advantage of it. Instead, we restrict ourselves to strips which make the proofs easier. We consider a complex strip around the torus: $\mathbb{T}_\sigma = \{\tau \in \mathbb{C}/2\pi\mathbb{Z}\colon |\Im\tau| \le \sigma\}$ with σ independent of ε and μ. In fact, since through the proof we have to reduce slightly the strip of analyticity, we consider $\sigma_1 > \sigma_2 > \cdots > \sigma_5 > 0$. Note that σ_1 will be given by Theorem 3.2 and will be independent of ε and μ and, therefore, we will be able to make this finite number of reductions.

Notation 3.1 From now on, in order to simplify the notation, if there is no danger of confusion, the dependence on μ or ε of all the functions will be omitted.

First, we state the existence of the periodic orbit in the complex extension of the torus.

Theorem 3.2 *There exists $\varepsilon_0 > 0$ such that for any $\bar\mu > 0$ there exists $\sigma_1 > 0$, such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \bar\mu)$, system (4) has a 2π-periodic orbit $(x_{\mathrm{p}}(\tau), y_{\mathrm{p}}(\tau))$: $\mathbb{T}_{\sigma_1} \to \mathbb{C}^2$ which is real-analytic and satisfies*

$$\sup_{\tau\in\mathbb{T}_{\sigma_1}} \left|x_{\mathrm{p}}(\tau) + \mu\sin\tau\right| \le b_0\mu\varepsilon^2, \tag{30}$$

where b_0 is a constant depending only on σ_1, $\bar\mu$ and ε_0.

The proof of this theorem, which is done in Fontich (1993) for μ small enough, is postponed to Sect. 4.

Remark 3.3 The strip of analyticity of the periodic orbit does not play any role in the results of this paper as it does in the case of a quasi-periodic perturbation (see Delshams et al. 1997). For this reason, even though it is possible to prove that the periodic orbit is analytic in a strip of size $\sigma_\varepsilon = \mathcal{O}(\ln(C/\mu))$ for a suitable constant $C > 0$ (see Seara and Villanueva 2000 for a related problem), we content ourselves to work in a strip of fixed width $\sigma_1 > 0$.

Once we know the existence of the hyperbolic periodic orbit by Theorem 3.2, with change (19), system (4) becomes a Hamiltonian system with Hamiltonian

$$K(q,p,\tau) = \varepsilon\left(\frac{p^2}{2} + (\cos q - 1)\cos x_{\mathrm{p}}(\tau) + (q - \sin q)\sin x_{\mathrm{p}}(\tau)\right) \tag{31}$$

which has a hyperbolic periodic orbit at the origin $p = q = 0$.

Since its local invariant manifolds are Lagrangian, they have a parameterization of the form $(q, \partial_q S^{\pm}(q,\tau))$, where $S^{\pm}$ are 2π-periodic in τ solutions of the nonautonomous Hamilton–Jacobi equation

$$\partial_\tau S + K\big(q, \partial_q S(q,\tau), \tau\big) = 0,$$

that is,

$$\partial_\tau S + \varepsilon\left(\frac{1}{2}(\partial_q S)^2 + (\cos q - 1)\cos x_{\mathrm{p}}(\tau) + (q - \sin q)\sin x_{\mathrm{p}}(\tau)\right) = 0, \tag{32}$$

with asymptotic conditions

$$\begin{cases} \lim\limits_{q\to 0} \partial_q S^-(q,\tau) = 0 & \text{(for the unstable manifold)}, \\ \lim\limits_{q\to 2\pi} \partial_q S^+(q,\tau) = 0 & \text{(for the stable manifold)}. \end{cases}$$

Note that the separatrix of the unperturbed system is given by $(q, \partial_q S_0(q))$ where $S_0(q) = 4(1 - \cos(q/2))$, which is a solution of (32) for $\mu = 0$.

Recalling that $x_{\mathrm{p}}(\tau) = -\mu \sin\tau + \mathcal{O}(\mu\varepsilon^2)$, (32) is approximated up to first order in ε by

$$\partial_\tau S + \varepsilon\left(\frac{1}{2}(\partial_q S)^2 + (\cos q - 1)\cos(\mu\sin\tau) - (q - \sin q)\sin(\mu\sin\tau)\right) = 0, \tag{33}$$

which is the Hamilton–Jacobi equation of the Hamiltonian

$$K_0(q,p,\tau) = \varepsilon\left(\frac{1}{2}p^2 + (\cos q - 1)\cos(\mu\sin\tau) - (q - \sin q)\sin(\mu\sin\tau)\right),$$

whose associated system of differential equations is

$$\begin{cases} q' = \varepsilon p, \\ p' = \varepsilon\big(\sin q\cos(\mu\sin\tau) + (1 - \cos q)\sin(\mu\sin\tau)\big). \end{cases}$$

Even if this system is a perturbation of the pendulum

$$\begin{cases} q' = \varepsilon p, \\ p' = \varepsilon \sin q, \end{cases} \tag{34}$$

for μ small enough with respect to ε, this is not true for general values of μ. On the other hand, as it is nonautonomous and periodic in time, averaging theory ensures that it is close to the averaged system

$$\begin{cases} q' = \varepsilon p, \\ p' = \varepsilon J_0(\mu)\sin q, \end{cases} \tag{35}$$

where $J_0(\mu)$ is the zero order Bessel function of first type defined in (10). Recall that when μ is small, $J_0(\mu)$ has the asymptotic expansion

$$J_0(\mu) = 1 - \frac{\mu^2}{4} + \mathcal{O}\big(\mu^4\big),$$

so that (35) is an $\mathcal{O}(\varepsilon^2)$-perturbation of the pendulum (34) provided $\mu^2 = \mathcal{O}(\varepsilon)$.

The Hamilton–Jacobi equation of (35) is

$$\frac{1}{2}(\partial_q S)^2 + J_0(\mu)(\cos q - 1) = 0.$$

Due to the coefficient $J_0(\mu)$, this equation does not correspond exactly to the Hamilton–Jacobi equation of the unperturbed pendulum (34). In fact, the solution of this equation is $\sqrt{J_0(\mu)}S_0(q) = S_0(q) + \mathcal{O}(\mu^2)$.

In order to have the classical pendulum as a first order approximation of (32), we perform the change

$$S(q,\tau) = \sqrt{J_0(\mu)} \cdot \bar{S}(q,\tau) \tag{36}$$

for $\mu \in (0, \mu_0)$, where μ_0 is the first zero of the Bessel function. This change is well defined since $J_0(\mu) > 0$ for $\mu \in (0, \mu_0)$ (see Remark 2.3). From now on, we consider μ belonging to compact sets $[0, \bar{\mu}_0]$ for $\bar{\mu}_0 < \mu_0$. Therefore, all the results which are stated in the rest of this section and Sects. 5 to 8 are only valid provided this condition holds.

In order to simplify the notation, from now on, we write J instead of $J_0(\mu)$. Moreover, we consider as a new small parameter

$$\delta = \varepsilon\sqrt{J},$$

so that the transformed Hamilton–Jacobi equation reads

$$\partial_\tau \bar{S} + \delta\left(\frac{1}{2}(\partial_q \bar{S})^2 + \frac{1}{J}(\cos q - 1)\cos x_{\mathrm{p}}(\tau) + \frac{1}{J}(q - \sin q)\sin x_{\mathrm{p}}(\tau)\right) = 0.$$

Remark 3.4 With this new parameter δ and taking $\mu \in (0, \bar{\mu}_0]$, Theorem 3.2 is still valid if we replace ε by δ, for $\delta < \delta_0 = \varepsilon_0/J$.

Following the idea given by Poincaré in Poincaré (1892–1899), we reparameterize the invariant manifolds with the time through the homoclinic orbit, namely

$$q = q_0(u) = 4\arctan\left(\mathrm{e}^u\right). \tag{37}$$

Taking

$$T^{\pm}(u,\tau) = \bar{S}^{\pm}\left(q_0(u), \tau\right) \tag{38}$$

and defining

$$\psi(u) = q_0(u) - \sin q_0(u), \tag{39}$$

we obtain that the stable and unstable manifolds $T^{\pm}(u,\tau)$ are 2π-periodic in τ solutions of the equation

$$\begin{aligned}\partial_\tau T(u,\tau)& \\ &+ \delta\left(\frac{\cosh^2 u}{8}\left(\partial_u T(u,\tau)\right)^2 - \frac{2}{J\cosh^2 u}\cos x_{\mathrm{p}}(\tau) + \frac{1}{J}\psi(u)\sin x_{\mathrm{p}}(\tau)\right) = 0,\end{aligned} \tag{40}$$

with asymptotic conditions

$$\begin{cases} \lim_{u\to-\infty} \cosh u \cdot \partial_u T^-(u,\tau) = 0, \\ \lim_{u\to+\infty} \cosh u \cdot \partial_u T^+(u,\tau) = 0. \end{cases} \tag{41}$$

In order to understand the behavior of $T^{\pm}$, we expand them formally in power series

$$T^{\pm}(u,\tau) \sim \sum_{k=0}^{\infty} \delta^k T_k^{\pm}(u,\tau), \tag{42}$$

where $T_k^{\pm}$ are 2π-periodic in τ and satisfy asymptotic conditions (41). Next proposition, whose proof is straightforward, gives the first terms in these asymptotic expansions.

Proposition 3.5 *Except for constant terms, the first terms in* (42) *are*

$$T_0(u) = 4\frac{\mathrm{e}^u}{\cosh u}, \tag{43}$$

$$T_1(u,\tau) = -\frac{2}{\cosh^2 u} f_1(\tau) + \psi(u) f_2(\tau), \tag{44}$$

where ψ is the function defined in (39) *and $f_1(\tau)$ and $f_2(\tau)$ are the primitives of $l_1(\tau) = (J - \cos(\mu\sin\tau))/J$ and $l_2(\tau) = \sin(\mu\sin\tau)/J$ which are 2π-periodic and have zero mean. Moreover, we remark that for $\tau \in \mathbb{T}_{\sigma_0}$, they satisfy*

$$\left|f_1(\tau)\right|, \left|f_2(\tau)\right| \le \mathcal{O}(\mu). \tag{45}$$

One can see that $T_k^+ = T_k^- = T_k$ for all $k \in \mathbb{N}$, so that the formal expansion $\tilde{T} = \sum_{k=0}^{\infty} \delta^k T_k$ is the same for both manifolds and is of Gevrey type (in suitable domains, there exist $M > 0$ and $\rho > 0$ such that $|T_k| < M\rho^k k!$, see Balser 1994). This fact, as it is well known, leads to the exponential smallness of the difference between T^+ and T^-.

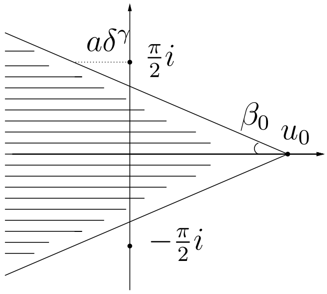
The first term of these asymptotic expansions, T_0, corresponds to the homoclinic orbit for the unperturbed problem, namely for $\mu = 0$. In other words, for the unperturbed system, the homoclinic orbit in the new variable u can be parameterized by $(u, \frac{\cosh u}{2}\partial_u T_0(u))$. This term has polar singularities at $u = \mathrm{i}\pi/2 + k\pi\mathrm{i}$ for $k \in \mathbb{Z}$ which propagate to all the terms in the series (42).

On the other hand, in order to compute the exponentially small splitting of the perturbed invariant manifolds $T^{\pm}(u,\tau)$, it is a crucial step to prove their existence for u in a complex strip as wide as possible. In fact, following the idea given by Lazutkin (2003), we study the invariant manifolds up to a distance of order $\mathcal{O}(\delta\ln(1/\delta))$ of the singularities $u = \pm\mathrm{i}\pi/2$, which are the closest to the real axis.

3.2 Existence and Approximation of the Invariant Manifolds in the Outer Domains

As a first step, we prove the existence of $T^{\pm}$ in certain domains that are called *outer domains*, which correspond to sectorial neighborhoods of $\pm\infty$ which are far from the

Fig. 4 The outer domain D^{u}_{γ}



singularities of $T_0(u)$, that is, at a distance $\mathcal{O}(\delta^\gamma)$ with $\gamma \in (0,1)$. In Sect. 3.3, we study the invariant manifolds when u is closer to $\pm i\pi/2$.

The outer domains, as it can be seen in Fig. 4, are given by

$$\begin{cases} D^{\mathrm{u}}_{\gamma} = \left\{ u \in \mathbb{C}\colon\ |\Im u| < -\dfrac{\pi}{2(u_0 + a\delta^\gamma)}\left(\Re u + a\delta^\gamma\right) + \dfrac{\pi}{2} \right\}, \\ D^{\mathrm{s}}_{\gamma} = \left\{ u \in \mathbb{C}\colon\ -u \in D^{\mathrm{u}}_{\gamma} \right\}, \end{cases} \tag{46}$$

where $a \in (0, \pi/4)$, $\gamma \in [0,1)$ and $u_0 > 0$ are fixed. Throughout the article, we will also consider the constant β_0 related to the slopes of the boundaries of the outer domains, which can be obtained from u_0, a and δ as

$$\beta_0 = \arctan\left(\frac{\pi}{2(u_0 + a\delta^\gamma)} \right) < \frac{\pi}{2}. \tag{47}$$

It is necessary to split these domains into two parts in order to study the behavior of the invariant manifolds whether $\Re u \to \pm\infty$ or near the singularities $u = \pm i\pi/2$. Thus, we fix real constants $\rho > 0$, small but independent of δ, and $U > u_0$, and we consider the domains (see Fig. 5),

$$D^{\mathrm{u,c}}_{\gamma} = \left\{ u \in D^{\mathrm{u}}_{\gamma} \mid \Re u > -U - \rho \right\},$$
$$\text{where “c” is written for close to the singularity,} \tag{48}$$
$$D^{\mathrm{u,f}}_{\gamma} = \left\{ u \in D^{\mathrm{u}}_{\gamma} \mid \Re u < -U + \rho \right\},$$
$$\text{where “f” is written for far from the singularity.} \tag{49}$$

Let us note that $D^{\mathrm{u,f}}_{\gamma} \cap D^{\mathrm{u,c}}_{\gamma} \neq \emptyset$ provided $\rho > 0$. We define analogously $D^{\mathrm{s,c}}_{\gamma}$ and $D^{\mathrm{s,f}}_{\gamma}$.

Moreover, for technical reasons, throughout the article, we have to reduce slightly the domains a finite number $N = 14$ of times. These reductions, which are called $D^{\mathrm{u}(i)}_{\gamma}$, $D^{\mathrm{u,c}(i)}_{\gamma}$, and $D^{\mathrm{u,f}(i)}_{\gamma}$ for $0 \le i \le N$, are done in such a way that they preserve the shape of the original ones, as can be seen in Fig. 6. We take parameters $0 = h_0 < \cdots < h_N < 1$. Then for fixed δ and γ, we consider the modified parameters

$$a^{(i)} = (1 + h_i)a,$$
$$u_0^{(i)} = u_0 - h_i a\delta^\gamma, \qquad u_0^{(0)} = u_0,$$
$$\rho^{(i)} = (1 - h_i)\rho,$$

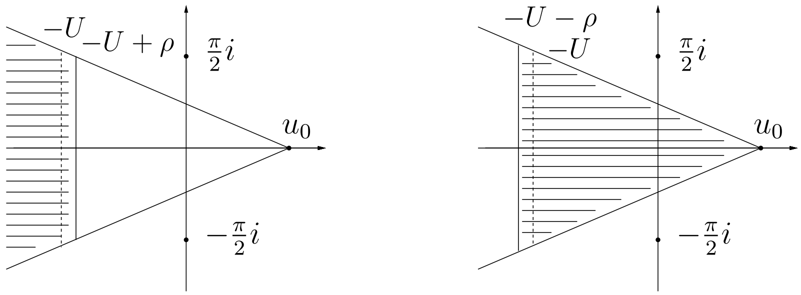


Fig. 5 The domain far from the singularity and close to the singularity: $D_\gamma^{\mathrm{u,f}}$, $D_\gamma^{\mathrm{u,c}}$

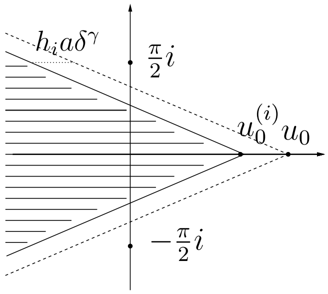


Fig. 6 Reduction of the outer domain, $D_\gamma^{\mathrm{u}(i)} \subset D_\gamma^{\mathrm{u}}$

in such a way that the new domains are defined as

$$D_\gamma^{\mathrm{u}(i)} = \left\{u \in \mathbb{C}\colon\ |\Im u| < -\tan\beta_0\left(\Re u + a^{(i)}\delta^\gamma\right) + \pi/2\right\} \tag{50}$$

and analogously for $D_\gamma^{*,\mathrm{c}(i)}$ and $D_\gamma^{*,\mathrm{f}(i)}$, where $*$ denotes either s or u.

With these reductions, writing $*$ for either s or u, we obtain the following inclusions:

$$D_\gamma^{*(N)} \subset \cdots \subset D_\gamma^{*(1)} \subset D_\gamma^{*(0)}.$$

Moreover, for $u \in D_\gamma^{\mathrm{u}(i)}$,

$$\mathrm{d}\left(u, \partial D_\gamma^{\mathrm{u}(i-1)}\right) > \mathcal{O}\left(\delta^\gamma\right). \tag{51}$$

Observe that these properties also hold for $D_\gamma^{*,\mathrm{f}(i)}$ and $D_\gamma^{*,\mathrm{c}(i)}$.

The next theorem gives the existence and the asymptotic expressions of the invariant manifolds in the outer domains.

Theorem 3.6 *Let σ_1, $\bar\mu_0$ and γ be real numbers such that $0 < \sigma_1 < \sigma_0$, $0 < \bar\mu_0 < \mu_0$ and $\gamma \in [0,1)$. Then there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $\mu \in (0, \bar\mu_0]$, (40) has unique (module an additive constant) solutions in $D_\gamma^{\mathrm{u}(1)} \times \mathbb{T}_{\sigma_1}$ and $D_\gamma^{\mathrm{s}(1)} \times \mathbb{T}_{\sigma_1}$ of the form*

$$T^\pm(u,\tau) = T_0(u) + \delta T_1(u,\tau) + Q^\pm(u,\tau)$$

satisfying asymptotic conditions (41).

Moreover, there exists a real constant $b_1 > 0$ *independent of* δ *and* μ, *such that*

$$\left|\partial_u^{i+1} Q^-(u,\tau)\right| \le \begin{cases} \frac{b_1\mu\delta^{2-i\gamma}}{|\cosh^4 u|} & \textit{for } (u,\tau)\in D_\gamma^{\mathrm{u,c}(1)}\times\mathbb{T}_{\sigma_1}, \\ \frac{b_1\mu\delta^{2-i\gamma}}{|\cosh^2 u|} & \textit{for } (u,\tau)\in D_\gamma^{\mathrm{u,f}(1)}\times\mathbb{T}_{\sigma_1} \end{cases}$$

for $i = 0, 1$. *Analogous bounds for* Q^+ *also hold.*

This theorem is proved in Sect. 5. It shows that the invariant manifolds in the outer domains are well approximated by the unperturbed separatrix. In fact, using (44), as a consequence of this theorem we have that,

$$\left|\partial_u T^\pm(u,\tau) - \partial_u T_0(u)\right| \le \frac{1}{|\cosh^3 u|}\mathcal{O}(\mu\delta)$$

for $u \in D_\gamma^{\mathrm{u,c}(1)}$. However, in the proofs of Theorems 3.9 and 3.11, it will be convenient to have explicitly the order δ term T_1 in the expansion of $T^\pm$.

3.3 Existence and Approximation of the Invariant Manifolds in the Inner Domains

In order to prove the exponentially small difference between T^+ and T^-, it is necessary to obtain good approximations of the invariant manifolds close to the singularities $u = \pm\mathrm{i}\pi/2$. In fact, from Theorem 3.6 and formulas (43) and (44), near these singularities the invariant stable and unstable manifolds grow considerably. In order to study the unstable manifold T^- close to the singularities $\pm\mathrm{i}\pi/2$, following Lazutkin (1984), we extend the unstable outer domain to the inner domain defined as

$$\begin{aligned} D_{\delta,+}^{\mathrm{u}} = \big\{u\in\mathbb{C}\colon\ &\Im u > -\tan\beta_0\big(\Re u + \tilde a\delta^\gamma\big) + \pi/2,\ \Im u < \pi/2 - c\delta\ln(1/\delta), \\ &\Im u < -\tan\beta_1\Re u + \pi/2 - c\delta\ln(1/\delta)\big\}, \\ D_{\delta,-}^{\mathrm{u}} = \big\{u\in\mathbb{C}\colon\ &\bar u\in D_{\delta,+}^{\mathrm{u}}\big\}, \end{aligned} \tag{52}$$

where β_0 is the parameter defined in (47), c is a parameter that we choose such that $0 < c < 1$ and $\beta_1 > \beta_0$ is taken such that $\beta_1 - \beta_0 = \mathcal{O}(1)$ (see Fig. 7).

As for the outer domain, the inner one is reduced, obtaining

$$\begin{aligned} D_{\delta,+}^{\mathrm{u}(i)} = \big\{u\in\mathbb{C}\colon\ &\Im u > -\tan\beta_0\big(\Re u + \tilde a^{(i)}\delta^\gamma\big) + \pi/2,\ \Im u < \pi/2 - c^{(i)}\delta\ln(1/\delta), \\ &\Im u < -\tan\beta_1\Re u + \pi/2 - c^{(i)}\delta\ln(1/\delta)\big\}, \end{aligned} \tag{53}$$

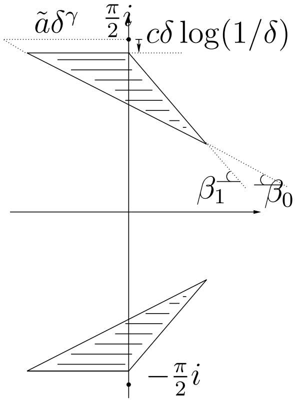
where

$$\begin{aligned} \tilde a^{(i)} &= (1-h_i)\tilde a, \\ c^{(i)} &= (1+h_i)c \end{aligned} \tag{54}$$

and hence

$$c^{(i)} < 2 \quad \text{for } i = 1,\dots,N. \tag{55}$$

Fig. 7 The inner domain for T^-, $D^{\mathrm{u}}_{\delta,+} \cup D^{\mathrm{u}}_{\delta,-}$



For the other domains the reduction is done analogously. Moreover, this reduction guarantees that for $u \in D^{*(i)}_{\delta,\pm}$ it holds that

$$\mathrm{d}\big(u, \partial D^{*(i-1)}_{\delta,\pm}\big) > \mathcal{O}\big(\delta \ln(1/\delta)\big) \tag{56}$$

and

$$D^{*(N)}_{\delta,\pm} \subset \cdots \subset D^{*(1)}_{\delta,\pm} \subset D^{*(0)}_{\delta,\pm}.$$

Furthermore, since we need an overlapping domain between the inner and outer domains, we take $\tilde{a} > a$ in (46), such that for all i, j with $1 \leq i, j \leq N$, $D^{\mathrm{u}(i)}_{\delta,\pm} \cap D^{\mathrm{u}(j)}_{\gamma} \neq \emptyset$.

Since the behavior near both singularities is analogous, we deal only with the proof for $D^{\mathrm{u}(i)}_{\delta,+}$. In order to study the unstable manifold in this domain, we consider the change

$$z = \delta^{-1}(u - \mathrm{i}\pi/2). \tag{57}$$

The variable z is called the *inner variable*, in contraposition to the *outer variable* u. Formulas (43) and (44) suggest the change

$$\phi^-(z, \tau) = \delta T^-(\delta z + \mathrm{i}\pi/2, \tau). \tag{58}$$

We will prove the existence of ϕ^- in

$$\begin{aligned}\mathcal{D}^{\mathrm{u}(i)}_{\delta,+} = \big\{z \in \mathbb{C} \colon\ & \Im z > -\tan\beta_0\big(\Re z + \tilde{a}^{(i)}\delta^{\gamma-1}\big),\ \Im z < -c^{(i)}\ln(1/\delta), \\ & \Im z < -\tan\beta_1 \Re z - c^{(i)}\ln(1/\delta)\big\}\end{aligned} \tag{59}$$

which corresponds to $D^{\mathrm{u}(i)}_{\delta,+}$ expressed in the inner variable. Let us observe that for $z \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+}$,

$$\mathrm{d}\big(z, \partial\mathcal{D}^{\mathrm{u}(i-1)}_{\delta,+}\big) > \mathcal{O}\big(\ln(1/\delta)\big). \tag{60}$$

By (40), the unstable manifold in the inner variables, $\phi^-(z,\tau)$, has to satisfy

$$\partial_\tau\phi+\frac{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}{8\delta^2}(\partial_z\phi)^2-\frac{2\delta^2}{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}\frac{\cos x_{\mathrm{p}}(\tau)}{J} + \delta^2\psi\left(\mathrm{i}\frac{\pi}{2}+\delta z\right)\frac{\sin x_{\mathrm{p}}(\tau)}{J}=0, \tag{61}$$

where ψ is the function defined in (39).

Using (43), (44), and Theorem 3.6, one expects that for $\Re z<0$, $\Im z<0$ with $|\delta z|\ll 1$, the following equality holds

$$\begin{aligned}\phi^-(z,\tau)&=\delta T_0\left(\mathrm{i}\frac{\pi}{2}+\delta z\right)+\delta^2 T_1\left(\mathrm{i}\frac{\pi}{2}+\delta z,\tau\right)+\cdots\\&=\frac{4}{z}+\mathcal{O}\left(\frac{1}{z^2}\right).\end{aligned}$$

So, we look for the solution of (61) which is 2π-periodic in τ and satisfies the asymptotic condition

$$\lim_{\Re z\to-\infty}\phi^-(z,\tau)=0. \tag{62}$$

Analogously, for the stable manifold, we define the domains $D^{\mathrm{s}(i)}_{\delta,\pm}$ and $\mathcal{D}^{\mathrm{s}(i)}_{\delta,\pm}$, and the function ϕ^+ defined in $\mathcal{D}^{\mathrm{s}(i)}_{\delta,+}$ which has to satisfy (61) and

$$\lim_{\Re z\to+\infty}\phi^+(z,\tau)=0. \tag{63}$$

As a first step, we have to study (61) for the first order terms in δ. Taking $\delta=0$, we obtain the so-called *inner equation*

$$\partial_\tau\phi_0-\frac{z^2}{8}(\partial_z\phi_0)^2+\frac{2}{Jz^2}\big(\cos(\mu\sin\tau)+\mathrm{i}\sin(\mu\sin\tau)\big)=0 \tag{64}$$

and we look for 2π-periodic in τ solutions ϕ_0^+ and ϕ_0^- satisfying asymptotic conditions (63) and (62), defined for z in domains of the form

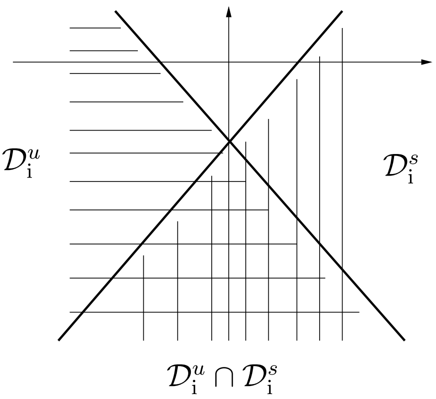
$$\begin{aligned}\mathcal{D}^{\mathrm{u}}_{\mathrm{i}}&=\{z\in\mathbb{C}\colon\ \Im z<-\tan\beta_1\Re z-d\},\\ \mathcal{D}^{\mathrm{s}}_{\mathrm{i}}&=\{z\in\mathbb{C}\colon\ \Im z<\tan\beta_1\Re z-d\},\end{aligned}$$

where β_1 was introduced to define the domains $D^{\mathrm{u,s}}_{\delta,\pm}$ in (52) and $d>0$ is a real constant (see Fig. 8).

Remark 3.7 Let us observe that the inner equation (64) is the Hamilton–Jacobi equation of the Hamiltonian system with Hamiltonian

$$\mathcal{H}(z,\omega,\tau)=-\frac{z^2}{8}w^2+\frac{2}{Jz^2}\big(\cos(\mu\sin\tau)+\mathrm{i}\sin(\mu\sin\tau)\big).$$

Fig. 8 The inner domain for ϕ_0^+ and ϕ_0^-, $\mathcal{D}_{\mathrm{i}}^{\mathrm{s,u}}$



The dynamical system associated to this Hamiltonian was called *reference system* in Gelfreich (1997b).

Theorem 3.8 *The following statements hold*:

- *For each* $\mu \in \mathbb{C}$, (64) *has a unique formal solution*

$$\tilde{\phi}_0(z,\tau) = \sum_{n\geq 0} \frac{c_n(\tau)}{z^{n+1}}, \tag{65}$$

where $c_0 = 4$, $c_1(\tau) = 2(f_1(\tau) - \mathrm{i} f_2(\tau))$ (*see Proposition* 3.5) *and* $c_n(\tau)$ *are* 2π-*periodic entire functions*.

- *Let us consider the constant* σ_1 *defined in Theorem* 3.6 *and any* $\bar{\mu} > 0$. *Then there exists a constant* $d > 0$ *such that*, *for* $\mu \in (0, \bar{\mu})$, (64) *has unique* 2π-*periodic in* τ *solutions* $\phi_0^{\pm}(z,\tau)$ *asymptotic to* $\tilde{\phi}_0$ *in the corresponding domains* $(z,\tau) \in \mathcal{D}_{\mathrm{i}}^{\mathrm{u,s}} \times \mathbb{T}_{\sigma_1}$. *In particular*, *they satisfy the asymptotic conditions* (63) *and* (62), *respectively.*

  Furthermore, *their difference is exponentially small. There exists a constant* $b_2 > 0$, *such that*, *for* $z \in \mathcal{D}_{\mathrm{i}}^{\mathrm{s}} \cap \mathcal{D}_{\mathrm{i}}^{\mathrm{u}}$, $\tau \in \mathbb{T}_{\sigma_1}$ *and* $\mu \in (0, \bar{\mu})$:

$$\left|\phi_0^+(z,\tau) - \phi_0^-(z,\tau) - \mu f(\mu) \mathrm{e}^{-\mathrm{i}(z-\tau)}\right| \leq b_2 \mu \left(|z|^{-1} \mathrm{e}^{\Im z} + \mathrm{e}^{a_1 \Im z}\right), \tag{66}$$

where f *is a real-analytic even function which satisfies* $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$ *for* μ *small and* a_1 *is any constant such that* $a_1 \in (1,2)$.

- *There exists a change of variables defined in* $\mathcal{D}_{\mathrm{i}}^{\mathrm{u}} \times \mathbb{T}_{\sigma_1}$

$$z = x + R^-(x,\tau) \tag{67}$$

which conjugates

$$\bar{\mathcal{L}} = \partial_\tau - \frac{1}{4} z^2 \partial_z \phi_0^-(z,\tau) \partial_z \tag{68}$$

with

$$\mathcal{L} = \partial_\tau + \partial_x. \tag{69}$$

Moreover, there exists $\delta_0 > 0$, *such that for* $\delta \in (0, \delta_0)$, $\mathcal{D}^{\mathrm{u}(i)}_{\delta,+} \subset \mathcal{D}^{\mathrm{u}}_{\mathrm{i}}$ *for* $i = 0, \ldots, N$ *and*:

(i) *If* $(x, \tau) \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$, $z = x + R^-(x, \tau) \in \mathcal{D}^{\mathrm{u}(i-1)}_{\delta,+}$ *for* $i = 1, \ldots, N$.

(ii) *There exists a constant* $b_3 > 0$, *such that for* $(x, \tau) \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$ *with* $i = 1, \ldots, N$, *it satisfies*

$$\left|\partial_x^j R^-(x, \tau)\right| \le \frac{b_3}{|x|^{j+1}} \quad \textit{for } j = 0, 1, 2.$$

- *There exists the inverse change of variables* $x = z + S^-(z, \tau)$ *which holds*:

(i) *If* $(z, \tau) \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$, $x = z + S^-(z, \tau) \in \mathcal{D}^{\mathrm{u}(i-1)}_{\delta,+}$ *for* $i = 1, \ldots, N$.

(ii) *There exists a constant* $b_4 > 0$, *such that for* $(z, \tau) \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$ *with* $i = 1, \ldots, N$, *it holds*

$$\left|\partial_z^j S^-(z, \tau)\right| \le \frac{b_4}{|z|^{j+1}} \quad \textit{for } j = 0, 1, 2.$$

Outline of the proof The paper Olivé et al. (2003) provides a complete and detailed proof of the results of this theorem in the case $\phi_0^\pm$ are solutions of the equation

$$\partial_\tau \phi_0 - \frac{z^2}{8}(\partial_z \phi_0)^2 + \frac{2}{z^2}(1 - \mu \sin \tau) = 0, \tag{70}$$

which corresponds to the inner equation of

$$x'' = \varepsilon^2 \sin x + \mu \varepsilon^2 \sin x \sin \tau.$$

Let us observe that both (64) and (70) are of the form

$$\partial_\tau \phi_0 - \frac{z^2}{8}(\partial_z \phi_0)^2 + \frac{1}{z^2} P_\mu(\tau) = 0,$$

where $P_\mu(\tau)$ is an entire 2π-periodic function.

As it is stated in Remark 1 of that paper, the proof of these results for general 2π-periodic functions $P_\mu(\tau)$ can be handled by the same method with little effort. For this reason, we will just give the main ideas of the proof and we refer the reader to that paper to check the technical details.

The proof in Olivé et al. (2003) is done using resurgence theory of Écalle (Écalle 1981a, 1981b; Candelpergher et al. 1993).

To stress the symmetries of the solutions and of function $f(\mu)$ in formula (66), we will write the dependence on μ of all the functions which appear in this sketch.

First, we look for a formal solution

$$\tilde{\phi}_0(z, \tau, \mu) = \sum_{n \ge 0} \frac{c_n(\tau, \mu)}{z^{n+1}}$$

of (64) by inserting it in the equation and obtaining a recurrence relation for the coefficients $c_n(\tau, \mu)$. This recurrence can be easily solved by entire periodic functions and are unique provided $c_0 = 4$ (see Lemma 1 of Olivé et al. 2003). Moreover, as $P_\mu(\tau) = P_\mu(\pi - \tau)$, this formal solution has the symmetry

$$\tilde{\phi}_0(-z, \pi - \tau, \mu) = -\tilde{\phi}_0(z, \tau, \mu).$$

On the other hand, since it also satisfies $P_\mu(\tau) = P_{-\mu}(\tau - \pi)$, the formal solution has also the symmetry

$$\tilde{\phi}_0(z, \tau, -\mu) = \tilde{\phi}_0(z, \pi + \tau, \mu).$$

This last symmetry is the cause of the evenness of function $f(\mu)$ in formula (66).

Once we have the formal solution $\tilde{\phi}_0$, the solutions $\phi_0^{\pm}$ are obtained using the Borel resummation method.

First, one considers the Borel transform of $\tilde{\phi}_0(z, \tau, \mu)$ with respect to the variable z, namely

$$\hat{\phi}_0(\zeta, \tau, \mu) = \sum_{n \geq 0} c_n(\tau, \mu) \frac{\zeta^n}{n!},$$

which satisfies:

$$\hat{\phi}_0(-\zeta, \pi - \tau, \mu) = \hat{\phi}_0(\zeta, \tau, \mu), \tag{71}$$

$$\hat{\phi}_0(\zeta, \tau, -\mu) = \hat{\phi}_0(\zeta, \pi + \tau, \mu). \tag{72}$$

The first result, which concerns $\hat{\phi}_0(\zeta, \tau, \mu)$ is that it converges when ζ is near the origin (uniformly in μ and τ) and defines a holomorphic function of ζ with analytic continuation along any path of $\mathbb{C}$ which starts from the origin and avoids $\mathrm{i}\mathbb{Z}$. This fact is proved studying the Borel transform $\mathcal{B}$ of (64), which using that

$$\mathcal{B}(\tilde{\phi}\, \tilde{\psi}) = \mathcal{B}(\tilde{\phi}) * \mathcal{B}(\tilde{\psi}),$$

where

$$(\hat{\phi} * \hat{\psi})(\zeta) = \int_0^\zeta \hat{\phi}(\zeta_1) \hat{\psi}(\zeta - \zeta_1)\, \mathrm{d}\zeta_1$$

and that $\mathcal{B}(z \partial_z \tilde{\phi}) = -\partial_\zeta(\zeta \hat{\phi})$, reads:

$$\partial_\tau \hat{\phi}_0 - \frac{1}{8}(\hat{\phi}_0 + \zeta\, \partial_\zeta \hat{\phi}_0)^{*2} + \frac{2\zeta}{J}\big(\cos(\mu \sin\tau) + \mathrm{i} \sin(\mu \sin\tau)\big) = 0, \tag{73}$$

where

$$\hat{\phi}^{*2} = \hat{\phi} * \hat{\phi}.$$

Studying this equation, one can see that the only possible singularities of the analytic continuation of $\hat{\phi}_0$ lie on $\mathrm{i}\mathbb{Z}$. Considering the Riemann surface $\mathcal{R}$ consisting of all homotopy classes of paths issuing from the origin and lying on $\mathbb{C} \setminus \mathrm{i}\mathbb{Z}$ (except for their origin), one can see that $\hat{\phi}_0$ is holomorphic in $\mathcal{R}^0$, the main sheet of $\mathcal{R}$, which consists on points ζ that can be represented by a straight segment $[0, \zeta]$. And also in $\mathcal{R}^1$, which consists on the union of $\mathcal{R}^0$ and of the "nearby half-sheets": this is the subset of $\mathcal{R}$ consisting of the homotopy classes of paths issuing from 0 and lying in $\mathbb{C}\setminus \mathrm{i}\mathbb{Z}$ but crossing the imaginary axis at most once. These results are contained in Theorem 1 of the paper.

Once we know that the function $\hat{\phi}_0$ can be extended to $\mathbb{C}\setminus \mathrm{i}\mathbb{Z}$, one can define its Laplace transform along straight lines avoiding $\mathrm{i}\mathbb{Z}$:

$$\mathcal{L}^\theta(\hat{\phi}_0)(z,\tau,\mu)=\int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty}\mathrm{e}^{-z\zeta}\hat{\phi}_0(\zeta,\tau,\mu)\,\mathrm{d}\zeta.$$

By Cauchy's theorem, this process defines two analytic functions $\phi_0^\pm$ in suitable domains $\mathcal{D}^\pm$, taking $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ in the case of ϕ_0^+ and $\theta\in(\frac{\pi}{2},\frac{3\pi}{2})$ in the case of ϕ_0^-.

An analogous process gives us information about the solutions of the "formal" linearized equation around $\tilde{\phi}_0$,

$$\partial_\tau Y-\frac{1}{4}z^2\partial_z\tilde{\phi}_0(z,\tau,\mu)\partial_z Y=0,$$

and about the solutions of the linearized equation around ϕ_0^-,

$$\partial_\tau Y-\frac{1}{4}z^2\partial_z\phi_0^-(z,\tau,\mu)\partial_z Y=0.$$

One can first find a "formal solution" of the formal linearized equation (see Lemma 3 in Olivé et al. 2003) of the form $\tilde{Y}(z,\tau,\mu)=z-\tau+\tilde{S}(z,\tau,\mu)$ and then using the Borel resummation method, one can obtain the analytic solution $S^-(z,\tau,\mu)$. Once we have proved the existence of $S^-(z,\tau,\mu)$, one can easily see the existence and properties of R^- using the fact that $z=x+R^-(x,\tau,\mu)$ is the inverse change of $x=z+S^-(z,\tau,\mu)$. These results are summarized in Proposition 5 of Olivé et al. (2003).

To obtain information about the difference $\phi_0^+-\phi_0^-$, one needs to have more precise information about the singularities of $\hat{\phi}_0$. In fact, it is enough to study the nearest singularities to the real axis $\pm i$, and, by symmetry (71), it is enough to study the point $\zeta=i$.

The study of this singularity is done by using the so called "alien derivatives." The result is given in Theorem 2 of Olivé et al. (2003) where one can see that $\hat{\phi}_0$ has a simply ramified singularity at $\zeta=i$, that is, it only has a polar part (which corresponds to a simple pole) and a logarithmic part:

$$\hat{\phi}_0(i+\xi,\tau,\mu)=\frac{f_0^{[i]}(\mu)\mathrm{e}^{\mathrm{i}\tau}}{2\pi\mathrm{i}\xi}+\hat{\psi}(\xi,\tau,\mu)\frac{\log\xi}{2\pi\mathrm{i}}+\hat{r}(\xi,\tau,\mu)\tag{74}$$

for suitable holomorphic functions $\hat{\psi}(\xi,\tau,\mu)$, $\hat{r}(\xi,\tau,\mu)$, 2π-periodic in τ and that have analytic continuations for $\zeta\in\mathcal{R}^0$. The function $f_0^{[i]}(\mu)$ is analytic and, as a consequence of symmetry (72), odd in μ.

This theorem is proved using Écalle's theory and, mainly, its concept of "alien derivative" Δ_ω, which is an operator that gives the singular part of the function $\hat{\phi}_0$ in any point $\omega \in \mathbb{C}$. In terms of the resurgence theory, (74) reads

$$\Delta_i \tilde{\phi}_0 = f_0^{[i]}(\mu) e^{i\tau} + \tilde{\psi}(z, \tau)$$

where $\tilde{\psi}(z, \tau)$ is the formal Laplace transform of $\hat{\psi}(\zeta, \tau)$. Let us just say here that a crucial tool in the resurgence theory is what is known as the "bridge equation." In our system, the bridge equation relates the alien derivative at the singularity i, $\Delta_i \tilde{\phi}_0$, with the solutions of the linearized formal equation around $\tilde{\phi}_0$

$$\partial_\tau Y - \frac{1}{4} z^2 (\partial \tilde{\phi}_0) \partial_z Y = 0$$

and knowing that any solution of this equation are the composition of any analytic function with $\tilde{Y}(z, \tau, \mu) = z - \tau + \tilde{S}(z, \tau, \mu)$, one can get expressions for it. We refer to the paper to make this argument rigorous.

Finally, writing $f_0^{[i]}(\mu) = \mu f(\mu)$, $f(\mu)$ is even and analyzing the linear terms in μ of the equation one can easily see that $f(\mu) = 2\pi + \mathcal{O}(\mu^2)$.

Once we know the behavior of $\hat{\phi}_0$ near its singularity i, we can easily compute the difference between the functions ϕ_0^+ and ϕ_0^-, just using Cauchy's theorem to compute the difference between the integrals defining them.

Another proof of the first part of Theorem 3.8 (in particular of (66)) can be found in Baldomá (2006). Let us remark that in that paper the result is obtained without the use of the resurgence theory but with direct methods of functional analysis. Moreover, it applies to more general Hamiltonians which only are $\mathcal{C}^1$ in τ. □

Theorem 3.8 gives the existence and behavior of $\phi_0^{\pm}$. The next theorem proves the existence of the invariant manifolds $\phi^{\pm}$ in $\mathcal{D}_{\delta,+}^{s,u(7)}$, considering them as a perturbation of $\phi_0^{\pm}$.

Theorem 3.9 *Let $\bar{\mu}_0$ be the constant considered in Theorem 3.6 and σ_3 and γ be real numbers such that $0 < \sigma_3 < \sigma_1$ and $\gamma \in (1/3, 1/2)$. Then there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ and $\mu \in [0, \bar{\mu}_0]$, there exists functions $\phi^{\pm}(z, \tau)$ defined in $\mathcal{D}_{\delta,+}^{s,u(7)} \times \mathbb{T}_{\sigma_3}$ which are solutions of (61) and satisfy asymptotic conditions (63) and (62), respectively.*

The functions $\delta^{-1} \phi^{\pm}(\frac{u - i\pi/2}{\delta})$ are the analytic continuation of the invariant manifolds $T^{\pm}(u, \tau)$ given by Theorem 3.6, to the corresponding inner domains $D_{\delta,+}^{u,s(7)}$.

Moreover, there exists $b_5 > 0$ independent of δ and μ such that for $(z, \tau) \in \mathcal{D}_{\delta,+}^{u(7)} \times \mathbb{T}_{\sigma_3}$,

$$\left|\partial_z \phi^{\pm}(z, \tau) - \partial_z \phi_0^{\pm}(z, \tau)\right| \le b_5 \delta^2,$$

$$\left|\partial_z^2 \phi^{\pm}(z, \tau) - \partial_z^2 \phi_0^{\pm}(z, \tau)\right| \le b_5 \frac{\delta^2}{\ln^2(1/\delta)}.$$

This theorem is proved in Sect. 6 using complex matching techniques. That is, using a characteristic-like method which consists in integrating the Hamilton–Jacobi

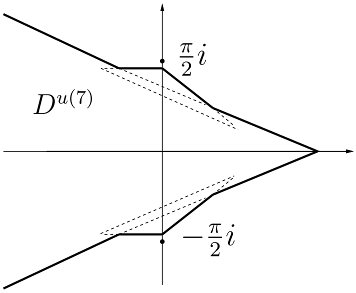


Fig. 9 The domains for T^- and T^+

equation (61) from the overlapping zone between the inner and outer domains where the invariant manifolds are already defined by Theorem 3.6. This procedure ensures that the functions we obtain are the analytic continuation of the invariant manifolds defined in the outer domains.

In conclusion, Theorems 3.6 and 3.9 give the existence of the invariant manifolds $T^\pm$ in the domains $D^{\mathrm{u}(7)}$ and $D^{\mathrm{s}(7)}$ (see Fig. 9) defined as

$$\begin{aligned} D^{\mathrm{u}(i)} &= D_\gamma^{\mathrm{u}(i)} \cup D_{\delta,+}^{\mathrm{u}(i)} \cup D_{\delta,-}^{\mathrm{u}(i)}, \\ D^{\mathrm{s}(i)} &= D_\gamma^{\mathrm{s}(i)} \cup D_{\delta,+}^{\mathrm{s}(i)} \cup D_{\delta,-}^{\mathrm{s}(i)}. \end{aligned} \tag{75}$$

3.4 Study of the Difference Between the Invariant Manifolds

Next step is to compute $\Delta T(u,\tau) = T^+(u,\tau) - T^-(u,\tau)$ in $D^{(7)} \subset D^{\mathrm{s}(7)} \cap D^{\mathrm{u}(7)}$ which is of the form

$$D^{(i)} = \left\{ u \in \mathbb{C}\colon\ |\Im u| + \tan\beta_2 |\Re u| < \frac{\pi}{2} - c^{(i)}\delta \ln(1/\delta) \right\}, \tag{76}$$

where $c^{(i)}$ are the constants defined in (54). The angle $\beta_2 > \beta_1$, satisfying $|\beta_2| = \mathcal{O}(1)$, has been chosen in such a way that the domains $D^{(i)}$ can be split as

$$D^{(i)} = D_\gamma^{(i)} \cup D_{\delta,+}^{(i)} \cup D_{\delta,-}^{(i)}, \tag{77}$$

where $D_\gamma^{(i)} = D^{(i)} \cap D_\gamma^{\mathrm{u}(i)} \cap D_\gamma^{\mathrm{s}(i)}$, and the same for $D_{\delta,\pm}^{(i)}$ (see Fig. 10).

Let us consider in $D^{(7)}$ the linear differential operator

$$\tilde{\mathcal{L}}_\delta = \delta^{-1}\partial_\tau + \left(\frac{\cosh^2 u}{8} \big(\partial_u T^+(u,\tau) + \partial_u T^-(u,\tau) \big) \right) \partial_u \tag{78}$$

which satisfies $\tilde{\mathcal{L}}_\delta(\Delta T) = 0$. To deal with $\tilde{\mathcal{L}}_\delta$, we observe that, heuristically, as $\partial_u T^+$ and $\partial_u T^-$ are close to $\partial_u T_0$, $\tilde{\mathcal{L}}_\delta$ is close to the linear differential operator with constants coefficients

$$\mathcal{L}_\delta = \delta^{-1}\partial_\tau + \partial_u. \tag{79}$$

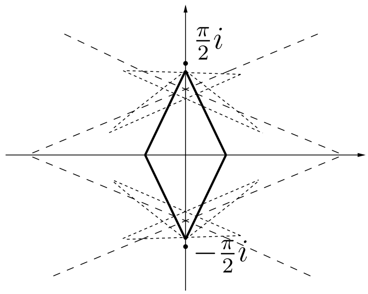


Fig. 10 The domain for $T^+ - T^-$, $D^{(7)}$

The following lemma that will be crucial to prove the exponentially small asymptotic expansion of ΔT, shows that analytic solutions $\zeta(u,\tau)$ of $\mathcal{L}_\delta \zeta = 0$ defined for $(u,\tau) \in (-\mathrm{i}r_0, \mathrm{i}r_0) \times \mathbb{T}_{\sigma_4}$ are exponentially small for real (u,τ).

Lemma 3.10 *Let us consider a function $\zeta(u,\tau)$ analytic in $(-\mathrm{i}r_0, \mathrm{i}r_0) \times \mathbb{T}_{\sigma_4}$ which is solution of $\mathcal{L}_\delta \zeta = 0$. Then ζ can be extended analytically to $\{|\Im u| < r_0\} \times \mathbb{T}_{\sigma_4}$ and its mean value*

$$\langle \zeta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \zeta(u,\tau)\,\mathrm{d}\tau$$

does not depend on u. Moreover, for $r \in (0, r_0)$ and $\sigma \in (0, \sigma_4)$, we define

$$M_r = \max_{(u,\tau)\in[-\mathrm{i}r,\mathrm{i}r]\times\overline{\mathbb{T}}_\sigma} \left|\partial_u^2 \zeta(u,\tau)\right|. \tag{80}$$

Then provided δ is small enough, the following inequalities hold

$$\forall (u,\tau) \in \mathbb{R} \times \mathbb{T}_\sigma, \qquad \begin{cases} \left|\partial_u^2 \zeta(u,\tau)\right| \le 4M_r \mathrm{e}^{-\frac{r}{\delta}}, \\ \left|\partial_u \zeta(u,\tau)\right| \le 4\delta M_r \mathrm{e}^{-\frac{r}{\delta}}, \\ \left|\zeta(u,\tau) - \langle \zeta \rangle\right| \le 4\delta^2 M_r \mathrm{e}^{-\frac{r}{\delta}}. \end{cases}$$

Proof Since ζ is periodic in τ, we can write

$$\zeta(u,\tau) = \sum_{k\in\mathbb{Z}} \zeta^k(u)\mathrm{e}^{\mathrm{i}k\tau}.$$

On the other hand, being ζ solution of $\mathcal{L}_\delta \zeta = 0$, there exists a 2π-periodic function $\Lambda(s)$ such that

$$\zeta(u,\tau) = \Lambda\big(\tau - \delta^{-1}u\big).$$

Thus, it is straightforward to see that $\langle \zeta \rangle$ does not depend on u and it can be extended to $u \in \{|\Im u| < r_0\}$ as

$$\zeta(u,\tau) = \Lambda\big(\tau - \delta^{-1}\Re u - \mathrm{i}\delta^{-1}\Im u\big) = \zeta\big(\mathrm{i}\delta^{-1}\Im u, \tau - \delta^{-1}\Re u\big).$$

Considering $\overline{M_r}$ defined in (80), one has that $|\partial_u^2\zeta(u,\tau)| \leq M_r$ for $(u,\tau) \in \{|\Im u| \leq r\} \times \overline{\mathbb{T}_\sigma}$. Moreover since ζ is analytic for $(u,\tau) \in \{|\Im u| \leq r\} \times \overline{\mathbb{T}_\sigma}$,

$$\left|\partial_u^2\zeta^k(u)\right| \leq M_r \mathrm{e}^{-|k|\sigma}.$$

On the other hand, it can be seen that for $k \in \mathbb{Z}$

$$\partial_u^2\zeta^k(u) = -\frac{k^2}{\delta^2}\Lambda^k \mathrm{e}^{-\mathrm{i}ku/\delta} \tag{81}$$

and taking $u = \pm\mathrm{i}r$,

$$\Lambda^k = -\frac{\delta^2}{k^2}\partial_u^2\zeta^k(\pm\mathrm{i}r)\mathrm{e}^{\pm kr/\delta}.$$

Hence, with this equality and (81), we obtain that for $u \in \mathbb{R}$,

$$\left|\partial_u^2\zeta^k(u)\right| \leq \left|\partial_u^2\zeta^k\bigl(-\operatorname{sgn}(k)\mathrm{i}r\bigr)\right|\mathrm{e}^{-|k|r/\delta} \leq M_r\mathrm{e}^{-|k|\sigma}\mathrm{e}^{-|k|r/\delta}.$$

With this bound and recalling that $\partial_u^2\zeta^0(u) = \partial_u^2\langle\zeta\rangle = 0$, for $u \in \mathbb{R}$ and $|\Im\tau| < \sigma$,

$$\left|\partial_u^2\zeta(u,\tau)\right| \leq \sum_{k\neq 0}\left|\partial_u^2\zeta^k(u)\right|\mathrm{e}^{|k||\Im\tau|} \leq 2M_r\sum_{k>0}\bigl(\mathrm{e}^{|\Im\tau|-\sigma-r/\delta}\bigr)^k \leq 4M_r\mathrm{e}^{-r/\delta}.$$

The other two bounds are obtained proceeding analogously. □

In order to apply Lemma 3.10 to ΔT, it is natural to look for a change of variables that conjugates operators (78) and (79). The operator $\tilde{\mathcal{L}}_\delta$ can be split as

$$\tilde{\mathcal{L}}_\delta = \tilde{\mathcal{L}}_\delta^- + \left(\frac{\cosh^2 u}{8}\bigl(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau)\bigr)\right)\partial_u,$$

where

$$\tilde{\mathcal{L}}_\delta^- = \partial_\tau + \frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)\partial_u. \tag{82}$$

As a first step, we prove the existence of a change which straightens $\tilde{\mathcal{L}}_\delta^-$.

Theorem 3.11 *Let σ_3 and $\bar{\mu}_0$ be the constants considered in Theorem 3.9. Then for any fixed $\gamma \in (1/3, 1/2)$, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$ and $\mu \in (0,\bar{\mu}_0]$, there exists a real-analytic function $\mathcal{C}^-(w,\tau)$ defined in $D^{\mathrm{u}(10)} \times \mathbb{T}_{\sigma_3}$, such that the change*

$$(u,\tau) = \bigl(w + \mathcal{C}^-(w,\tau),\tau\bigr)$$

conjugates the operators $\tilde{\mathcal{L}}_\delta^-$ and $\mathcal{L}_\delta$ defined in (82) and (79).

Moreover, there exists a constant $b_6>0$, *independent of* δ *and* μ, *such that*:

- *For* $(w,\tau)\in D^{\mathrm{u}(10)}_{\gamma}\times\mathbb{T}_{\sigma_3}$, *it holds that* $u=w+\mathcal{C}^-(w,\tau)\in D^{\mathrm{u}(9)}_{\gamma}$ *and*

$$\left|\mathcal{C}^-(w,\tau)\right|\le b_6\big(\delta^2|w|+\delta^{2-2\gamma}\big),$$
$$\left|\partial_w^j\mathcal{C}^-(w,\tau)\right|\le b_6\delta^{2-(j+2)\gamma}\quad \textit{for } j=1,2.$$

- *For* $(w,\tau)\in D^{\mathrm{u}(10)}_{\delta,\pm}\times\mathbb{T}_{\sigma_3}$, *it holds that* $u=w+\mathcal{C}^-(w,\tau)\in D^{\mathrm{u}(9)}_{\delta,\pm}$ *and*

$$\left|\partial_u^j\mathcal{C}^-(w,\tau)\right|\le b_6\delta^{1-j}\ln(1/\delta)^{-1-j}\quad \textit{for } j=0,1,2.$$

The proof of this theorem is given in Sect. 7. We note that we will need to proceed as in the proof of the existence of the invariant manifolds in Sects. 5 and 6. First, in Sect. 7.1 it is proved the existence of the change of variables in the outer domain $D^{\mathrm{u}(10)}_{\gamma}$ and afterwards it is extended to the inner ones $D^{\mathrm{u}(10)}_{\delta,\pm}$ using complex matching techniques in Sect. 7.2.

Applying the change $\mathcal{C}^-$ to $\tilde{\mathcal{L}}_\delta$ (in (78)) as a first step, it can be found the global change which conjugates $\tilde{\mathcal{L}}_\delta$ to $\mathcal{L}_\delta$.

Theorem 3.12 *Let* $\bar\mu_0$ *be the constant considered in Theorem* 3.11 *and* σ_4 *and* γ *real numbers such that* $0<\sigma_4<\sigma_3$ *and* $\gamma\in(1/3,1/2)$. *Then there exists* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$ *and* $\mu\in(0,\bar\mu_0]$, *there exists a real-analytic function* $\mathcal{U}(v,\tau)$ *in* $D^{(12)}\times\mathbb{T}_{\sigma_4}$, *such that the change*

$$(u,\tau)=\big(v+\mathcal{U}(v,\tau),\tau\big)$$

conjugates the operators $\tilde{\mathcal{L}}_\delta$ *and* $\mathcal{L}_\delta$ *defined in* (78) *and* (79). *Moreover, there exist constants* $b_7>0$ *and* $\nu_1>0$ *independent of* δ *and* μ *such that*:

- *For all* $(v,\tau)\in D^{(12)}_{\gamma}\times\mathbb{T}_{\sigma_4}$, *it holds that* $v+\mathcal{U}(v,\tau)\in D^{(11)}_{\gamma}$ *and*

$$\left|\partial_v^j\mathcal{U}(v,\tau)\right|\le b_7\delta^{1+\nu_1-j\gamma}\quad \textit{for } j=0,1,2.$$

- *For all* $(v,\tau)\in D^{(12)}_{\delta,\pm}\times\mathbb{T}_{\sigma_4}$, *it holds that* $v+\mathcal{U}(v,\tau)\in D^{(11)}_{\delta,\pm}$ *and*

$$\left|\partial_v^j\mathcal{U}(v,\tau)\right|\le b_7\delta^{1-j}\big(\ln(1/\delta)\big)^{-1-j}\quad \textit{for } j=0,1,2.$$

It will also be necessary to consider the inverse change of variables in the outer domain $D^{(17)}_{\gamma}$. The existence and properties of it are stated in the following theorem whose proof is straightforward considering a fixed point argument.

Theorem 3.13 *Let* $\bar\mu_0$, σ_4 *and* ν_1 *be the constants defined in Theorem* 3.12 *and* γ *be any fixed constant* $\gamma\in(1/3,1/2)$. *Then there exists* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$ *and* $\mu\in(0,\bar\mu_0]$, *there exists a real-analytic function* $\mathcal{V}(u,\tau)$ *in* $D^{(14)}_{\gamma}\times\mathbb{T}_{\sigma_4}$, *such that*

$$(v,\tau)=\big(u+\mathcal{V}(u,\tau),\tau\big)$$

is the inverse of the change given in Theorem 3.12. Moreover, there exists a constant $b_8 > 0$ independent of δ and μ such that for all $(u,\tau) \in D_\gamma^{(14)} \times \mathbb{T}_{\sigma_4}$, it holds that $u + \mathcal{V}(u,\tau) \in D_\gamma^{(13)}$ and

$$\left|\partial_u^j \mathcal{V}(u,\tau)\right| \le b_8 \delta^{1+\nu_1 - j\gamma} \quad \textit{for } j = 0,1,2.$$

With Lemma 3.10 and the change of variables obtained in Theorem 3.12, we have all the tools which will be necessary to obtain the exponentially small asymptotic expression of the splitting of separatrices.

In fact, in the following theorem, we prove that the first asymptotic term of ΔT is given essentially by the function

$$\begin{aligned} F(u,\tau) &= 2\delta^{-1}\mu f(\mu) \mathrm{e}^{-\frac{\pi}{2\delta}} \cos\left(\tau - \delta^{-1} u\right) \\ &= \delta^{-1}\mu f(\mu)\left(\mathrm{e}^{\mathrm{i}\tau}\mathrm{e}^{-\mathrm{i}\frac{u-\mathrm{i}\pi/2}{\delta}} + \mathrm{e}^{-\mathrm{i}\tau}\mathrm{e}^{\mathrm{i}\frac{u+\mathrm{i}\pi/2}{\delta}}\right) \end{aligned} \tag{83}$$

which comes from the difference between the solutions of the inner equation $\phi_0^{\pm}(z,\tau)$ (see (66)) stated in Theorem 3.8.

Theorem 3.14 *Let $\bar{\mu}_0$ be the constant defined in Theorem 3.13, then for any fixed $0 < \bar{\mu} < \bar{\mu}_0$ there exist $\delta_0 > 0$ and $b_9 > 0$ independent of δ and μ such that, for $\delta \in (0,\delta_0)$ and $\mu \in (0,\bar{\mu}]$, $u \in (-u_1,u_1) := D^{(14)} \cap \mathbb{R}$ and $\tau \in \mathbb{T}$, the following bounds hold*:

$$\left|\Delta T(u,\tau) - \alpha(\mu,\delta) - F(u,\tau)\right| \le b_9 \mu \frac{1}{\ln(1/\delta)} \delta^{-1} \mathrm{e}^{-\frac{\pi}{2\delta}},$$

$$\left|\partial_u \Delta T(u,\tau) - \partial_u F(u,\tau)\right| \le b_9 \mu \frac{1}{\ln(1/\delta)} \delta^{-2} \mathrm{e}^{-\frac{\pi}{2\delta}},$$

$$\left|\partial_u^2 \Delta T(u,\tau) - \partial_u^2 F(u,\tau)\right| \le b_9 \mu \frac{1}{\ln(1/\delta)} \delta^{-3} \mathrm{e}^{-\frac{\pi}{2\delta}},$$

where $\alpha(\mu,\delta) = \langle \Delta T(v + \mathcal{U}(v,\tau),\tau)\rangle$ (which is independent of v) and $\mathcal{U}$ is the change given in Theorem 3.12.

Proof of Theorem 2.2 After the statement of Theorem 3.14, we are ready to prove Theorem 2.2. Changes (38), (37), (36), and (19), Theorems 3.2 and 3.6, and formula (44) give the two first statements. For the last two ones, we also need Theorem 3.14. □

In order to prove Theorem 3.14, we consider the change of variables found in Theorem 3.12 and we split $\Delta T(u,\tau) - F(u,\tau)$ as

$$\Delta T\left(v + \mathcal{U}(v,\tau),\tau\right) - F\left(v + \mathcal{U}(v,\tau),\tau\right) = \zeta_1(v,\tau) + \zeta_2(v,\tau),$$

where

$$\begin{aligned} \zeta_1(v,\tau) &= \Delta T\left(v + \mathcal{U}(v,\tau),\tau\right) - F(v,\tau), \\ \zeta_2(v,\tau) &= F(v,\tau) - F\left(v + \mathcal{U}(v,\tau),\tau\right). \end{aligned} \tag{84}$$

In the following two propositions, whose proofs are delayed to the end of this section, we obtain the desired bounds of ζ_1 and ζ_2 which will lead to the proof of Theorem 3.14.

Proposition 3.15 *Let $\bar{\mu}_0$ be the constant defined in Theorem 3.13, then for any fixed $0<\bar{\mu}<\bar{\mu}_0$ there exist $\delta_0>0$ and $b_{10}>0$ independent of δ and μ such that for $\delta\in(0,\delta_0)$, $\mu\in(0,\bar{\mu}]$, $v\in D^{(13)}\cap\mathbb{R}$ and $\tau\in\mathbb{T}$, the function ζ_1 defined in (84) satisfies*:

$$\left|\zeta_1(v,\tau)-\langle\zeta_1\rangle\right|\le b_{10}\frac{\mu}{\ln(1/\delta)}\delta^{-1}\mathrm{e}^{-\frac{\pi}{2\delta}},$$

$$\left|\partial_v^j\zeta_1(v,\tau)\right|\le b_{10}\frac{\mu}{\ln(1/\delta)}\delta^{-(j+1)}\mathrm{e}^{-\frac{\pi}{2\delta}}\quad \textit{for } j=1,2,$$

where $\langle\zeta_1\rangle$ is independent of u.

Proposition 3.16 *Let $\bar{\mu}_0$ and ν_1 be the constants defined in Theorem 3.12, then there exist $\delta_0>0$ and $b_{11}>0$ independent of δ and μ such that for $\delta\in(0,\delta_0)$, $\mu\in(0,\bar{\mu}_0)$, $v\in D^{(12)}\cap\mathbb{R}$ and $\tau\in\mathbb{T}$,*

$$\left|\partial_v^j\zeta_2(v,\tau)\right|\le b_{11}\mu\delta^{\nu_1-j-1}\mathrm{e}^{-\frac{\pi}{2\delta}}\quad \textit{for } j=1,2. \tag{85}$$

Proof of Theorem 3.14 In order to recover the bounds stated in Theorem 3.14 from the bounds obtained in Propositions 3.15 and 3.16, it is enough to perform the real-analytic change of variables obtained in Theorem 3.13. □

In the rest of the paper, we will use the following convention.

Notation 3.17 In order to make the proofs more readable, we will use the following convention: we will say that $g_1\le\mathcal{O}(g_2)$ in some domain D if there exists some constant $C>0$ that may depend on other constants that will be defined through the article, but does not depend on ε neither μ (and, therefore, neither on δ), such that $|g_1|\le C|g_2|$ in D. Moreover, we will say that $g_1=\mathcal{O}(g_2)$ if $g_1\le\mathcal{O}(g_2)$ and $g_2\le\mathcal{O}(g_1)$.

Proof of Proposition 3.15 By Theorem 3.12 and definition of F in (83), $\mathcal{L}_\delta\zeta_1(v,\tau)=0$. Then Lemma 3.10 can be applied and it gives that $\langle\zeta_1\rangle$ does not depend on v. In order to bound ζ_1, we recall that $\Delta T=T^+-T^-=Q^+-Q^-$. Differentiating ζ_1:

$$\begin{aligned}\partial_v^2\zeta_1(v,\tau)&=\partial_u^2\Delta T\big(v+\mathcal{U}(v,\tau),\tau\big)\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2\\&\quad+\partial_u\Delta T\big(v+\mathcal{U}(v,\tau),\tau\big)\partial_v^2\mathcal{U}(v,\tau)-\partial_v^2F(v,\tau).\end{aligned}$$

The next step is to bound $|\partial_v^2\zeta_1(v,\tau)|$ for $(v,\tau)\in[-\mathrm{i}r,\mathrm{i}r]\times\overline{\mathbb{T}_{\sigma_5}}$ for any fixed $\sigma_5<\sigma_4$ and $r=\pi/2-c^{(13)}\delta\ln(1/\delta)$.

For $v\in\overline{D_\gamma^{(13)}}$ we apply Theorems 3.6 and 3.12, and the expression of F in (83), obtaining, since $\gamma>1/3$,

$$\left|\partial_v^2\zeta_1(v,\tau)\right|\le\mathcal{O}\big(\mu\delta^{2-5\gamma}\big)\le\mathcal{O}\big(\delta^{-1}\big). \tag{86}$$

For $v \in \overline{D^{(13)}_{\delta,+}} \subset \overline{D^{(12)}_{\delta,+}}$, we have that

$$z = \big(v + \mathcal{U}(v,\tau) - \mathrm{i}\pi/2\big)/\delta \in \overline{\mathcal{D}^{(11)}_{\delta,+}}.$$

We split ζ_1 using the functions $\phi^{\pm}$ and $\phi_0^{\pm}$ whose existence has been proved in Theorems 3.8 and 3.9

$$\begin{aligned}\delta\zeta_1(v,\tau) &= \big(\phi^+ - \phi_0^+\big)(z,\tau) - (\phi^- - \phi_0^-)(z,\tau)\\ &\quad + \big(\phi_0^+ - \phi_0^-\big)(z,\tau) - \delta F(v,\tau).\end{aligned}$$

Thus, differentiating it twice, we obtain

$$\delta^3\partial_z^2\zeta_1(v,\tau) = \partial_z^2\big(\phi^+ - \phi_0^+\big)(z,\tau)\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2 \tag{87}$$

$$- \partial_z^2(\phi^- - \phi_0^-)(z,\tau)\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2 \tag{88}$$

$$+ \delta\partial_z\big(\phi^+ - \phi_0^+\big)(z,\tau)\partial_v^2\mathcal{U}(v,\tau) \tag{89}$$

$$- \delta\partial_z(\phi^- - \phi_0^-)(z,\tau)\partial_v^2\mathcal{U}(v,\tau) \tag{90}$$

$$+ \partial_z^2\big(\phi_0^+ - \phi_0^-\big)(z,\tau)\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2 - \delta^3\partial_v^2 F(v,\tau) \tag{91}$$

$$+ \delta\partial_z\big(\phi_0^+ - \phi_0^-\big)(z,\tau)\partial_v^2\mathcal{U}(v,\tau). \tag{92}$$

Applying Theorems 3.9 and 3.12, we obtain that terms (87) and (88) can be bounded by $\mathcal{O}(\delta^2\ln^{-2}(1/\delta))$ and terms (89) and (90) by $\mathcal{O}(\delta^2\ln^{-3}(1/\delta))$.

To bound (91), we differentiate twice formula (66) for $\phi_0^+ - \phi_0^-$ and (83) for F:

$$\begin{aligned}&\Big|\partial_z^2\big(\phi_0^+ - \phi_0^-\big)\big(\big(v+\mathcal{U}(v,\tau)-\mathrm{i}\pi/2\big)/\delta,\tau\big)\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2 - \delta^3\partial_v^2 F(v,\tau)\Big|\\ &\quad\le \mathcal{O}\Big(\Big(\mu e^{\Im(\frac{v+\mathcal{U}(v,\tau)-\mathrm{i}\pi/2}{\delta})}\big[-1+\mathcal{O}\big(\ln^{-1}(1/\delta)\big)\big] + \mathcal{O}\big(\delta^{a_1 c^{(13)}}\big)\Big)\big|1+\partial_v\mathcal{U}(v,\tau)\big|^2\Big)\\ &\qquad + \mathcal{O}\big(\mu e^{\Im(\frac{v-\mathrm{i}\pi/2}{\delta})} + \mu e^{-\Im(\frac{v+\mathrm{i}\pi/2}{\delta})}\big)\\ &\quad\le \mathcal{O}\Big(\mu e^{\Im(\frac{v-\mathrm{i}\pi/2}{\delta})}\Big(e^{\frac{|\mathcal{U}(v,\tau)|}{\delta}}\big[-1+\mathcal{O}\big(\ln^{-1}(1/\delta)\big)\big]\big(1+\partial_v\mathcal{U}(v,\tau)\big)^2+1\Big)\Big)\\ &\qquad + \mathcal{O}\big(\mu e^{-\Im(\frac{v+\mathrm{i}\pi/2}{\delta})} + \mathcal{O}\big(\delta^{a_1 c^{(13)}}\big)\big|1+\partial_v\mathcal{U}(v,\tau)\big|^2\big).\end{aligned}$$

Applying the bounds obtained in Theorem 3.12 and recalling that $0 \le \Im v \le \pi/2 - c^{(13)}\delta\ln(1/\delta)$, we obtain the following statements:

$$\begin{aligned}e^{\frac{|\mathcal{U}(v,\tau)|}{\delta}} &= 1+\mathcal{O}\big(\ln^{-1}(1/\delta)\big),\\ e^{\Im(\frac{v-\mathrm{i}\pi/2}{\delta})} &\le \mathcal{O}\big(\delta^{c^{(13)}}\big),\\ e^{-\Im(\frac{v+\mathrm{i}\pi/2}{\delta})} &\le \mathcal{O}\big(e^{-\frac{\pi}{2\delta}}\big).\end{aligned}$$

From these bounds and applying again Theorem 3.12, it can be seen that term (91) is bounded by $\mathcal{O}(\delta^{c^{(13)}}\ln^{-1}(1/\delta))$.

The last term (92) can be bounded analogously by $\mathcal{O}(\delta^{c^{(13)}}(\ln(1/\delta))^{-3})$ differentiating formula (66) once and using Theorem 3.12.

Therefore, since $0 < c^{(13)} < 2$ (see (55)), for $(v,\tau) \in \overline{D^{(13)}_{\delta,+}} \times \overline{\mathbb{T}_{\sigma_5}}$,

$$\left|\partial_v^2 \zeta_1(v,\tau)\right| \leq \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Thus, joining this bound with (86), for $(v,\tau) \in \overline{D^{(13)}} \times \overline{\mathbb{T}_{\sigma_5}}$,

$$\left|\partial_v^2 \zeta_1(v,\tau)\right| \leq \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Furthermore, taking $m(v,\tau,\delta,\mu) = \partial_z^2 \zeta_1(v,\tau)$, m is analytic in μ and holds that $m(v,\tau,\delta,0) = 0$. Thus, considering an arbitrary $\bar{\mu} \in (0,\bar{\mu}_0)$ such that $\rho = |\bar{\mu}_0 - \bar{\mu}|$ is independent of δ, and applying Cauchy integral formula, for $\mu < \bar{\mu}$

$$\left|\partial_\mu m(v,\tau,\delta,\mu)\right| = \left|\frac{1}{2\pi}\int_{|\xi-\mu|=\rho} \frac{m(v,\tau,\delta,\xi)}{(\xi-\mu)^2}\,\mathrm{d}\xi\right| \leq \mathcal{O}\left(\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right).$$

Therefore,

$$\begin{aligned}\left|\partial_v^2 \zeta_1(v,\tau)\right| &= \left|m(v,\tau,\delta,\mu)\right| \\ &= \left|\int_0^\mu \partial_\mu m(v,\tau,\delta,\xi)\,\mathrm{d}\xi\right| \leq \mathcal{O}\left(\mu\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right). \qquad (93)\end{aligned}$$

Finally, we apply Lemma 3.10 to ζ_1 with $r = \pi/2 - c^{(13)}\delta\ln(1/\delta)$ and

$$M_r = \mathcal{O}\left(\mu\delta^{-3}\frac{\delta^{c^{(13)}}}{\ln(1/\delta)}\right),$$

and we obtain the bounds stated in the proposition having into account that

$$\mathrm{e}^{-\frac{r}{\delta}} = \delta^{-c^{(13)}}\mathrm{e}^{-\frac{\pi}{2\delta}}. \qquad \square$$

Proof of Proposition 3.16 It is enough to apply carefully the mean value theorem to the function F given in (83) using the bounds of Theorem 3.12. $\square$

Notation 3.18 In the forthcoming sections, we will use several Banach spaces $(\mathcal{X}, \|\cdot\|)$. We will denote the ball of center 0 and radius r as

$$B(r) = \left\{x \in \mathcal{X}\colon \|x\| < r\right\}.$$

4 Proof of Theorem 2.1 and 3.2

As it was noticed in Delshams and Seara (1992), the periodic orbit x_{p} of system (3) holds the symmetry $x_{\mathrm{p}}(-\tau) = -x_{\mathrm{p}}(\tau)$, and thus it has zero mean. On the other hand, it has to hold the functional equation

$$\mathcal{L}_0(x_\mathrm{p}) = \varepsilon^2 \sin(x_\mathrm{p}) + \mu \sin\tau,$$

where $\mathcal{L}_0 x = x''$.

However, for a fixed value of $\mu > 0$ independent of ε, $x_\mathrm{p}(\tau)$ is not small. Expanding formally the periodic orbit in powers of ε is obtained that

$$x_\mathrm{p}(\tau) = -\mu \sin\tau + \mathcal{O}\big(\varepsilon^2\big).$$

Thus, in order to apply a fixed point argument, we consider $z_\mathrm{p}(\tau) = x_\mathrm{p}(\tau) + \mu\sin\tau$. Hence, we will look for a solution of the equation

$$\mathcal{L}_0 z_\mathrm{p} = \varepsilon^2 \sin(z_\mathrm{p} - \mu\sin\tau)$$

in the Banach space

$$\mathcal{X}_0 = \big\{z : \mathbb{T}_{\sigma_0} \to \mathbb{C}\text{: real-analytic, } 2\pi\text{-periodic, } z(\tau) = -z(-\tau) = 0,\ \|z\|_{\sigma_0} < \infty\big\}, \tag{94}$$

where the norm $\|\cdot\|_{\sigma_0}$ is defined using the Fourier series of z and is given by

$$\|z\|_{\sigma_0} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \big|z^{[k]}\big| \mathrm{e}^{|k|\sigma_0}.$$

Moreover, we take σ depending on μ, since the strip of analyticity of the periodic orbit seems to shrink when $\mu \to \infty$. Indeed, one has to take the width of the strip in such a way that the function $b(\tau) = \sin(\mu\sin\tau)$ holds $|b(\tau)| \leq \mathcal{O}(1)$. As it will be seen in the proof of Proposition 4.1, this bound holds provided $\mu\sinh\sigma \leq \mathcal{O}(1)$. Therefore, one has to take

$$\sigma_0(\mu) = \ln\left(\frac{1}{\mu} + \sqrt{\frac{1}{\mu^2} + 1}\right). \tag{95}$$

Notice that when $\mu \to 0$ it holds that $\sigma_0(\mu) \sim \ln(1/\mu)$ whereas as $\mu \to \infty$ it holds that $\sigma_0(\mu) \sim 1/\mu$.

On the other hand, for functions belonging to $\mathcal{X}_0$, the operator $\mathcal{L}_0$ has an inverse which is defined through the Fourier series

$$\mathcal{G}_0(z) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{z^{[k]}}{k^2} \mathrm{e}^{ik\tau}$$

and thus it can be bounded by $\|\mathcal{G}_0\|_{\sigma_0} \leq \mathcal{O}(1)$. Using this operator, first statement of Theorem 2.1 will be a straightforward consequence of the following proposition.

Proposition 4.1 *There exists ε_0 such that for $\varepsilon \in (0, \varepsilon_0)$ and any $\mu > 0$ there exists a 2π-periodic real-analytic function $z_\mathrm{p}(\tau) \in \mathcal{X}_0$ defined in $\mathbb{T}_{\sigma_0}$, which is a fixed point* *of the functional*

$$\mathcal{F}_0(z) = \mathcal{G}_0\big(\varepsilon^2 \sin(z + \mu \sin\tau)\big). \tag{96}$$

Moreover, it satisfies $\|z_{\mathrm{p}}(\tau)\|_{\sigma_0} \le \mathcal{O}(\varepsilon^2)$.

From this proposition, we obtain the following corollary which has Theorem 3.2, and thus also the second statement of Theorem 2.1, as a consequence.

Corollary 4.2 *There exists ε_0 such that, for any fixed $\bar{\mu}$, there exists $\sigma_1 > 0$ (independent of μ and ε), such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \bar{\mu})$, system (3) has a periodic orbit, which holds the following bound,*

$$\big\|x_{\mathrm{p}}(\tau) + \mu \sin\tau\big\|_{\sigma_1} \le \mathcal{O}\big(\mu\varepsilon^2\big).$$

In order to prove the proposition, we need the following technical lemma.

Lemma 4.3 *There exists constants $c_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and any $\mu > 0$:*

1. $\|\mathcal{F}_0(0)\|_{\sigma_0} \le \frac{c_1}{2}\varepsilon^2$.
2. *For all $z_1, z_2 \in B(c_1\mu\varepsilon^2) \subset \mathcal{X}_0$, $\|\mathcal{F}_0(z_1) - \mathcal{F}_0(z_2)\|_{\sigma_0} \le \mathcal{O}(\varepsilon^2)\|z_1 - z_2\|_{\sigma_0}$.*

Proof For the first statement, we take $b(\tau) = \sin(\mu \sin\tau)$, and thus

$$\mathcal{F}_0(0) = \mathcal{G}_0\big(-\varepsilon^2 b(\tau)\big) = \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{b^{[k]}}{k^2} \mathrm{e}^{\mathrm{i}k\tau}.$$

As a first step, we bound $b(\tau) = \sin(\mu \sin\tau)$. Writing τ as $\tau = \zeta + \mathrm{i}\theta$ with $|\theta| \le \sigma_0(\mu)$,

$$\begin{aligned}
\big|b(\tau)\big| &= \big|\sin(\mu \sin\tau)\big| \\
&\le \big|\sin(\mu \sin\zeta \cosh\theta)\big|\big|\cosh(\mu \cos\zeta \sinh\theta)\big| \\
&\quad + \big|\cos(\mu \sin\zeta \cosh\theta)\big|\big|\sinh(\mu \cos\zeta \sinh\theta)\big| \\
&\le c_0
\end{aligned}$$

for certain constant $c_0 > 0$ independent of ε and μ. Therefore, $|b^{[k]}| \le c_0 \mathrm{e}^{-|k|\sigma_0}$. Thus, taking $c_1 = 2c_0\pi^2/3$,

$$\big\|\mathcal{F}_0(0)\big\|_{\sigma_0} \le \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|b^{[k]}|}{|k|^2} \mathrm{e}^{|k|\sigma_0} \le \varepsilon^2 \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{c_0}{|k|^2} \le \frac{c_1}{2}\varepsilon^2.$$

For the second statement,

$$\begin{aligned}
\big\|\mathcal{F}_0(z_1) - \mathcal{F}(z_2)\big\|_{\sigma_0} &\le \mathcal{O}\big(\varepsilon^2\big)\big\|\sin\big(z_1 - \mu\sin(\tau)\big) - \sin\big(z_2 - \mu\sin(\tau)\big)\big\|_{\sigma_0} \\
&\le \mathcal{O}\big(\varepsilon^2\big)\|z_1 - z_2\|_{\sigma_0}.
\end{aligned}$$

□

Proof of Proposition 4.1 Considering the bounds of Lemma 4.3, $\mathcal{F}_0$ is a contraction from $B(c_1\varepsilon^2)\subset\mathcal{X}_0$ to itself. Thus, it has a fixed point $z_{\mathrm{p}}(\tau)$ that gives the periodic orbit $x_{\mathrm{p}}(\tau)=z_{\mathrm{p}}(\tau)-\mu\sin\tau$ and holds the wanted bound. □

5 Invariant Manifolds in the Outer Domains: Proof of Theorem 3.6

5.1 Banach Spaces and Technical Lemmas

We define several norms which will be used throughout the proof. Since we work with functions that near the singularities $\pm\mathrm{i}\pi/2$ behave like $h(u,\tau)\sim\cosh^{-n}u$ and near infinity like $h(u,\tau)\sim\mathrm{e}^{\mp mu}$, we will use certain weighted norms in the variable u. However, for simplicity and in order to have symmetry between $+\infty$ and $-\infty$, we will also use $\cosh^m u$ as weight at infinity. Moreover, since we work with analytic functions 2π-periodic in τ, Fourier norms are considered, as well as weighted supremum norms.

In this way, we define for analytic functions $h(u)$ the following norms in the domains defined in (48), (49), and (50).

(i) For $h: D_\gamma^{\mathrm{u,c}(j)}\to\mathbb{C}$: $\|h\|_{n,\mathrm{c}}=\sup_{u\in D_\gamma^{\mathrm{u,c}(j)}}|\cosh^n u\cdot h(u)|$.
(ii) For $h: D_\gamma^{\mathrm{u,f}(j)}\to\mathbb{C}$: $\|h\|_{m,\mathrm{f}}=\sup_{u\in D_\gamma^{\mathrm{u,f}(j)}}|\cosh^m u\cdot h(u)|$.
(iii) For $h: D_\gamma^{\mathrm{u}(j)}\to\mathbb{C}$: $\|h\|_{n,m}=\|h\|_{n,\mathrm{c}}+\|h\|_{m,\mathrm{f}}$.

Given $h(u,\tau)$ an analytic function of 2 variables 2π-periodic in τ, we consider the corresponding Fourier norms

(i) For $h: D_\gamma^{\mathrm{u,c}(j)}\times\mathbb{T}_{\sigma_i}\to\mathbb{C}$: $\|h\|_{n,\mathrm{c},\sigma_i}=\sum_{k\in\mathbb{Z}}\|h^{[k]}\|_{n,\mathrm{c}}\mathrm{e}^{|k|\sigma_i}$.
(ii) For $h: D_\gamma^{\mathrm{u,f}(j)}\times\mathbb{T}_{\sigma_i}\to\mathbb{C}$: $\|h\|_{m,\mathrm{f},\sigma_i}=\sum_{k\in\mathbb{Z}}\|h^{[k]}\|_{m,\mathrm{f}}\mathrm{e}^{|k|\sigma_i}$.
(iii) For $h: D_\gamma^{\mathrm{u}(j)}\times\mathbb{T}_{\sigma_i}\to\mathbb{C}$: $\|h\|_{n,m,\sigma_i}=\|h\|_{n,\mathrm{c},\sigma_i}+\|h\|_{m,\mathrm{f},\sigma_i}$.

On the other hand, it will be used also the Fourier supremum norm defined by $\|h\|_{\infty,\sigma_i}=\sum_{k\in\mathbb{Z}}\|h^{[k]}\|_\infty\mathrm{e}^{|k|\sigma_i}$ where a subindex c or f will be added if we are restricted to the domains $D_\gamma^{\mathrm{u,f}(j)}$ or $D_\gamma^{\mathrm{u,c}(j)}$. Note that when h does not depend on u, this norm coincides with the Fourier norm $\|\cdot\|_{\sigma_i}$ introduced in Sect. 4. In order to clarify the notation, we will denote the classical supremum norm for functions defined in $D_\gamma^{\mathrm{u}(j)}\times\mathbb{T}_{\sigma_i}$ as $\|\cdot\|_{\infty,i}$. All these norms are defined analogously for the domains where the stable manifold is defined. Using the Fourier expansion properties, one can see the following relations between these norms:

Lemma 5.1 *The following inequalities hold*:

1. $\|h\|_{\infty,i}\le\|h\|_{\infty,\sigma_i}$.
2. $\|\cosh^n u\cdot h\|_{\infty,*,i}\le\|h\|_{n,*,\sigma_i}$ *for* $*=\mathrm{f},\mathrm{c}$.
3. $\|h\|_{n,*,\sigma_{i+1}}\le\mathcal{O}(\frac{1}{|\sigma_{i+1}-\sigma_i|})\|\cosh^n u\cdot h\|_{\infty,*,i}$ *for* $*=\mathrm{f},\mathrm{c}$ *and for any* $0<\sigma_{i+1}<\sigma_i$.
4. $\|h_1\cdot h_2\|_{n,*,\sigma_i}\le\|h_1\|_{\infty,*,\sigma_i}\|h_2\|_{n,*,\sigma_i}$ *for* $*=\mathrm{f},\mathrm{c}$.

5. $\|h_1 \cdot h_2\|_{n,m,\sigma_i} \leq \|h_1\|_{\infty,\sigma_i} \|h_2\|_{n,m,\sigma_i}$.
6. $\|\cosh^n u \cdot h_1 h_2\|_{n,*,\sigma_i} \leq \|h_1\|_{n,*,\sigma_i} \|h_2\|_{n,*,\sigma_i}$ *for* $* = \mathrm{f}, \mathrm{c}$.

Since the proof for both invariant manifolds is analogous, we will deal only with the unstable case and then for the proof of Theorem 3.6, we will look for a solution of (40) with a fixed point argument in the Banach space

$$\mathcal{E}_{n,m} = \left\{h(u,\tau) \mid h : D_\gamma^{\mathrm{u}(0)} \times \mathbb{T}_{\sigma_1} \to \mathbb{C},\ \text{real-analytic},\ \|h\|_{n,m,\sigma_1} < \infty\right\} \tag{97}$$

for certain naturals n and m.

For the proof of Theorem 5.9, the Banach space will be needed:

$$\mathcal{E}_m = \left\{h(u,\tau) \mid h : D_\gamma^{\mathrm{u},\mathrm{f}(0)} \times \mathbb{T}_{\sigma_1} \to \mathbb{C},\ \text{real-analytic},\ \|h\|_{m,\mathrm{f},\sigma_1} < \infty\right\}. \tag{98}$$

We state some previous technical lemmas.

Lemma 5.2 *For* $\delta > 0$ *small enough and* $u \in D_\gamma^{\mathrm{u}(j)}$:

1. $|\cosh^{-1} u| \leq \mathcal{O}(\delta^{-\gamma})$.
2. $|\tanh u| \leq \mathcal{O}(\delta^{-\gamma})$.
3. $\forall \beta \in [0, \beta_0/2]$ (*see* (47)) *and* $\forall \xi \in \mathbb{R}^-$, $\left|\frac{\cosh u}{\cosh(u+\xi \mathrm{e}^{\pm \mathrm{i}\beta})}\right| \leq \mathcal{O}(1)$.
4. *For* $m \geq 1$, $\int_{-\infty}^0 \left|\frac{\cosh u}{\cosh(u+t)}\right|^m \mathrm{d}t \leq \mathcal{O}(1)$.

Lemma 5.3 *The following inequalities hold*:

1. (*Cauchy inequalities*) *Considering the nested domains* $D_\gamma^{\mathrm{u}(j+1)} \subset D_\gamma^{\mathrm{u}(j)} \subset D_\gamma^{\mathrm{u}(0)}$, *and denoting the corresponding norms* $\|\cdot\|_{n,m,\sigma_i}^{(j+1)}$ *and* $\|\cdot\|_{n,m,\sigma_i}^{(j)}$, *for* $h \in \mathcal{E}_{n,m}$,
$$\|\partial_u h\|_{n,m,\sigma_i}^{(j+1)} \leq \mathcal{O}\big(\delta^{-\gamma}\big) \|h\|_{n,m,\sigma_i}^{(j)}.$$
2. *For* $h \in \mathcal{E}_{n,m}$ *or* $h \in \mathcal{E}_m$ *with* $m > 0$, *and* $l < m$, $\lim_{\Re u \to -\infty} \cosh^l u \cdot h(u,\tau) = 0$ *and* $\int_{-\infty}^u h(v,\tau)\,\mathrm{d}v$ *exists and is bounded.*

Proof For the first statement, we consider $h(u,\tau) = \sum_{k\in\mathbb{Z}} h^{[k]}(u)\mathrm{e}^{\mathrm{i}k\tau}$ and we have:

$$\cosh^n u \cdot \partial_u h^{[k]}(u) = \partial_u\big(\cosh^n u \cdot h^{[k]}(u)\big) - n\cosh^{n-1} u \sinh u \cdot h^{[k]}(u)$$

and we bound each summand. For the first one, it is enough to apply Cauchy estimates to $D^{\mathrm{u}(j+1)}$:

$$\left\|\partial_u\big(\cosh^n u \cdot h^{[k]}(u)\big)\right\|_{\infty,\mathrm{c}}^{(j+1)} \leq \mathcal{O}\big(\delta^{-\gamma}\big)\left\|h^{[k]}\right\|_{n,\mathrm{c}}^{(j)}.$$

The second summand, using the second statement of Lemma 5.2, can be bounded as:

$$\begin{aligned}&\left\|n\cosh^{n-1} u \sinh u \cdot h^{[k]}(u)\right\|_{\infty,\mathrm{c}}^{(j+1)}\\ &\quad \leq \mathcal{O}(1)\|\tanh u\|_{\infty,\mathrm{c}}^{(j)} \cdot \left\|h^{[k]}\right\|_{n,\mathrm{c}}^{(j)} \leq \mathcal{O}\big(\delta^{-\gamma}\big)\left\|h^{[k]}\right\|_{n,\mathrm{c}}^{(j)}.\end{aligned}$$

Moreover, analogous bounds hold in $D^{\mathrm{u,f}}_{\gamma}$. Thus, $\|\partial_u h^{[k]}\|^{(j+1)}_{n,m} \leq \mathcal{O}(\delta^{-\gamma})\|h^{[k]}\|^{(j)}_{n,m}$, and hence $\|\partial_u h\|^{(j+1)}_{n,m,\sigma_i} \leq \mathcal{O}(\delta^{-\gamma})\|h\|^{(j)}_{n,m,\sigma_i}$.

The proof of the second statement is enough to consider the relation between supremum and weighted norms. □

Throughout this section, we are going to solve equations of the form $\mathcal{L}_\delta(h) = g$ where $\mathcal{L}_\delta = \delta^{-1}\partial_\tau + \partial_u$ is the differential operator defined in (79). Despite not being invertible, this operator has a right inverse (in fact, more than one) which will be used in several sections of this paper:

$$\mathcal{G}_\delta(h) = \int_{\mathbb{R}^-} h\big(u+t, \tau + \delta^{-1}t\big)\,\mathrm{d}t. \tag{99}$$

Thus, we state some technical lemmas which will be needed in the proofs of Theorems 3.6 and 5.9.

Lemma 5.4 *For all $h \in \mathcal{E}_{n,m}$ or $h \in \mathcal{E}_m$ with $m > 0$, we have*

$$\partial_u \int_{\mathbb{R}^-} h(u+t,\tau)\,\mathrm{d}t = \int_{\mathbb{R}^-} \partial_u h(u+t,\tau)\,\mathrm{d}t = h(u,\tau), \tag{100}$$

$$\partial_u \int_{\mathbb{R}^-} h(u+t,\tau+t)\,\mathrm{d}t = \int_{\mathbb{R}^-} \partial_u h(u+t,\tau+t)\,\mathrm{d}t. \tag{101}$$

Proof Since all the statements are proved analogously, we deal only with the first one. In order to prove that the integral and ∂_u commute for $h \in \mathcal{E}_{n,m}$, it is enough to prove the uniform convergence of the integral at ∞ since $h \in \mathcal{E}_{n,m}$ guarantees that $\partial_u h$ is differentiable in $D^{\mathrm{u}(i)}_{\gamma}$. Thus, we split the integral as

$$\int_{\mathbb{R}^-} \partial_u h(u+t,\tau)\,\mathrm{d}t = \int_{-\infty}^{-T} \partial_u h(u+t,\tau)\,\mathrm{d}t + \int_{-T}^{0} \partial_u h(u+t,\tau)\,\mathrm{d}t$$

taking $T > 0$ big enough such that $u + t \in D^{\mathrm{u,f}(i+1)}_{\gamma}$ for $t \in (-\infty, -T)$ and we have to bound the first one. Using the first statement of Lemma 5.3,

$$\left|\int_{-\infty}^{-T} \partial_u h(u+t,\tau)\,\mathrm{d}t\right| \leq \int_{-\infty}^{-T} \big|\cosh^{-m}(u+t)\big|\,\|\partial_u h\|^{(i+1)}_{n,m}\,\mathrm{d}t \leq \mathcal{O}\big(\delta^{\gamma}\big)\|h\|^{(i)}_{n,m}. \qquad \square$$

Lemma 5.5 *The operators $\mathcal{G}_\delta$ defined in (99) and $\bar{\mathcal{G}}_\delta = \partial_u \mathcal{G}_\delta$ are well defined in $\mathcal{E}_{n,m}$ and $\mathcal{E}_m$ for $m > 0$. Moreover,*

1. *$\mathcal{G}_\delta$ is linear from $\mathcal{E}_{n,m}$ to itself, commutes with ∂_u and $\mathcal{L}_\delta \circ \mathcal{G}_\delta = \mathrm{Id}$.*
2. *$\bar{\mathcal{G}}_\delta$ is linear from $\mathcal{E}_{n,m}$ to itself.*
3. *For $h \in \mathcal{E}_{n,m}$, $\|\mathcal{G}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(1)\|h\|_{n,m,\sigma_i}$. Furthermore, if*

$$\langle h\rangle = \frac{1}{2\pi}\int_0^{2\pi} h(u,\tau)\,\mathrm{d}\tau = 0,$$

then

$$\left\|\mathcal{G}_\delta(h)\right\|_{n,m,\sigma_i} \leq \mathcal{O}(\delta)\|h\|_{n,m,\sigma_i}.$$

4. *For* $h \in \mathcal{E}_{n,m}$, $\|\bar{\mathcal{G}}_\delta(h)\|_{n,m,\sigma_i} \leq \mathcal{O}(1)\|h\|_{n,m,\sigma_i}$.
5. *The same properties and bounds hold for* $h \in \mathcal{E}_m$.

Proof We take

$$g(u,\tau) = \mathcal{G}_\delta(h)(u,\tau) = \sum_{k\in\mathbb{Z}} g^{[k]}(u)\mathrm{e}^{\mathrm{i}k\tau},$$

where

$$g^{[k]}(u) = \int_{-\infty}^0 h^{[k]}(u+t)\mathrm{e}^{\mathrm{i}k\delta^{-1}t}\,\mathrm{d}t$$

which is integrable for $h \in \mathcal{E}_{n,m}$. In order to prove the lemma, we want bounds of $g^{[k]}$ in terms of $h^{[k]}$. These bounds are computed in different ways depending on the domain and whether $k \neq 0$ or $k = 0$.

We consider first the case $k \neq 0$. For $u \in D_\gamma^{\mathrm{u,f}(i)}$:

$$\cosh^m u \cdot g^{[k]}(u) = \int_{-\infty}^0 h^{[k]}(u+t)\cosh^m u \cdot \mathrm{e}^{\mathrm{i}k\delta^{-1}t}\,\mathrm{d}t.$$

Since the integrand has exponential decay for $\Re t \to -\infty$, we change the integration path to the line from 0 to $-\infty$ with angle $\mp\beta_0/2$, where the sign $+$ is taken for $k > 0$ and $-$ for $k < 0$. Moreover, using the third statement of Lemma 5.2, we obtain

$$\begin{aligned}\left|\cosh^m u \cdot g^{[k]}(u)\right| &\leq \mathcal{O}(1)\int_{-\infty}^0 \left\|h^{[k]}\right\|_{m,\mathrm{f}}\left|\mathrm{e}^{\mathrm{i}k\delta^{-1}\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2}} \cdot \mathrm{e}^{\mp\mathrm{i}\beta_0/2}\right|\mathrm{d}\xi \\ &\leq \mathcal{O}(\delta)\frac{\|h^{[k]}\|_{m,\mathrm{f}}}{|k|\sin(\beta_0/2)} \leq \mathcal{O}(\delta)\frac{1}{|k|}\left\|h^{[k]}\right\|_{n,m}.\end{aligned}$$

For $u \in D_\gamma^{\mathrm{u,c}(i)}$, we have to change the path of integration in the same way. Moreover, since $u+t$ can belong either to $D_\gamma^{\mathrm{u,c}(i)}$ or $D_\gamma^{\mathrm{u,f}(i)}$ and we use different norms in both domains, the integral must be split in two parts $(-\infty, T)$ and $(T, 0)$ where T is chosen such that $u+t \in D_\gamma^{\mathrm{u,f}(i)}$ for t with $\Re t < \Re T$. We recall that, in this case, $u \in D_\gamma^{\mathrm{u,c}(i)}$ and then $\cosh u$ is bounded.

$$\begin{aligned}\left|\cosh^n u \cdot g^{[k]}(u)\right| &= \left|\int_{-\infty}^T \cosh^n u \cdot h^{[k]}\big(u+\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2}\big)\cdot\mathrm{e}^{\mathrm{i}k\delta^{-1}\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2}}\mathrm{e}^{\mp\mathrm{i}\beta_0/2}\,\mathrm{d}\xi\right. \\ &\quad\left.+\int_T^0 \cosh^n u \cdot h^{[k]}\big(u+\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2}\big)\cdot\mathrm{e}^{\mathrm{i}k\delta^{-1}\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2}}\mathrm{e}^{\mp\mathrm{i}\beta_0/2}\,\mathrm{d}\xi\right| \\ &\leq \mathcal{O}(1)\left\|h^{[k]}\right\|_{m,\mathrm{f}}\int_{-\infty}^T \mathrm{e}^{|k|\sin(\beta_0/2)\delta^{-1}\xi}\,\mathrm{d}\xi\end{aligned}$$

$$
\begin{aligned}
&+\left\|h^{[k]}\right\|_{n,\mathrm{c}}\int_T^0\left(\frac{\cosh u}{\cosh(u+\xi\mathrm{e}^{\mp\mathrm{i}\beta_0/2})}\right)^n\mathrm{e}^{|k|\sin(\beta_0/2)\delta^{-1}\xi}\,\mathrm{d}\xi\\
&\le\mathcal{O}(\delta)\frac{1}{|k|}\left\|h^{[k]}\right\|_{m,\mathrm{f}}+\mathcal{O}(\delta)\frac{1}{|k|}\left\|h^{[k]}\right\|_{n,\mathrm{c}}\le\mathcal{O}(\delta)\frac{1}{|k|}\left\|h^{[k]}\right\|_{n,m},
\end{aligned}
$$

where we have used the third statement of Lemma 5.2.

Thus, for $k\neq 0$, we have obtained

$$
\left\|g^{[k]}\right\|_{n,m}\le\mathcal{O}(\delta)\frac{\|h^{[k]}\|_{n,m}}{|k|}. \tag{102}
$$

For $k=0$ and $u\in D_\gamma^{\mathrm{u,f}(i)}$, using the fourth statement of Lemma 5.2:

$$
\left|\cosh^m u\cdot g^{[0]}(u)\right|\le\left\|h^{[0]}\right\|_{m,\mathrm{f}}\int_{-\infty}^0\left|\frac{\cosh u}{\cosh(u+t)}\right|^m\mathrm{d}t\le\mathcal{O}(1)\left\|h^{[0]}\right\|_{n,m},
$$

and splitting the integral again for $u\in D_\gamma^{\mathrm{u,c}(i)}$:

$$
\begin{aligned}
\left|\cosh^n u\cdot g^{[0]}(u)\right|&\le\int_{-\infty}^T\left|\cosh^n u\cdot h^{[0]}(u+t)\right|\mathrm{d}t+\int_T^0\left|\cosh^n u\cdot h^{[0]}(u+t)\right|\mathrm{d}t\\
&\le\mathcal{O}(1)\left\|h^{[0]}\right\|_{m,\mathrm{f}}+\mathcal{O}(1)\left\|h^{[0]}\right\|_{n,\mathrm{c}}\le\mathcal{O}(1)\left\|h^{[0]}\right\|_{n,m}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\|\mathcal{G}_\delta(h)\right\|_{n,m,\sigma_i}&=\sum_{k\in\mathbb{Z}}\left\|g^{[k]}\right\|_{n,m}\mathrm{e}^{|k|\sigma_i}\le\mathcal{O}(1)\left\|h^{[0]}\right\|_{n,m}+\mathcal{O}(\delta)\sum_{k\neq 0}\left\|h^{[k]}\right\|_{n,m}\frac{1}{|k|}\mathrm{e}^{|k|\sigma_i}\\
&\le\mathcal{O}(1)\|h\|_{n,m,\sigma_i}.
\end{aligned}
$$

Moreover, if $\langle h\rangle=h^{[0]}(u)=0$, then

$$
\left\|\mathcal{G}_\delta(h)\right\|_{n,m,\sigma_i}\le\mathcal{O}(\delta)\|h\|_{n,m,\sigma_i}.
$$

For the fourth statement, we consider the derivative of the Fourier coefficients and Lemmas 5.3 and 5.4:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}g^{[k]}(u)&=\int_{-\infty}^0\frac{\mathrm{d}}{\mathrm{d}u}h^{[k]}(u+t)\mathrm{e}^{\mathrm{i}k\delta^{-1}t}\,\mathrm{d}t\\
&=\int_{-\infty}^0\frac{\mathrm{d}}{\mathrm{d}t}\left(h^{[k]}(u+t)\mathrm{e}^{\mathrm{i}k\delta^{-1}t}\right)\mathrm{d}t-\mathrm{i}k\delta^{-1}\int_{-\infty}^0h^{[k]}(u+t)\mathrm{e}^{\mathrm{i}k\delta^{-1}t}\,\mathrm{d}t\\
&=h^{[k]}(u)-\lim_{t\to-\infty}\left(h^{[k]}(u+t)\mathrm{e}^{\mathrm{i}k\delta^{-1}t}\right)-\mathrm{i}k\delta^{-1}g^{[k]}(u)\\
&=h^{[k]}(u)-\mathrm{i}k\delta^{-1}g^{[k]}(u).
\end{aligned} \tag{103}
$$

Hence, using (102), it is clear that for $k \neq 0$, $\|\frac{\mathrm{d}}{\mathrm{d}u} g^{[k]}\|_{n,m} \leq \mathcal{O}(1)\|h^{[k]}\|_{n,m}$. Moreover, recalling that $\frac{\mathrm{d}}{\mathrm{d}u} g^{[0]} = h^{[0]}$, we obtain

$$\left\|\partial_u \mathcal{G}_\delta(h)\right\|_{n,m,\sigma_i} = \|\partial_u g\|_{n,m\sigma_i} = \sum_{k\in\mathbb{Z}} \left\|\frac{\mathrm{d}}{\mathrm{d}u} g^{[k]}\right\|_{n,m} \mathrm{e}^{|k|\sigma_i} \leq \mathcal{O}(1)\|h\|_{n,m,\sigma_i}.$$

Therefore, we have seen that if $h \in \mathcal{E}_{n,m}$, then both g and $\partial_u g$ belong to $\mathcal{E}_{n,m}$. In order to prove $\mathcal{L}_\delta \circ \mathcal{G}_\delta = \mathrm{Id}$, we use (103):

$$\begin{aligned}\mathcal{L}_\delta \circ \mathcal{G}_\delta(h)(u,\tau) &= \sum_{k\in\mathbb{Z}} \mathcal{L}_\delta\left(g^{[k]}(u)\mathrm{e}^{ik\tau}\right) = \sum_{k\in\mathbb{Z}} \left(\mathrm{i}k\delta^{-1} g^{[k]}(u) + \frac{\mathrm{d}}{\mathrm{d}u} g^{[k]}(u)\right)\mathrm{e}^{ik\tau} \\ &= h(u,\tau).\end{aligned}$$

For functions belonging to $\mathcal{E}_m$, all the statements are proved analogously. □

5.2 Proof of Theorem 3.6

In order to prove Theorem 3.6 through a fixed point argument, we work with $Q = T - T_0 - \delta T_1$. Replacing it in (40), using (39), and considering the linear operator $\mathcal{L}_\delta$ defined in (79), we obtain

$$\mathcal{L}_\delta Q = \mathcal{F}(\partial_u Q), \tag{104}$$

where

$$\begin{aligned}\mathcal{F}(h) = &-\frac{\cosh^2 u}{8} h^2 \\ &- \delta\left(\tanh u f_1(\tau) + \frac{1}{\cosh u} f_2(\tau)\right) h + \frac{2}{J\cosh^2 u}\left(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau)\right) \\ &- \frac{1}{J}\psi(u)\left(\sin x_{\mathrm{p}}(\tau) + \sin(\mu\sin\tau)\right) - 4\delta\left(\frac{\sinh u}{\cosh^3 u} f_1(\tau) + \frac{1}{\cosh^3 u} f_2(\tau)\right) \\ &- 2\delta^2\left(\left(f_1(\tau)\right)^2 \frac{\sinh^2 u}{\cosh^4 u} + 2 f_1(\tau) f_2(\tau)\frac{\sinh u}{\cosh^4 u} + \left(f_2(\tau)\right)^2 \frac{1}{\cosh^4 u}\right)\end{aligned} \tag{105}$$

and f_1 and f_2 are the functions introduced in Proposition 3.5.

Using the notation introduced in Sect. 5.1 and considering a fixed point argument in $\partial_u Q^-$ instead of Q^-, Theorem 3.6 can be rewritten as the following proposition.

Proposition 5.6 *There exists δ_0 such that for $\delta \in (0, \delta_0)$, there exists a function $Q^-(u,\tau)$ for $(u,\tau) \in D_\gamma^{\mathrm{u}(1)} \times \mathbb{T}_{\sigma_1}$ such that $\partial_u Q^- \in \mathcal{E}_{4,2}$ is a fixed point of the functional*

$$\bar{\mathcal{F}}(h) = \bar{\mathcal{G}}_\delta \mathcal{F}(h), \tag{106}$$

where $\bar{\mathcal{G}}_\delta$ is the operator defined in Lemma 5.5. Furthermore,

$$\|\partial_u Q^-\|_{4,2,\sigma_1} \le \mathcal{O}\big(\mu\delta^2\big),$$
$$\big\|\partial_u^2 Q^-\big\|_{4,2,\sigma_1} \le \mathcal{O}\big(\mu\delta^{2-\gamma}\big).$$

Considering the expected behavior of $\partial_u Q^-$ at infinity and near the singularities in $u = \pm i\pi/2$ of T_0, we look for it in the Banach space $\mathcal{E}_{4,2}$ defined in (97) using the corresponding norm.

The next lemma gives the main properties of the functional $\bar{\mathcal{F}}$.

Lemma 5.7 *There exists a constant $b_1 > 0$ such that the functional $\bar{\mathcal{F}}$ defined in (106) holds the following properties*:

1. *If $h \in \mathcal{E}_{4,2}$, $\bar{\mathcal{F}}(h) \in \mathcal{E}_{4,2}$.*
2. $\|\bar{\mathcal{F}}(0)\|_{4,2,\sigma_1} \le \frac{b_1}{2}\mu\delta^2$.
3. *For $h_1, h_2 \in B(b_1\mu\delta^2) \subset \mathcal{E}_{4,2}$, $\|\bar{\mathcal{F}}(h_2) - \bar{\mathcal{F}}(h_1)\|_{4,2,\sigma_1} \le \mathcal{O}(\mu\delta^{1-\gamma}) \times \|h_2 - h_1\|_{4,2,\sigma_1}$.*

Proof The first statement is clear recalling that all the terms of $\mathcal{F}$ are analytic in $D_\gamma^{\mathrm{u}(0)}$, considering their behavior at infinity and applying the fourth statement of Lemma 5.5.

For the second statement, we split $\mathcal{F}(0)$ as $\mathcal{F}(0)(u,\tau) = q_1(u,\tau) + q_2(u,\tau)$ where

$$\begin{aligned} q_1(u,\tau) = {}& \frac{2}{J\cosh^2 u}\big(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau)\big) - \frac{1}{J}\psi(u)\big(\sin x_{\mathrm{p}}(\tau) + \sin(\mu\sin\tau)\big) \\ & + \delta^2\left(2\big(f_1(\tau)\big)^2\frac{\sinh^2 u}{\cosh^4 u} + 4f_1(\tau)f_2(\tau)\frac{\sinh u}{\cosh^4 u} + 2\big(f_2(\tau)\big)^2\frac{1}{\cosh^4 u}\right), \\ q_2(u,\tau) = {}& -4\delta\left(\frac{\sinh u}{\cosh^3 u}f_1(\tau) + \frac{1}{\cosh^3 u}f_2(\tau)\right). \end{aligned}$$

Using Corollary 4.2 and statement (45), one can see that $\|q_1\|_{4,2,\sigma_1} \le \mathcal{O}(\mu\delta^2)$. Therefore, this bound and the fourth statement of Lemma 5.5, give $\|\bar{\mathcal{G}}_\delta(q_1)\|_{4,2,\sigma_1} \le \mathcal{O}(\mu\delta^2)$.

On the other hand, q_2 has zero mean. Thus, commuting ∂_u with $\mathcal{G}_\delta$, applying statement three of Lemma 5.5 and statement (45), we obtain $\|\bar{\mathcal{G}}_\delta(q_2)\|_{4,2,\sigma_1} \le \mathcal{O}(\delta)\|\partial_u q_2\|_{4,2,\sigma_1} \le \mathcal{O}(\mu\delta^2)$.

Taking $b_1 = 2\|\bar{\mathcal{F}}\|_{4,2}/\mu\delta^2$, we obtain the second statement.

For the third statement, for $h_1, h_2 \in B(b_1\mu\delta^2) \subset \mathcal{E}_{4,2}$,

$$\begin{aligned} &\big\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\big\|_{4,2,\sigma_1} \\ &\quad \le \left(\frac{\mathcal{O}(1)}{|\cosh^2 u|}\|h_1 + h_2\|_{4,2,\sigma_1} + \mathcal{O}(\delta)\left\|\frac{\sinh u}{\cosh u}f_1(\tau) + \frac{1}{\cosh u}f_2(\tau)\right\|_{\infty,\sigma_1}\right) \\ &\qquad \times \|h_1 - h_2\|_{4,2,\sigma_1} \\ &\quad \le \mathcal{O}\big(\mu\delta^{1-\gamma}\big)\|h_1 - h_2\|_{4,2,\sigma_1}. \end{aligned}$$

Thus, applying the fourth statement of Lemma 5.5, we obtain the result. □

With this lemma, we are ready to prove the existence of the unstable manifold in the outer domain.

Proof of Proposition 5.6 Lemma 5.7 gives that the functional $\bar{\mathcal{F}}$ is a contraction from $B(b_1\mu\delta^2)\subset\mathcal{E}_{4,2}$ to itself, and hence it has a fixed point h^-. Furthermore, calling $Q^-=\mathcal{G}_\delta\mathcal{F}(h^-)$, we have that $h^-=\partial_u Q^-$ and Q^- is solution of (104).

In order to bound $\partial_u^2 Q^-$ it is enough to apply the first statement of Lemma 5.3 to the nested domains $D_\gamma^{\mathrm{u}(1)}$ and $D_\gamma^{\mathrm{u}(0)}$. □

For the stable manifold, the proof can be done analogously.

Corollary 5.8 *Equation* (40) *has a solution of the form* $T^+=T_0+\delta T_1+Q^+$ *defined in* $D_\gamma^{\mathrm{s}(1)}\times\mathbb{T}_{\sigma_1}$ *such that* $\lim_{\Re u\to+\infty}\cosh u\cdot Q^+(u,\tau)=0$. *Moreover, the following bounds hold*

$$\left\|\partial_u Q^+\right\|_{4,2,\sigma_1}\le\mathcal{O}\left(\mu\delta^2\right),$$
$$\left\|\partial_u^2 Q^+\right\|_{4,2,\sigma_1}\le\mathcal{O}\left(\mu\delta^{2-\gamma}\right).$$

5.3 Behavior of the Invariant Unstable Manifold at Infinity

Even though the existence of the invariant manifolds in the outer domains has already been proved in Sect. 5.2, in order to prove Theorem 3.11 it will be necessary to have more information about the behavior of the unstable invariant manifold T^- as $\Re u\to-\infty$.

In fact, the use of the weighted norm in the proof of Theorem 3.6, has showed that $Q^-\sim \mathrm{e}^{2u}$ as $\Re u\to-\infty$. Replacing $Q^-(u,\tau)$ by $Q^-(u,\tau)=8\lambda(\tau)\mathrm{e}^{2u}+\mathcal{O}(\mathrm{e}^{3u})$ in (104) shows (107) which $\lambda(\tau)$ has to satisfy. The solution of this equation is studied in Lemma 5.10 and the behavior of $Q^-(u,\tau)$ as $\Re u\to-\infty$ is given in next theorem.

Theorem 5.9 *There exists* $\delta_0>0$ *such that for all* $\delta\in(0,\delta_0)$, *the differential equation*

$$\begin{aligned}\lambda' = {} & -2\delta\lambda+2\delta^2 f_1(\tau)\lambda-\delta\lambda^2+\frac{\delta}{J}\left(\cos x_{\mathrm{p}}(\tau)-\cos(\mu\sin\tau)\right)\\ & +2\delta^2 f_1(\tau)-\delta^3\left(f_1(\tau)\right)^2,\end{aligned}\tag{107}$$

where f_1 *is the function defined in Proposition* 3.5, *has a unique real-analytic solution defined in* $\mathbb{T}_{\sigma_0}$, *which holds* $|\lambda(\tau)|\le\mathcal{O}(\mu\delta^2)$.

Moreover, for all $(u,\tau)\in D_\gamma^{\mathrm{u,f}(0)}\times\mathbb{T}_{\sigma_1}$,

$$\left|\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau)-\lambda(\tau)\right|\le|\cosh u|^{-1}\mathcal{O}\left(\mu\delta^2\right).$$

The proof of this theorem is done in several steps.

Lemma 5.10 *There exists* $\delta_0 > 0$ *such that for* $\delta \in (0, \delta_0)$, (107) *has a unique real-analytic solution defined in* $\mathbb{T}_{\sigma_0}$. *Moreover, it holds* $|\lambda(\tau)| \le \mathcal{O}(\mu\delta^2)$.

Proof Equation (107) can be rewritten as $\lambda' = -2\delta\lambda - \delta F(\tau, \lambda(\tau))$, with

$$F(\tau, z) = -2\delta f_1(\tau)z + z^2 - \frac{1}{J}\bigl(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau)\bigr) - 2\delta f_1(\tau) + \delta^2\bigl(f_1(\tau)\bigr)^2.$$

Thus, λ is a 2π-periodic solution of (107) provided it holds:

$$\begin{aligned}\lambda(\tau) &= \mathcal{F}(\lambda)(\tau)\\ &= -\delta e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s} F\bigl(s, \lambda(s)\bigr)\,ds + \frac{1}{e^{4\pi\delta} - 1}\int_0^{2\pi} e^{2\delta s} F\bigl(s, \lambda(s)\bigr)\,ds\right).\end{aligned} \tag{108}$$

We will prove that, for certain $C_1 > 0$, $\mathcal{F}$ is a contraction from $B(C_1\mu\delta^2) \subset \mathcal{X}_{\mathrm{p}}$ to itself, where $\mathcal{X}_{\mathrm{p}}$ is the Banach space

$$\mathcal{X}_{\mathrm{p}} = \bigl\{z : \mathbb{T}_{\sigma_0} \to \mathbb{C}\text{: real-analytic, } 2\pi\text{-periodic, } \|z\|_\infty < \infty\bigr\}.$$

Note that $F(\tau, 0) = \tilde{F}(\tau) - 2\delta f_1(\tau)$ where $\tilde{F}(\tau) = -(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau))/J + \delta^2(f_1(\tau))^2$. We split $\mathcal{F}(0)$ as $\mathcal{F}(0)(\tau) = h_1(\tau) + h_2(\tau)$ where

$$\begin{aligned}h_1(\tau) &= -\delta e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s}\tilde{F}(s)\,ds + \frac{1}{e^{4\pi\delta} - 1}\int_0^{2\pi} e^{2\delta s}\tilde{F}(s)\,ds\right),\\ h_2(\tau) &= 2\delta^2 e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s} f_1(s)\,ds + \frac{1}{e^{4\pi\delta} - 1}\int_0^{2\pi} e^{2\delta s} f_1(s)\,ds\right)\end{aligned}$$

and we bound each term in different ways. For the first one, by Corollary 4.2 and statement (45), since $|\tilde{F}(\tau)| \le \mathcal{O}(\mu\delta^2)$,

$$\bigl|h_1(\tau)\bigr| \le \mathcal{O}\bigl(\mu\delta^3\bigr)e^{-2\delta\tau}\left(\int_0^\tau ds + \frac{1}{e^{4\pi\delta} - 1}\int_0^{2\pi} ds\right) \le \mathcal{O}\bigl(\mu\delta^2\bigr).$$

For the second one, we define $\hat{f}_1$ such that $\hat{f}_1' = f_1$ and $\langle \hat{f}_1 \rangle = 0$ and we integrate it by parts. From statement (45), it is clear that $\hat{f}_1 \le \mathcal{O}(\mu)$. Using this bound and the 2π-periodicity of $\hat{f}_1$, it is obtained

$$h_2(\tau) = 2\delta^2\hat{f}_1(\tau) - 4\delta^3 e^{-2\delta\tau}\left(\int_0^\tau e^{2\delta s}\hat{f}_1(s)\,ds + \frac{1}{e^{4\pi\delta} - 1}\int_0^{2\pi} e^{2\delta s}\hat{f}_1(s)\,ds\right).$$

Hence, reasoning as for h_1 it is clear that $|h_2(\tau)| \le \mathcal{O}(\mu\delta^2)$. Thus, there exists C_1 such that $\|\mathcal{F}(0)\|_\infty \le \frac{C_1}{2}\mu\delta^2$.

Moreover, taking $\lambda_1, \lambda_2 \in B(C_1\mu\delta^2) \subset \mathcal{X}_{\mathrm{p}}$,

$$\begin{aligned}
&\mathcal{F}(\lambda_1)(\tau) - \mathcal{F}(\lambda_2)(\tau)\\
&\quad \leq \delta e^{-2\delta\tau} \int_0^\tau e^{2\delta s}\bigl(-2\delta f_1(\tau)\bigl(\lambda_1(s)-\lambda_2(s)\bigr) + \bigl(\lambda_1(s)-\lambda_2(s)\bigr)^2\bigr)\,ds\\
&\qquad + \frac{\delta e^{-2\delta\tau}}{e^{4\pi\delta}-1} \int_0^{2\pi} e^{2\delta s}\bigl(-2\delta f_1(\tau)\bigl(\lambda_1(s)-\lambda_2(s)\bigr) + \bigl(\lambda_1(s)-\lambda_2(s)\bigr)^2\bigr)\,ds.
\end{aligned}$$

Thus,

$$\begin{aligned}
\bigl|\mathcal{F}(\lambda_1)(\tau) - \mathcal{F}(\lambda_2)(\tau)\bigr| &\leq \mathcal{O}\bigl(\mu\delta^2\bigr)\|\lambda_1-\lambda_2\|_\infty \int_0^\tau ds + \mathcal{O}(\mu\delta)\|\lambda_1-\lambda_2\|_\infty \int_0^{2\pi} ds\\
&\leq \mathcal{O}(\mu\delta)\|\lambda_1-\lambda_2\|_\infty.
\end{aligned}$$

Therefore, reducing δ if it was necessary, $\mathcal{F}$ is a contraction from $B(C_1\mu\delta^2) \subset \mathcal{X}_{\mathrm{p}}$ to itself and there exists a unique fixed point $\lambda(\tau)$ which is a solution of (107). $\square$

In order to see that $Q^- = 8\lambda(\tau)e^{2u} + \mathcal{O}(e^{3u})$, we replace $Q^-(u,\tau) = 8\lambda(\tau)e^{2u} + P^-(u,\tau)$ in (104). Using also (107) for $\lambda(\tau)$, we obtain

$$\mathcal{L}_\delta P^- = \mathcal{H}_2\bigl(\partial_u P^-, u, \tau\bigr), \tag{109}$$

where

$$\begin{aligned}
&\mathcal{H}_2(h,u,\tau)\\
&\quad = -\frac{\cosh^2 u}{8}h^2 - 4\cosh^2 u e^{2u}\lambda(\tau)h - \delta\left(\frac{\sinh u}{\cosh u} f_1(\tau) + \frac{1}{\cosh u} f_2(\tau)\right)h\\
&\qquad + 8\bigl(\lambda(\tau)\bigr)^2 e^{2u}\bigl(1 - 4\cosh^2 u e^{2u}\bigr)\\
&\qquad - 16\lambda(\tau)e^{2u}\delta\left((\tanh u + 1)f_1(\tau) + \frac{1}{\cosh u} f_2(\tau)\right)\\
&\qquad + \frac{2}{J}\bigl(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau)\bigr)\left(\frac{1}{\cosh^2 u} - 4e^{2u}\right)\\
&\qquad + \frac{1}{J}\psi(u)\bigl(\sin x_{\mathrm{p}}(\tau) + \sin(\mu\sin\tau)\bigr)\\
&\qquad - 4\delta\left(\left(\frac{\sinh u}{\cosh^3 u} + 4e^{2u}\right)f_1(\tau) + \frac{1}{\cosh^3 u} f_2(\tau)\right)\\
&\qquad - 2\delta^2\left(\left(\frac{\sinh^2 u}{\cosh^4 u} - 4e^{2u}\right)\bigl(f_1(\tau)\bigr)^2 + 2f_1(\tau)f_2(\tau)\frac{\sinh u}{\cosh^4 u}\right.\\
&\qquad \left. + \bigl(f_2(\tau)\bigr)^2\frac{1}{\cosh^4 u}\right)
\end{aligned}$$

and f_1 and f_2 are the functions defined in Proposition 3.5.

In order to solve (109), using the operator $\bar{\mathcal{G}}_\delta$ defined in Lemma 5.5, we observe that $\partial_u P^-$ will be a fixed point of

$$\bar{\mathcal{H}}_2 = \bar{\mathcal{G}}_\delta \circ \mathcal{H}_2. \tag{110}$$

Then we will look for it in the Banach space $\mathcal{E}_3$ defined in (98). We will need the following technical lemma.

Lemma 5.11 *There exists a constant $C_2 > 0$ such that the functional $\bar{\mathcal{H}}_2$ defined in* (110) *holds the following properties*:

1. *If $h \in \mathcal{E}_3$, $\bar{\mathcal{H}}_2(h, \cdot, \cdot) \in \mathcal{E}_3$.*
2. *There exists $C_2 > 0$ such that $\|\bar{\mathcal{H}}_2(0, \cdot, \cdot)\|_{3,f,\sigma_1} \le \frac{C_2}{2}\mu\delta^2$.*
3. *For all $h_1, h_2 \in B(C_2\mu\delta^2) \subset \mathcal{E}_3$, $\|\bar{\mathcal{H}}_2(h_2, \cdot, \cdot) - \bar{\mathcal{H}}_2(h_1, \cdot, \cdot)\|_{3,f,\sigma_1} \le \mathcal{O}(\mu\delta) \times \|h_2 - h_1\|_{3,f,\sigma_1}$.*

Proof The first statement is straightforward recalling that all the terms of $\mathcal{H}_2$ are real-analytic in $D_\gamma^{\mathrm{u,f(0)}}$, considering their behavior at infinity (taking advantage of the cancellations) and applying Lemma 5.5. For the second statement, $\mathcal{H}_2(0, u, \tau)$ is split as $\mathcal{H}_2(0, u, \tau) = p_1(u, \tau) + p_2(u, \tau)$ where

$$\begin{aligned}
p_1(u, \tau) = {} & 8\big(\lambda(\tau)\big)^2 \mathrm{e}^{2u}\big(1 - 4\cosh^2 u \mathrm{e}^{2u}\big) \\
& - 16\lambda(\tau)\mathrm{e}^{2u}\delta\left((\tanh u + 1) f_1(\tau) + \frac{1}{\cosh u} f_2(\tau)\right) \\
& + \frac{2}{J}\big(\cos x_{\mathrm{p}}(\tau) - \cos(\mu \sin\tau)\big)\left(\frac{1}{\cosh^2 u} - 4\mathrm{e}^{2u}\right) \\
& + \frac{1}{J}\psi(u)\big(\sin x_{\mathrm{p}}(\tau) + \sin(\mu\sin\tau)\big) \\
& - 2\delta^2\left(\left(\frac{\sinh^2 u}{\cosh^4 u} - 4\mathrm{e}^{2u}\right)\big(f_1(\tau)\big)^2 + 2 f_1(\tau) f_2(\tau)\frac{\sinh u}{\cosh^4 u}\right. \\
& \left. + \big(f_2(\tau)\big)^2 \frac{1}{\cosh^4 u}\right), \\
p_2(u, \tau) = {} & -4\delta\left(\left(\frac{\sinh u}{\cosh^3 u} + 4\mathrm{e}^{2u}\right) f_1(\tau) + \frac{1}{\cosh^3 u} f_2(\tau)\right).
\end{aligned}$$

Using the properties of the norm given in Lemma 5.1, statement (45), Corollary 4.2, and the bound of λ obtained in Lemma 5.10, it is clear that $\|p_1(u, \tau)\|_{3,\mathrm{f},\sigma_1} \le \mathcal{O}(\mu\delta^2)$, and thus by Lemma 5.5,

$$\big\|\bar{\mathcal{G}}_\delta\big(p_1(u, \tau)\big)\big\|_{3,\mathrm{f},\sigma_1} \le \mathcal{O}\big(\mu\delta^2\big).$$

On the other hand, since $p_2(u, \tau)$ has zero mean, commuting $\mathcal{G}_\delta$ and ∂_u and using again Lemmas 5.5 and 5.1 and statement (45), we obtain

$$\big\|\bar{\mathcal{G}}_\delta\big(p_1(u, \tau)\big)\big\|_{3,\mathrm{f},\sigma_1} \le \mathcal{O}(\delta)\big\|\partial_u p_2(u, \tau)\big\|_{3,\mathrm{f},\sigma_1} \le \mathcal{O}\big(\mu\delta^2\big),$$

which gives the second statement of the lemma with a suitable constant C_2.

The proof of the third statement is analogous to the proof of the third statement of Lemma 5.7. $\square$

Proof of Theorem 5.9 Using the results obtained in Lemma 5.11, it is clear that the functional $\bar{\mathcal{H}}_2$ defined in (110) is a contraction from $B(C_2\mu\delta^2)\subset\mathcal{E}_{3,\mathrm{f}}$ to itself. Thus, it has a unique fixed point h^- which holds $h^-=\partial_u P^-$ with $P^-=\mathcal{G}_\delta\mathcal{H}_2(h^-,u,\tau)$ and then $\|\partial_u P^-\|_{3,\mathrm{f},\sigma_1}\le C_2\mu\delta^2$.

Therefore, since $|\partial_u P^-(u,\tau)|\le|\cosh^{-3}u|\mathcal{O}(\mu\delta^2)$, we obtain

$$\begin{aligned}\left|\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau)-\lambda(\tau)\right| &\le \left|\frac{\cosh^2 u}{4}\partial_u P^-(u,\tau)\right|+\left|4\cosh^2 u e^{2u}-1\right|\left|\lambda(\tau)\right|\\ &\le \left|\cosh^{-1}u\right|\mathcal{O}\big(\mu\delta^2\big).\end{aligned}$$

$\square$

6 Invariant Manifolds in the Inner Domains: Proof of Theorem 3.9

In this section, we will prove the existence of the unstable invariant manifold in the inner domain (52) (the proof of the existence of the stable one is analogous). As we will work in the inner variables (z,τ) (see (57)), the manifold $\phi^-(z,\tau)$ (see (58)) is solution of (61). Moreover, we will have to impose that the solution has a given initial condition, which comes from the information that we have obtained from the outer domain. This will allow us to guarantee that the solution we will find in this section is the analytic continuation of the invariant manifold $T^-(u,\tau)$ obtained in Theorem 3.6.

We recall that we have constructed the inner and outer domains in such a way that, for all $0\le i,j\le N$, $D_\gamma^{\mathrm{u}(i)}\cap D_{\delta,\pm}^{\mathrm{u}(j)}\ne\emptyset$. Therefore, it will be in this intersecting zone, usually called matching domain, where we will use the information given by the invariant manifold in the outer domain in order to continue it to the inner one. We recall that in this section we will work in the inner variables (z,τ), and hence we will consider the domains $\mathcal{D}_{\delta,\pm}^{\mathrm{u}(i)}\times\mathbb{T}_{\sigma_1}$ defined in (59), so that we will take the initial conditions in curves of the form

$$\begin{aligned}\Gamma_{\gamma,+}^{\mathrm{u}(i)}=\Big\{&z\in\mathbb{C}:\ \Im z=-\tan\beta_0\big(\Re z+\tilde{a}^{(i)}\delta^{\gamma-1}\big),\\ &\frac{\tan\beta_0}{\tan\beta_1-\tan\beta_0}\big(c^{(i)}\ln(1/\delta)-\tan\beta_1\tilde{a}^{(i)}\delta^{\gamma-1}\big)<\Im z<-c^{(i)}\ln(1/\delta)\Big\},\end{aligned}\tag{111}$$

which belong to the matching domain (see Fig. 11).

It is expected that in the inner domains the unstable invariant manifold ϕ^- is well approximated by the solution ϕ_0^- of (64) given in Theorem 3.8. Thus, we consider the corresponding partial differential equation for $\varphi^-(z,\tau)=\phi^-(z,\tau)-\phi_0^-(z,\tau)$ for $(z,\tau)\in\mathcal{D}_{\delta,+}^{\mathrm{u}(0)}\times\mathbb{T}_{\sigma_1}$:

$$\partial_\tau\varphi-\frac{1}{4}z^2\partial_z\phi_0^-\partial_z\varphi=a(z,\tau)\partial_z\varphi+b(z,\tau)(\partial_z\varphi)^2+c(z,\tau),\tag{112}$$

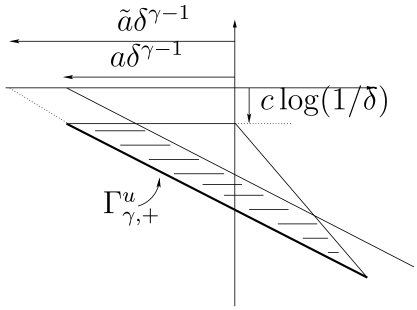


Fig. 11 The matching domain and $\mathit{\Gamma}^{\mathrm{u}}_{\gamma,+}$ in variable z

where

$$a(z,\tau) = -\left(\frac{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}{4\delta^2} + \frac{z^2}{4}\right)\partial_z\phi_0^-(z,\tau), \tag{113}$$

$$b(z,\tau) = -\frac{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}{8\delta^2}, \tag{114}$$

$$\begin{aligned} c(z,\tau) = {} & -\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}\cos x_{\mathrm{p}}(\tau) - \frac{2}{Jz^2}\cos(\mu\sin\tau) \\ & + \frac{\delta^2}{J}\psi\left(\frac{\pi}{2}\mathrm{i}+\delta z\right)\sin x_{\mathrm{p}}(\tau) - \frac{2\mathrm{i}}{Jz^2}\sin(\mu\sin\tau) \\ & + \left(\frac{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}{8\delta^2} + \frac{z^2}{8}\right)\left(\partial_z\phi_0^-(z,\tau)\right)^2. \end{aligned} \tag{115}$$

We know that the function $\varphi^-(z,\tau)$ is an analytic solution of (112) in a neighborhood of $\mathit{\Gamma}^{\mathrm{u}(0)}_{\gamma,+}$ contained in the matching domain. So, we look for a solution of (112) with prescribed initial conditions $\varphi^-(z,\tau) = \phi^-(z,\tau) - \phi_0^-(z,\tau)$ where (z,τ) are in a suitable curve in the matching domain (close to $\mathit{\Gamma}^{\mathrm{u}(0)}_{\gamma,+}$), which a posteriori will be the analytic continuation of φ^-. In fact, as we are interested in $\partial_z\varphi^-$, the initial condition we will take is $\partial_z\varphi^- = \partial_z\phi^- - \partial_z\phi_0^-$.

In order to solve (112), the first step is to perform the change of variables $z = x + R^-(x,\tau)$, given in Theorem 3.8, which conjugates the linear differential operator $\partial_\tau - \frac{1}{4}z^2\partial_z\phi_0^-\partial_z$ to $\partial_\tau + \partial_x$ and is defined from $\mathcal{D}^{\mathrm{u}(1)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$ to $\mathcal{D}^{\mathrm{u}(0)}_{\delta,+}$.

The second step is to extend the function $\mathit{\Phi}^-(x,\tau) = \varphi^-(x+R^-(x,\tau),\tau)$ already defined in a neighborhood of $\mathit{\Gamma}^{\mathrm{u}(1)}_{\gamma,+}$ to the domain $\mathcal{D}^{\mathrm{u}(1)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$, that satisfies

$$\mathcal{L}\mathit{\Phi} = A(x,\tau)\partial_x\mathit{\Phi} + B(x,\tau)(\partial_x\mathit{\Phi})^2 + C(x,\tau), \tag{116}$$

where

$$A(x,\tau) = a\left(x+R^-(x,\tau),\tau\right)\frac{1}{1+\partial_x R^-(x,\tau)}, \tag{117}$$

$$B(x,\tau)=b\bigl(x+R^-(x,\tau),\tau\bigr)\frac{1}{(1+\partial_x R^-(x,\tau))^2},\tag{118}$$

$$C(x,\tau)=c\bigl(x+R^-(x,\tau),\tau\bigr)\tag{119}$$

and $\mathcal{L}$ is the operator defined in (69).

The next proposition shows the existence of $\Phi^-(x,\tau)$ in a reduced inner domain $\mathcal{D}^{\mathrm{u}(4)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$ as a solution of (116) with prescribed initial condition expressed in the new variables (x,τ):

$$\begin{aligned}\partial_x\Phi^-\bigl(x^*,\tau\bigr)&=\eta\bigl(x^*,\tau\bigr)\\&=\Bigl(\partial_z\phi^-\bigl(x^*+R^-\bigl(x^*,\tau\bigr),\tau\bigr)-\partial_z\phi_0^-\bigl(x^*+R^-\bigl(x^*,\tau\bigr),\tau\bigr)\Bigr)\\&\quad\times\frac{1}{1+\partial_x R^-(x^*,\tau)}\end{aligned}\tag{120}$$

when $(x^*,\tau)\in\Gamma^{\mathrm{u}(1)}_{\gamma,+}$ defined in (111).

Proposition 6.1 *Let us consider a constant $\sigma_2<\sigma_1$ and $\gamma\in(1/3,1/2)$. Then there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there exists an analytic function $\Phi^-(x,\tau)$ for $(x,\tau)\in\mathcal{D}^{\mathrm{u}(3)}_{\delta,+}\times\mathbb{T}_{\sigma_2}$, which holds the partial differential equation (116) and the initial condition $\partial_x\Phi^-(x^*,\tau)=\eta(x^*,\tau)$ for $(x^*,\tau)\in\Gamma^{\mathrm{u}(1)}_{\gamma,+}\times\mathbb{T}_{\sigma_2}$. Moreover,*

$$\|\partial_x\Phi\|_\infty\le\mathcal{O}\bigl(\delta^2\bigr).$$

Considering $\Psi=\partial_x\Phi^-$ and differentiating expression (116) we obtain

$$\tilde{\mathcal{L}}\Psi=\partial_x A(x,\tau)\Psi+\partial_x B(x,\tau)\Psi^2+\partial_x C(x,\tau),\tag{121}$$

where

$$\tilde{\mathcal{L}}=\partial_\tau+\bigl(1-A(x,\tau)-2B(x,\tau)\Psi(x,\tau)\bigr)\partial_x.\tag{122}$$

Since this partial differential equation is quasilinear it can be solved through a characteristics-like method.

The initial condition $\eta(x^*,\tau)$ defined in (120), which is given in terms of the invariant manifold ϕ^- and ϕ_0^- is bounded in the next lemma.

Lemma 6.2 *For any fixed $\gamma\in(0,1/2)$, there exists δ_0 such that for all $\delta\in(0,\delta_0)$, the following bound holds*

$$\forall(x,\tau)\in\Gamma^{\mathrm{u}(1)}_{\gamma,+}\times\mathbb{T}_{\sigma_1},\quad\bigl|\eta(x,\tau)\bigr|\le\mathcal{O}\bigl(\delta^2\bigr).$$

Proof By definition

$$\Phi^-(x,\tau)=\varphi^-\bigl(x+R^-(x,\tau),\tau\bigr)=\phi^-\bigl(x+R^-(x,\tau),\tau\bigr)-\phi_0^-\bigl(x+R^-(x,\tau),\tau\bigr),$$

and since $x \in \Gamma^{\mathrm{u}(1)}_{\gamma,+}$, $|\delta x| \le \mathcal{O}(\delta^\gamma)$. Using (58) and the definition of T_0 and T_1 given in (43) and (44), we bound first $\partial_z \varphi^-(z,\tau)$:

$$\begin{aligned}\left|\partial_z\varphi^-(z,\tau)\right| \le \delta^2 &\left|\partial_u T^-\left(\frac{\pi}{2}\mathrm{i}+\delta z,\tau\right) - \partial_u T_0\left(\frac{\pi}{2}\mathrm{i}+\delta z\right) - \delta\partial_u T_1\left(\frac{\pi}{2}\mathrm{i}+\delta z,\tau\right)\right| \\ &+ \left|\delta^2\partial_u T_0\left(\frac{\pi}{2}\mathrm{i}+\delta z\right) + 4z^{-2}\right| \\ &+ \left|\delta^3\partial_u T_1\left(\frac{\pi}{2}\mathrm{i}+\delta z,\tau\right) + 4\big(f_1(\tau)-\mathrm{i} f_2(\tau)\big)z^{-3}\right| \\ &+ \left|\partial_z\phi_0^-(z,\tau) + 4z^{-2} + 4\big(f_1(\tau)-\mathrm{i} f_2(\tau)\big)z^{-3}\right|.\end{aligned}$$

Applying the bound obtained in Theorem 3.6 and using that $\gamma > 1/2$, one can see that the first summand is of order $\mathcal{O}(\mu\delta^2)$. For the second one, we have to recall that

$$\partial_u T_0\left(\frac{\pi}{2}\mathrm{i}+\delta z\right) = \frac{-4}{\delta^2 z^2} + \mathcal{O}(1),$$

and thus $|\delta^2\partial_u T_0(\frac{\pi}{2}\mathrm{i}+\delta z) + 4z^{-2}| \le \mathcal{O}(\delta^2)$. Reasoning in the same way, the third term can be bounded by $\mathcal{O}(\delta^2)$. Since by Theorem 3.8, the last one holds

$$\partial_z\phi_0^-(z,\tau) + 4z^{-2} + 4\big(f_1(\tau)-\mathrm{i} f_2(\tau)\big)z^{-3} = \mathcal{O}\big(z^{-4}\big),$$

it is of order $\mathcal{O}(\delta^{4-4\gamma}) \le \mathcal{O}(\delta^2)$. To obtain the bound of $\partial_x\Phi^-(x,\tau)$, we use, by Theorem 3.8, the function R^- holds $\partial_x R^-(x,\tau) = \mathcal{O}(\delta^{2-2\gamma})$ for $(x,\tau) \in \Gamma^{\mathrm{u}(1)}_{\delta,+}$. □

To solve (121), we need to consider a change of variables which straightens the differential operator (122). We look for it by applying the following lemma, whose proof is straightforward.

Lemma 6.3 *Let us consider a function $W(x,\tau)$, then if the function $g(y,\tau)$ is a solution of*

$$\mathcal{L}g(y,\tau) = W\big(g(y,\tau),\tau\big),$$

where $\mathcal{L}$ is the operator defined in (69), the change of variables $x = g(y,\tau)$ conjugates the operator $\partial_\tau + W(x,\tau)\partial_x$ to $\mathcal{L}$.

If we knew a function $g(y,\tau)$ such that the change $x = g(y,\tau)$ conjugated $\tilde{\mathcal{L}}$ in (122) with $\mathcal{L}$, it would be natural to apply this change to (121) obtaining

$$\begin{aligned}\mathcal{L}\xi(y,\tau) = \partial_x A\big(g(y,\tau),\tau\big)\xi(y,\tau) + \partial_x B\big(g(y,\tau),\tau\big)\big(\xi(y,\tau)\big)^2 \\ + \partial_x C\big(g(y,\tau),\tau\big).\end{aligned} \tag{123}$$

Once we knew $\xi(y,\tau)$ satisfying this equation, the solution of (121) would be given by the implicit relation

$$\Psi\big(g(y,\tau),\tau\big) = \xi(y,\tau).$$

However, by Lemma 6.3, the equation that $g(y,\tau)$ has to satisfy to conjugate $\tilde{\mathcal{L}}$ and $\mathcal{L}$ is

$$\mathcal{L}g(y,\tau)=1-A\bigl(g(y,\tau),\tau\bigr)-2B\bigl(g(y,\tau),\tau\bigr)\xi(y,\tau)$$

which has ξ on the right-hand side. So that, we will look for the solutions of both equations at the same time.

Remark 6.4 Let us observe that the method that we will use to construct g, ξ, and g^{-1} will provide functions of class $\mathcal{C}^1$ in x (but analytic in τ). Nevertheless, since the partial differential equation (121) is quasi-linear and analytic and the initial condition η in (120) is also analytic, the solution $\Psi(x,\tau)$ thus obtained will be analytic in both variables for $(x,\tau)\in\mathcal{D}^{\mathrm{u}(4)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$.

We look for a function of the form $g(y,\tau)=y+\hat{g}(y,\tau)$ with $|\hat{g}|\le\mathcal{O}(\delta^s)$ for certain $s>0$ in such a way that for $(y,\tau)\in\mathcal{D}^{\mathrm{u}(2)}_{\delta,+}$, it holds that $y+\hat{g}(y,\tau)\in\mathcal{D}^{\mathrm{u}(1)}_{\delta,+}$. Thus, $\hat{g}$ has to be a solution of the partial differential equation

$$\mathcal{L}\hat{g}(y,\tau)=-A\bigl(y+\hat{g}(y,\tau),\tau\bigr)-2B\bigl(y+\hat{g}(y,\tau),\tau\bigr)\xi(y,\tau),\qquad(124)$$

and (123) reads:

$$\begin{aligned}\mathcal{L}\xi(y,\tau)=\partial_xA\bigl(y+\hat{g}(y,\tau),\tau\bigr)\xi(y,\tau)+\partial_xB\bigl(y+\hat{g}(y,\tau),\tau\bigr)\bigl(\xi(y,\tau)\bigr)^2&\\ +\partial_xC\bigl(y+\hat{g}(y,\tau),\tau\bigr).&\qquad(125)\end{aligned}$$

In respect to the initial conditions for $\hat{g}$ and ξ in $\Gamma^{\mathrm{u}(2)}_{\gamma,+}$, we choose

$$\hat{g}\bigl(y^*,\tau\bigr)=0\quad\text{for } y^*\in\Gamma^{\mathrm{u}(2)}_{\gamma,+}\qquad(126)$$

and consequently, using (120),

$$\xi\bigl(y^*,\tau\bigr)=\Psi\bigl(y^*+\hat{g}\bigl(y^*,\tau\bigr),\tau\bigr)=\Psi\bigl(y^*,\tau\bigr)=\eta\bigl(y^*,\tau\bigr)\quad\text{for } y^*\in\Gamma^{\mathrm{u}(2)}_{\gamma,+}.\qquad(127)$$

The next lemma shows that both (124) and (125) with initial conditions (126) and (127), can be expressed as integral equations.

Lemma 6.5 *For $(y,\tau)\in\mathcal{D}^{(2)}_{\delta,+}\times\mathbb{T}_{\sigma_1}$, it can be considered the following integral operator, which is a left inverse of $\mathcal{L}$:*

$$\mathcal{G}(h)(y,\tau)=\int_{y^*-y}^{0}h(y+t,\tau+t)\,\mathrm{d}t,\qquad(128)$$

where given $y\in\mathcal{D}^{\mathrm{u}(2)}_{\delta,+}$, we consider $y^*=y^*(y)\in\Gamma^{\mathrm{u}(2)}_{\gamma,+}$ such that $\Im y^*=\Im y$.

Then $(\hat{g},\xi)$ is a solution $\mathcal{C}^1$ in y and analytic in τ of (124) and (125) with initial conditions (126) and (127), provided they hold

$$\left\{\begin{aligned}\hat{g}&=\mathcal{G}\bigl(-A(y+\hat{g},\tau)-2B(y+\hat{g},\tau)\xi\bigr),\\ \xi&=\mathcal{G}\bigl(\partial_xA(y+\hat{g},\tau)\xi+\partial_xB(y+\hat{g},\tau)\xi^2+\partial_xC(y+\hat{g},\tau)\bigr)\\ &\qquad+\eta\bigl(y^*,\tau+y^*-y\bigr),\end{aligned}\right.\qquad(129)$$

and are continuous in y and analytic in τ.

Proof As (129) is an integral equation, its solutions $(\hat{g}, \xi)$ are $\mathcal{C}^1$ in y and analytic in τ. Thus, it is enough to differentiate by $\mathcal{L}$ in both sides of the equations in order to prove that they are classical solutions of (124) and (125). For the initial condition, we only have to evaluate the equations for $y = y^*$. □

To solve (129), it is necessary to perform the change

$$\hat{\xi}(y,\tau) = \xi(y,\tau) - \eta\big(y^*, \tau + y^* - y\big) \tag{130}$$

which lead to the following equations:

$$\begin{cases} \hat{g} = \mathcal{G}\big(-A(y+\hat{g},\tau) - 2B(y+\hat{g},\tau)\big(\hat{\xi} + \eta\big(y^*, \tau + y^* - y\big)\big)\big), \\ \hat{\xi} = \mathcal{G}\big(\partial_x A(y+\hat{g},\tau)\big(\hat{\xi} + \eta\big(y^*, \tau + y^* - y\big)\big) \\ \qquad + \partial_x B(y+\hat{g},\tau)\big(\hat{\xi} + \eta\big(y^*, \tau + y^* - y\big)\big)^2 + \partial_x C(y+\hat{g},\tau)\big), \end{cases} \tag{131}$$

whose solutions will be found in the following proposition.

Proposition 6.6 *For any fixed $\gamma \in (1/3, 1/2)$, there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists a solution $(\hat{g}, \hat{\xi})$ of (131) defined in $\mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$, $\mathcal{C}^1$ in y and analytic in τ. Moreover, for all $(y,\tau) \in \mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_1}$,*

$$\big|\hat{g}(y,\tau)\big| \le \mathcal{O}\big(\delta^{3\gamma-1}\big),$$
$$\big|y\hat{\xi}(y,\tau)\big| \le \mathcal{O}\big(\delta^2\big).$$

The proof of this proposition will be done in Sect. 6.2, but before it, we need some technicalities which will be explained in the following subsection.

6.1 Banach Spaces and Technical Lemmas

Since we expect that $\hat{g} \sim \delta^{3\gamma-1}$ and $\hat{\xi} \sim y^{-1}\delta^2$, we consider the norms

1. For $\hat{g}$: $\llbracket \hat{g} \rrbracket_1 = \|\hat{g}\|_\infty$.
2. For $\hat{\xi}$: $\llbracket \hat{\xi} \rrbracket_2 = \|\delta^{3\gamma-3} y \hat{\xi}\|_\infty$.

Therefore, for the pairs $(\hat{g}, \hat{\xi})$, we will consider the norm $\llbracket (\hat{g}, \hat{\xi}) \rrbracket = \sup\{\llbracket \hat{g} \rrbracket_1, \llbracket \hat{\xi} \rrbracket_2\}$ and the corresponding Banach space

$$\hat{\mathcal{X}} = \big\{(\hat{g}, \hat{\xi})\colon \mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_1} \to \mathbb{C}^2, \text{ continuous in } y \text{ and analytic in } \tau, \llbracket (\hat{g}, \hat{\xi}) \rrbracket < \infty\big\}. \tag{132}$$

Lemma 6.7 *The following bounds hold for all $(x,\tau) \in \mathcal{D}^{\mathrm{u}(1)}_{\delta,+}$:*

$$\big|A(x,\tau)\big| \le \mathcal{O}\big(\delta^{2\gamma}\big), \qquad \big|B(x,\tau)\big| \le \mathcal{O}(1)|x|^2,$$
$$\big|\partial_x A(x,\tau)\big| \le \mathcal{O}\big(\delta^{1+\gamma}\big), \qquad \big|\partial_x B(x,\tau)\big| \le \mathcal{O}(1)|x|,$$
$$\big|\partial_x^2 A(x,\tau)\big| \le \mathcal{O}\big(\delta^2\big), \qquad \big|\partial_x^2 B(x,\tau)\big| \le \mathcal{O}(1).$$

Moreover,

$$\partial_x C(x,\tau) = -\mathcal{L}\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2(x+R^-(x,\tau))}\right) + \mathcal{O}(\delta^2)|x|^{-2},$$

where f_2 is the function defined in Proposition 3.5.

Proof For the bounds of function A and its derivatives, one can check that $a(z,\tau) = \mathcal{O}(\varepsilon z)^2$ (see (113)). Differentiating, one can obtain $\partial_z a(z,\tau) = \varepsilon\mathcal{O}(\varepsilon z)$ and $\partial_z^2 a(z,\tau) = \varepsilon^2\mathcal{O}(1)$, and considering the properties of the change R^- stated in Theorem 3.8 it is straightforward to obtain the wanted bounds.

Since $b(z,\tau) = \frac{1}{4}z^2(1+\mathcal{O}(\varepsilon z)^2)$ (see (114)), proceeding in the same way one can obtain the bounds of B and its derivatives.

For the last expression, we consider first $c(z,\tau)$ (see (115)) and we split it in several terms. We will call these terms $c_i(z,\tau)$, and we will also deal with the corresponding functions in the new variable $C_i(x,\tau)$. We recall that we are not interested in bounding $C(x,\tau)$ but its derivative $\partial_x C(x,\tau)$. We consider $c(z,\tau) = c_1(z,\tau) + c_2(z,\tau) + c_3(z,\tau)$, where

$$\begin{aligned}
c_1(z,\tau) &= -\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}\big(\cos x_{\mathrm{p}}(\tau) - \cos(\mu\sin\tau)\big)\\
&\quad + \frac{\delta^2}{J}\psi\left(\frac{\pi}{2}\mathrm{i}+\delta z\right)\big(\sin x_{\mathrm{p}}(\tau) - \sin(-\mu\sin\tau)\big)\\
&\quad + \left(\frac{\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)}{8\delta^2} + \frac{z^2}{8}\right)(\partial_z\phi_0)^2,\\
c_2(z,\tau) &= -\left(\frac{2\delta^2}{J\cosh^2(\frac{\pi}{2}\mathrm{i}+\delta z)} + \frac{2}{Jz^2}\right)\cos(\mu\sin\tau),\\
c_3(z,\tau) &= -\left(\frac{\delta^2}{J}\psi\left(\frac{\pi}{2}\mathrm{i}+\delta z\right) + \frac{2\mathrm{i}}{Jz^2}\right)\sin(\mu\sin\tau).
\end{aligned}$$

In order to bound the first and the third terms of c_1, we need to know the behavior of $\partial_u\psi(u) = \frac{4}{\cosh^3 u}$ (see (39)) in the inner domain:

$$\delta^2\partial_z\left[\psi\left(\frac{\pi}{2}\mathrm{i}+\delta z\right)\right] = \frac{4\mathrm{i}}{z^3} + \frac{\mathrm{i}\delta^2}{2z} + \delta^3\mathcal{O}(\delta z). \tag{133}$$

With this fact and Corollary 4.2, the first term holds $|\partial_z c_1(z,\tau)| \le |z|^{-2}\mathcal{O}(\delta^2)$.

For c_2, since $|z| \le \mathcal{O}(\delta^{\gamma-1})$ and $\gamma < 1/3$, $|\partial_z c_2(z,\tau)| = \delta^3\mathcal{O}(\delta z) \le |z|^{-2}\mathcal{O}(\delta^2)$.

The third one, c_3, has to be treated more carefully. Using (133), $\partial_z c_3$ can be split as

$$\partial_z c_3(z,\tau) = \partial_z c_{31}(z,\tau) + \partial_z c_{32}(z,\tau),$$

where $\partial_z c_{31}(z,\tau) = -\frac{\mathrm{i}\delta^2}{2Jz}\sin(\mu\sin\tau)$ and $\partial_z c_{32}(z,\tau) = \delta^3\mathcal{O}(\delta z)$. For this last term, since holds $|c_{32}(z,\tau)| \le \mathcal{O}(\delta^{3+\gamma}) \le |z|^{-2}\mathcal{O}(\delta^2)$, we can proceed as for the previous ones.

For c_{31}, we consider the operator $\bar{\mathcal{L}}$ defined in Theorem 3.8, and we recall that $\bar{\mathcal{L}} = \partial_\tau + \mathcal{O}(1)\partial_z$,

$$\partial_z c_{31}(z,\tau) = -\partial_\tau\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2z}\right) = -\bar{\mathcal{L}}\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2z}\right) + |z|^{-2}\mathcal{O}(\delta^2).$$

Performing the change of variables defined in (67) and using the properties stated in Theorem 3.8, it is obtained

$$\begin{aligned}\partial_x C_{31}(x,\tau) &= \partial_z c_{31}\big(x + R^-(x,\tau),\tau\big)\big(1+\partial_x R^-(x,\tau)\big)\\ &= -\mathcal{L}\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2(x+R^-(x,\tau))}\right) + |x|^{-2}\mathcal{O}(\delta^2),\end{aligned}$$

obtaining the desired bound for $\partial_x C$. □

Lemma 6.8 *Let us consider $\gamma \in (0,1)$. Then, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$, for $y \in \mathcal{D}^{\mathrm{u}(i)}_{\delta,+}$ and $y^* \in \Gamma^{\mathrm{u}(i)}_{\delta,+}$ with $\Im y = \Im y^*$, it holds that*

$$\int_{y^*-y}^0 |y+t|^n\,\mathrm{d}t \le \begin{cases} \mathcal{O}(1)|y|^{n+1} & \text{if } n \le -2,\\ \mathcal{O}(\ln(1/\delta)) & \text{if } n=-1,\\ \mathcal{O}(\delta^{(n+1)(\gamma-1)}) & \text{if } n \ge 0\end{cases}$$

and for $n \ge -1$,

$$\int_{y^*-y}^0 |y+t|^n\,\mathrm{d}t \le \begin{cases} \mathcal{O}(\ln(1/\delta)\delta^{\gamma-1})|y|^{-1} & \text{if } n=-1,\\ \mathcal{O}(\delta^{(n+2)(\gamma-1)})|y|^{-1} & \text{if } n \ge 0.\end{cases}$$

Proof It is straightforward. □

6.2 Proof of Propositions 6.6 and 6.1 and Theorem 3.9

First, we prove Proposition 6.6. We define the functional $\mathcal{K} = (\mathcal{K}_1,\mathcal{K}_2)$ as

$$\begin{aligned}\mathcal{K}_1(\hat{g},\hat{\xi}) &= -A(y+\hat{g},\tau) - 2B(y+\hat{g},\tau)\big(\hat{\xi} + \eta\big(y^*,\tau+y^*-y\big)\big),\\ \mathcal{K}_2(\hat{g},\hat{\xi}) &= \partial_x A(y+\hat{g},\tau)\big(\hat{\xi}+\eta\big(y^*,\tau+y^*-y\big)\big)\\ &\quad + \partial_x B(y+\hat{g},\tau)\big(\hat{\xi}+\eta\big(y^*,\tau+y^*-y\big)\big)^2 + \partial_x C(y+\hat{g},\tau)\end{aligned}$$

and $\bar{\mathcal{K}}_i = \mathcal{G}\circ\mathcal{K}_i$ for $i=1,2$ (where $\mathcal{G}$ is the operator defined in (128)).

Lemma 6.9 *For any fixed $\gamma \in (1/3,1/2)$, there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$, the functional $\bar{\mathcal{K}}$ holds the following properties*

- *If $(\hat{g},\hat{\xi}) \in \hat{\mathcal{X}}$, $\bar{\mathcal{K}}(\hat{g},\hat{\xi}) \in \hat{\mathcal{X}}$, where $\hat{\mathcal{X}}$ is the Banach space in (132).*
- *There exists $C_3 > 0$ such that $\llbracket \bar{\mathcal{K}}(0,0)\rrbracket \le \frac{C_3}{2}\delta^{3\gamma-1}$.*

- *For all* $(\hat{g}_1,\hat{\xi}_1),(\hat{g}_2,\hat{\xi}_2)\in B(C_3\delta^{3\gamma-1})\subset\hat{\mathcal{X}}$,

$$\big\|\bar{\mathcal{K}}(\hat{g}_1,\hat{\xi}_1)-\bar{\mathcal{K}}(\hat{g}_2,\hat{\xi}_2)\big\|\le\mathcal{O}\big(\delta^{3\gamma-1}\big)\big\|(\hat{g}_1,\hat{\xi}_1)-(\hat{g}_2,\hat{\xi}_2)\big\|.$$

Proof The first statement is straightforward. For the second statement, since $\hat{g}=0$, we can apply Lemmas 6.2 for η and 6.7. Recalling that $\mathcal{O}(\ln(1/\delta))\le|y|\le\mathcal{O}(\delta^{\gamma-1})$ and $\gamma>1/3$,

$$\begin{aligned}\big|\mathcal{K}_1(0,0)\big|&=\big|A(y,\tau)\big|+2\big|B(y,\tau)\big|\big|\eta\big(y^*,\tau+y^*-y\big)\big|\le\mathcal{O}\big(\delta^{2\gamma}\big)+\mathcal{O}\big(\delta^2\big)|y|^2\\&\le\mathcal{O}\big(\delta^{2\gamma}\big)\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{K}_2(0,0)-\mathcal{L}\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2(y+R^-(y,\tau))}\right)\right|&=\big|\partial_x A(y,\tau)\big|\big|\eta\big(y^*,\tau+y^*-y\big)\big|\\&\quad+\big|\partial_x B(y,\tau)\big|\big|\big(\eta\big(y^*,\tau+y^*-y\big)\big)\big|^2\\&\quad+\left|\partial_x C(y,\tau)-\mathcal{L}\left(\frac{\mathrm{i}\delta^2 f_2(\tau)}{2(y+R^-(y,\tau))}\right)\right|\\&\le\mathcal{O}\big(\delta^{3+\gamma}\big)+\mathcal{O}\big(\delta^4\big)|y|+\mathcal{O}\big(\delta^2\big)|y|^{-2}\\&\le\mathcal{O}\big(\delta^2\big)|y|^{-2}.\end{aligned}$$

Therefore, using that $\mathcal{G}\circ\mathcal{L}=\mathrm{Id}$ and Lemma 6.8:

$$\begin{aligned}\big|\bar{\mathcal{K}}_1(0,0)\big|&=\big|\mathcal{G}\circ\mathcal{K}_1(0,0)\big|\le\mathcal{O}\big(\delta^{3\gamma-1}\big),\\\big|\bar{\mathcal{K}}_2(0,0)\big|&=\big|\mathcal{G}\circ\mathcal{K}_2(0,0)\big|\le\mathcal{O}\big(\delta^2\big)\int_{y^*-y}^0|y+t|^{-2}\,\mathrm{d}t+\left|\frac{\mathrm{i}\delta^2 f_2(\tau)}{2(y+R^-(y,\tau))}\right|\\&\le\mathcal{O}\big(\delta^2\big)|y|^{-1}.\end{aligned}$$

Thus, $\|\bar{\mathcal{K}}(0,0)\|\le\mathcal{O}(\delta^{3\gamma-1})$, and then there exists a constant $C_3>0$ such that the second statement holds.

Finally, for the third statement, we consider $(\hat{g}_1,\hat{\xi}_1),(\hat{g}_2,\hat{\xi}_2)\in B(C_3\delta^{3\gamma-1})\subset\hat{\mathcal{X}}$ and we use the mean value theorem. In order to apply it, we consider $\hat{g}_\rho=\rho\hat{g}_1+(1-\rho)\hat{g}_2$ for some $\rho\in(0,1)$, obtaining:

$$\begin{aligned}&\big|\mathcal{K}_1(\hat{g}_1,\hat{\xi}_1,y,\tau)-\mathcal{K}_1(\hat{g}_2,\hat{\xi}_2,y,\tau)\big|\\&\quad\le\big|A(y+\hat{g}_1,\tau)-A(y+\hat{g}_2,\tau)\big|+2\big|B(y+\hat{g}_1,\tau)\big|\cdot|\hat{\xi}_1-\hat{\xi}_2|\\&\qquad+2\big|B(y+\hat{g}_1,\tau)-B(y+\hat{g}_2,\tau)\big|\cdot\big|\hat{\xi}_2+\eta(y^*,\tau+y^*-y)\big|\\&\quad\le\big|\partial_x A(y+\hat{g}_\rho,\tau)\big|\cdot|\hat{g}_1-\hat{g}_2|+2\big|B(y+\hat{g}_1,\tau)\big||\hat{\xi}_1-\hat{\xi}_2|\\&\qquad+2\big|\partial_x B(y+\hat{g}_\rho,\tau)\big|\cdot|\hat{g}_1-\hat{g}_2|\cdot\big|\hat{\xi}_2+\eta\big(y^*,\tau+y^*-y\big)\big|\end{aligned}$$

and

$$\begin{aligned}
&\big|\mathcal{K}_2(\hat g_1,\hat\xi_1,y,\tau)-\mathcal{K}_2(\hat g_2,\hat\xi_2,y,\tau)\big|\\
&\quad\le\big|\partial_x A(y+\hat g_1,\tau)\big|\cdot|\hat\xi_1-\hat\xi_2|\\
&\qquad+\big|\hat\xi_2+\eta\big(y^*,\tau+y^*-y\big)\big|\cdot\big|\partial_x A(y+\hat g_1,\tau)-\partial_x A(y+\hat g_2,\tau)\big|\\
&\qquad+\big|\partial_x B(y+\hat g_1,\tau)\big|\cdot\big|\hat\xi_1+\hat\xi_2+2\eta\big(y^*,\tau+y^*-y\big)\big|\cdot|\hat\xi_1-\hat\xi_2|\\
&\qquad+\big|\hat\xi_2+\eta\big(y^*,\tau+y^*-y\big)\big|^2\cdot\big|\partial_x B(y+\hat g_1,\tau)-\partial_x B(y+\hat g_2,\tau)\big|\\
&\qquad+\big|\partial_x C(y+\hat g_1,\tau)-\partial_x C(y+\hat g_2,\tau)\big|.
\end{aligned}$$

Now, using the bounds of Lemma 6.7 and recalling that for $i=1,2$, $|\hat\xi_i|=|y|^{-1}\mathcal{O}(\delta^2)\le\mathcal{O}(\delta^2)$, $|y+\hat g_\rho|\le\mathcal{O}(1)|y|$ and the definition of the norms:

$$\begin{aligned}
&\big|\mathcal{K}_1(\hat g_1,\hat\xi_1,y,\tau)-\mathcal{K}_1(\hat g_2,\hat\xi_2,y,\tau)\big|\\
&\quad\le\mathcal{O}\big(\delta^{1+\gamma}\big)\|\hat g_1-\hat g_2\|_1+\mathcal{O}\big(\delta^{3-3\gamma}\big)|y|\|\hat\xi_1-\hat\xi_2\|_2+\mathcal{O}\big(\delta^2\big)|y|\|\hat g_1-\hat g_2\|_1,\\
&\big|\mathcal{K}_2(\hat g_1,\hat\xi_1,y,\tau)-\mathcal{K}_2(\hat g_2,\hat\xi_2,y,\tau)\big|\\
&\quad\le\mathcal{O}\big(\delta^{1+\gamma}\big)|\hat\xi_1-\hat\xi_2|+\mathcal{O}\big(\delta^4\big)|\hat g_1-\hat g_2|+\mathcal{O}\big(\delta^2\big)|y||\hat\xi_1-\hat\xi_2|\\
&\qquad+\mathcal{O}\big(\delta^4\big)|\hat g_1-\hat g_2|+\mathcal{O}\big(\delta^2\big)|y|^{-2}|\hat g_1-\hat g_2|\\
&\quad\le\mathcal{O}\big(\delta^{4-2\gamma}\big)|y|^{-1}\|\hat\xi_1-\hat\xi_2\|_2+\mathcal{O}\big(\delta^4\big)\|\hat g_1-\hat g_2\|_1\\
&\qquad+\mathcal{O}\big(\delta^{5-3\gamma}\big)\|\hat\xi_1-\hat\xi_2\|_2+\mathcal{O}\big(\delta^2\big)|y|^{-2}\|\hat g_1-\hat g_2\|_1.
\end{aligned}$$

Moreover, taking into account that $\gamma\in(1/2,1/3)$ and $|y|\le\mathcal{O}(\delta^{\gamma-1})$, we obtain that

$$\begin{aligned}
&\big|\mathcal{K}_1(\hat g_1,\hat\xi_1,y,\tau)-\mathcal{K}_1(\hat g_2,\hat\xi_2,y,\tau)\big|\le\mathcal{O}\big(\delta^{2-2\gamma}\big)\big\|(\hat g_1,\hat\xi_1)-(\hat g_2,\hat\xi_2)\big\|,\\
&\big|\mathcal{K}_2(\hat g_1,\hat\xi_1,y,\tau)-\mathcal{K}_2(\hat g_2,\hat\xi_2,y,\tau)\big|\le\mathcal{O}\big(\delta^2\big)|y|^{-2}\big\|(\hat g_1,\hat\xi_1)-(\hat g_2,\hat\xi_2)\big\|.
\end{aligned}$$

Using these bounds, Lemma 6.8 and recalling the definition of the norms, we obtain the third statement. □

With the bounds of the previous lemma we can prove Proposition 6.6.

Proof of Proposition 6.6 Since $\bar{\mathcal{K}}$ is a contraction from $B(C_3\varepsilon^{3\gamma-1})\subset\hat{\mathcal{X}}$ to itself, there exists a unique fixed point of this functional which is indeed a solution of (131) with the desired bounds. □

In order to prove Proposition 6.1, the original variables have to be recovered. So, we have to find the inverse change of variables of $x=y+\hat g(y,\tau)$.

Lemma 6.10 *Let us consider any constants $\sigma_2<\sigma_1$ and $\gamma\in(1/3,1/2)$. Then, for $(y,\tau)\in\mathcal{D}^{\mathrm{u}(2)}_{\delta,+}\times\mathbb{T}_{\sigma_2}$, the function $g(y,\tau)=y+\hat g(y,\tau)$ is invertible and its inverse* *can be written as* $y = x + \hat{h}(x,\tau)$ *and it is defined for* $(x,\tau) \in \mathcal{D}^{\mathrm{u}(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$. *Moreover, it is* $\mathcal{C}^1$ *and holds* $y = x + \hat{h}(x,\tau) \in \mathcal{D}^{\mathrm{u}(2)}_{\delta,+}$.

Proof It can be seen that $\hat{h}$ is solution of the functional equation

$$\hat{h}(x,\tau) = \mathcal{F}(\hat{h})(x,\tau) = -\hat{g}\big(x + \hat{h}(x,\tau),\tau\big).$$

We prove its existence with a fixed point argument in the Banach space:

$$\mathcal{Y} = \big\{\hat{h}\colon \mathcal{D}^{\mathrm{u}(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2} \to \mathbb{C},\ \text{continuous in } y \text{ and analytic in } \tau,\ \|\hat{h}\|_\infty < +\infty\big\}.$$

A bound of $\partial_y \hat{g}$ in $\mathcal{D}^{\mathrm{u}(2)}_{\delta,+}$ will be required in the proof. Since we will obtain it from (124) (recall definition of $\mathcal{L}$ in (79)), we have to bound $\partial_\tau \hat{g}$. For this purpose, we reduce slightly the domain in τ taking any constant $\sigma_2 \in (0,\sigma_1)$ and we apply Cauchy estimates. Hence, we obtain that for $(y,\tau) \in \mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$\big|\partial_\tau \hat{g}(y,\tau)\big| \le \mathcal{O}(1)\big|\hat{g}(y,\tau)\big| \le \mathcal{O}\big(\delta^{3\gamma-1}\big).$$

Using (124) and the bounds obtained in Lemma 6.9, for $(y,\tau) \in \mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$\begin{aligned}\big|\partial_y \hat{g}(y,\tau)\big| &\le \big|\partial_\tau \hat{g}(y,\tau)\big| + \big|A\big(y + \hat{g}(y,\tau),\tau\big)\big| + 2\big|B\big(y + \hat{g}(y,\tau),\tau\big)\big|\big|\xi(y,\tau)\big| \\ &\le \mathcal{O}\big(\delta^{3\gamma-1}\big). \end{aligned} \tag{134}$$

The bound obtained in Proposition 6.6 leads to $\|\mathcal{F}(0)\|_\infty \le C_3\delta^{3\gamma-1}$. Moreover, considering $\hat{h}_1, \hat{h}_2 \in B(2C_3\delta^{3\gamma-1}) \subset \mathcal{Y}$, we take $\hat{h}_\rho = \rho\hat{h}_1 + (1-\rho)\hat{h}_2$ for $\rho \in (0,1)$, and thus

$$\begin{aligned}&\big\|\mathcal{F}(\hat{h}_1) - \mathcal{F}(\hat{h}_2)\big\|_\infty \\ &\quad\le \big\|\partial_y \hat{g}\big(x + \hat{h}_\rho(x,\tau),\tau\big)\big\|_\infty \|\hat{h}_1 - \hat{h}_2\|_\infty \le \mathcal{O}\big(\delta^{3\gamma-1}\big)\|\hat{h}_1 - \hat{h}_2\|_\infty.\end{aligned}$$

Hence, this functional is a contraction from $B(2C_3\delta^{3\gamma-1}) \subset \mathcal{Y}$ to itself, and thus there exists a function $\hat{h}$ continuous in x such that $y = x + \hat{h}(x,\tau)$ is the inverse change of $x = g(y,\tau)$. In order to prove it is $\mathcal{C}^1$, it is enough to recall that considering (134), for $(y,\tau) \in \mathcal{D}^{\mathrm{u}(2)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$,

$$\partial_y g(y,\tau) = 1 + \mathcal{O}\big(\delta^{3\gamma-1}\big) \ne 0$$

and, therefore, g is a diffeomorphism. □

Proof of Proposition 6.1 From Lemma 6.10, Proposition 6.6, and (130), we can recover

$$\begin{aligned}\partial_x \Phi^-(x,\tau) = \Psi(x,\tau) &= \xi\big(x + \hat{h}(x,\tau),\tau\big) \\ &= \hat{\xi}\big(x + \hat{h}(x,\tau),\tau\big) + \eta\big(x^*, \tau + x^* - x - \hat{h}(x,\tau)\big),\end{aligned}$$

for $(x,\tau) \in \mathcal{D}^{\mathrm{u}(3)}_{\delta,+} \times \mathbb{T}_{\sigma_2}$ and $\|\partial_x \Phi(x,\tau)\|_\infty \le \mathcal{O}(\delta^2)$.

As we have observed in Remark 6.4, this function is the analytic continuation of $\partial_x \Phi^-$ to the inner domain.

Moreover, we can recover Φ taking

$$\forall (x,\tau)\in \mathcal{D}^{\mathrm{u}(3)}_{\delta,+}\times \mathbb{T}_{\sigma_2},\quad \Phi(x,\tau)=\Phi\bigl(x^*,\tau\bigr)+\int_{x^*}^{x}\Psi(r,\tau)\,\mathrm{d}r. \qquad \square$$

One can see that Cauchy estimates to $\partial_x\Phi^-$ in $\mathcal{D}^{\mathrm{u}(4)}_{\delta,+}$ lead to

$$\bigl|\partial_x^2\Phi^-(x,\tau)\bigr|\le \mathcal{O}\bigl(\delta^2\ln^{-1}(1/\delta)\bigr). \tag{135}$$

However, a better estimate will be required. To this end, an analogous fixed point procedure is needed. In fact, differentiating both sides of (121) and taking $\Theta(x,\tau)=\partial_x^2\Phi(x,\tau)$:

$$\begin{aligned}\tilde{\mathcal{L}}_2\Theta &= \bigl(2\partial_x A(x,\tau)+4\partial_x B(x,\tau)\partial_x\Phi(x,\tau)\bigr)\Theta+2B(x,\tau)\Theta^2\\ &\quad +\partial_x^2 A(x,\tau)\partial_x\Phi(x,\tau)+\partial_x^2 B(x,\tau)\bigl(\partial_x\Phi(x,\tau)\bigr)^2+\partial_x^2 C(x,\tau),\end{aligned} \tag{136}$$

where

$$\tilde{\mathcal{L}}_2=\partial_\tau+\bigl(1-A(x,\tau)-2B(x,\tau)\partial_x\Phi(x,\tau)\bigr)\partial_x. \tag{137}$$

Hence, we want to solve this partial differential equation with initial condition

$$\begin{aligned}\eta_2\bigl(x^*,\tau\bigr)&=\partial_x^2\Phi^-\bigl(x^*,\tau\bigr)\\ &=\partial_x\bigl[\phi^-\bigl(x+R^-(x,\tau),\tau\bigr)-\phi_0^-\bigl(x+R^-(x,\tau),\tau\bigr)\bigr]\big|_{x=x^*}\end{aligned} \tag{138}$$

for $(x^*,\tau)\in\Gamma^{\mathrm{u}(5)}_{\gamma,+}\times\mathbb{T}_{\sigma_2}$ which can be bounded analogously to Lemma 6.2 as

$$\eta_2\bigl(x^*,\tau\bigr)\le\mathcal{O}\bigl(\delta^3+\delta^{5-5\gamma}\bigr).$$

Proposition 6.11 *Let us fix any constants $\sigma_3<\sigma_2$ and $\gamma\in(1/3,1/2)$. Then there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there exists an analytic function $\Theta(x,\tau)$ for $(x,\tau)\in\mathcal{D}^{\mathrm{u}(6)}_{\delta,+}\times\mathbb{T}_{\sigma_3}$, which holds the partial differential equation (136) and the initial condition $\Theta(x^*,\tau)=\eta_2(x^*,\tau)$ for $(x^*,\tau)\in\Gamma^{\mathrm{u}(5)}_{\gamma,+}\times\mathbb{T}_{\sigma_3}$. Moreover, $\Theta=\partial_x^2\Phi$ and $\|\Theta\|_\infty\le|x|^{-1}\mathcal{O}(\delta^2\ln^{-1}(1/\delta))$.*

Proof We proceed as in Proposition 6.1. However, since the differential operator of the left-hand side of the equation does not depend on Θ, it will be easier.

In fact, as a first step, we solve equation

$$\mathcal{L}\tilde{g}=-A(x,\tau)-2B(x,\tau)\partial_x\Phi(x,\tau)|_{x=y+\tilde{g}(y,\tau)}$$

using Lemma 6.7, Proposition 6.1, and bound (135). Then there exists a change of variables $x=y+\tilde{g}(y,\tau)$ defined in $(y,\tau)\in\mathcal{D}^{\mathrm{u}(5)}_{\delta,+}\times\mathbb{T}_{\sigma_2}$, which holds $x=y+$

$\tilde{g}(y,\tau)\in\mathcal{D}^{\mathrm{u}(4)}_{\delta,+}$ and conjugates the operator $\tilde{\mathcal{L}}_2$ defined in (137) to $\mathcal{L}$, and moreover holds $|\tilde{g}(y,\tau)|\leq\mathcal{O}(\delta^{2\gamma})$ and for $y^*\in\Gamma^{\mathrm{u}(5)}_{\gamma,+}$, $\tilde{g}(y^*,\tau)=0$.

Hence, performing this change to (136) and considering $\bar{\Theta}(y,\tau)=\Theta(y+\tilde{g}(y,\tau),\tau)$, we obtain an equation of the form

$$\mathcal{L}\bar{\Theta}=L(y,\tau)\bar{\Theta}+M(y,\tau)\bar{\Theta}^2+N(y,\tau).$$

Thus, considering again the operator $\mathcal{G}$ defined in Lemma 6.5 and the initial condition η_2 in (138), a fixed point argument can be used in certain Banach space with norm $\llbracket\bar{\Theta}\rrbracket_1=\|y\bar{\Theta}\|_\infty$ obtaining the wanted $\bar{\Theta}$ with bound $|\bar{\Theta}|\leq|y|^{-1}\mathcal{O}(\delta^2\ln(1/\delta))$.

Proceeding as in Lemma 6.10 (and thus taking from now on $\tau\in\mathbb{T}_{\sigma_3}$ for any fixed $\sigma_3<\sigma_2$), we can find the inverse change of $x=y+\tilde{g}(y,\tau)$ so that we recover function $\Theta(x,\tau)$ defined in $\mathcal{D}^{\mathrm{u}(6)}_{\delta,+}\times\mathbb{T}_{\sigma_3}$ with the desired bound. Analogously to Remark 6.4, the solution thus obtained is analytic in its domain. □

With Propositions 6.1 and 6.11 is straightforward to prove Theorem 3.9:

Proof We consider $x=z+S^-(z,\tau)$ the inverse change of variables of $z=x+R^-(x,\tau)$ defined in Theorem 3.8 which holds that for $(z,\tau)\in\mathcal{D}^{\mathrm{u}(7)}_{\delta,+}\times\mathbb{T}_{\sigma_3}$, $x=z+S^-(z,\tau)\in\mathcal{D}^{\mathrm{u}(6)}_{\delta,+}$. Hence,

$$\phi^-(z,\tau)=\phi_0^-(z,\tau)+\Phi^-\big(z+S^-(z,\tau),\tau\big)$$

and

$$\begin{aligned}\partial_z(\phi-\phi_0)(z,\tau)&=\partial_x\Phi\big(z+S^-(z,\tau),\tau\big)\big(1+\partial_zS^-(z,\tau)\big),\\ \partial_z^2(\phi-\phi_0)(z,\tau)&=\partial_x^2\Phi\big(z+S^-(z,\tau),\tau\big)\big(1+\partial_zS^-(z,\tau)\big)^2\\ &\quad+\partial_x\Phi\big(z+S^-(z,\tau),\tau\big)\partial_z^2S^-(z,\tau).\end{aligned}$$

To end the proof of Theorem 3.9, it is enough to use the bounds of Propositions 6.1 and 6.11 and the bound of ∂_z^iS in Theorem 3.8. □

7 First Approximation of the Change of the Variables: Proof of Theorem 3.11

The proof of Theorem 3.11 is divided into two parts. First, in Proposition 7.1, we find the change of variables $u=w+\mathcal{C}^-(w,\tau)$ in the outer domain $D^{\mathrm{u}(9)}_\gamma$. Later, in Proposition 7.2, we find its analytic continuation to the inner domains $D^{\mathrm{u}(11)}_{\delta,\pm}$, using matching techniques as we did for the proof of the existence of the invariant manifolds.

Within this section, γ will be any fixed constant $\gamma\in(1/3,1/2)$.

Proposition 7.1 *There exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$, there exists a change of variables $u=w+\mathcal{C}^-(w,\tau)$ which conjugates $\tilde{\mathcal{L}}_\delta^-$ (in 82) to $\mathcal{L}_\delta=\delta^{-1}\partial_\tau+\partial_w$, such* *that for* $(w,\tau)\in D_\gamma^{\mathrm{u}(9)}\times\mathbb{T}_{\sigma_3}$, *it holds that* $u=w+\mathcal{C}^-(w,\tau)\in D_\gamma^{\mathrm{u}(8)}$ *and*

$$\begin{aligned}\left|\mathcal{C}^-(w,\tau)\right| &\le \mathcal{O}\left(\delta^2\right)|w|+\mathcal{O}\left(\delta^{2-2\gamma}\right),\\ \left|\partial_w\mathcal{C}^-(w,\tau)\right| &\le \mathcal{O}\left(\delta^{2-3\gamma}\right),\\ \left|\partial_w^2\mathcal{C}^-(w,\tau)\right| &\le \mathcal{O}\left(\delta^{2-4\gamma}\right).\end{aligned}$$

Proposition 7.2 *There exists* $\delta_0>0$ *such that for* $\delta\in(0,\delta_0)$, *there exists an analytic continuation of the change of variables found in Proposition* 7.1 *to the inner domains* $D_{\delta,\pm}^{\mathrm{u}(10)}$, *such that for* $(w,\tau)\in D_{\delta,\pm}^{\mathrm{u}(10)}\times\mathbb{T}_{\sigma_3}$, *it holds that* $u=w+\mathcal{C}^-(w,\tau)\in D_{\delta,\pm}^{\mathrm{u}(9)}$ *and for* $j=0,1,2$,

$$\left|\partial_w^j\mathcal{C}^-(w,\tau)\right|\le\mathcal{O}\left(\delta^{1-j}\ln(1/\delta)^{-1-j}\right).$$

7.1 Change of Variables in the Outer Domain: Proof of Proposition 7.1

Due to Lemma 6.3, if we call $u=g(w,\tau)$ to the change that conjugates $\tilde{\mathcal{L}}_\delta^-$ and $\mathcal{L}_\delta$, the function g has to hold the following equation:

$$\mathcal{L}_\delta g(w,\tau)=\frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)\bigg|_{u=g(w,\tau)}. \tag{139}$$

Let us observe that

$$\frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)=1+\delta\frac{\cosh^2 u}{4}\partial_u T_1(u,\tau)+\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau),$$

where T_1 is the function defined in (44). Thus, by Theorem 3.6,

$$\frac{\cosh^2 u}{4}\partial_u T^-(u,\tau)$$

does not tend to 1 when $\Re u\to-\infty$. In fact, by Theorem 5.9 its limit as $\Re u\to-\infty$ is $1-\delta f_1(\tau)+\lambda(\tau)$, where f_1 and λ are the functions defined in Proposition 3.5 and Theorem 5.9, respectively.

Hence, if we look for a change of the form $u=g(w,\tau)=g_0w+g_1(\tau)+g_2(w,\tau)$, we obtain that

$$g_0=1+\langle\lambda\rangle \tag{140}$$

and g_1 satisfies

$$\delta^{-1}g_1'=\lambda-\langle\lambda\rangle-\delta f_1(\tau)\quad\text{and}\quad\langle g_1\rangle=0. \tag{141}$$

Moreover, by the definitions of f_1 and λ stated in Proposition 3.5 and Theorem 5.9 respectively, one has that

$$|g_0-1|\le\mathcal{O}\left(\mu\delta^2\right), \tag{142}$$

$$\left|g_1(\tau)\right|\le\mathcal{O}\left(\mu\delta^2\right). \tag{143}$$

Replacing these terms in (139), we look for a solution of

$$\mathcal{L}_\delta g_2 = \mathcal{H}(g_2, w, \tau), \tag{144}$$

where

$$\mathcal{H}(g_2, w, \tau) = \left(\frac{\cosh^2 u}{4}\partial_u Q^- - \lambda(\tau) + \delta(\tanh u + 1) f_1(\tau) + \frac{\delta}{\cosh u} f_2(\tau)\right)\bigg|_{u=g_0 w + g_1(\tau) + g_2(w,\tau)}. \tag{145}$$

In fact, we look for a fixed point of equation $g_2(w,\tau) = \bar{\mathcal{H}}(g_2) = \mathcal{G}_\delta(\mathcal{H}(g_2, w, \tau))$ where $\mathcal{G}_\delta$ is the operator defined in (99).

Although the change of variables g is not close to the identity, in the following lemma, we will see that $u = g(w,\tau) \in D_\gamma^{(7)}$ provided $(w,\tau) \in D_\gamma^{(8)} \times \mathbb{T}_{\sigma_3}$.

Lemma 7.3 *Let us consider the change of variables $u = g(w,\tau) = g_0 w + g_1(\tau) + g_2(w,\tau)$ where g_0 and g_1 are the functions defined in (140) and (141) and g_2 is any function defined in $(w,\tau) \in D_\gamma^{\mathrm{u}(8)} \times \mathbb{T}_{\sigma_3}$ satisfying $|g_2(w,\tau)| \le \mathcal{O}(\delta^{2-2\gamma})$. Then there exists $\delta_0 > 0$ such that for $\delta < \delta_0$, $u = g(w,\tau) \in D_\gamma^{\mathrm{u}(7)}$ provided $(w,\tau) \in D_\gamma^{\mathrm{u}(8)} \times \mathbb{T}_{\sigma_3}$.*

Proof Recalling that $g_0 \in \mathbb{R}$ and $\gamma < 2 - 2\gamma$, for $(w,\tau) \in D_\gamma^{\mathrm{u}(8)} \times \mathbb{T}_{\sigma_3}$:

$$\begin{aligned}|\Im u| < \left|\Im g(w,\tau)\right| &< g_0|\Im w| + \mathcal{O}\big(\delta^{2-2\gamma}\big)\\ &< -\tan\beta_0\big(\Re(g_0 w) + a^{(8)}\delta^\gamma\big) + \frac{\pi}{2} + \mathcal{O}\big(\delta^{2-2\gamma}\big)\\ &< -\tan\beta_0\big(\Re u + a^{(7)}\delta^\gamma\big) + \frac{\pi}{2}\end{aligned}$$

and, therefore, $u = g(w,\tau) \in D_\gamma^{(7)}$. □

In order to prove the existence of the function g_2, we will prove that the operator $\bar{\mathcal{H}}$ is a contraction in a certain subset of the Banach space

$$\mathcal{Y}_1 = \left\{g_2(w,\tau) \mid g_2 : D_\gamma^{\mathrm{u}(8)} \times \mathbb{T}_{\sigma_3} \to \mathbb{C},\ \text{real-analytic},\ \|g_2\|_1 < \infty\right\}$$

where $\|h\|_1 = \|\cosh(g_0 w) \cdot h\|_\infty$ and g_0 is the constant defined in (140).

Proposition 7.4 *There exists a constant $C_4 > 0$ such that $\bar{\mathcal{H}}$ is a contraction from $B(C_4\delta^{2-\gamma}) \subset \mathcal{Y}_1$ to itself. Therefore, there exists a function $g_2 \in B(C_4\delta^{2-\gamma}) \subset \mathcal{Y}_1$ which is a solution of* (144).

In order to prove this proposition some technical lemmas are required.

Lemma 7.5 *The operator $\mathcal{G}_\delta$ defined in* (99) *is linear and is well defined from $\mathcal{Y}_1$ to itself. Moreover, if $h \in \mathcal{Y}_1$, then*

$$\left\|\mathcal{G}_\delta(h)\right\|_1 \leq \mathcal{O}(1)\|h\|_1.$$

Proof Applying the third statement of Lemma 5.2 adapted to the norm $\|\cdot\|_1$ we are dealing with, we obtain that

$$\begin{aligned}\left|\cosh(g_0 w)\mathcal{G}_\delta(h)(w,\tau)\right| &\leq \left|\cosh(g_0 w)\int_{-\infty}^0 h\big(w+t,\tau+\delta^{-1}t\big)\,\mathrm{d}t\right| \\ &\leq \|h\|_1 \int_{-\infty}^0 \left|\frac{\cosh(g_0 w)}{\cosh(g_0(w+t))}\right| \mathrm{d}t \leq \mathcal{O}(1)\|h\|_1.\end{aligned}$$

As a consequence, if $h \in \mathcal{Y}_1$, then $\mathcal{G}_\delta(h) \in \mathcal{Y}_1$. □

Lemma 7.6 *The following inequality holds*: $\|\bar{\mathcal{H}}(0)\|_1 = \|\mathcal{G}_\delta(\mathcal{H}(0,\cdot,\cdot))\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma})$.

Proof We split $\mathcal{H}$ defined in (145) in several terms in order to simplify the proof

- $\mathcal{H}_1(g_2,w,\tau) = \frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau) - \lambda(\tau)\big|_{u=g_0w+g_1(\tau)+g_2(w,\tau)}$.
- $\mathcal{H}_2(g_2,w,\tau) = \delta(\tanh(g_0 w + g_1(\tau) + g_2(w,\tau)) + 1) f_1(\tau)$.
- $\mathcal{H}_3(g_2,w,\tau) = \delta f_2(\tau)(\cosh(g_0 w + g_1(\tau)) + g_2(w,\tau))^{-1}$.

First, using (143), it is straightforward to see that

$$\frac{\cosh(g_0 w)}{\cosh(g_0 w + g_1(\tau))} = \mathcal{O}(1) \quad \text{for } (w,\tau) \in D_\gamma^{\mathrm{u}(8)} \times \mathbb{T}_{\sigma_3}.$$

Since we want to use the bounds obtained in Proposition 5.6 and Theorem 5.9, we split the domain $D_\gamma^{\mathrm{u}(8)} = D_\gamma^{\mathrm{u,c}(8)} \cup D_\gamma^{\mathrm{u,f}(8)}$ (see (48) and (49)). For these domains, we define two auxiliary norms

$$\|h\|_{1,\mathrm{c}} = \left\|\cosh(g_0 w)\cdot h\right\|_{\infty,\mathrm{c}} \quad \text{and} \quad \|h\|_{1,\mathrm{f}} = \left\|\cosh(g_0 w)\cdot h\right\|_{\infty,\mathrm{f}},$$

which are the weighted supremum norm in the corresponding domains, in such a way that $\|h\|_1 = \sup\{\|h\|_{1,\mathrm{c}}, \|h\|_{1,\mathrm{f}}\}$.

Since $g_0 w \in D_\gamma^{\mathrm{u}(7)}$, then by Proposition 5.6,

$$\begin{aligned}\left\|\mathcal{H}_1(0,w,\tau)\right\|_{1,\mathrm{c}} &\leq \frac{1}{4}\left|\frac{1}{\cosh(g_0 w)}\right|\left|\frac{\cosh(g_0 w)}{\cosh(g_0 w + g_1(\tau))}\right|^2 \|\partial_u Q^-\|_{4,2,\sigma_1} \\ &\quad + \left\|\cosh(g_0 w)\lambda(\tau)\right\|_{\infty,\mathrm{c}} \leq \mathcal{O}\big(\mu\delta^{2-\gamma}\big)\end{aligned}$$

and using Theorem 5.9

$$\begin{aligned}\left\|\mathcal{H}_1(0,w,\tau)\right\|_{1,\mathrm{f}} &\leq \left\|\cosh(g_0 w)\left(\frac{\cosh^2 u}{4}\partial_u Q^-(u,\tau) - \lambda(\tau)\bigg|_{u=g_0w+g_1(\tau)}\right)\right\|_{\infty,\mathrm{f}} \\ &\leq \mathcal{O}\big(\mu\delta^2\big)\end{aligned}$$

and, applying Lemma 7.5, we obtain $\|\mathcal{G}_\delta(\mathcal{H}_1(0,w\tau))\|_1 \leq \mathcal{O}(\mu\delta^{2-\gamma})$.

For the other two terms integrating by parts, considering 2π-periodic functions $\hat{f}_i(\tau)$ such that $\hat{f}_i' = f_i$ and $\langle \hat{f}_i\rangle = 0$, and using the properties of g_0 and g_1:

$$\begin{aligned}
\mathcal{G}_\delta\mathcal{H}_2(0,w,\tau) &= \delta\int_{-\infty}^0 \big(\tanh\big(g_0(w+t)+g_1\big(\tau+\delta^{-1}t\big)\big)+1\big) f_1\big(\tau+\delta^{-1}t\big)\,\mathrm{d}t\\
&= \delta^2\big(\tanh\big(g_0w+g_1(\tau)\big)+1\big)\hat{f}_1(\tau)\\
&\quad -\delta^2\mathcal{G}_\delta\bigg(\frac{g_0+\delta^{-1}g_1'(\tau)}{\cosh^2(g_0w+g_1(\tau))}\hat{f}_1(\tau)\bigg),\\
\mathcal{G}_\delta\mathcal{H}_3(0,w,\tau) &= \delta\int_{-\infty}^0 \frac{1}{\cosh(g_0(w+t)+g_1(\tau+\delta^{-1}t)}f_2\big(t+\delta^{-1}t\big)\,\mathrm{d}t\\
&= \delta^2\frac{1}{\cosh(g_0w+g_1(\tau))}\hat{f}_2(\tau)\\
&\quad +\delta^2\mathcal{G}_\delta\bigg(\frac{(g_0+\delta^{-1}g_1'(\tau))\sinh(g_0w+g_1(\tau))}{\cosh^2(g_0w+g_1(\tau))}\hat{f}_2(\tau)\bigg).
\end{aligned}$$

Hence, using that $|\hat{f}_i(\tau)|\le\mathcal{O}(\mu)$, statement (143) and Lemma 7.5,

$$\begin{aligned}
\big\|\mathcal{G}_\delta\mathcal{H}_2(0,w,\tau)\big\|_1 &\le \mathcal{O}\big(\mu\delta^2\big)+\mathcal{O}\big(\delta^2\big)\bigg\|\frac{g_0+\delta^{-1}g_1'(\tau)}{\cosh^2(g_0w+g_1(\tau))}\hat{f}_1(\tau)\bigg\|_1 \le \mathcal{O}\big(\mu\delta^{2-\gamma}\big),\\
\big\|\mathcal{G}_\delta\mathcal{H}_3(0,w,\tau)\big\|_1 &\le \mathcal{O}\big(\mu\delta^2\big)+\mathcal{O}\big(\delta^2\big)\bigg\|\frac{(g_0+\delta^{-1}g_1'(\tau))\sinh(g_0w+g_1(\tau))}{\cosh^2(g_0w+g_1(\tau))}\hat{f}_2(\tau)\bigg\|_1\\
&\le \mathcal{O}\big(\mu\delta^{2-\gamma}\big).
\end{aligned}$$

□

Lemma 7.7 *For all $g_2,\tilde{g}_2$ such that $\|g_2\|_1,\|\tilde{g}_2\|_1\le\mathcal{O}(\mu\delta^{2-\gamma})$, it holds that*

$$\big\|\mathcal{G}_\delta\mathcal{H}(g_2,w,\tau)-\mathcal{G}_\delta\mathcal{H}(\tilde{g}_2,w,\tau)\big\|_1\le\mathcal{O}\big(\mu\delta^{1-2\gamma}\big)\|g_2-\tilde{g}_2\|_1.$$

Proof In order to apply the mean value theorem, we split $\mathcal{H}$ again and we define $g_2^\rho=\rho g_2+(1-\rho)\tilde{g}_2$ and $u^\rho=g_0w+g_1(\tau)+g_2^\rho(w,\tau)$ for $\rho\in(0,1)$. In fact, it is clear that there exists $\rho\in(0,1)$ such that

$$\begin{aligned}
\mathcal{H}_1(g_2,w,\tau)-\mathcal{H}_1(\tilde{g}_2,w,\tau) &= \partial_u\bigg(\frac{\cosh^2u}{4}\partial_u Q^-\bigg)\bigg|_{u=u^\rho}\big(g_2(w,\tau)-\tilde{g}_2(w,\tau)\big),\\
\mathcal{H}_2(g_2,w,\tau)-\mathcal{H}_2(\tilde{g}_2,w,\tau) &= \frac{\delta f_1(\tau)}{\cosh^2u^\rho}\big(g_2(w,\tau)-\tilde{g}_2(w,\tau)\big),\\
\mathcal{H}_3(g_2,w,\tau)-\mathcal{H}_3(\tilde{g}_2,w,\tau) &= \frac{\delta f_2(\tau)\sinh u^\rho}{\cosh^2u^\rho}\big(g_2(w,\tau)-\tilde{g}_2(w,\tau)\big),
\end{aligned}$$

where we have used $u^\rho\in D_\gamma^{\mathrm{u}(7)}$ for $(w,\tau)\in D_\gamma^{\mathrm{u}(8)}\times\mathbb{T}_{\sigma_3}$. Applying Proposition 5.6, we obtain:

$$\begin{aligned}
&\left\|\mathcal{H}_1(g_2,\cdot,\cdot)-\mathcal{H}_1(\tilde{g}_2,\cdot,\cdot)\right\|_1\\
&\quad\leq\left(\left\|\frac{\cosh^2 u^\rho}{4}\partial_u^2 Q^-\big(u^\rho,\tau\big)\right\|_\infty+\left\|\frac{\cosh u^\rho\sinh u^\rho}{2}\partial_u Q^-\big(u^\rho,\tau\big)\right\|_\infty\right)\|g_2-\tilde{g}_2\|_1\\
&\quad\leq\mathcal{O}\big(\mu\delta^{2-3\gamma}\big)\|g_2-\tilde{g}_2\|_1,\\
&\left\|\mathcal{H}_2(g_2,\cdot,\cdot)-\mathcal{H}_2(\tilde{g}_2,\cdot,\cdot)\right\|_1\\
&\quad\leq\left\|\frac{\delta f_1(\tau)}{\cosh^2 u^\rho}\right\|_\infty\|g_2-\tilde{g}_2\|_1\leq\mathcal{O}\big(\mu\delta^{1-2\gamma}\big)\|g_2-\tilde{g}_2\|_1,\\
&\left\|\mathcal{H}_3(g_2,\cdot,\cdot)-\mathcal{H}_3(\tilde{g}_2,\cdot,\cdot)\right\|_1\\
&\quad\leq\left\|\frac{\delta f_2(\tau)\sinh u^\rho}{\cosh^2 u^\rho}\right\|_\infty\|g_2-\tilde{g}_2\|_1\leq\mathcal{O}\big(\mu\delta^{1-2\gamma}\big)\|g_2-\tilde{g}_2\|_1.
\end{aligned}$$

Thus, to prove the lemma, it is enough to join all the bounds and apply Lemma 7.5. □

Thus, we are ready to prove Proposition 7.4.

Proof of Proposition 7.4 Joining the results of Lemmas 7.6 and 7.7, the conclusion is that there exists a constant $C_4>0$ such that $\|\bar{\mathcal{H}}(0)\|_1\leq C_4\mu\delta^{2-\gamma}/2$ and for all $g_2,\tilde{g}_2\in B(C_4\mu\delta^{2-\gamma})$, $\|\bar{\mathcal{H}}(g_2)-\bar{\mathcal{H}}(\tilde{g}_2)\|_1\leq\mathcal{O}(\mu\delta^{1-2\gamma})\|g_2-\tilde{g}_2\|_1$. Thus, reducing δ if it is necessary, $\bar{\mathcal{H}}$ is a contraction from $B(C_4\mu\delta^{2-\gamma})\subset\mathcal{Y}_1$ to itself and, therefore, it has a unique fixed point. □

Hence, for the proof of Proposition 7.1, it is sufficient to consider $\mathcal{C}^-(w,\tau)=(g_0-1)w+g_1(\tau)+g_2(w,\tau)$. Moreover, the first bound holds and we only need to reduce slightly the domain to $\mathcal{D}_\gamma^{\mathrm{u}(9)}$ and apply Cauchy estimates to obtain the bounds of the derivatives.

7.2 Change of Variables in the Inner Domain: Proof of Proposition 7.2

In the last subsection, we obtained a change of variables $\mathcal{C}^-$ such that conjugates $\tilde{\mathcal{L}}_\delta^-$ to $\mathcal{L}_\delta$ in the outer domain $D_\gamma^{\mathrm{u}(9)}$. We will now see that there exists an analytical continuation of it to the inner domains $D_{\delta,\pm}^{\mathrm{u}(9)}$. Since the procedure is analogous for both regions, we will only deal with $D_{\delta,+}^{\mathrm{u}(9)}$.

Proceeding as in the outer domain, we know that the change $u=w+g(w,\tau)$ has to hold (139) for $(w,\tau)\in D_{\delta,\pm}^{\mathrm{u}(9)}\times\mathbb{T}_{\sigma_3}$. This equation will be solved using complex matching techniques like we did in Sect. 6. For this purpose, we consider inner variables and as initial condition in a curve $\Gamma_{\gamma,+}^{\mathrm{u}(9)}$ (see (111)) we take the change of variables $u=w+\mathcal{C}^-(w,\tau)$ already obtained in the outer domain. Then performing a characteristics-like method, we will obtain the continuation of this change to the inner domains which will be analytic *a posteriori*.

In order to translate (139) to inner variables, we consider

$$z=\delta^{-1}\left(u-\mathrm{i}\frac{\pi}{2}\right),\qquad x=\delta^{-1}\left(w-\mathrm{i}\frac{\pi}{2}\right),\qquad g_{\mathrm{i}}(x,\tau)=\delta^{-1}g\left(\mathrm{i}\frac{\pi}{2}+\delta x,\tau\right).\tag{146}$$

Thus, the change in inner variables is given by $z = x + g_{\mathrm{i}}(x,\tau)$, which is a solution of

$$\mathcal{L} g_{\mathrm{i}}(x,\tau) = \frac{\cosh^2(\mathrm{i}\frac{\pi}{2} + \delta z)}{4\delta^2} \partial_z \phi^-(z,\tau) - 1 \Big|_{z = x + g_{\mathrm{i}}(x,\tau)}, \tag{147}$$

where $\mathcal{L} = \partial_\tau + \partial_x$.

We will consider as a first approximation of g_{i} the change of variables $z = x + R^-(x,\tau)$ given by Theorem 3.8. Since this change conjugates the operators $\bar{\mathcal{L}}$ defined in (68) and $\mathcal{L}$, using Lemma 6.3, the function R^- holds the partial differential equation

$$\partial_\tau R^-(x,\tau) + \partial_x R^-(x,\tau) + 1 = -\frac{1}{4} z^2 \partial_z \phi_0^-(z,\tau) \Big|_{z = x + R^-(x,\tau)}.$$

We will look for a function g_{i} of the form $g_{\mathrm{i}}(x,\tau) = R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)$ with $|\hat{g}_{\mathrm{i}}| \le \mathcal{O}(\delta^{\nu_0})$ for certain $\nu_0 > 0$. As a consequence, using (60), $z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau) \in \mathcal{D}^{(8)}_{\delta,+}$ provided $(x,\tau) \in \mathcal{D}^{\mathrm{u}(9)}_{\delta,+} \times \mathbb{T}_{\sigma_3}$.

Hence, we will look for a solution of the following equation:

$$\mathcal{L} \hat{g}_{\mathrm{i}}(x,\tau) = F_1\big(\hat{g}_{\mathrm{i}}(x,\tau), x, \tau\big) + F_2\big(\hat{g}_{\mathrm{i}}(x,\tau), x, \tau\big), \tag{148}$$

where

$$\begin{aligned} & F_1\big(\hat{g}_{\mathrm{i}}(x,\tau), x, \tau\big) \\ &\quad = \frac{\cosh^2(\mathrm{i}\frac{\pi}{2} + \delta z)}{4\delta^2} \big(\partial_z \phi^-(z,\tau) - \partial_z \phi_0^-(z,\tau)\big) \\ &\qquad + \left(\frac{\cosh^2(\mathrm{i}\frac{\pi}{2} + \delta z)}{4\delta^2} + \frac{z^2}{4}\right) \partial_z \phi_0^-(z,\tau) \Big|_{z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)} \\ &\quad = -2b(z,\tau)\big(\partial_z \phi^-(z,\tau) - \partial_z \phi_0^-(z,\tau)\big) - a(z,\tau) \big|_{z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)}, \\ & F_2\big(\hat{g}_{\mathrm{i}}(x,\tau), x, \tau\big) \\ &\quad = -\frac{z^2}{4} \partial_z \phi_0^-(z,\tau) \Big|_{z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)} + \frac{z^2}{4} \partial_z \phi_0^-(z,\tau) \Big|_{z = x + R^-(x,\tau)} \\ &\quad = d\big(x + R^-(x,\tau), \tau\big) - d\big(x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau), \tau\big) \end{aligned}$$

and a and b are the functions defined in (113) and (114) and $d(z,\tau) = \frac{z^2}{4} \partial_z \phi_0^-(z,\tau)$.

As in Sect. 6, in order to solve (148), we will use we know the initial condition in a curve $\varGamma^{\mathrm{u}(9)}_{\gamma,+}$ defined in (111).

Lemma 7.8 *For all* $(x^*,\tau) \in \varGamma^{\mathrm{u}(9)}_{\gamma,+} \times \mathbb{T}_{\sigma_3}$, $|\hat{g}_{\mathrm{i}}(x^*,\tau)| \le \mathcal{O}(\delta^{1-2\gamma})$.

Proof We consider the change obtained in Proposition 7.1 and we express it in inner variables using (146):

$$z = x + \delta^{-1} \mathcal{C}^-(\mathrm{i}\pi/2 + \delta x, \tau).$$

Thus, for $(x^*, \tau) \in \Gamma^{\mathrm{u}(9)}_{\gamma,+} \times \mathbb{T}_{\sigma_3}$, since these points belong to the intersection between the inner and the outer domains, $R^- + \hat{g}_\mathrm{i}$ must coincide with the function $\delta^{-1}\mathcal{C}^-$ and so $\hat{g}_\mathrm{i}(x^*, \tau) = \delta^{-1}\mathcal{C}^-(\mathrm{i}\frac{\pi}{2} + \delta x^*, \tau) - R^-(x^*, \tau)$. Finally, to obtain the statement of the lemma, it is enough to use the bounds obtained in Proposition 7.1 and Theorem 3.8. □

With this initial condition, we write equation (148) as an integral equation using the operator $\mathcal{G}$ defined in Lemma 6.5 and defining a new operator $\mathcal{F}_\mathrm{i}$:

$$\hat{g}_\mathrm{i}(x,\tau) = \mathcal{F}_\mathrm{i}(\hat{g}_\mathrm{i}) = \mathcal{G}\big(F_1\big(\hat{g}_\mathrm{i}(x,\tau), x, \tau\big) + F_2\big(\hat{g}_\mathrm{i}(x,\tau), x, \tau\big)\big) + \hat{g}_\mathrm{i}\big(x^*, \tau + x^* - x\big), \tag{149}$$

where we have used $\mathcal{L}\hat{g}_\mathrm{i}(x^*, \tau + x^* - x) = 0$. We will solve this equation with a fixed point argument in the Banach space:

$$\mathcal{Y}_\infty = \big\{\hat{g}_\mathrm{i}\colon \mathcal{D}^{\mathrm{u}(9)}_{\delta,+} \times \mathbb{T}_{\sigma_3} \to \mathbb{C}, \text{continuous}, \ \|\hat{g}_\mathrm{i}\|_\infty \le +\infty\big\}.$$

Lemma 7.9 *The operator $\mathcal{G}$ defined in Lemma 6.5 is linear from $\mathcal{Y}_\infty$ to itself, and*

$$\big\|\mathcal{G}(h)\big\|_\infty \le \mathcal{O}\big(\delta^{\gamma-1}\big)\|h\|_\infty.$$

Proof It is straightforward recalling that $y^* - y = \mathcal{O}(\delta^{\gamma-1})$. □

Lemma 7.10 *There exists $C_5 > 0$ and $\nu_0 > 0$ such that the following inequalities hold*

1. $\|\mathcal{F}_\mathrm{i}(0)\|_\infty \le \frac{C_5}{2}\delta^{\nu_0}$.
2. *For all* $\hat{g}^1_\mathrm{i}, \hat{g}^2_\mathrm{i} \in B(C_5\delta^{\nu_0}) \subset \mathcal{Y}_\infty$, $\|\mathcal{F}_\mathrm{i}(\hat{g}^1_\mathrm{i}) - \mathcal{F}_\mathrm{i}(\hat{g}^2_\mathrm{i})\|_\infty \le \mathcal{O}(\ln^{-1}(1/\delta)) \times \|\hat{g}^1_\mathrm{i} - \hat{g}^2_\mathrm{i}\|_\infty$.

Proof Since $F_2(0, x, \tau) = 0$, $\mathcal{F}_\mathrm{i}(0) = \mathcal{G}(F_1(0, x, \tau)) + \hat{g}_\mathrm{i}(x^*, \tau + x^* - x)$. Moreover, using the functions $A(x, \tau)$ and $B(x, \tau)$ defined in (117) and (118):

$$F_1(0, x, \tau) = -2B(x,\tau)(\partial_z\phi^- - \partial_z\phi^-_0)|_{z = x + R^-(x,\tau)} - A(x,\tau).$$

Thus, using the bounds obtained in Lemma 6.7 and Theorem 3.9, it holds that $\|F_1(0, x, \tau)\|_\infty \le \mathcal{O}(\delta^{2\gamma})$. Therefore, applying Lemmas 7.8 and 7.9, and taking $\nu_0 = \min\{1 - 2\gamma, 3\gamma - 1\}$,

$$\big\|\mathcal{F}_\mathrm{i}(0)\big\|_\infty \le \mathcal{O}\big(\delta^{\gamma-1}\big)\big\|F_1(0, x, \tau)\big\|_\infty + \big|\hat{g}_\mathrm{i}\big(x^*, \tau + x^* - x\big)\big| \le \mathcal{O}\big(\delta^{\nu_0}\big).$$

For the second statement,

$$\begin{aligned}\big\|\mathcal{F}_\mathrm{i}\big(\hat{g}^1_\mathrm{i}\big) - \mathcal{F}_\mathrm{i}\big(\hat{g}^2_\mathrm{i}\big)\big\|_\infty &\le \big\|\mathcal{G}\big(F_1\big(\hat{g}^1_\mathrm{i}, x, \tau\big) - F_1\big(\hat{g}^2_\mathrm{i}, x, \tau\big)\big)\big\|_\infty \\ &\quad + \big\|\mathcal{G}\big(F_2\big(\hat{g}^1_\mathrm{i}, x, \tau\big) - F_2\big(\hat{g}^1_\mathrm{i}, x, \tau\big)\big)\big\|_\infty.\end{aligned}$$

Thus, we study each term separately. For the first one, we consider $\hat{g}_{\mathrm{i}}^{\rho} = \rho \hat{g}_{\mathrm{i}}^1 + (1-\rho)\hat{g}_{\mathrm{i}}^2$ in order to apply the mean value theorem:

$$\begin{aligned}
&F_1\big(\hat{g}_{\mathrm{i}}^1, x, \tau\big) - F_1\big(\hat{g}_{\mathrm{i}}^2, x, \tau\big) \\
&\quad = \big(\hat{g}_{\mathrm{i}}^1(x,\tau) - \hat{g}_{\mathrm{i}}^2(x,\tau)\big)\big(-2\partial_z b(z,\tau)\big(\partial_z \phi^-(z,\tau) - \partial_z \phi_0^-(z,\tau)\big) \\
&\qquad - 2b(z,\tau)\big(\partial_z^2 \phi^-(z,\tau) - \partial_z^2 \phi_0^-(z,\tau)\big) - \partial_z a(z,\tau)\big)\big|_{z = x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}^{\rho}(x,\tau)}.
\end{aligned}$$

Therefore, applying again the bounds of Lemma 6.7 and Theorem 3.9, since $\gamma < 1/2$,

$$\big\| F_1\big(\hat{g}_{\mathrm{i}}^1, x, \tau\big) - F_1\big(\hat{g}_{\mathrm{i}}^2, x, \tau\big)\big\|_\infty \le \mathcal{O}\big(\delta^{2\gamma} \ln^{-2}(1/\delta)\big)\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty,$$

and applying Lemma 7.9,

$$\big\|\mathcal{G}\big(F_1\big(\hat{g}_{\mathrm{i}}^1, x, \tau\big) - F_1\big(\hat{g}_{\mathrm{i}}^2, x, \tau\big)\big)\big\|_\infty \le \mathcal{O}\big(\delta^{3\gamma-1} \ln^{-2}(1/\delta)\big)\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty.$$

For the second term, using Theorem 3.9, we have $\partial_z d(z,\tau) = \mathcal{O}(z^{-2})$. Considering again $\hat{g}_{\mathrm{i}}^{\rho}$:

$$\big|\partial_z d\big(x + R^-(x,\tau) + \hat{g}_{\mathrm{i}}^{\rho}(x,\tau)\big)\big| \le \mathcal{O}(1)\frac{1}{|x|^2}.$$

Therefore, applying mean value theorem

$$\big\| F_2\big(\hat{g}_{\mathrm{i}}^1, x, \tau\big) - F_2\big(\hat{g}_{\mathrm{i}}^2, x, \tau\big)\big\|_\infty \le \mathcal{O}(1)|x|^{-2}\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty.$$

Hence, considering Lemma 6.8, we obtain,

$$\begin{aligned}
\big\|\mathcal{G}\big(F_2\big(\hat{g}_{\mathrm{i}}^1, x, \tau\big) - F_2\big(\hat{g}_{\mathrm{i}}^2, x, \tau\big)\big)\big\|_\infty &\le \mathcal{O}(1)\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty \int_{x^*-x}^0 |x+t|^{-2}\,\mathrm{d}t \\
&\le \mathcal{O}\big(\ln^{-1}(1/\delta)\big)\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty,
\end{aligned}$$

and considering the bound of each term

$$\big\|\mathcal{F}_{\mathrm{i}}\big(\hat{g}_{\mathrm{i}}^1\big) - \mathcal{F}_{\mathrm{i}}\big(\hat{g}_{\mathrm{i}}^2\big)\big\|_\infty \le \mathcal{O}\big(\ln^{-1}(1/\delta)\big)\big\|\hat{g}_{\mathrm{i}}^1 - \hat{g}_{\mathrm{i}}^2\big\|_\infty$$

that is the second statement of the lemma. □

Thus, now we are ready to prove Proposition 7.2.

Proof of Proposition 7.2 The bounds obtained in Lemma 7.10 give us that $\mathcal{F}_{\mathrm{i}}$ is a contraction from $B(C_5\delta^{\nu_0}) \subset \mathcal{Y}_\infty$ to itself. Hence, there exists a unique fixed point $\hat{g}_{\mathrm{i}}$ which holds $|\hat{g}_{\mathrm{i}}(x,\tau)| \le \mathcal{O}(\delta^{\nu_0})$.

Therefore, we have obtained the desired change of variables, which is given in the inner variables by

$$g_{\mathrm{i}}(x,\tau) = R^-(x,\tau) + \hat{g}_{\mathrm{i}}(x,\tau)$$

and then $\|g_{\mathrm{i}}\|_\infty \le \mathcal{O}(\ln^{-1}(1/\delta))$. Moreover, reasoning as in Sect. 6, it can be guaranteed that it is analytic. Translating it to outer variables, for $w \in D^{\mathrm{u}(9)}_{\delta,+}$

$$\mathcal{C}^-(w,\tau) = \delta g_{\mathrm{i}}\left(\delta^{-1}\left(w - \mathrm{i}\frac{\pi}{2}\right), \tau\right)$$

and then $|\mathcal{C}^-(w,\tau)| \le \mathcal{O}(\delta \ln^{-1}(1/\delta))$. For the bounds of the derivatives is enough to reduce the domain to $D^{\mathrm{u}(10)}_{\delta,+}$ and apply Cauchy estimates. □

8 The Global Change of Variables

Considering the change of variables $u = w + \mathcal{C}^-(w,\tau)$ defined in Theorem 3.11, we will find in this section the change of variables which conjugates $\tilde{\mathcal{L}}_\delta$ and $\mathcal{L}_\delta$.

Recalling that the operator $\tilde{\mathcal{L}}_\delta$ is defined by

$$\tilde{\mathcal{L}}_\delta = \tilde{\mathcal{L}}_\delta^- + \left(\frac{\cosh^2 u}{8}\big(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau)\big)\right)\partial_u$$

and applying the change of variables $u = w + \mathcal{C}^-(w,\tau)$ defined in Theorem 3.11 we see that this change conjugates $\tilde{\mathcal{L}}_\delta$ with

$$\delta^{-1}\partial_\tau + \partial_w + H(w,\tau)\partial_w, \tag{150}$$

where

$$H(w,\tau) = \frac{1}{1+\partial_w\mathcal{C}^-(w,\tau)}\left(\frac{\cosh^2 u}{8}\big(\partial_u T^+(u,\tau) - \partial_u T^-(u,\tau)\big)\right)\bigg|_{u=w+\mathcal{C}^-(w,\tau)}. \tag{151}$$

Thus, we look for a new change of variables $w = v + \mathcal{C}(v,\tau)$ which conjugates (150) to $\mathcal{L}_\delta$ for $(w,\tau) \in D^{(10)} \times \mathbb{T}_{\sigma_4}$ where $D^{(10)}$ is the domain defined in (76) where both manifolds are defined. Moreover, we will need $|\mathcal{C}| \le \mathcal{O}(\delta^{1+\nu_1})$ for certain $\nu_1 > 0$, since from (51) and (56), this will guarantee that this change of variables is well defined from $D^{(11)}$ to $D^{(10)}$, and also from $D^{(11)}_\gamma$ to $D^{(10)}_\gamma$ and from $D^{(11)}_{\delta,\pm}$ to $D^{(10)}_{\delta,\pm}$.

Applying Lemma 6.3, we will look for a function $\mathcal{C}$ solution of the partial differential equation

$$\mathcal{L}_\delta\mathcal{C} = H\big(v + \mathcal{C}(v,\tau),\tau\big). \tag{152}$$

In this way, the change which will conjugate $\tilde{\mathcal{L}}_\delta$ with $\mathcal{L}_\delta$ will be the composition of both changes, namely

$$u = v + \mathcal{C}(v,\tau) + \mathcal{C}^-\big(v + \mathcal{C}(v,\tau),\tau\big) = v + \mathcal{U}(v,\tau).$$

8.1 Banach Spaces and Technical Lemmas

In order to prove the existence of this change of variables, we will work with the following Banach spaces:

$$\mathcal{Z}_\infty = \{\mathcal{C}(v,\tau) \mid \mathcal{C}: D^{(11)} \times \mathbb{T}_{\sigma_4} \to \mathbb{C} \text{ real-analytic}, \|\mathcal{C}\|_{\infty,\sigma_4} < \infty\}, \tag{153}$$

$$\mathcal{Z}_1 = \{\mathcal{C}(v,\tau) \mid \mathcal{C}: D^{(11)} \times \mathbb{T}_{\sigma_4} \to \mathbb{C} \text{ real-analytic}, \|\mathcal{C}\|_{1,\sigma_4} < \infty\}, \tag{154}$$

where $D^{(11)}$ is the domain defined in (76) and $\|\cdot\|_{1,\sigma_4}$ is a Fourier norm defined analogously to the norm $\|\cdot\|_{1,c,\sigma_4}$ in Sect. 5.1 but in this new domain.

In the following lemmas, we will consider also the norms

$$\|\cdot\|_{1,\sigma_4,\mathrm{o}} \quad \text{and} \quad \|\cdot\|_{1,\sigma_4,\mathrm{i}}, \tag{155}$$

which are the norm $\|\cdot\|_{1,\sigma_4}$ restricted to the inner and outer parts of the domain $D^{(11)}$ defined in (77).

We note that σ_4 is any real number such that $0<\sigma_4<\sigma_3$. This reduction of domain has been considered in order to apply the following lemma.

Lemma 8.1 *Let σ_4 be any number such that $0<\sigma_4<\sigma_3$, the following bound holds*

$$\begin{aligned} &\left\|\partial_w \mathcal{C}^-(w,\tau)\right\|_{\infty,\mathrm{o},\sigma_4} \le \mathcal{O}\left(\delta^{2-3\gamma}\right) \left\|\partial_w^2 \mathcal{C}^-(w,\tau)\right\|_{\infty,\mathrm{o},\sigma_4} \le \mathcal{O}\left(\delta^{2-4\gamma}\right),\\ &\left\|\partial_w \mathcal{C}^-(w,\tau)\right\|_{\infty,\mathrm{i},\sigma_4} \le \mathcal{O}\left(\ln^{-2}(1/\delta)\right) \left\|\partial_w \mathcal{C}^-(w,\tau)\right\|_{\infty,\mathrm{i},\sigma_4} \le \mathcal{O}\left(\delta^{-1}\ln^{-3}(1/\delta)\right). \end{aligned} \tag{156}$$

Proof It is enough to apply the third statement of Lemma 5.1 (adapted to the domain $D^{(11)}$) to the bounds obtained in Theorem 3.11. □

Corollary 8.2 *Let $\nu_1>0$ be any positive constant and $\mathcal{C}$ a function defined in $D^{(11)}$ holding $|\mathcal{C}| \le \mathcal{O}(\delta^{1+\nu_1})$. Then for $u = v + \mathcal{C}(v,\tau) + \mathcal{C}^-(v+\mathcal{C}(v,\tau),\tau)$ where $\mathcal{C}^-$ is the function found in Theorem 3.11, the following bound holds*:

$$\left|\frac{\cosh v}{\cosh u}\right| \le \mathcal{O}(1). \tag{157}$$

In order to find a solution of (152), we need to consider a right inverse of the operator $\mathcal{L}_\delta$.

Lemma 8.3 *Let us consider $\rho^* = \pi/2 - c^{(11)}\delta\ln(1/\delta)$. For all $f \in \mathcal{Z}_1$, we consider the operator $\mathcal{G}_\delta^{\rho^*}$, defined acting on the Fourier coefficients of f*:

$$\mathcal{G}_\delta^{\rho^*}(f)^{[k]}(v) = \begin{cases} \int_{-\mathrm{i}\rho^*}^{v} \mathrm{e}^{\mathrm{i}k\delta^{-1}(r-v)} f^{[k]}(r)\,\mathrm{d}r & \text{if } k<0,\\ \int_0^v f^{[k]}(r)\,\mathrm{d}r & \text{if } k=0,\\ -\int_v^{\mathrm{i}\rho^*} \mathrm{e}^{\mathrm{i}k\delta^{-1}(r-v)} f^{[k]}(r)\,\mathrm{d}r & \text{if } k>0. \end{cases}$$

Then $\mathcal{G}_\delta^{\rho^*}$ is well defined from $\mathcal{Z}_1$ to $\mathcal{Z}_\infty$ and has the following properties:

1. $\mathcal{G}_\delta^{\rho^*}$ *is linear.*
2. $\mathcal{L}_\delta \circ \mathcal{G}_\delta^{\rho^*} = \mathrm{Id}$.
3. *If* δ *is small enough*, $\|\mathcal{G}_\delta^{\rho^*}(f)\|_{\infty,\sigma_4} \le \mathcal{O}(\ln(1/\delta))\|f\|_{1,\sigma_4}$.

Proof The first two statements are straightforward. For the third one, we consider as a path of integration the line between points v and $\pm i\rho^*$, so that $-k\Im(r-v) \le 0$ and, therefore, for all $k \in \mathbb{Z}$:

$$\left|e^{ik\delta^{-1}(r-v)}\right| \le 1.$$

We deal only with the case $k < 0$ since the other ones are analogous. Bounding the integral:

$$\left|\mathcal{G}_\delta^{\rho^*}(f)^{[k]}(v)\right| \le \left\|f^{[k]}(v)\right\|_1 \left|\int_{-i\rho^*}^{v} \frac{1}{|\cosh r|}\,dr\right|.$$

Since $|\cosh t|^{-1}$ has poles at $\pm i\pi/2$, we split the integral whether $\Im r > 0$ or $\Im r < 0$. In fact, for the case $\Im v > 0$, we consider r_0 as the value in the path of integration such that $\Im r_0 = 0$, we split the path of integration as $[-i\rho^*, r_0]$ and $[r_0, v]$ and we bound each integral.

$$\begin{aligned}
\left|\int_{-i\rho^*}^{v} \frac{1}{|\cosh r|}\,dr\right| &\le \mathcal{O}(1)\left|\int_{-i\rho^*}^{r_0} \frac{1}{|r + i\pi/2|}\,dr + \int_{r_0}^{v} \frac{1}{|r - i\pi/2|}\,dr\right| \\
&\le \mathcal{O}(1) \sup_{r\in[-i\rho^*, r_0]} \big|\ln|r + i\pi/2|\big| + \mathcal{O}(1) \sup_{r\in[r_0, v]} \big|\ln|r - i\pi/2|\big| \\
&\le \mathcal{O}\big(\ln(1/\delta)\big).
\end{aligned}$$

For the case $\Im v < 0$, the integral can be bounded directly as the first integral in the previous case. Both bounds give the third statement.

The third statement guarantees that $\mathcal{G}_\delta^{\rho^*}(f)$ is analytic. Thus, in order to check that $\mathcal{G}_\delta^{\rho^*}(f) \in \mathcal{Z}_\infty$, it only remains to check that it is real-analytic. To prove that, it is enough to use the reflection principle. □

8.2 Proof of Theorem 3.12

If we want a solution of (152), it is enough to find a solution of

$$\mathcal{C}(v,\tau) = \mathcal{G}_\delta^{\rho^*}\big(F_2(\mathcal{C}, v, \tau)\big), \tag{158}$$

where $\mathcal{G}_\delta^{\rho^*}$ is the operator given in Lemma 8.3,

$$F_2(\mathcal{C}, v, \tau) = H\big(v + \mathcal{C}(v,\tau), \tau\big) \tag{159}$$

and H is the function defined in (151). We will find this function $\mathcal{C}$ through a fixed point argument.

Proposition 8.4 *There exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$, there exists a function $\mathcal{C}(v, \tau)$ defined in $D^{(11)} \times \mathbb{T}_{\sigma_4}$, which is a fixed point of the functional*

$$\bar{F}_2 = \mathcal{G}_\delta^{\rho^*} \circ F_2. \tag{160}$$

The next lemma gives some bounds which will be needed.

Lemma 8.5 *There exists $\delta_0, \nu_1 > 0, \nu_2 > 0$ such that for all $\delta \in (0, \delta_0)$, the functional F_2 has the following properties*

1. *There exists $C_6 > 0$ such that $\|F_2(0, \cdot, \cdot)\|_{1,\sigma_4} \le \frac{C_6}{2}\delta^{1+\nu_1}$.*
2. *For all $\mathcal{C}_1, \mathcal{C}_2 \in B(C_6\delta^{1+\nu_1}) \subset \mathcal{Z}_\infty$,*

$$\left\| F_2(\mathcal{C}_1, \cdot, \cdot) - F_2(\mathcal{C}_2, \cdot, \cdot)\right\|_{1,\sigma_4} \le \mathcal{O}\left(\delta^{\nu_2}\right)\|\mathcal{C}_1 - \mathcal{C}_2\|_{\infty,\sigma_4}.$$

Proof For the first statement, we recall that

$$F_2(0, v, \tau) = \left(1 + \partial_w \mathcal{C}^-(w, \tau)\right)^{-1}\left(\frac{\cosh^2 u}{8}\left(\partial_u T^+ - \partial_u T^-\right)(u, \tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}\right).$$

Using bound (156), it is clear that $\|(1 + \partial_w \mathcal{C}^-(w, \tau))^{-1}\|_{\infty,\sigma_4} \le \mathcal{O}(1)$. In order to bound the second factor, we consider the auxiliary norms stated in (155). Recalling that $T^+ - T^- = Q^+ - Q^-$ and applying the bounds obtained in Theorem 3.6:

$$\left\| F_2(0, \cdot, \cdot)\right\|_{1,\sigma_4,\mathrm{o}} \le \mathcal{O}(1)\left\|\partial_u Q^+ - \partial_u Q^-\right\|_{3,c,\sigma_4} \le \mathcal{O}\left(\mu\delta^{2-\gamma}\right).$$

In the inner domain,

$$\begin{aligned}\left(\partial_u T^+ - \partial_u T^-\right)(u, \tau) = {}& \delta^{-2}\left(\partial_z\phi^+ - \partial_z\phi_0^+\right)\left(\delta^{-1}(u - \mathrm{i}\pi/2)\right)\\ &- \delta^{-2}\left(\partial_z\phi^- - \partial_z\phi_0^-\right)\left(\delta^{-1}(u - \mathrm{i}\pi/2)\right)\\ &+ \delta^{-2}\left(\partial_z\phi_0^+ - \partial_z\phi_0^-\right)\left(\delta^{-1}(u - \mathrm{i}\pi/2)\right).\end{aligned} \tag{161}$$

Applying Theorem 3.8,

$$\begin{aligned}&\left\|\cosh^2 u\delta^{-2}\left(\partial_z\phi_0^+ - \partial_z\phi_0^-\right)\left(\delta^{-1}(u - \mathrm{i}\pi/2)\right)\right\|_{1,\sigma_4,\mathrm{i}}\\ &\quad\le \mathcal{O}(\delta)|z|^3\mathrm{e}^{\Im z} \le \mathcal{O}\left(\delta^{1+c^{(12)}}\ln^3(1/\delta)\right)\end{aligned} \tag{162}$$

since $|z|^3\mathrm{e}^{\Im z}$ reaches its maximum for the biggest imaginary part of the inner domain. Thus, applying this bound and Theorem 3.9,

$$\left\| F_2(0, \cdot, \cdot)\right\|_{1,\sigma_4,\mathrm{i}} \le \mathcal{O}\left(\delta^{3\gamma} + \delta^{1+c^{(11)}}\ln^3(1/\delta)\right) \tag{163}$$

and thus, taking ν_1 such that $0 < \nu_1 < \min\{3\gamma - 1, c^{(11)}\}$

$$\left\| F_2(0, \cdot, \cdot)\right\|_{1,\sigma_4} \le \mathcal{O}\left(\delta^{1+\nu_1}\right).$$

For the second statement, consider $\mathcal{C}^\rho = \rho\mathcal{C}_1 + (1-\rho)\mathcal{C}_2$ for $\rho \in (0,1)$, we recall that

$$F_2(\mathcal{C}_1, v, \tau) - F_2(\mathcal{C}_2, v, \tau) = \partial_w H\big(v + \mathcal{C}^\rho(v,\tau)\big)\big(\mathcal{C}_1(v,\tau) - \mathcal{C}_2(v,\tau)\big),$$

where

$$\begin{aligned}\partial_w H(w,\tau) = {}& \frac{-\partial_w^2 \mathcal{C}^-(w,\tau)}{(1+\partial_w \mathcal{C}^-(w,\tau))^2}\bigg(\frac{\cosh^2 u}{8}\big(\partial_u T^+ - \partial_u T^-\big)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}\bigg)\\ &+ \frac{\cosh u \sinh u}{4}\big(\partial_u T^+ - \partial_u T^-\big)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}\\ &+ \frac{\cosh^2 u}{8}\big(\partial_u^2 T^+ - \partial_u^2 T^-\big)(u,\tau)\Big|_{u=w+\mathcal{C}^-(w,\tau)}.\end{aligned}$$

We bound it in different ways in the outer and inner domains. For the outer one, recalling again that $\partial_u^j T^+ - \partial_u^j T^- = \partial_u^j Q^+ - \partial_u^j Q^-$ and applying the bounds obtained in Lemma 8.1, Corollary 8.2, and Theorem 3.6,

$$\big\|\partial_w H\big(v + \mathcal{C}^\rho(v,\tau),\tau\big)\big\|_{1,\mathrm{o},\sigma_4} \le \mathcal{O}\big(\delta^{2-2\gamma}\big).$$

For the inner domain, applying Lemma 8.1, using (161), and

$$\begin{aligned}\big(\partial_u^2 T^+ - \partial_u^2 T^-\big)(u,\tau) = {}& \delta^{-3}\big(\partial_z^2\phi^+ - \partial_z^2\phi_0^+\big)\big(\delta^{-1}(u - \mathrm{i}\pi/2)\big)\\ &- \delta^{-3}\big(\partial_z^2\phi^- - \partial_z^2\phi_0^-\big)\big(\delta^{-1}(u - \mathrm{i}\pi/2)\big)\\ &+ \delta^{-3}\big(\partial_z^2\phi_0^+ - \partial_z^2\phi_0^-\big)\big(\delta^{-1}(u - \mathrm{i}\pi/2)\big)\end{aligned}$$

and proceeding analogously to (162),

$$\big\|\partial_w H\big(v + \mathcal{C}^\rho(v,\tau),\tau\big)\big\|_{1,\mathrm{i},\sigma_4} \le \mathcal{O}\big(\delta^{3\gamma-1}\ln^{-2}(1/\delta) + \delta^{c^{(11)}}\ln^3(1/\delta)\big).$$

Therefore, taking ν_2 such that $0 < \nu_2 < \min\{2-2\gamma, 3\gamma-1, c^{(11)}\}$,

$$\big\|F_2(\mathcal{C}_1, v, \tau) - F_2(\mathcal{C}_2, v, \tau)\big\|_{1,\sigma_4} \le \mathcal{O}\big(\delta^{\nu_2}\big)\|\mathcal{C}_1 - \mathcal{C}_2\|_{\infty,\sigma_4}. \qquad \square$$

With this lemma we are ready to prove Proposition 8.4.

Proof of Proposition 8.4 The results obtained in Lemmas 8.3 and 8.5 give that the operator $\bar{F}_2$ defined in (160) is a contraction from $B(C_6\delta^{\nu_1}) \subset \mathcal{Z}_\infty$ to itself. Hence, it has a fixed point $\mathcal{C}$ which holds that for $(w,\tau) \in D^{(11)} \times \mathbb{T}_{\sigma_4}$

$$\big|\mathcal{C}(w,\tau)\big| \le \mathcal{O}\big(\delta^{1+\nu_1}\big). \qquad \square$$

Having proved the proposition, we can now prove Theorem 3.12.

Proof of Theorem 3.12 It is sufficient to define $\mathcal{U}(v,\tau) = v + \mathcal{C}(v,\tau) + \mathcal{C}^-(v + \mathcal{C}(v,\tau),\tau)$ and consider the bounds of Theorem 3.11 and Proposition 8.4.

For the bounds of the derivatives it is enough to reduce the domain to $D^{(12)}$ and apply Cauchy estimates to the function $\mathcal{C}$ in the outer and inner domains of $D^{(12)}$, and then use again the bounds obtained in Theorem 3.11. $\square$

9 Proof of Theorem 2.7

In this section, we give the complete proof of Theorem 2.7 following the same scheme that was presented in Sect. 3 to prove Theorem 2.2. However, since in this case we are only looking for an upper bound of the splitting of separatrices, the proof will be easier. In fact, we will only need to study the invariant manifolds $T^{\pm}(u,\tau)$ defined in (38) in the outer domains (see (50)). The existence of these manifolds is obtained in Theorem 9.2 and its proof is analogous to the proof of Theorem 3.6 in Sect. 5. Later, we consider the equation $\tilde{\mathcal{L}}_\delta \Delta T = 0$ verified by the difference between the manifolds (see (78)) and in Theorem 9.3 we look for a change of variables $\mathcal{U}$ that straightens $\tilde{\mathcal{L}}_\delta$ in the intersection of the outer domain. This procedure is analogous to the one presented in Sect. 8. In this case, we are not going to consider the inner equation nor the matching procedure.

In this section, we restrict ourselves to $\mu = \varepsilon^p$ with $p \in (-4, 0)$ and $\mu \in I_i \in (\mu_{2i+1}, \mu_{2i+2})$ where μ_i are the zeros of the Bessel function $J_0(\mu)$, and thus μ tends to infinity as ε tends to 0. On the other hand, we recall that the existence of the periodic orbit has been proved in Proposition 4.1, which has the following corollary.

Corollary 9.1 *For any $p < 0$, there exists $\varepsilon_0 > 0$ such that for any s_1 holding $s_1\varepsilon^{-p} < \sigma_0(\varepsilon^p)$ (see (95)), system (4) with $\mu = \varepsilon^p$ has a periodic orbit $x_{\mathrm{p}}(\tau)$ defined in $\mathbb{T}_{\sigma_1}$ with $\sigma_1 = s_1\varepsilon^{-p}$, which holds*

$$\left\| x_{\mathrm{p}}(\tau) + \mu \sin\tau \right\|_{\sigma_1} \le \mathcal{O}\left(\varepsilon^2\right), \tag{164}$$

where $\|\cdot\|_{\sigma_1}$ is the Fourier norm defined in Sect. 4.

From now on, we restrict ourselves to $p \in (-4, 0)$. On the other hand, we want to point out that through this section we will have to reduce once again the analyticity strip in τ and, therefore, we will take $\sigma_2 = s_2\varepsilon^{-p}$ for any $s_2 > 0$ such that $\sigma_2 < \sigma_1$.

As we did in Sect. 1, we perform the change of variables (19) in order to translate the periodic orbit to the origin, and we obtain the Hamilton–Jacobi equation (32). Considering changes (36), (37), and (38) as in Sect. 3, we obtain again (40) with the corresponding asymptotic conditions (41).

We recall that in order to perform these changes, it is needed $J_0(\mu) > 0$. As it has been explained in Sect. 2.1, since for μ big,

$$J_0(\mu) \sim \sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4}\right)$$

(see Abramowitz and Stegun 1992), the zeros $\mu_0 < \mu_1 < \cdots$ of $J_0(\mu)$ are separated when $\mu \to +\infty$. Thus, from now on, for any fixed $p < 0$, we will consider ε arbitrarily small but defined in compact intervals such that $\mu = \varepsilon^p \in I_i$ for $i \in \mathbb{N}$, in such a way that $\cos(\mu - \pi/4)$ has a positive lower bound independent of ε and, therefore, $J_0(\mu) > 0$ holds. On the other hand, in these intervals I_i, we have the following bounds:

$$\left|J_0\left(\varepsilon^p\right)\right| \leq \mathcal{O}\left(\varepsilon^{-\frac{p}{2}}\right) \quad \text{and} \quad \left|J_0\left(\varepsilon^p\right)\right|^{-1} \leq \mathcal{O}\left(\varepsilon^{\frac{p}{2}}\right). \tag{165}$$

Therefore, in this case, the parameter $\delta = \varepsilon\sqrt{J}$ is not of the same order as ε but smaller: $\delta = \mathcal{O}(\varepsilon^{1-\frac{p}{4}})$. Hence, in terms of δ, bounds (165) read

$$\left|J_0\left(\varepsilon^p\right)\right| \leq \mathcal{O}\left(\delta^{-\frac{2p}{4-p}}\right) \quad \text{and} \quad \left|J_0\left(\varepsilon^p\right)\right|^{-1} \leq \mathcal{O}\left(\delta^{\frac{2p}{4-p}}\right). \tag{166}$$

On the other hand, δ will have to be chosen such that $\mu = (\delta/\sqrt{J})^p \in I_i$ for $i \in \mathbb{N}$. Thus, all the results stated from now on will hold provided this condition holds.

The next step is to prove the existence of the stable and unstable invariant manifolds in the outer domains (see (50)). We proceed as in Sect. 5 looking for solutions of the Hamilton–Jacobi equation (40) with the corresponding asymptotic conditions (41).

Theorem 9.2 *Let p and γ be real numbers such that $p \in (-4, 0)$ and $\gamma \in (0, \nu(p))$ where*

$$\nu(p) = \frac{4+p}{4-p} > 0. \tag{167}$$

Then there exists $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ holding $\mu = (\delta/\sqrt{J})^p \in I_i$, in such a way that $J_0(\mu) > 0$, (40) has unique (module an additive constant) solutions in $D_\gamma^{\mathrm{u}(1)} \times \mathbb{T}_{\sigma_1}$ and $D_\gamma^{\mathrm{s}(1)} \times \mathbb{T}_{\sigma_1}$, with $\sigma_1 = s_1\varepsilon^{-p}$, of the form

$$T^{\pm}(u,\tau) = T_0(u) + W^{\pm}(u,\tau)$$

holding asymptotic conditions (41).

Moreover, there exists a real constant $d_1 > 0$ independent of δ such that, for $i = 0, 1$,

$$\left\|\partial_u^{i+1} W^-\right\|_{3,2,\sigma_1} \leq d_1 \delta^{\nu(p)-\mathrm{i}\gamma} \ln(1/\delta),$$

where $\|\cdot\|_{3,2,\sigma_1}$ is the Fourier norm defined in Sect. 5.1. Analogous bounds for W^+ also hold.

Proof of Theorem 9.2 We deal only with the case of the unstable manifold. Analogously to the proof of Theorem 3.6 in Sect. 5, we prove the existence of the invariant manifold using a fixed point argument, but in this case for $W = T - T_0$ in the Banach space $\mathcal{E}_{3,2}$ defined in (97) with $\sigma_1 = s_1\varepsilon^{-p}$.

Thus, we replace $W = T - T_0$ in (40) and, using the operator $\bar{\mathcal{G}}_\delta$ defined in Lemma 5.5, we obtain the following equation:

$$\partial_u W = \bar{\mathcal{F}}(\partial_u W) = \bar{\mathcal{G}}_\delta\left(\mathcal{F}(\partial_u W)\right),$$

where

$$\mathcal{F}(h) = -\frac{\cosh^2 u}{8} h^2 + \frac{2}{J \cosh^2 u}\big(\cos x_{\mathrm{p}}(\tau) - J\big) + \frac{1}{J}\psi(u) \sin x_{\mathrm{p}}(\tau).$$

Although the width σ_1 of the strip of analyticity in τ depends on ε, the properties of operator $\bar{\mathcal{G}}_\delta$ stated in Lemma 5.5 still hold. On the other hand, it is straightforward to see that the operator $\bar{\mathcal{F}}$ is well defined from $\mathcal{E}_{3,2}$ to itself. Thus, in order to prove the existence of a fixed point of $\bar{\mathcal{F}}$, we have to obtain bounds analogous to the ones in Lemma 5.7.

First, we express bound (164) in terms of δ,

$$\big\| x_{\mathrm{p}}(\tau) + \varepsilon^p \sin \tau \big\|_{\sigma_1} \leq \mathcal{O}\big(\delta^{\frac{8}{4-p}}\big), \tag{168}$$

with $\sigma_1 = s_1 \varepsilon^{-p} = s_1 (\delta/\sqrt{J})^{-p}$ for any s_1 such that $s_1 \varepsilon^{-p} < \sigma(\mu)$.

We split $\mathcal{F}(0)(u,\tau) = p_1(u,\tau) + p_2(u,\tau)$ with

$$p_1(u,\tau) = \frac{2}{J \cosh^2 u}\big(\cos x_{\mathrm{p}}(\tau) - \cos\big(\varepsilon^p \sin\tau\big)\big) + \frac{1}{J}\psi(u)\big(\sin x_{\mathrm{p}}(\tau) + \sin\big(\varepsilon^p \sin\tau\big)\big)$$

and

$$p_2(u,\tau) = \frac{2}{J \cosh^2 u}\big(\cos\big(\varepsilon^p \sin\tau\big) - J\big) - \frac{1}{J}\psi(u) \sin\big(\varepsilon^p \sin\tau\big).$$

Using bounds (166) and (168), we obtain

$$\big\| p_1(u,\tau) \big\|_{3,2,\sigma_1} \leq \mathcal{O}\big(\delta^{\frac{2(p+4)}{4-p}}\big)$$

and applying Lemma 5.5, we have that $\|\bar{\mathcal{G}}(p_1)\|_{3,2,\sigma_1}$ has the same bound.

The term $\bar{\mathcal{G}}(p_2) = \partial_u \mathcal{G}(p_2)$ has to be bounded more carefully. First, we recall that p_2 has zero average, thus proceeding as in Lemma 5.7, we use that $\bar{\mathcal{G}}(p_2) = \mathcal{G}(\partial_u p_2)$. On the other hand, using bound (166) and that for $\tau \in \mathbb{T}_{\sigma(\mu)}$ (see (95)) $|\sin(\mu \sin\tau)| \leq \mathcal{O}(1)$ and $|\cos(\mu \sin\tau)| \leq \mathcal{O}(1)$ (see Sect. 4), we have that

$$\big\| \partial_u p_2^{[k]}(u) \big\|_{3,2} \leq \mathrm{e}^{-|k|\sigma(\mu)} \mathcal{O}\big(\delta^{\frac{2p}{4-p}}\big).$$

Therefore, using this bound and (102), we have that

$$\begin{aligned}
\big\| \mathcal{G}(\partial_u p_2) \big\|_{3,2,\sigma_1} &\leq \mathcal{O}(\delta) \sum_{k \in \mathbb{Z}\setminus\{0\}} \mathcal{O}\big(\delta^{\frac{2p}{4-p}}\big) \frac{1}{|k|} \mathrm{e}^{|k|(\sigma_1 - \sigma(\mu))} \\
&\leq \mathcal{O}\big(\delta^{\frac{4+p}{4-p}}\big) \ln\big(1 - \mathrm{e}^{(\sigma_1 - \sigma(\mu))}\big) \leq \mathcal{O}\big(\delta^{\nu(p)} \ln(1/\delta)\big).
\end{aligned}$$

Therefore, one can see that there exists a constant $d_1 > 0$ such that

$$\big\| \bar{\mathcal{F}}(0) \big\|_{3,2,\sigma_1} \leq d_1 \delta^{\nu(p)} \ln(1/\delta)/2.$$

Taking now $h_1, h_2 \in B(d_1\delta^{\nu(p)}\ln(1/\delta)) \subset \mathcal{E}_{3,2}$,

$$\left\|\bar{\mathcal{F}}(h_1) - \bar{\mathcal{F}}(h_2)\right\|_{3,2,\sigma_1} \leq \mathcal{O}\bigl(\delta^{\nu(p)-\gamma}\bigr)\|h_1 - h_2\|_{3,2,\sigma_1}.$$

Since $\gamma < \nu(p)$ and reducing δ if necessary, $\bar{\mathcal{F}}$ is contractive from $B(d_1\delta^{\nu(p)}) \subset \mathcal{E}_{3,2}$ to itself. Thus, reasoning as in the proof of Proposition 5.6, we obtain the existence of the invariant manifold $T^-(u,\tau)$ in $D_\gamma^{\mathrm{u}(0)}$. Finally, reducing the domain to $D_\gamma^{\mathrm{u}(1)}$, we get the wanted bounds for the second derivative. □

The last step is to bound the difference between the invariant manifolds. We consider $\Delta T(u,\tau) = T^+(u,\tau) - T^-(u,\tau)$ which holds $\tilde{\mathcal{L}}_\delta \Delta T = 0$ where $\tilde{\mathcal{L}}_\delta$ is the operator defined in (78). We recall that ΔT is defined in $D_\gamma^{(1)} = D_\gamma^{\mathrm{u}(1)} \cap D_\gamma^{\mathrm{s}(1)}$ which is of the form

$$D_\gamma^{(i)} = \left\{u \in \mathbb{C}\colon\ |\Im u| + \tan\beta_0|\Re u| < \frac{\pi}{2} - \tan\beta_0 a^{(i)}\delta^\gamma\right\}.$$

As we have done in Sect. 8, we look for a change of variables that conjugates the operator $\tilde{\mathcal{L}}_\delta$ to $\mathcal{L}_\delta$ (see (79)) in order to apply Lemma 3.10. As we are working in the outer domain, where the invariant manifolds $T^\pm$ are well approximated by T_0, the change $\mathcal{U}$ we look for is close to the identity.

Theorem 9.3 *Let s_1 be the constant defined in Theorem 9.2, and p and γ be real numbers such that $p \in (-4,0)$ and $\gamma \in (0,\nu(p))$ where $\nu(p)$ has been defined in (167). Then there exists $\delta_0 > 0$ such that for $\delta \in (0,\delta_0)$, there exists a real-analytic function $\mathcal{U}(v,\tau)$ in $D_\gamma^{(3)} \times \mathbb{T}_{\sigma_1}$, with $\sigma_1 = s_1\varepsilon^{-p}$, such that the change*

$$(u,\tau) = \bigl(v + \mathcal{U}(v,\tau),\tau\bigr)$$

conjugates the operators $\tilde{\mathcal{L}}_\delta$ and $\mathcal{L}_\delta$ defined in (78) and (79). Moreover, for $(v,\tau) \in D_\gamma^{(3)} \times \mathbb{T}_{\sigma_1}$,

(i) $v + \mathcal{U}(v,\tau) \in D_\gamma^{(2)}$.
(ii) $|\partial_v^j \mathcal{U}(v,\tau)| \leq \mathcal{O}(\delta^{\nu(p)-j\gamma}\ln^2(1/\delta))$ *for* $j = 0,1,2$.

There exists the inverse change of variables $(v,\tau) = (u + \mathcal{V}(u,\tau),\tau)$ which is real-analytic and for $(u,\tau) \in D_\gamma^{(4)} \times \mathbb{T}_{\sigma_1}$ holds:

(i) $u + \mathcal{V}(u,\tau) \in D_\gamma^{(3)}$.
(ii) $|\partial_u^j \mathcal{V}(u,\tau)| \leq \mathcal{O}(\delta^{\nu(p)-j\gamma}\ln^2(1/\delta))$ *for* $j = 0,1,2$.

Proof Applying Lemma 6.3, function $\mathcal{U}$ holds

$$\mathcal{L}_\delta \mathcal{U} = \frac{\cosh^2 u}{8}\bigl(\partial_u W^+(u,\tau) + \partial_u W^-(u,\tau)\bigr)\Big|_{u=v+\mathcal{U}(v,\tau)}. \tag{169}$$

We find $\mathcal{U}$ as in Sect. 8. We consider the Banach spaces $\mathcal{Z}_\infty$ and $\mathcal{Z}_1$ (see (153) and (154)) for functions defined in $D_\gamma^{(2)} \times \mathbb{T}_{\sigma_1}$ with $\sigma_1 = s_1\varepsilon^{-p}$.

Recalling the operator $\mathcal{G}_\delta^{\rho^*}$ defined in Lemma 8.3 with $\rho^* = \pi/2 - a^{(2)}\delta^\gamma$, one can see that Lemma 8.3 still holds. Thus, in order to prove the theorem, it is enough to look for a fixed point of the functional

$$\tilde{\mathcal{H}} = \mathcal{G}_\delta^{\rho^*} \circ \mathcal{H},$$

where

$$\mathcal{H}(\mathcal{U}) = \frac{\cosh^2 u}{8}\big(\partial_u W^+(u,\tau) + \partial_u W^-(u,\tau)\big)\Big|_{u=v+\mathcal{U}(v,\tau)}.$$

We will need that $|\mathcal{U}| \le \mathcal{O}(\delta^s)$ with $s > \gamma$ in order to guarantee that this change is well defined from $D_\gamma^{(2)}$ to $D_\gamma^{(1)}$, in such a way that $\mathcal{H}(\mathcal{U})$ can be defined.

Considering the bounds obtained in Theorem 9.2 and Lemma 8.3, it is straightforward to see that there exists $d_2 > 0$ such that

$$\big\|\tilde{\mathcal{H}}(0)\big\|_{\infty,\sigma_1} \le \mathcal{O}\big(\ln(1/\delta)\big)\big\|\mathcal{H}(0)\big\|_{1,\sigma_1} \le \frac{d_2}{2}\delta^{\nu(p)}\ln^2(1/\delta).$$

Thus, taking $h_1, h_2 \in B(d_2\delta^{\nu(p)}\ln^2(1/\delta)) \subset \mathcal{Z}_\infty$, we have that

$$\begin{aligned}\big\|\tilde{\mathcal{H}}(h_2) - \tilde{\mathcal{H}}(h_1)\big\|_{\infty,\sigma_1} &\le \mathcal{O}\big(\ln(1/\delta)\big)\big\|\mathcal{H}(h_2) - \mathcal{H}(h_1)\big\|_{1,\sigma_1} \\ &\le \mathcal{O}\big(\delta^{\nu(p)-\gamma}\ln^2(1/\delta)\big)\|h_1 - h_2\|_{\infty,\sigma_1}.\end{aligned}$$

Recalling that $\nu(p) - \gamma > 0$ and reducing δ if necessary, $\tilde{\mathcal{H}}$ is a contraction from $B(d_2\delta^{\nu(p)}\ln^2(1/\delta)) \subset \mathcal{Z}_\infty$ to itself and, therefore, there exists a unique fixed point $\mathcal{U}$ of (169) which gives the change of variables. In order to obtain the bounds of the derivatives, it is enough to reduce the domain to $D_\gamma^{(3)}$ and apply Cauchy estimates.

To obtain the existence and bounds of the inverse change, one has to use again a fixed point argument. □

With the existence of this change of variables, we are ready to prove Theorem 2.7.

Proof of Theorem 2.7 The last statement of this theorem follows applying Lemma 3.10 to the function

$$\zeta(v,\tau) = \Delta T\big(v + \mathcal{U}(v,\tau),\tau\big)$$

which satisfies $\mathcal{L}_\delta\zeta = 0$.

We observe that

$$\begin{aligned}\partial_v^2\zeta(v,\tau) &= \partial_u^2\Delta T\big(v+\mathcal{U}(v,\tau),\tau\big)\partial_v\big(1+\mathcal{U}(v,\tau)\big)^2 \\ &\quad + \partial_u\Delta T\big(v+\mathcal{U}(v,\tau),\tau\big)\partial_v^2\mathcal{U}(v,\tau).\end{aligned}$$

Recalling that $\Delta T = W^+ - W^-$ and using the bounds obtained in Theorems 9.2 and 9.3, it is straightforward to see that for $(v,\tau) \in \overline{D_\gamma^{(4)}} \times \overline{\mathbb{T}_{\sigma_2}}$, with $\sigma_2 = s_2\varepsilon^{-p}$ for any $s_2 < s_1$,

$$\big|\partial_v^2\zeta(v,\tau)\big| \le \mathcal{O}\big(\delta^{\nu(p)-4\gamma}\big).$$

Thus, applying Lemma 3.10 with $r=\pi/2-a^{(4)}\delta^\gamma$ and $M_r=\mathcal{O}(\delta^{\nu(p)-4\gamma})$, we obtain that for $v\in D_\gamma^{(4)}\cap\mathbb{R}$ and $\tau\in\mathbb{T}$

$$\left|\partial_v^2\zeta(v,\tau)\right|\le\mathcal{O}\left(\delta^{\nu(p)-4\gamma}\mathrm{e}^{-\frac{1}{\delta}(\frac{\pi}{2}-a^{(4)}\delta^\gamma)}\right)$$

and analogous bounds for $|\partial_v\zeta(v,\tau)|$ and $|\zeta(v,\tau)-\langle\zeta\rangle|$. Therefore, using the inverse change of variables $v=u+\mathcal{V}(u,\tau)$ obtained in Theorem 9.3, recalling that $\delta=\varepsilon\sqrt{J}$,

$$\delta^{\nu(p)}\le\mathcal{O}\left(\varepsilon^{1+\frac{p}{4}}\right)$$

(where $\kappa(p)$ is the function defined in Theorem 2.7) and taking $\bar{\gamma}=\gamma(4-p)/4>0$, we obtain the bounds of Theorem 2.7, after performing the changes (38), (37), (36), and (19). □

10 Conclusions

In this paper, we present some results and some open questions related to a classical problem: to obtain rigorously the measure of the splitting of separatrices of the rapidly forced pendulum

$$\ddot{x}=\sin x+\frac{\mu}{\varepsilon^2}\sin\frac{t}{\varepsilon},$$

where $\varepsilon>0$ is a small parameter.

This problem was considered firstly by Holmes, Marsden, and Scheurle in Holmes et al. (1988) and solved in the case that $\mu=\varepsilon^p$ with $p>10$. Basically, they showed that the prediction given by the Melnikov formula in this problem is exponentially small due to two facts: the homoclinic orbit of the pendulum has complex singularities (poles, in this case) and the perturbation is periodic with a small period $2\pi\varepsilon$. Then they showed that this prediction is valid for $p>10$.

Since then, several authors have tried (and succeeded) to weaken the hypothesis of the size of the perturbation, that is, to decrease the values of p, and give a rigorous proof of the splitting of the invariant manifolds arising from the periodic orbit of the system.

In this paper, we present three results about this problem.

First, accordingly to Gelfreich (2000), we rigorously show in Theorem 2.2 that the prediction of the Melnikov formula is valid if $\mu=\varepsilon^p$ with $p>1/2$ and we give an alternative formula for the splitting when $0\le p\le 1/2$. It is important to stress that if $0<p\le 1/2$ the alternative formula can be obtained if we apply the Melnikov method to the system (13), obtained after two steps of averaging and a suitable rescaling. Of course, after these changes, the system has a different period. This implies that the splitting has a different exponential size.

An important feature of the proof presented here relies on the fact that, as noticed in Sauzin (2001), Lochak et al. (2003), since the perturbed pendulum is a Hamiltonian system, the invariant manifolds can be written as graphs of the differential of certain functions $S^\pm$. Moreover, these functions satisfy the so-called Hamilton–Jacobi equation, which is a partial differential equation of first order. To obtain these functions, matching techniques in the complex plane are used to obtain different approximations for them. In particular, we use some results about the so-called "inner equation" in Baldomá (2006), Olivé et al. (2003).

It is important to stress that the methods used in this paper can be extended to any trigonometric polynomial Hamiltonian perturbation of the pendulum using the results about the corresponding inner equation in Baldomá (2006).

Second, we show that the problem still makes sense below the so called "singular case" $\mu = \mathcal{O}(1)$, since the system under consideration has a periodic orbit even if $\mu = \varepsilon^p$, for $p < 0$ and it is hyperbolic for $-4 < p < 0$. Moreover, in Theorem 2.7, we obtain an exponentially small upper bound for the splitting in this case. We also give some ideas about the possible size of the splitting in this case which mainly rely, again, on doing more steps of averaging to the system and obtain the singularities of the (new) homoclinic orbit.

Even if this "below the singular case" can seem quite "artificial" (the size of μ being too big), it is important to note that this case appears in lots of systems even if the perturbation is small. For instance, considering the perturbed pendulum:

$$\ddot{x} = \sin x + \mu \sin(nx) \sin \frac{t}{\varepsilon}$$

the singular case where the methods we present here can work would be $\mu = \varepsilon^p$ for $p = 2n - 2$. In the cases $p < 2n - 2$, even if the perturbation of the pendulum is small, the methods used up to now would not give any answer to the question of the splitting and other approximations to the problem, like the averaging procedure, should be applied to obtain the correct exponential size of the splitting.

Finally, we study the problem of splitting of separatrices when μ is close to the zeros of the Bessel function $J_0(\mu)$, namely $\mu = \mu_i - \varepsilon^r$ where μ_i is one of the zeros of this function. In that case, the averaged system obtained after two steps of averaging has different hyperbolic behavior, and hence splitting size. In this setting, we prove that the periodic orbit is still hyperbolic provided $r \in (0, 2)$ and give a non-sharp exponentially small upper bound of the size of the splitting in Theorem 2.9. For $r \geq 2$, we give some insight on the appearance of some bifurcation phenomena which substantially change the topology of the hyperbolic structure, even for the averaged

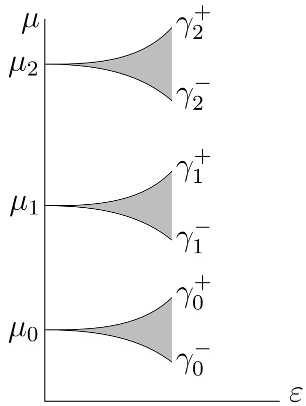


Fig. 12 Parameter space (μ, ε). In *white*, we show the domain in which we obtain exponentially small results and in *gray* the domain in which the problem remains open

system (which is integrable). In fact, $\mu(\varepsilon) = \mu_i \pm c_i^{\pm}\varepsilon^2 + o(\varepsilon^2)$ for certain $c_i^{\pm} > 0$, are codimension-1 curves $\gamma_i^{\pm}$ on which occurs a pitchfork bifurcations in which a parabolic critical point appears and then bifurcates to two elliptic and a hyperbolic points, giving birth to a figure eight made of two homoclinic connections. The study of the splitting in the domains in the parameter space delimited by the pairs of curves $\gamma_i^{\pm}$ in which both hyperbolic structures coexist remains open.

In Fig. 12, we illustrate the set of values of the parameters (μ, ε) for which we can provide results about the exponentially small splitting.

The authors think that this paper shows that the problem of the exponentially small splitting of separatrices is still a problem that is far from being completely understood and bring some new ideas that open new points of view to deal with it.

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References

Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover, New York (1992). Reprint of the 1972 edition

Angenent, S.: A variational interpretation of Mel'nikov's function and exponentially small separatrix splitting. In: Symplectic Geometry. London Math. Soc. Lecture Note Ser., vol. 192, pp. 5–35. Cambridge Univ. Press, Cambridge (1993)

Baldomá, I.: The inner equation for one and a half degrees of freedom rapidly forced Hamiltonian systems. Nonlinearity **19**(6), 1415–1445 (2006)

Baldomá, I., Fontich, E.: Exponentially small splitting of invariant manifolds of parabolic points. Mem. Am. Math. Soc. **167**(792), x+83 (2004)

Baldomá, I., Fontich, E.: Exponentially small splitting of separatrices in a weakly hyperbolic case. J. Differ. Equ. **210**(1), 106–134 (2005)

Baldomá, I., Seara, T.M.: Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity. J. Nonlinear Sci. **16**(6), 543–582 (2006)

Baldomá, I., Seara, T.M.: The inner equation for generic analytic unfoldings of the Hopf-zero singularity. Discrete Contin. Dyn. Syst. Ser. B **10**(2–3), 323–347 (2008)

Balser, W.: From Divergent Power Series to Analytic Functions. Lecture Notes in Mathematics, vol. 1582. Springer, Berlin (1994). Theory and application of multisummable power series

Benseny, A., Olivé, C.: High precision angles between invariant manifolds for rapidly forced Hamiltonian systems. In: Proceedings Equadiff91, pp. 315–319 (1993)

Bonet, C., Sauzin, D., Seara, T., València, M.: Adiabatic invariant of the harmonic oscillator, complex matching and resurgence. SIAM J. Math. Anal. **29**(6), 1335–1360 (1998) (electronic)

Candelpergher, B., Nosmas, J.-C., Pham, F.: Approche de la résurgence. Actualités Mathématiques [Current Mathematical Topics]. Hermann, Paris (1993)

Chierchia, L., Gallavotti, G.: Drift and diffusion in phase space. Ann. Inst. H. Poincaré Phys. Théor. **60**(1), 144 (1994)

Delshams, A., Gutiérrez, P.: Splitting potential and the Poincaré–Melnikov method for whiskered tori in Hamiltonian systems. J. Nonlinear Sci. **10**(4), 433–476 (2000)

Delshams, A., Ramírez-Ros, R.: Melnikov potential for exact symplectic maps. Commun. Math. Phys. **190**(1), 213–245 (1997)

Delshams, A., Ramírez-Ros, R.: Exponentially small splitting of separatrices for perturbed integrable standard-like maps. J. Nonlinear Sci. **8**(3), 317–352 (1998)

Delshams, A., Seara, T.M.: An asymptotic expression for the splitting of separatrices of the rapidly forced pendulum. Commun. Math. Phys. **150**(3), 433–463 (1992)

Delshams, A., Seara, T.M.: Splitting of separatrices in Hamiltonian systems with one and a half degrees of freedom. Math. Phys. Electron. J., 3, Paper 4, 40 pp. (electronic) (1997)

Delshams, A., Gelfreich, V., Jorba, À., Seara, T.M.: Exponentially small splitting of separatrices under fast quasiperiodic forcing. Commun. Math. Phys. **189**(1), 35–71 (1997)

Delshams, A., Gutiérrez, P., Seara, T.M.: Exponentially small splitting for whiskered tori in Hamiltonian systems: flow-box coordinates and upper bounds. Discrete Contin. Dyn. Syst. **11**(4), 785–826 (2004)

Écalle, J.: Les fonctions résurgentes. Tome I. Publications Mathématiques d'Orsay 81 [Mathematical Publications of Orsay 81], vol. 5. Université de Paris-Sud Département de Mathématique, Orsay (1981a). Les algèbres de fonctions résurgentes. [The algebras of resurgent functions], With an English foreword

Écalle, J.: Les fonctions résurgentes. Tome II. Publications Mathématiques d'Orsay 81 [Mathematical Publications of Orsay 81], vol. 6. Université de Paris-Sud Département de Mathématique, Orsay (1981b). Les fonctions résurgentes appliquées à l'itération. [Resurgent functions applied to iteration]

Eliasson, L.H.: Biasymptotic solutions of perturbed integrable Hamiltonian systems. Bol. Soc. Brasil. Mat. (N.S.) **25**(1), 57–76 (1994)

Ellison, J.A., Kummer, M., Sáenz, A.W.: Transcendentally small transversality in the rapidly forced pendulum. J. Dyn. Differ. Equ. **5**(2), 241–277 (1993)

Fiedler, B., Scheurle, J.: Discretization of homoclinic orbits, rapid forcing and "invisible" chaos. Mem. Am. Math. Soc. **119**(570), viii+79 (1996)

Fontich, E.: Exponentially small upper bounds for the splitting of separatrices for high frequency periodic perturbations. Nonlinear Anal. **20**(6), 733–744 (1993)

Fontich, E.: Rapidly forced planar vector fields and splitting of separatrices. J. Differ. Equ. **119**(2), 310–335 (1995)

Fontich, E., Simó, C.: The splitting of separatrices for analytic diffeomorphisms. Ergod. Theory Dyn. Syst. **10**(2), 295–318 (1990)

Gallavotti, G.: Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review. Rev. Math. Phys. **6**(3), 343–411 (1994)

Gallavotti, G., Gentile, G., Mastropietro, V.: Separatrix splitting for systems with three time scales. Commun. Math. Phys. **202**(1), 197–236 (1999)

Gelfreich, V.G.: Separatrices splitting for the rapidly forced pendulum. In: Seminar on Dynamical Systems, St. Petersburg, 1991. Progr. Nonlinear Differential Equations Appl., vol. 12, pp. 47–67. Birkhäuser, Basel (1994)

Gelfreich, V.G.: Melnikov method and exponentially small splitting of separatrices. Physica D **101**(3–4), 227–248 (1997a)

Gelfreich, V.G.: Reference systems for splittings of separatrices. Nonlinearity **10**(1), 175–193 (1997b)

Gelfreich, V.G.: A proof of the exponentially small transversality of the separatrices for the standard map. Commun. Math. Phys. **201**(1), 155–216 (1999)

Gelfreich, V.G.: Separatrix splitting for a high-frequency perturbation of the pendulum. Russ. J. Math. Phys. **7**(1), 48–71 (2000)

Gelfreich, V.G., Naudot, V.: Width of homoclinic zone in the parameter space for quadratic maps. Exp. Math. **18**(4), 409–427 (2009)

Gelfreich, V., Sauzin, D.: Borel summation and splitting of separatrices for the Hénon map. Ann. Inst. Fourier (Grenoble) **51**(2), 513–567 (2001)

Gelfreich, V.G., Lazutkin, V.F., Tabanov, M.B.: Exponentially small splittings in Hamiltonian systems. Chaos **1**(2), 137–142 (1991)

Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer, Berlin (1983)

Holmes, P.J., Marsden, J.E.: A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam. Arch. Ration. Mech. Anal. **76**(2), 135–165 (1981)

Holmes, P.J., Marsden, J.E.: Melnikov's method and Arnol'd diffusion for perturbations of integrable Hamiltonian systems. J. Math. Phys. **23**(4), 669–675 (1982)

Holmes, P.J., Marsden, J.E.: Horseshoes and Arnol'd diffusion for Hamiltonian systems on Lie groups. Indiana Univ. Math. J. **32**(2), 273–309 (1983)

Holmes, P.J., Marsden, J.E., Scheurle, J.: Exponentially small splittings of separatrices with applications to KAM theory and degenerate bifurcations. In: Hamiltonian Dynamical Systems. Contemp. Math., vol. 81 (1988)

Lazutkin, V.F.: Splitting of separatrices for the Chirikov standard map. VINITI 6372/82, 1984. Preprint (Russian)

Lazutkin, V.F.: Splitting of separatrices for the Chirikov standard map. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 300 (Teor. Predst. Din. Sist. Spets. Vyp. 8), 25–55, 285 (2003)

Lochak, P., Marco, J.-P., Sauzin, D.: On the splitting of invariant manifolds in multidimensional near-integrable Hamiltonian systems. Mem. Am. Math. Soc. **163**(775), viii+145 (2003)

Lombardi, E.: Oscillatory Integrals and Phenomena Beyond All Algebraic Orders. Lecture Notes in Mathematics, vol. 1741. Springer, Berlin (2000). With applications to homoclinic orbits in reversible systems

Melnikov, V.K.: On the stability of the center for time periodic perturbations. Trans. Mosc. Math. Soc. **12**, 1–57 (1963)

Neĭshtadt, A.I.: The separation of motions in systems with rapidly rotating phase. Prikl. Mat. Meh. **48**(2), 197–204 (1984)

Olivé, C.: Càlcul de l'escissió de separatrius usant tècniques de matching complex i ressurgència aplicades a l'equació de Hamilton–Jacobi. http://www.tdx.cat/TDX-0917107-125950 (2006)

Olivé, C., Sauzin, D., Seara, T.M.: Resurgence in a Hamilton–Jacobi equation. In: Proceedings of the International Conference in Honor of Frédéric Pham (Nice, 2002), vol. 53(4), pp. 1185–1235 (2003)

Poincaré, H.: Sur le problème des trois corps et les équations de la dynamique. Acta Math. **13**, 1–270 (1890)

Poincaré, H.: Les méthodes nouvelles de la mécanique céleste, vols. 1, 2, 3. Gauthier-Villars, Paris (1892–1899)

Sanders, J.A.: Melnikov's method and averaging. Celest. Mech. **28**(1–2), 171–181 (1982)

Sauzin, D.: Résurgence paramétrique et exponentielle petitesse de l'écart des séparatrices du pendule rapidement forcé. Ann. Inst. Fourier **45**(2), 453–511 (1995)

Sauzin, D.: A new method for measuring the splitting of invariant manifolds. Ann. Sci. Éc. Norm. Super. (4) **34** (2001)

Scheurle, J.: Chaos in a Rapidly Forced Pendulum Equation. Contemp. Math., vol. 97. Am. Math. Soc., Providence (1989)

Scheurle, J., Marsden, J.E., Holmes, P.J.: Exponentially small estimates for separatrix splittings. In: Asymptotics Beyond All Orders, La Jolla, CA, 1991. NATO Adv. Sci. Inst. Ser. B, Phys., vol. 284, pp. 187–195. Plenum, New York (1991)

Seara, T.M., Villanueva, J.: Asymptotic behaviour of the domain of analyticity of invariant curves of the standard map. Nonlinearity **13**(5), 1699–1744 (2000)

Simó, C.: Averaging under fast quasiperiodic forcing. In: Hamiltonian Mechanics, Toruń, 1993. NATO Adv. Sci. Inst. Ser. B, Phys., vol. 331, pp. 13–34. Plenum, New York (1994)

Smale, S.: Diffeomorphisms with many periodic points. In: Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), pp. 63–80. Princeton Univ. Press, Princeton (1965)

Treschev, D.: Hyperbolic tori and asymptotic surfaces in Hamiltonian systems. Russ. J. Math. Phys. **2**(1), 93–110 (1994)

Treschev, D.: Separatrix splitting for a pendulum with rapidly oscillating suspension point. Russ. J. Math. Phys. **5**(1), 63–98 (1997)

Treschev, D.: Width of stochastic layers in near-integrable two-dimensional symplectic maps. Physica D **116**(1–2), 21–43 (1998)